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Author(s)	Shimogaki, Tetsuya
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ON AN EQUIVALENCE RELATION ON SEMI-ORDERED LINEAR SPACES

By

Tetsuya SHIMOGAKI

§ 1. Let (E, Ω, μ) be a finite measure space with a countably additive non-negative measure μ defined on a σ -field Ω . Two real-valued μ -measurable functions $f(t)$ and $g(t)$ on E are called *mutually equi-measurable* [14], if $\mu\{t; f(t) > r\} = \mu\{t; g(t) > r\}$ holds for each real number r . If we write $f \sim g$, when f and g are mutually equi-measurable, it is observed easily that the relation $\sim$ is an equivalence relation on the space $\mathfrak{M}$ of all measurable functions on E . As is shown in [14], the concept of equi-measurability plays an important rôle in the theory of functions of real variables. Now let X be a linear space consisting of real-valued measurable functions, which is *semi-normal* in the sense of Nakano [11], i.e.

$$(1.1) \quad 0 \leq f \in X, \quad |g| \leq f, \quad g \in \mathfrak{M} \text{ implies } g \in X,$$

where $0 \leq f$ means that $0 \leq f(t)$ holds almost everywhere. Evidently the function space X is considered as a universally continuous semi-ordered linear space¹⁾ by this order.

We say that a function space X has the *weak rearrangement invariant property* (*w-RIP*), if $f \in X$, $f \sim g$ always implies $g \in X$, i.e. X is closed under the relation defined by equi-measurability. In the sequel, a function space X on E is termed to be a *Banach function space*²⁾ on E , if it is semi-normal and has a complete norm satisfying

$$(1.2) \quad \|f\| = \sup_{\lambda \in \Lambda} \|f_\lambda\|, \quad \text{whenever } 0 \leq f_\lambda \uparrow_{\lambda \in \Lambda} f.$$

A Banach function space X is said to have the *strong rearrangement invariant property* (*s-RIP*), if $f \in X$, $f \sim g$ implies $g \in X$ and $\|g\| \leq A\|f\|$, where A is a fixed constant independent on f and g . $L^p(E)$ spaces with $1 \leq p$, Orlicz spaces $L_\phi(E)$ and $\Lambda(\phi)$ -spaces established by G. G. Lorentz [5, 6] and I. Halperin

1) A semi-ordered linear space R is called *universally continuous*, if $0 \leq a_\lambda$ ($\lambda \in \Lambda$) implies $\bigcap_{\lambda \in \Lambda} a_\lambda \in R$, i.e. a conditionally complete vector lattice in Birkhoff's sense or a K -space in the sense of Vulich [12].

2) For the detailed properties of Banach function spaces see [7] or [13].

[1] independently with much regard to this property, have all s -RIP with the majorant 1 obviously. The subject of this note concerns with RIP of function spaces, but we deal with abstract semi-ordered linear spaces in the first place, since the theory of semi-ordered linear spaces can throw light on this subject by formalization and by use of representation theory of the spaces.

In §2 we generalize axiomatically the relation of equi-measurability on function spaces, to an equivalence relation (called an $\mathcal{E}$ -relation) on abstract semi-ordered linear spaces R . Theorem 1 shows, however, that in case the space R is discrete, the equivalence relation generalized is essentially the same one as is given by equi-measurability on R considered as a discrete measure space. In the next section 3, we treat about a semi-ordered linear space R which has a certain functional ρ together with an $\mathcal{E}$ -relation. Utilizing some topological properties of the proper space $\mathfrak{C}$ of R , we derive a result showing that the functional ρ is uniformly bounded with respect to the $\mathcal{E}$ -relation in a sense (Theorem 1). In §4 we return to function spaces and applying this result, we show that if a Banach function space has w -RIP, then it must have s -RIP, in case E is a non-atomic finite measure space (Theorem 3). Furthermore, as another application of this, we state a theorem characterizing Orlicz spaces among modularized function spaces $L_{M(\xi, \iota)}(E)$ in terms of RIP, i. e. we prove that if a modularized function space $L_{M(\xi, \iota)}(E)$ has w -RIP it reduces to an Orlicz space $L_\phi(E)$ (Theorem 4).

At the end of this paper we extend the equi-measurability relation on finite measure spaces to the relation between two integrable functions on ρ -finite measure spaces. It is then noted that for function spaces on ρ -finite measure spaces, the above results concerning w -RIP and s -RIP hold all to be valid.

§ 2. It will be assumed, in the sequel, that R is a *universally continuous semi-ordered linear space* and S^+ ($S \subset R$) denotes the set of all positive parts of S , i. e. $S^+ = \{x \cup 0; x \in S\}$. A linear lattice manifold M of R is called a P -manifold, if $[p]M \subset M$ for any projector $[p]$ ³⁾ ($p \in R$). A P -manifold M is called *full*, if $M \perp x$ ⁴⁾ implies $x=0$. It is obvious that if M is a full P -manifold, $0 \leq x$ is represented as $x = \bigcup_{\lambda \in A} x_\lambda$, where $x_\lambda \in M$ ($\lambda \in A$). A system $\{x_\lambda\}_{\lambda \in A}$ of elements of M is said to be *M-fundamental with respect to* $x \in R$, if $x = \bigcup_{\lambda \in A} x_\lambda$ and $[p]x = [p]x_\lambda$ holds for each $\lambda \in A$, whenever $[p]x \in M$. Now we introduce an equivalence relation on R which can be considered as a generalization of that of equi-measurability in function spaces.

An equivalence relation $\sim$ on R^+ is called an $\mathcal{E}$ -relation, if it satisfies the

3) A projector $[p]$ is a projection operator on R onto the normal manifold $\{p^\perp\}^\perp$.

4) $M \perp x$ means $|x| \cap |y| = 0$ for all $y \in M$. We write $x = x_1 \oplus x_2$, if $x = x_1 + x_2$ and $x_1 \perp x_2$.

following conditions (R.1)–(R.4):

(R.1) $x \sim y, x, y \in R^+$ implies $\alpha x \sim \alpha y$ for each $\alpha > 0$;

(R.2) if $0 \leq x_\lambda \uparrow_{\lambda \in A}$ and $0 \leq y_\lambda \uparrow_{\lambda \in A}$ and $x_\lambda \sim y_\lambda$ for each $\lambda \in A$, then $\bigcup_{\lambda \in A} x_\lambda \in R^+$ implies $\bigcup_{\lambda \in A} y_\lambda \in R^+$ with $\bigcup_{\lambda \in A} x_\lambda \sim \bigcup_{\lambda \in A} y_\lambda$;

(R.3) if $x = x_1 \oplus x_2, y = y_1 \oplus y_2$ and $x_i \sim y_i$ ($i=1, 2$), then $x \sim y$;

(R.4) there exists a full P -manifold $M \subset R$ satisfying the following properties:

(i) if $x \sim y, x, y \in R^+$, then there exists a pair of M -fundamental systems $\{x_\lambda\}_{\lambda \in A}$ and $\{y_\lambda\}_{\lambda \in A}$ with respect to x and y respectively such that $x_\lambda \sim y_\lambda$ holds for each $\lambda \in A$;

(ii) if $x \sim y, x, y \in M^+$ and $\{[p_\lambda]\}_{\lambda \in A}$ is a mutually orthogonal system of projectors with $\sum_{\lambda \in A} [p_\lambda] = [x]$ there exists also a mutually orthogonal system of projectors $\{[q_\lambda]\}_{\lambda \in A}$ such that

$$\sum_{\lambda \in A} [q_\lambda] = [y] \quad \text{and} \quad [p_\lambda]x \sim [q_\lambda]y \quad (\lambda \in A) \text{ hold.}$$

The P -manifold M satisfying the conditions (i) and (ii) in (R.4) is called the D -manifold of the $\mathcal{E}$ -relation $\sim$.

It is clear that the $\mathcal{E}$ -relation on R^+ can be extended to R canonically, i. e. we now induce the relation $\sim$ to be defined on R in such a way that for any $x, y \in R$

$$(2.1) \quad x \sim y \text{ if and only if } x^+ \sim y^+ \text{ and } x^- \sim y^-.$$

This extended $\mathcal{E}$ -relation $\sim$ is called an $\mathcal{E}$ -relation on R , and R associated with an $\mathcal{E}$ -relation is called shortly a *space with an $\mathcal{E}$ -relation*. It follows from the condition (R.4) that if $x \sim y, x, y \in M$ and $[p]$ is an arbitrary projector, then we can find a projector $[q]$ for which $[p]x \sim [q]y$ and $(1 - [p])x \sim (1 - [q])y$ hold simultaneously, that is, we may say that the $\mathcal{E}$ -relation $\sim$ is decomposable on M .

In what follows, $\sim$ stands for an $\mathcal{E}$ -relation on R always.

Lemma 1. $x \sim 0$ implies $x = 0$.

Proof. If $x \sim 0$ and $x \in R^+$, then $nx \sim 0$ for each natural number n by (R.1). Putting $x_n = 0$ and $y_n = nx$ ($n=1, 2, \dots$), we obtain increasing sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \sim y_n$ ($n \geq 1$). Since $0 = \bigcup_{n=1}^{\infty} x_n \in R^+$, we have $\bigcup_{n=1}^{\infty} y_n \in R^+$ on account of (R.2), which implies $x = 0$, because R is Archimedean⁵⁾. From the

5) Since R is universally continuous, $\bigcap_{n=1}^{\infty} \frac{1}{n} a = 0$ must hold for any $a \in R^+$.

formula (2.1) it is now evident that Lemma 1 holds.

Lemma 2. *If $x \sim \alpha x$ for some $0 < \alpha \neq 1$, then $x \sim 0$.*

Proof. If $0 \leq x \sim \alpha x$ for some α with $1 < \alpha$, then $x \sim \alpha x \sim \alpha^2 x \sim \dots \sim \alpha^n x \sim \dots$. Thus, from (R.2) again we have $\bigcup_{n=1}^{\infty} \alpha^n x \in R$. Hence $x=0$ must hold. On account of (R.1) and (2.1) it is obvious that the lemma holds. Q.E.D.

Lemma 3. *If $x \sim y$ and x is an atomic element,⁶⁾ then y is also such a one.*

Proof. Assume that $x \sim y$, $x, y \in R^+$ and x is an atomic element. If y is decomposed into $y = z_1 \oplus z_2$ with $z_i \neq 0$ ($i=1,2$), then for each M -fundamental system $\{y_\lambda\}_{\lambda \in A}$ with respect to y , we can find an index $\lambda \in A$ with $[z_i]y_\lambda \neq 0$ ($i=1,2$). On the other hand, as M is full, $x \in M$ and also $x \sim y_\lambda = [z_1]y_\lambda \oplus [z_2]y_\lambda$. This implies that one of the elements $[z_i]y_{\lambda(i=1,2)}$, say $[z_1]y_\lambda$, must be equivalent to 0 by virtue of (R.4, (ii)) and the assumption that x is an atomic element. It follows from Lemma 1 that $[z_1]y_\lambda = 0$ and it is a contradiction. Q.E.D.

Lemma 4. *If $x \sim y$ and x is of finite dimension,⁷⁾ then y is also of the same dimension.*

Proof. Since P -manifold M is full, each atomic element, hence each element of finite dimension belongs to M . Now the proof is easily derived similarly from Lemma 3 and (R.4).

Lemma 5. *If $x \sim y$, $x, y \in R$ and $[p]x$ is of finite dimension, then there exists a projector $[q]$ such that $[p]x \sim [q]y$ and $(1-[p])x \sim (1-[q])y$ hold simultaneously.*

Proof. First suppose that $x, y \in R$ and $[p]x$ is an atomic element. Then in view of (R.4), we can find a pair of M -fundamental systems $\{x_\lambda\}_{\lambda \in A}$, $\{y_\lambda\}_{\lambda \in A}$ with respect to x and y and a system of projectors $\{[q_\lambda]\}_{\lambda \in A}$ such that $[q_\lambda] \leq [y_\lambda]$, $[p]x = [p]x_\lambda \sim [q_\lambda]y_\lambda$ and $(1-[p])x_\lambda \sim (1-[q_\lambda])y_\lambda$ hold for all $\lambda \in A$. By Lemma 3 $[q_\lambda]y_\lambda$ is an atomic element, hence $[q_\lambda]y$ is also such a one, and á fortiori $[q_\lambda]y \in M$ and $[q_\lambda]y = [q_\lambda]y_\lambda$ for all $\lambda \in A$. If $[q_\lambda] \neq [q_{\lambda_1}]$ holds for a fixed $\lambda_1 \in A$, we have by (R.3)

$$\begin{aligned} (1-[p])x_\lambda &\sim (1-[q_\lambda])y_\lambda = (1-[q_{\lambda_1}]-[q_\lambda])y_\lambda + [q_{\lambda_1}]y_\lambda \sim \\ &(1-[q_{\lambda_1}]-[q_\lambda])y_\lambda + [q_\lambda]y_\lambda = (1-[q_{\lambda_1}])y_\lambda \end{aligned}$$

and $[p]x_\lambda \sim [q_{\lambda_1}]y_{\lambda_1}$. Consequently both $[p]x \sim [q_{\lambda_1}]y_{\lambda_1}$ and $(1-[p])x_\lambda \sim (1-[q_{\lambda_1}])y_{\lambda_1}$

6) An element $x \in R$ is called an *atomic element*, if $x = y \oplus z$, implies always $y=0$ or $z=0$.

7) An element $x \in R$ is called to be of *finite dimension*, if it is represented as $x = \sum_{\nu=1}^n \xi_\nu e_\nu$, where e_ν is an atomic element for each $1 \leq \nu \leq n$.

$[q_{\lambda_1}]y_{\lambda}$ hold for all $\lambda \in A$. Then from (R.2) it follows that $(1 - [p])x = \bigcup_{\lambda \in A} (1 - [p])x_{\lambda} \sim \bigcup_{\lambda \in A} (1 - [q_{\lambda_1}]y_{\lambda}) = (1 - [q_{\lambda_1}])y$. In case x is n -dimensional, the proof is similarly obtained by use of induction and the condition (R.3). Q. E. D.

Now we establish a theorem which reveals the structure of an $\mathcal{E}$ -relation in the case of discrete spaces. Suppose that R is discrete. Then there exists a mutually orthogonal system of positive atomic elements $\{e_r\}_{r \in \Gamma}$ such that each element $x \in R^+$ is uniquely represented as $x = \sum_{r \in \Gamma} \xi_r e_r$, where $\xi_r \geq 0$. We call the system $\{e_r\}_{r \in \Gamma}$ above the *natural basis* of R . We put further $I(x, \xi) = \{r; r \in \Gamma, \xi_r = \xi\}$. For any subset $J \subset \Gamma$ we denote by $n(J)$ the number of elements belonging to J , that is, $n(J) = k$ ($k = 0, 1, 2, \dots$) or $+\infty$ if J contains an infinite number of elements.

Theorem 1. *Let R be a discrete space with an $\mathcal{E}$ -relation $\sim$. Then there exists a natural basis $\{d_r\}_{r \in \Gamma}$ and a partition of the index set $\Gamma = \sum_{\alpha \in \mathfrak{A}} \Gamma_{\alpha}$, $\Gamma_{\alpha} \cap \Gamma_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$, such that $x = \sum_{r \in \Gamma} \xi_r d_r$ and $y = \sum_{r \in \Gamma} \eta_r d_r$ stand in the relation if and only if $n(\Gamma_{\alpha} \cap I(x, \xi)) = n(\Gamma_{\alpha} \cap I(y, \xi))$ for all real number ξ and $\alpha \in \mathfrak{A}$.*

Proof. Let $\{e_r\}_{r \in \Gamma}$ be an arbitrary natural basis. Since $\sim$ is an equivalence relation, we can classify Γ as $\Gamma = \sum_{\alpha \in \mathfrak{A}} \Gamma_{\alpha}$, $\Gamma_{\alpha} \cap \Gamma_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$ in such a way that for any $r, r' \in \Gamma_{\alpha}$, $e_r \sim \xi e_{r'}$ holds with some $\xi > 0$, and there exists no real number $\xi > 0$ for which $e_r \sim \xi e_{r'}$ holds, whenever r and r' do not belong to the same class. Then, by Choice Axiom, we can find a subsystem $\{e_{r_{\alpha}}\}_{\alpha \in \mathfrak{A}}$ of $\{e_r\}_{r \in \Gamma}$ with $e_{r_{\alpha}} \in \Gamma_{\alpha}$ for every $\alpha \in \mathfrak{A}$. Here for any fixed α , we define d_r ($r \in \Gamma_{\alpha}$) by

$$(2.2) \quad d_r = \xi e_r,$$

where ξ is a positive number satisfying $e_{r_{\alpha}} \sim \xi e_r$. d_r is atomic and uniquely determined for each $r \in \Gamma_{\alpha}$ on account of Lemma 2 and the construction of Γ_{α} . Repeating this process to whole $\alpha \in \mathfrak{A}$, we obtain a natural basis $\{d_r\}_{r \in \Gamma}$ for which $d_r \sim d_{r'}$ stands if and only if r and r' belong to the same class Γ_{α} . Assume now $x = \sum_{r \in \Gamma} \xi_r d_r$, $y = \sum_{r \in \Gamma} \eta_r d_r$ and $x \sim y$. For any fixed $\alpha_0 \in \mathfrak{A}$, $[\{d_{r_1}, d_{r_2}, \dots, d_{r_k}\}]x \in M$ ($r_i \in \Gamma_{\alpha_0}$, $1 \leq i \leq k$; $k = 1, 2, \dots$), and furthermore there exists a collection of elements of Γ_{α_0} : $r'_1, r'_2, \dots, r'_k$ such that both $\sum_{i=1}^k \xi_{r_i} d_{r_i} \sim \sum_{i=1}^k \eta_{r'_i} d_{r'_i}$ and $(1 - [\{d_{r_1}, \dots, d_{r_k}\}])x \sim (1 - [\{d_{r'_1}, \dots, d_{r'_k}\}])y$ hold at the same time in accordance with Lemmas 4 and 5. Then, on account of the construction of the basis $\{d_r\}_{r \in \Gamma}$, we can infer from Lemma 4 that $n(\Gamma_{\alpha_0} \cap I(x, \xi)) = n(\Gamma_{\alpha_0} \cap I(y, \xi))$ holds for each real number ξ and $\alpha \in \mathfrak{A}$. Q. E. D.

Conversely suppose that a partition of Γ exists. We obtain an equivalence relation $\sim$ on R in such a way that $x = \sum_{r \in \Gamma} \xi_r d_r$ and $y = \sum_{r \in \Gamma} \eta_r d_r$ stand in the

relation $\sim$ if and only if $n(\Gamma_\alpha \cap I(x, \xi)) = n(\Gamma_\alpha \cap I(y, \xi))$ for each ξ and $\alpha \in \mathfrak{A}$. For the D -manifold we take a linear subset S of all elements of finite dimension. Now it is evident that the equivalence relation thus defined is an $\mathcal{E}$ -relation with the D -manifold S , i. e. it satisfies the conditions (R. 1)–(R. 4).

§ 3. Here we deal with R which has a certain functional ρ together with an $\mathcal{E}$ -relation $\sim$. The end of this section is to show, in a sense, uniform boundedness of a ρ -functional with respect to an $\mathcal{E}$ -relation (Theorem 2).

A functional ρ defined on R is called a ρ -functional, if it satisfies

$$(\rho. 1) \quad 0 \leq \rho(x) = \rho(|x|) \leq +\infty \quad \text{for all } x \in R;$$

$$(\rho. 2) \quad \rho(x+y) \leq \rho(x) + \rho(y), \quad \text{if } x \perp y;$$

$$(\rho. 3) \quad \inf_{\alpha > 0} \rho(\alpha x) < +\infty \quad \text{for every } x \in R;$$

$$(\rho. 4) \quad 0 \leq x_\lambda \uparrow_{\lambda \in A} x \quad \text{implies } \rho(x) = \sup_{\lambda \in A} \rho(x_\lambda);$$

$$(\rho. 5) \quad \text{if } \{x_\nu\}_{\nu=1}^\infty \text{ is a mutually orthogonal sequence with } \sum_{\nu=1}^\infty \rho(x_\nu) < +\infty, \text{ then } x_0 = \sum_{\nu=1}^\infty x_\nu \text{ belongs to } R.$$

From the definition it follows immediately

$$(3. 1) \quad \rho([p]x) \leq \rho(x) \quad \text{for every } x, p \in R;$$

and

$$(3. 2) \quad [p_\lambda] \uparrow_{\lambda \in A} [p] \quad \text{implies } \rho([p]x) = \sup_{\lambda \in A} \rho([p_\lambda]x).$$

ρ -functionals thus defined are sufficiently general to include known functionals on semi-ordered linear spaces. For instance, the following functionals are all ρ -functionals respectively.

(i) a semi-continuous and complete norm or quasi-norm on R ;

(ii) a monotone complete modular in the sense of Nakano [11] or of Orlicz and Musielak [9], a concave modular of Nakano [10] and a quasi-modular in [2, 3]. We shall establish the following basic result on R with a ρ -functional and an $\mathcal{E}$ -relation:

Theorem 2. *Let ρ -functional be defined on R with an $\mathcal{E}$ -relation. Then there exist positive numbers $\alpha, \gamma, \varepsilon$ and a finite co-dimensional⁸⁾ normal manifold N of R such that*

$$(3. 3) \quad \rho(x) \leq \varepsilon \quad \text{implies } \rho(\alpha y) \leq \gamma$$

8) A linear manifold $N \subset R$ is called a *normal manifold*, if each $x \in R$ is uniquely represented as $x = x_1 + x_2$, $x_1 \in N$ and $x_2 \in N^\perp$. A normal manifold N is called to be finite co-dimensional if $N^\perp$ is of finite dimension.

for any $x, y \in N$ with $x \sim y$.

For the proof of this theorem, we need to prove a number of auxiliary lemmas whose proofs are based on the topological properties of the proper space of semi-ordered linear spaces. In the sequel, $\mathfrak{G}$ denotes the *proper space* of R , i.e. the Boolean lattice of all maximal ideals $\mathfrak{p}$ consisting of normal manifolds $N \subset R$, equipped with the topology generated by the neighbourhood system $\{U_{[N]}\}_{N \subset R}$, where $U_{[N]}$ is the set of all $\mathfrak{p} \in \mathfrak{G}$ such that $N \in \mathfrak{p}$. $U_{[N]}$ is both open and compact in $\mathfrak{G}$ for any normal manifold N , hence $\mathfrak{G}$ is itself compact, because $\mathfrak{G} = U_{[R]}$. An element $\mathfrak{p} \in \mathfrak{G}$ is called *non-atomic*, if for any $N \in \mathfrak{p}$, there exists $M \subset N$ such that $M \in \mathfrak{p}$.

Lemma 6. *Let $x, y \in R$ satisfy $\rho(x) < +\infty$, $\rho(y) < +\infty$ and $x \sim y$. Then for any non-atomic $\mathfrak{p}_0, \mathfrak{p}'_0 \in \mathfrak{G}$ and $\varepsilon > 0$, there exist two elements $x_0, y_0 \in R$ such that (i) $\rho(x_0) > \rho(x) - \varepsilon$, $\rho(y_0) > \rho(y) - \varepsilon$; (ii) $[x_0]R \notin \mathfrak{p}_0$, $[y_0]R \notin \mathfrak{p}'_0$; and (iii) $x_0 \sim y_0$ hold.*

Proof. It is sufficient to prove the lemma, when $[x]R \in \mathfrak{p}_0$ and $[y]R \in \mathfrak{p}'_0$. Assume $x \sim y$, $x, y \in R^+$ and $\varepsilon > 0$. In view of the conditions (R.4) and (ρ .4), there exist elements x', y' of the D -manifold M such that $0 \leq x' \leq x$, $0 \leq y' \leq y$ and $\rho(x') > \rho(x) - \varepsilon$, $\rho(y') > \rho(y) - \varepsilon$ with $x' \sim y'$. Since $\mathfrak{p}_0$ is non-atomic, we can find a system of mutually orthogonal projectors $\{[p_r]\}_{r \in \Gamma}$ with $\bigcup_{r \in \Gamma} [p_r] = [x']$ and $[p_r]R \notin \mathfrak{p}_0$ ($r \in \Gamma$). On account of (R.4 (ii)), there exists an orthogonal system $\{[q_r]\}_{r \in \Gamma}$ such that $\bigcup_{r \in \Gamma} [q_r] = [y']$ and $[p_r]x' \sim [q_r]y'$ ($r \in \Gamma$) hold. By virtue of (ρ .4) we have for suitable chosen $\gamma_1, \gamma_2, \dots, \gamma_k$ ($\gamma_i \in \Gamma$)

$$\rho\left(\sum_{i=1}^k [p_{\gamma_i}]x'\right) > \rho(x) - \varepsilon \text{ and } \rho\left(\sum_{i=1}^k [q_{\gamma_i}]y'\right) > \rho(y) - \varepsilon.$$

Putting $x'' = \sum_{i=1}^k [p_{\gamma_i}]x'$ and $y'' = \sum_{i=1}^k [q_{\gamma_i}]y'$, we now have by (R.3)

$$x'' \sim y'', \quad \rho(x'') > \rho(x) - \varepsilon, \quad \rho(y'') > \rho(y) - \varepsilon$$

and $[x'']R \notin \mathfrak{p}_0$. If $[y'']R \in \mathfrak{p}'_0$, then applying the quite same argument (only changing the rôle of x' and y' into y'' and x'' respectively), we can show that we get two elements x_0, y_0 which fulfil the requirement of Lemma 6. Q.E.D.

Lemma 7. *For any non-atomic $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{G}$ there exist normal manifolds $N_{\mathfrak{p}} \in \mathfrak{p}$ and $N_{\mathfrak{p}'} \in \mathfrak{p}'$ and positive numbers $\alpha, \gamma, \varepsilon > 0$ such that $x \in N_{\mathfrak{p}}$, $\rho(x) \leq \varepsilon$ implies*

$$(3.3) \quad \rho(\alpha y) \leq \gamma \text{ for each } y \in N_{\mathfrak{p}'} \text{ with } x \sim y.$$

Proof. Assume that the lemma is not valid. Then for a pair of non-

atomic maximal ideals $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{G}$, there exists no pair of normal manifolds $(N_{\mathfrak{p}}, N_{\mathfrak{p}'})$ and positive numbers α, γ and ε which satisfies (3.3) above. Now we can start with a pair of elements (x', y') with $x' \sim y', \rho(x') < \frac{1}{2}, \rho(y') > 1, [x']R \in \mathfrak{p}$ and $[y']R \in \mathfrak{p}'$. From Lemma 6 it follows that there exist elements x_1, y_1 ($x_1, y_1 \in M$) such that $\rho(x_1) < \frac{1}{2}, \rho(y_1) > 1, [x_1]R \notin \mathfrak{p}, [y_1]R \notin \mathfrak{p}'$ and $x_1 \sim y_1$. Since $\mathfrak{p}$ and $\mathfrak{p}'$ are maximal ideals of normal manifolds of R , $(1 - [x_1])R \in \mathfrak{p}$ and $(1 - [y_1])R \in \mathfrak{p}'$ stand. Again we can find also $x'', y'' \in R$ with $x'' \sim y'', x'' \in (1 - [x_1])R$ and $y'' \in (1 - [y_1])R$ satisfying $\rho(x'') < \frac{1}{2^2}$ together with $\rho\left(\frac{1}{2}y''\right) > 2$ by the assumption. In view of Lemma 6 again, there exists a pair of elements (x_2, y_2) such that $\rho(x_2) < \frac{1}{2^2}, \rho\left(\frac{1}{2}y_2\right) > 2, [x_2]R \notin \mathfrak{p}, [y_2]R \notin \mathfrak{p}'$ and $x_2 \sim y_2$. Proceeding this argument, we obtain two sequences of mutually orthogonal positive elements $\{x_\nu\}_{\nu=1}^\infty$ and $\{y_\nu\}_{\nu=1}^\infty$, for which $x_\nu \sim y_\nu, [x_\nu]R \notin \mathfrak{p}, [y_\nu]R \notin \mathfrak{p}', \rho(x_\nu) \leq \frac{1}{2^\nu}$ and $\rho\left(\frac{1}{\nu}y_\nu\right) \geq \nu$ hold for each $\nu \geq 1$. From (p. 4) it follows

$$\bigcup_{\nu=1}^{\infty} x_\nu \in R,$$

which implies $\bigcup_{\nu=1}^{\infty} y_\nu \in R$ on account of (R.2). This is, however, a contradiction, since $\rho\left(\frac{1}{n} \bigcup_{\nu=1}^{\infty} y_\nu\right) \geq \rho\left(\frac{1}{n}y_n\right) \geq n$ holds and it is inconsistent with (p. 3).

Q. E. D.

Lemma 8. *For any non-atomic $\mathfrak{p}_0 \in \mathfrak{G}$, there exists a finite number of normal manifolds $N_0, N_1, \dots, N_k, N'$ such that $N_0 \in \mathfrak{p}_0, R = N_1 \oplus N_2 \oplus \dots \oplus N_k \oplus N', N'$ is of finite dimension and for any $x \in N_0$ with $\rho(x) \leq \varepsilon$*

$$(3.4) \quad \text{Max} \left\{ \sup_{1 \leq i \leq k} \sup_{x \sim y, y \in N_i} \rho(\alpha y) \right\} \leq \gamma$$

holds, where α, γ and ε are all fixed positive constants.

Proof. Let $N_{\mathfrak{p}_0, \mathfrak{p}}$ and $N_{\mathfrak{p}}$ be two normal manifolds and $\alpha_{\mathfrak{p}}, \gamma_{\mathfrak{p}}$ and $\varepsilon_{\mathfrak{p}} > 0$ be positive numbers which satisfy the formula (3.3) corresponding to non-atomic maximal ideals $\mathfrak{p}_0$ and $\mathfrak{p}$. Let $\mathfrak{G}$ denote the set of all non-atomic elements of $\mathfrak{G}$. As the set $(\sum_{\mathfrak{p} \in \mathfrak{G}} U_{[N_{\mathfrak{p}}]})^-$ is both open and compact in $\mathfrak{G}$, $(\sum_{\mathfrak{p} \in \mathfrak{G}} U_{[N_{\mathfrak{p}}]})^- = U_{[N]}$ holds for a normal manifold $N \subset R$ and clearly $(1 - [N])R$ is of finite dimension.

On the other hand, if $\mathfrak{p}$ belongs to the set $U_{[N]} - \sum_{\mathfrak{p} \in \mathfrak{G}} U_{[N_{\mathfrak{p}}]}$ it must be non-

atomic as easily seen, whence it follows $U_{[N]} = \sum_{p \in \mathfrak{C}} U_{[N_p]}$. Thus we can find a finite number of $p \in \mathfrak{C}$, say $p_1, p_2, \dots, p_k$, such that $U_{[N]} = \sum_{\nu=1}^k U_{[N_{p_\nu}]}$ holds. Now we put $N_0 = \bigcap_{\nu=1}^k N_{p_\nu}$, $N' = (1 - [N])R$, $\varepsilon = \text{Min}_{1 \leq \nu \leq k} \{\varepsilon_{p_\nu}\}$, $\alpha = \text{Min}_{1 \leq \nu \leq k} \{\alpha_{p_\nu}\}$ and $\gamma = \text{Max}_{1 \leq \nu \leq k} \{\gamma_{p_\nu}\}$, and also we choose a set of mutually orthogonal normal manifolds: $\{N_\nu\}_{\nu=1}^{k'}$ such that $N = N_1 \oplus N_2 \oplus \dots \oplus N_{k'}$ and $N_\nu \subset N_{p_\nu}$ for each ν with $1 \leq \nu \leq k' \leq k$, by use of the usual orthogonalization method. It is now clear that (3.4) is valid for normal manifolds and the positive numbers thus constructed.

Q. E. D.

Lemma 9. For any non-atomic $p_0 \in \mathfrak{C}$ there exists a normal manifold N_{p_0} , a finite co-dimensional normal manifold N'_{p_0} and positive numbers $\alpha, \gamma, \varepsilon$ such that $x \in N_{p_0}$, $\rho(x) \leq \varepsilon$ implies

$$(3.5) \quad \sup_{x \sim y, y \in N'_{p_0}} \rho(\alpha y) \leq \gamma'.$$

Proof. Let $N_0, N_1, \dots, N_k, N'$ be normal manifolds and α, γ and ε be positive numbers which satisfy (3.4) in the preceding lemma. Suppose $x \sim y$, $x \in [N_0]M$, $y \in [N']M$ and $\rho(x) \leq \varepsilon$. Then, on account of (R.4), we can find a mutually orthogonal set of projectors $\{[p_\nu]\}_{\nu=1}^k$ for which $[p_\nu]x \sim [N_\nu]y$ holds ($1 \leq \nu \leq k$). Since $\rho([p_\nu]x) \leq \varepsilon$ for all $1 \leq \nu \leq k$, we have $\rho(\alpha[N_\nu]y) \leq \gamma$ and

$$\rho(\alpha y) \leq \sum_{\nu=1}^k \rho([N_\nu]y) \leq k \cdot \gamma.$$

Since ρ is semi-continuous and M is full, putting $\gamma' = k\gamma$ and $N'_{p_0} = (N')^\perp$, we obtain the proof.

Q. E. D.

Proof of Theorem 2. For any $p \in \mathfrak{C}$, we denote by $N_p, N'_p, \alpha_p, \gamma_p$ and ε_p be the same as in Lemma 9, corresponding with p . Then, as above, there exists a finite co-dimensional normal manifold N_0 satisfying $U_{[N_0]} = \sum_{p \in \mathfrak{C}} U_{[N_p]}$. Hence we can find a finite number of N_{p_ν} , $p_\nu \in \mathfrak{C}$ ($\nu = 1, 2, \dots, k$) for which $U_{[N_0]} = \sum_{\nu=1}^k U_{[N_{p_\nu}]}$ hold. Now we put $N = N_0 \cap \bigcap_{\nu=1}^k N'_{p_\nu}$, $\alpha = \text{Min}_{1 \leq \nu \leq k} \{\alpha_{p_\nu}\}$, $\gamma = \sum_{\nu=1}^k \gamma_{p_\nu}$ and $\varepsilon = \text{Min}_{1 \leq \nu \leq k} \{\varepsilon_{p_\nu}\}$ respectively. It is evident that N is a finite co-dimensional normal manifold, and we can find again normal manifolds $M_1, M_2, \dots, M_{k'}$ such that $N = M_1 \oplus M_2 \oplus \dots \oplus M_{k'}$, $M_\nu \subset N_{p_\nu}$ and $M_\nu \cap M_\mu = \{0\}$ for $\nu \neq \mu$. Now suppose that $x \sim y$, $x, y \in [N]M$ and $\rho(x) \leq \varepsilon$. Since $x = \sum_{\nu=1}^{k'} [M_\nu]x$ and $[M_\nu]x \in N_{p_\nu}$, there exists mutually orthogonal projectors $\{[p_\nu]\}_{\nu=1}^{k'}$ such that $[M_\nu]x \sim [p_\nu]y$, $\sum_{\nu=1}^{k'} [p_\nu] = [y]$. As $[M_\nu]x \in N_{p_\nu}$ and $[p_\nu]y \in N'_{p_\nu}$, we have by the preceding lemma

$$\rho(\alpha[p_\nu]y) \leq \rho(\alpha_{p_\nu}[p_\nu]y) \leq r_{p_\nu} \quad (1 \leq \nu \leq k').$$

Therefore we get

$$\rho(\alpha y) \leq \sum_{\nu=1}^{k'} \rho(\alpha_{p_\nu}[p_\nu]y) \leq \sum_{\nu=1}^{k'} r_{p_\nu} \leq r,$$

which implies (3.3), because of the semi-continuity of ρ .

Q. E. D.

Remark 1. If R is non-atomic,⁹⁾ there is no finite dimensional normal manifold. Hence the formula (3.3) in Theorem 2 holds valid in the whole space R in this case.

Corollary 1. Let R have a complete semi-continuous¹⁰⁾ norm $\|\cdot\|$ together with an $\mathcal{E}$ -relation. Then there exists a positive number γ such that

$$(3.6) \quad \frac{1}{\gamma} \|y\| \leq \|x\| \leq \gamma \|y\|$$

holds for each pair of elements $x, y \in R$ with $x \sim y$.

Proof. A complete semi-continuous norm $\|\cdot\|$ is a ρ -functional, hence by virtue of Theorem 2 there exist a finite co-dimensional normal manifold N and a positive number γ_1 , for which $\frac{1}{\gamma_1} \|y\| \leq \|x\| \leq \gamma_1 \|y\|$ holds for $x, y \in N$ with $x \sim y$. On the other hand, $N^\perp$ being of finite dimension, there exists also $\gamma_2 > 0$ such that $\frac{1}{\gamma_2} \|y\| \leq \|x\| \leq \gamma_2 \|y\|$ holds for $x, y \in N^\perp$ with $x \sim y$. Let $\{e_1, \dots, e_n\}$ be a natural basis of $N^\perp$ with $\|e_\nu\| = 1$ ($1 \leq \nu \leq n$). We put now $\alpha_\nu = \inf_{x \in N, x \sim e_\nu} \|x\|$, and $\beta_\nu = \sup_{x \in N, x \sim e_\nu} \|x\|$. It is evident that both $\alpha_\nu > 0$ and $\beta_\nu < +\infty$ hold for all ν ($1 \leq \nu \leq n$) from above. If $x \sim y$, $x \in N^\perp$ and $y \in N$, we can verify easily that $\frac{1}{\gamma_3} \|y\| \leq \|x\| \leq \gamma_3 \|y\|$ holds, where $\gamma_3 = n \cdot \max_{1 \leq \nu \leq n} \left\{ \frac{1}{\alpha_\nu}, \beta_\nu \right\}$. From these facts it follows immediately that there exists $\gamma > 0$ which satisfies (3.6) in the whole space.

Q. E. D.

Corollary 2. Let R be a modular (quasi-modular) semi-ordered linear space with a monotone complete modular¹¹⁾ (quasi-modular) m . If R is non-atomic and has an $\mathcal{E}$ -relation, then we can find positive numbers α , γ' and ε such that

9) R is called to be non-atomic, if R has no atomic element.

10) A norm $\|\cdot\|$ on R is called semi-continuous, if $0 \leq x_\lambda \uparrow x$ implies $\|x\| = \sup_{\lambda \in A} \|x_\lambda\|$.

11) For the definition of a modular see [11]. Here we use the term of modular in the sense of Nakano.

$$(3.7) \quad m(x) > \varepsilon \quad \text{implies} \quad m(\alpha y) \leq \gamma' m(x)$$

for any pair of elements $x, y \in R$ with $x \sim y$.

Proof. Since a monotone complete modular (or quasi-modular) m satisfies the conditions (ρ.1)–(ρ.5) [11, 3], it is a ρ -functional. Thus there exist positive numbers α , γ and ε such that $m(x) \leq \varepsilon$, $x \sim y$ yields $m(\alpha x) \leq \gamma$. If $x \sim y$, $x, y \in M$ and $m(x) > \varepsilon$, then we can find two sets of mutually orthogonal projectors $\{[p_\nu]\}_{\nu=1}^{k+1}$, $\{[q_\nu]\}_{\nu=1}^{k+1}$ such that $\sum_{\nu=1}^{k+1} [p_\nu] = [x]$, $\sum_{\nu=1}^{k+1} [q_\nu] = [y]$, $[p_\nu]x \sim [q_\nu]y$ ($1 \leq \nu \leq k+1$), $m([p_\nu]x) = \varepsilon$ ($1 \leq \nu \leq k$) and $m([p_{k+1}]x) < \varepsilon$ on account of the non-atomicity of R and the condition (R.4, (ii)). Hence we get

$$m(\alpha y) = \sum_{\nu=1}^{k+1} m(\alpha [q_\nu]y) \leq \gamma(k+1) \leq 2 \frac{\gamma}{\varepsilon} m(x),$$

which yields (3.7), since a modular (or quasi-modular) is semi-continuous.

Q. E. D.

§ 4. Throughout this section let E be a non-atomic finite measure space and $\mathbf{X}(E)$ be a Banach function space with a semi-continuous norm $\|\cdot\|$. It is well known that $\mathbf{X}$ constitutes a superuniversally continuous semi-ordered linear space¹²⁾ by the usual order and addition of measurable functions. When $\mathbf{X}$ has w -RIP, the relation of equi-measurability between two functions belonging to $\mathbf{X}$ can be regarded as an $\mathcal{E}$ -relation on the space $\mathbf{X}$. Indeed, the conditions (R.1) and (R.3) are evidently satisfied. Since μ is assumed to be countably additive and $\mathbf{X}$ has w -RIP, the condition (R.2) is fulfilled. As a D -manifold M , we can take the set of all simple functions¹³⁾ on E and it is now clear that the relation of equi-measurability satisfies also (R.4), hence an $\mathcal{E}$ -relation on $\mathbf{X}$.

Consequently, in view of Corollary 1 we have

Theorem 3. *In order that a Banach function space $\mathbf{X}(E)$ on a finite non-atomic measure space E has w -RIP, it is necessary and sufficient that $\mathbf{X}(E)$ has s -RIP, that is,*

$$(4.1) \quad \frac{1}{\gamma} \|g\| \leq \|f\| \leq \gamma \|g\|$$

for any two mutually equi-measurable functions f and $g \in \mathbf{X}$, where γ is a

12) R is called *superuniversally continuous*, if for any system of positive elements $\{a_\lambda\}_{\lambda \in A}$ there exists a sequence of elements: $\{a_{\lambda_\nu}\}_{\nu=1}^\infty \subset \{a_\lambda\}_{\lambda \in A}$ such that $\bigcap_{\nu=1}^\infty a_{\lambda_\nu} = \bigcap_{\lambda \in A} a_\lambda$ holds.

13) A function on E is called a *simple function* if it is represented as $\sum_{\nu=1}^n \xi_\nu x_{e_\nu}$ where x_{e_ν} is the characteristic function of a measurable set $e_\nu \subset E$ for each ν with $1 \leq \nu \leq n$.

positive number.

Corollary 3. *If a Banach function space $\mathbf{X}(E)$ has w -RIP, there exists an equivalent norm $\|\cdot\|_1$ on $\mathbf{X}(E)$ having the rearrangement majorant 1, i. e. $\|f\|_1 = \|g\|_1$ for $f, g \in \mathbf{X}$ with $f \sim g$.*

Proof. On account of Theorem above, we can define a finite valued functional $\|\cdot\|_1$ as

$$(4.2) \quad \|f\|_1 = \sup_{f \sim g} \|g\| \quad (f \in \mathbf{X}).$$

It is now evident from the definition that the functional $\|\cdot\|_1$ satisfies all the conditions of semi-continuous norm except for the subadditivity. For any simple functions $f, g: f = \sum_{\nu=1}^k \xi_\nu \chi_{e_\nu}, g = \sum_{\nu=1}^k \eta_\nu \chi_{e_\nu}$ with $e_\nu \cap e_\mu = \emptyset$ for $\nu \neq \mu$, we have

$$\begin{aligned} \|f+g\|_1 &= \sup_{e'_1 \oplus \dots \oplus e'_k = E, \mu(e_i) = \mu(e'_i)} \left\| \sum_{\nu=1}^k (\xi_\nu + \eta_\nu) \chi_{e'_\nu} \right\|. \\ &\quad (1 \leq \nu \leq k) \end{aligned}$$

Let h be a simple function such that $h = \sum_{\nu=1}^k (\xi_\nu + \eta_\nu) \chi_{e'_\nu}$ and $\mu(e'_\nu) = \mu(e_\nu), e'_\nu \cap e'_\mu = \emptyset$ for $\nu \neq \mu$. Then $|h| \leq \sum_{\nu=1}^k |\xi_\nu| \chi_{e'_\nu} + \sum_{\nu=1}^k |\eta_\nu| \chi_{e'_\nu}$ and $|f| \sim \sum_{\nu=1}^k |\xi_\nu| \chi_{e'_\nu}, |g| \sim \sum_{\nu=1}^k |\eta_\nu| \chi_{e'_\nu}$, which implies

$$\|h\|_1 \leq \|f\|_1 + \|g\|_1.$$

Consequently we have $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$ for arbitrary $f, g \in \mathbf{X}$ on account of the semi-continuity of $\|\cdot\|_1$. Q. E. D.

Next we turn to prove a theorem concerning characterization of Orlicz spaces among the classes of modularized function spaces.

Now let $M(\xi, t)$ be a *modular function*, i. e. $M(\xi, t)$ be a real-valued function on $[0, +\infty) \times E$ satisfying (i) it is a non-decreasing convex function of $\xi \geq 0$ which is left hand continuous for each $t \in E$; (ii) it is measurable on E for each $\xi \geq 0$; (iii) $\lim_{\xi \rightarrow 0} M(\xi, t) = 0, \sup_{0 \leq \xi} M(\xi, t) = +\infty$ and $M(0, t) = 0$ for all $t \in E$. Then a *modularized function space* $\mathbf{L}_{M(\xi, t)}$ is the set of all measurable functions f on E such that $\int_E M(\xi |f(t)|, t) d\mu(t) < +\infty$ for some $\xi > 0$. $\mathbf{L}_{M(\xi, t)}$ is a modularized space with the modular m :

$$(4.3) \quad m(f) = \int_E M(|f(t)|, t) d\mu(t) \quad (f \in \mathbf{L}_{M(\xi, t)}),$$

hence, as is well known, it is a Banach function space with the norm:

$\|f\| = \inf_{m(\xi f) \leq 1} \frac{1}{|\xi|} \quad (f \in L_{M(\xi, t)}).$ Evidently Orlicz spaces¹⁴⁾ constitute a special class in the modularized function spaces.

Now we have

Theorem 4. *If a modularized function space $L_{M(\xi, t)}(E)$ on a non-atomic finite measure space E has w -RIP, then it reduces to an Orlicz space $L_\phi(E)$.*

Proof. It is obvious that we may assume $\mu(E)=1$, without loss of generality. Putting $\Phi(\xi) = m(\xi \chi_E)$ for $\xi \geq 0$, we obtain a non-decreasing left hand continuous convex function $\Phi(\xi)$ on $[0, +\infty)$ satisfying $\lim_{\xi \rightarrow +\infty} \Phi(\xi) = +\infty$, $\inf_{0 < \xi} \Phi(\xi) = 0$ and $\Phi(0)=0$. Here we shall show that $L_{M(\xi, t)}$ coincides with the Orlicz space¹⁵⁾ L_ϕ defined by the function Φ as a Banach function space.

By virtue of Theorem 2 and Corollary 2 (3.7), we can find positive numbers γ_1, γ_2 and α satisfying both the conditions:

- (i) $m(f) \leq \varepsilon, f \sim g$ implies $m(\alpha g) \leq \gamma_1$;
- (ii) $m(f) > \varepsilon, f \sim g$ implies $m(\alpha g) \leq \gamma_2 m(f)$.

We put further for any $f \in L_{M(\xi, t)}$

$$m^*(f) = \sup_{f \sim g} m(g) \quad \text{and} \quad m_*(f) = \inf_{f \sim g} m(g).$$

It follows from above that for each $f \in L_{M(\xi, t)}$

$$(4.4) \quad m_*(\alpha f) \leq m(\alpha f) \leq m^*(\alpha f) \leq \gamma_2 m_*(f) + \gamma_1$$

holds. Let $\mathfrak{M}_0$ be the set of all simple functions $h = \sum_{\nu=1}^k \xi_\nu \chi_{e_\nu}$ such that

$$(4.5) \quad e_\nu \wedge e_\mu = \emptyset \quad \text{for } \nu \neq \mu, \quad E = \sum_{\nu=1}^k e_\nu \quad \text{and} \quad \mu(e_\nu) = \frac{1}{k} \quad \text{for all } \nu \geq 1.$$

Here we denote by $P_n(\nu)$ a permutation of the set: $\{1, 2, \dots, k\}$ defined by $P_n(\nu) = \nu + n \pmod{k}$ for each n . Then, for any $h \in \mathfrak{M}_0$ we put $h^{(n)} = \sum_{\nu=1}^k \xi_\nu \chi_{e_\nu^n}$ ($0 \leq n \leq k-1$), where $e_\nu^n = e_{P_n(\nu)}$. Evidently we have $h = h^{(0)} \sim h^{(1)} \sim \dots \sim h^{(k-1)}$ and $\sum_{n=0}^{k-1} m(h^{(n)}) = \sum_{n=0}^{k-1} \sum_{\nu=1}^k m(\xi_\nu \chi_{e_\nu^n}) = \sum_{\nu=1}^k \sum_{n=0}^{k-1} m(\xi_\nu \chi_{e_\nu^n}) = \sum_{\nu=1}^k m(\xi_\nu \chi_E) = \sum_{\nu=1}^k \Phi(\xi_\nu) = k \cdot m_\phi(h)$. Therefore there exists at least a pair of integers (m_0, n_0) ($0 \leq m_0, n_0 \leq k-1$) such that

$$m(h^{(m_0)}) \leq m_\phi(h) \leq m(h^{(n_0)}),$$

14) For the details of Orlicz spaces see [4], [7] or [13].

15) $m_\phi(f)$ denotes the modular of the space L_ϕ , i.e. for $f \in L_\phi$ $m_\phi(f) = \int_E \Phi(|f(t)|) d\mu(t)$.

Since $L_{M(\xi, t)}$ has w -RIP, $1 \in L_{M(\xi, t)}$.

which implies

$$m_*(h) \leq m_\phi(h) \leq m^*(h).$$

From this and (4.4) it follows that

$$(4.6) \quad m(\alpha h) \leq \gamma_2 m_\phi(h) + \gamma_1 \quad \text{and} \quad m_\phi(\alpha h) \leq \gamma_2 m(h) + \gamma_1.$$

Since E is non-atomic, for any $f \in L_{M(\xi, t)}$ there exists a sequence $\{h_n\}_{n=1}^\infty$ of elements of $\mathfrak{M}_0$ such that $h_n \uparrow_{n=1}^\infty |f|$ holds. Consequently, by the semi-continuity of m and m_ϕ , (4.6) implies

$$(4.7) \quad m(\alpha f) \leq \gamma_2 m_\phi(f) + \gamma_1 \quad \text{and} \quad m_\phi(\alpha f) \leq \gamma_2 m(f) + \gamma_1$$

for any $f \in L_{M(\xi, t)}$. It is now evident that the Banach function spaces $L_{M(\xi, t)}$ and L_ϕ coincide. Q. E. D.

Remark 2. As this proof shows, the convexity of modular m and m_ϕ is not used. Therefore, it is verified in the quite same way, that if a (non-convex) quasi-modular function space $L_{N(\xi, t)}$ [2] has w -RIP, then it reduces to a generalized Orlicz space L_N considered by S. Mazur and W. Orlicz in [8].

Lastly let E be a σ -finite (or locally finite) measure space with a countably additive measure μ . The relation defined by equi-measurability has essentially the sense on the set of finite measure only, in fact, it can not be extended naturally to the whole space of all measurable functions on E without loss of the original significance. Only we can define an equivalence relation $\sim$ on the set $\mathfrak{F}$ of all integrable functions on E in the following way. Two positive functions f, g belonging to $\mathfrak{F}$ are called equi-measurable if $\mu\{t; f(t) > r\} = \mu\{t; g(t) > r\}$ holds for every positive number r . Next two functions f, g of $\mathfrak{F}$ is called equi-measurable (in the extended sense) and written as $f \sim g$, if both f^+ and $f^{-16)}$ are equi-measurable to g^+ and g^- respectively. Then the relation $\sim$ comes to be an equivalence relation on the space $\mathfrak{F}$. Thus, if a Banach function space X consisting of integrable functions on E has w -RIP with respect to the relation $\sim$ of equi-measurability in the extended sense, the relation $\sim$ is an $\mathcal{E}$ -relation on X as is easily seen. Hence, on account of Theorem 2, we have as similarly as Theorem 3

Theorem 3'. *If a Banach function space X consisting of integrable functions on a σ -finite (or locally finite) measure space E has w -RIP, then it has s -RIP.*

We obtain also

Theorem 4'. *Let $L_{M(\xi, t)}(E)$ be a modular function space consisting*

16) $f^+(t) = \text{Max}(f(t), 0)$ and $f^-(t) = \text{Max}(-f(t), 0)$ for all $t \in E$.

of integrable functions on a non-atomic σ -finite measure space E . If $L_{M(\xi, t)}$ has w -RIP, then it reduces to an Orlicz space L_ϕ .

Proof. Let $\{E_\nu\}_{\nu=1}^\infty$ be a sequence of measurable sets of finite measure such that $E_\nu \uparrow_{\nu=1}^\infty E$ holds. Now we put

$$\Phi^*(\xi) = \sup_{0 \leq \gamma < \xi} \lim_{\nu \rightarrow \infty} \frac{m(\gamma \chi_{E_\nu})}{\mu(E_\nu)} \quad \text{and} \quad \Phi_*(\xi) = \lim_{\nu \rightarrow \infty} \frac{m(\xi \chi_{E_\nu})}{\mu(E_\nu)}.$$

Then, by virtue of Corollary 2 in §3 and the non-atomicity of E , we can find positive numbers α and γ for which $\Phi_*(\alpha\xi) \leq \Phi^*(\alpha\xi) \leq \gamma\Phi_*(\xi)$ holds for each $\xi \geq 0$. From this we can verify as similarly as in Theorem 4 that $L_{M(\xi, t)}$ coincides with the Orlicz space L_{ϕ^*} . Q. E. D.

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Department of Mathematics,
Hokkaido University

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