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ON QUASI-GALOIS EXTENSIONS OF DIVISION RINGS

Dedicated to Prof. Kinjiro Kunugi on his 60th birthday

By

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Throughout the present paper, R be always a division ring, and S a division subring of R . And, we use the following conventions: $C = V_R(R)$, $V = V_R(S)$, $H = V_R^2(S) = V_R(V_R(S))$, and further for any subrings $R_1 \supseteq R_2$ of R the set of all R_2 -(ring) isomorphisms of R_1 into R will be denoted as $\Gamma(R_1/R_2)$. As to other notations and terminologies used in this paper, we follow the previous one [3]. We consider here the following conditions:

(I) If S' is a subring of R properly containing S with $[S':S]_l < \infty$ then $\Gamma(S'/S) \neq 1$.

(I₀) If S' is a subring of R properly containing S with $[S':S]_r < \infty$ then $\Gamma(S'/S) \neq 1$.

(I') H/S is Galois.

(II) If $S_1 \supseteq S_2$ are intermediate rings of R/S with $[S_1:S]_l < \infty$ then $\Gamma(S_1/S) |_{S_2} = \Gamma(S_2/S)$.

(II₀) If $S_1 \supseteq S_2$ are intermediate rings of R/S with $[S_1:S]_r < \infty$ then $\Gamma(S_1/S) |_{S_2} = \Gamma(S_2/S)$.

(II') If $T_1 \supseteq T_2$ are intermediate rings of R/H with $[T_1:H]_l < \infty$ then $\Gamma(T_1/S) |_{T_2} = \Gamma(T_2/S)$ ¹⁾.

(II'₀) If $T_1 \supseteq T_2$ are intermediate rings of R/H with $[T_1:H]_r < \infty$ then $\Gamma(T_1/S) |_{T_2} = \Gamma(T_2/S)$.

Following [5], R/S is said to be (*left*-) *quasi-Galois* when (I) and (II) are fulfilled. Symmetrically, if (I₀) and (II₀) are done, we shall say R/S is *right-quasi-Galois*. In [5], we can find some fundamental theorems of quasi-Galois extensions. The purpose of the present paper is to expose several additional theorems concerning such extensions. At first, we shall recall the following lemmas which have been obtained in [4] and [5].

Lemma 1. *If S' is an intermediate ring of R/S then $[V:V_R(S')]_r \leq$*

1) In [5], the condition that if T is an intermediate ring of R/H with $[T:H]_l < \infty$ then $\Gamma(T/S) |_{H} = \Gamma(H/S)$ was cited as (II'). However, it will be rather natural to alter it like above.

$[S':S]_i$, and particularly in case $V_R^2(S)=S$ the equality holds (provided we do not distinguish between two infinite dimensions). If $[S':S]_i < \infty$ then $V_R^2(S')=H[S']$, and if R/S is (left-) locally finite then so is R/H .

Lemma 2. *Let R/S be locally finite. In order that R/S is quasi-Galois it is necessary and sufficient that (I') and (II) are satisfied, and if (I') and (II') are satisfied then R/S is quasi-Galois.*

Lemma 3. *Let R be locally finite and quasi-Galois over S . If T is an intermediate ring of H/S then $T\Gamma(T/S) \subseteq H$, whence it follows $\Gamma(H/S) = \mathfrak{G}(H/S)$.*

Lemma 4. *Let R be locally finite and quasi-Galois over S . If S' is an intermediate ring of R/S with $[S':S]_i < \infty$ then R/S' is quasi-Galois, $V_R^2(S')/S'$ is outer Galois and $\mathfrak{G}(V_R^2(S')/S') \approx \mathfrak{G}(H/H \cap S')$ by contraction, and $\Gamma(V_R^2(S')/S) \upharpoonright S' = \Gamma(S'/S)$.*

By Lemma 4, in the same way as in the proof of [3, Lemma 3.5], we can prove that if R is locally finite and quasi-Galois over S and R' an intermediate ring of R/S with $[H[R']:H]_i < \infty$ then $H[R']$ is locally finite and outer Galois over R' and $\mathfrak{G}(H[R']/R') \approx \mathfrak{G}(H/H \cap R')$ by contraction. Accordingly, we can apply the same argument as in the proof of [4, Lemma 4] to obtain the next

Theorem 1. *Let R be locally finite and quasi-Galois over S . If R' is an intermediate ring of R/S , and H' an intermediate ring of H/S that is Galois over S , then $H'[R']$ is locally finite and outer Galois over R' and $\mathfrak{G}(H'[R']/R') \approx \mathfrak{G}(H'/H' \cap R')$ (algebraically and topologically) by contraction.*

The proof of the next lemma will be easy from that of [3, Lemma 3.2].

Lemma 5. *Let T be an intermediate division ring of R/S , and $\mathfrak{G}$ an automorphism group of $H[T]$. If $J(\mathfrak{G}, H[T])=T$ and $H\mathfrak{G}=H$ then $[H^* \cdot T : H^*]_i^{(2)} = [T : H \cap T]_i$ and $[T \cdot H^* : H^*]_r = [T : H \cap T]_r$ for each intermediate division ring H^* of $H/H \cap T$.*

Now, Lemmas 4 and 5 enable us to apply the argument used in the proof of [3, Lemma 3.2] to obtain the next lemma.

Lemma 6. *Let R be locally finite and quasi-Galois over S . If S' is an intermediate ring of R/S with $[S':S]_i < \infty$ then $[H^*[S'] : H^*]_i = [R^* : H \cap R^*]_i = [S' : H \cap S']_i$ for each intermediate rings H^* of $H/H \cap S'$ and R^* of $H[S']/S'$.*

By the validity of Lemma 6, the proof of the next theorem proceeds evidently just like that of [3, Theorem 3.2] did.

2) $H^* \cdot T$ means the module product of H^* and T .

Theorem 2. *Let R be locally finite and quasi-Galois over S . If T is an f -regular intermediate ring of R/S then $[T:H \cap T]_l = [V:V_R(T)]_r < \infty$.*

Lemma 7. *If R/H is locally finite and R' is an intermediate ring of R/H with $[R':H]_l < \infty$ then R/H is right-locally finite and $[R':H]_r = [R':H]_l$.*

Proof. Although the first assertion is [2, Lemma 4] itself, we shall prove here both. Let X be an arbitrary finite subset of V that is linearly left-independent over $V' = V_R(R')$, and let $R_1 = R'[X]$, that is evidently left-finite over H . We set here $V_1 = V_{R_1}(H)$, $V'_1 = V_{R_1}(R')$, and $C_1 = V_{R_1}(R_1)$. Then, $[V_1:C_1] \leq [R_1:H]_l < \infty$ by Lemma 1, whence it follows $[V_1:V'_1]_l = [V_1:V'_1]_r < \infty$. On the other hand, Lemma 1 yields also $[V_1:V'_1]_r \leq [R':H]_l$, whence we obtain $[V_1:V'_1]_l \leq [R':H]_l$. Recalling here that $X \subseteq V_1$ and $V'_1 \subseteq V'$, we obtain $\#X \leq [V_1:V'_1]_l \leq [R':H]_l$, that is, $[V:V']_l \leq [R':H]_l$. Lemma 1 yields therefore $[R':H]_r = [V:V']_l \leq [R':H]_l$, because $V_R^2(V') = V'$. Now, the right-local finiteness of R/H is evident, and so it follows symmetrically $[R':H]_l \leq [R':H]_r$. We have proved therefore that $[R':H]_r = [R':H]_l$.

The next corollary has been stated in [2, Theorem 2], whose proof was essentially due to [1, Theorem 7.9.2]. However, we have recently found that the proof of [1, Theorem 7.9.2] would be open to doubt—we are afraid that the proof of [1, Theorem 7.8.1] was no longer efficient in that of [1, Theorem 7.9.2]. Because of this reason, we should like to present a new proof without making use of [1, Theorem 7.9.2] to our corollary.

Corollary 1. *Let R be Galois over S and locally finite over H . If S' is an intermediate ring of R/S with $[S':S]_l < \infty$ then $[S':S]_r = [S':S]_l$.*

Proof. At first, if R/S is outer Galois, [3, Lemma 1.3] yields at once $\infty > [S':S]_l = [(\mathbb{G}|S')R_r:R_r]_r^{3)} = [(\mathbb{G}|S')C_r:C_r]_r = [(\mathbb{G}|S')C_l:C_l]_r = [(\mathbb{G}|S')R_l:R_l]_r = [S':S]_r$. Next, for general case, R/S' is Galois by [2, Theorem 1]³⁾ and there holds $\infty > [H[S']:H]_l = [H[S']:H]_r$ by Lemma 7. And so, by Lemmas 1 and 5, we obtain $\infty > [H[S']:H]_r \geq [S' \cdot H:H]_r = [S':H \cap S']_r \geq [V:V_R(S')]_l = [H[S']:H]_r$ and $\infty > [H[S']:H]_l \geq [H \cdot S':H]_l = [S':H \cap S']_l \geq [V:V_R(S')]_r = [H[S']:H]_l$. Accordingly, it follows $[S':H \cap S']_r = [H[S']:H]_r = [H[S']:H]_l = [S':H \cap S']_l < \infty$. Recalling here that H/S is outer Galois, as is noted above, there holds $[H \cap S':S]_r = [H \cap S':S]_l < \infty$. Now, combining these equalities, our assertion $[S':S]_r = [S':S]_l$ will be evident.

Now, we shall prove the next theorem.

Theorem 3. *The following conditions are equivalent to each other:*

3) Since the division ring R is $\mathbb{G}R_r$ -irreducible and $V_{\text{Hom}(R,R)}(\mathbb{G}R_r) = Si$, $\mathbb{G}R_r$ is dense in $\text{Hom}_{S_l}(R, R)$ by JACOBSON's density theorem [1, p. 28].

(1) R/S is locally finite and quasi-Galois, (1₀) R/S is right-locally finite and right-quasi-Galois, (2) R/S is locally finite and (I'), (II) are fulfilled, (2₀) R/S is right-locally finite and (I'), (II₀) are fulfilled, (3) R/S is locally finite and (I'), (II') are fulfilled, and (3₀) R/S is right-locally finite and (I'), (II₀) are fulfilled.

Proof. In virtue of Lemma 2, one will readily see that only the implications (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (1₀) are left to be shown.

(1) $\Rightarrow$ (3). Let $T_1 \supseteq T_2$ be intermediate rings of R/H with $[T_1:H]_l < \infty$. Choose an intermediate ring S'_2 of T_2/S such that $[S'_2:S]_l < \infty$ and $T_2 = H[S'_2]$ and an intermediate ring S_1 of T_1/S'_2 such that $[S_1:S'_2]_l < \infty$ and $T_1 = H[S_1] = V_R^2(S_1)$. If we set $S_2 = T_2 \cap S_1 (\supseteq S'_2)$, then $[S_2:S]_l < \infty$ and $T_2 = H[S_2] = V_R^2(S_2)$ evidently. As R/S_2 is quasi-Galois, $\mathfrak{G}(T_2/S_2) = \mathfrak{G}(T_1/S_1)|_{T_2}$ by Lemma 4. Noting that $\Gamma(T_2/S)|_{S_2} \subseteq \Gamma(S_1/S)|_{S_2} = \Gamma(T_1/S)|_{S_2}$ by Lemma 4, for each $\sigma \in \Gamma(T_2/S)$ we can find some $\rho \in \Gamma(T_1/S)$ such that $\sigma|_{S_2} = \rho|_{S_2}$. By Lemma 3, $T_2\sigma = H[S_2\sigma] \subseteq H[S_1\rho] = T_1\rho$ and $\sigma\rho^{-1} \in \Gamma(T_2/S_2) = \mathfrak{G}(T_2/S_2) = \mathfrak{G}(T_1/S_1)|_{T_2}$. Accordingly, σ is contained in $\Gamma(T_1/S)|_{T_2}$ obviously.

(1) $\Rightarrow$ (1₀). Let S' be an intermediate ring of R/S with $[S':S]_l < \infty$. Since $\mathfrak{G}(H[S']/S') \approx \mathfrak{G}(H/H \cap S')$ by contraction (Lemma 4), Lemmas 1, 5 and 7 yield $[S':H \cap S']_r = [S' \cdot H:H]_r \leq [H[S']:H]_r < \infty$. On the other hand, recalling that H/S is outer Galois by Lemma 2, we obtain $[H \cap S':S]_r = [H \cap S':S]_l < \infty$. (See the proof of Corollary 1.) Combining those, we obtain $[S':S]_r < \infty$, which proves evidently the right-local finiteness of R/S . Now, our assertion will be obvious.

Corollary 2. *Let R be locally finite and quasi-Galois over S . If S' is an intermediate ring of R/S finitely generated over S then $[S':S]_r = [S':S]_l$.*

Proof. As R/H is locally finite by Lemma 1 and R is right-locally finite and right-quasi-Galois over S by Theorem 3, Lemmas 6 and 7 together with their symmetries yield $[S':H \cap S']_l = [H[S']:H]_l = [H[S']:H]_r = [S':H \cap S']_r$. Hence, we readily obtain $[S':S]_r = [S':S]_l$. (Cf. the proof of Corollary 1.)

The following corollary is [3, Corollary 3.5] itself. However, its proof contained a gap. In fact, in order to be able to apply the argument used in the proof of [3, Lemma 3.9], we had to prove the validity of (II'). This fact requested is now secured by Theorem 3.

Corollary 3. *If R is locally finite and quasi-Galois over S and $[R:H]_l \leq \aleph_0$, then R/S is Galois.*

Further, for the sake of completeness, we shall give here the proof of the following theorem [5, Theorem 2].

Theorem 4. *If R/S is locally finite and quasi-Galois then so is R/T for each f -regular intermediate ring T of R/S .*

Proof. Obviously, by Lemma 4, we may restrict our proof to the case that $T \subseteq H$. Let F be an arbitrary finite subset of R , and set $S' = S[F]$, $H' = T[H \cap S']$, $R' = H'[S'] = T[F]$. Then, $[R':H']_i = [S':H \cap S']_i < \infty$ by Lemma 6. On the other hand, noting that H is locally finite and outer Galois over S , there holds $[H':T] < \infty$ by [3, Conclusion 2.1]. Hence, we have $[T[F]:T]_i = [R':H']_i \cdot [H':T] < \infty$, which means evidently the local finiteness of R/T . Moreover, as $V_R^2(T) = H$ and the condition (II') holds by Theorem 3, our assertion is a consequence of Theorem 3.

Lemma 8. *Let R be locally finite and quasi-Galois over S . If T is an f -regular intermediate ring of R/S then $\Gamma(V_R^2(T)/S) \mid T = \Gamma(T/S)$.*

Proof. Take an intermediate ring S' of T/S such that $[S':S]_i < \infty$ and $V_R(S') = V_R(T)$. Then, $T' = V_R^2(T) = V_R^2(S') = H[S']$ and $[T':H]_i < \infty$ by Lemma 1. As $\mathfrak{G}(T'/S') \approx \mathfrak{G}(H/H \cap S')$ by contraction (Lemma 4), [3, Conclusion 2.1] will yield at once $T = (H \cap T)[S']$. Now, let σ be an arbitrary element of $\Gamma(T/S)$. Then, by Lemma 4 $\sigma|S' = \tau|S'$ for some $\tau \in \Gamma(T'/S)$, and by Lemma 3 we see that $T\sigma = ((H \cap T)\sigma)[S'\sigma] \subseteq H[S'\sigma] = H[S'\tau] = T'\tau$. And so, $\sigma\tau^{-1} \in \Gamma(T/S') = \mathfrak{G}(T'/S') \mid T$ by Lemmas 3, 4 and [3, Conclusion 2.1], whence we have $\sigma = (\sigma\tau^{-1})\tau \in \Gamma(T'/S) \mid T$.

By the light of Lemma 8, we can prove the following extension theorem of isomorphisms that corresponds to [3, Theorem 3.5].

Theorem 5. *Let R be locally finite and quasi-Galois over S , and $T_1 \supseteq T_2$ intermediate rings of R/S . If T_1 is f -regular then $\Gamma(T_2/S) = \Gamma(T_1/S) \mid T_2$.*

Proof. Setting here $T'_i = V_R^2(T_i)$ ($i=1, 2$), we have $T'_1 \supseteq T'_2 \supseteq H$, $T'_1 \supseteq T_1 \supseteq T_2$, $[T'_1:H]_i < \infty$ by Lemma 1, and so $\Gamma(T_i/S) = \Gamma(T'_i/S) \mid T_i$ ($i=1, 2$) and $\Gamma(T'_2/S) = \Gamma(T'_1/S) \mid T'_2$ by Lemma 8 and Theorem 3 respectively. It follows therefore $\Gamma(T_2/S) = (\Gamma(T'_1/S) \mid T'_2) \mid T_2 = \Gamma(T'_1/S) \mid T_2 = (\Gamma(T'_1/S) \mid T_1) \mid T_2 = \Gamma(T_1/S) \mid T_2$.

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