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# ON THE DISTRIBUTION OF ALMOST PRIMES IN AN ARITHMETIC PROGRESSION

By

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**1. Introduction.** An almost prime is a positive integer the number of whose prime divisors is bounded by a certain constant. The purpose of this paper is to deal with an existence problem of almost primes in a short arithmetic progression of integers. We shall prove the following

**Theorem.** *Let  $k$  and  $l$  be two integers with  $k \geq 1$ ,  $0 \leq l \leq k-1$ ,  $(k, l) = 1$ . There exists a numerical constant  $c_1 > 0$  such that for every real number  $x \geq c_1 k^{3.5}$  there is at least one integer  $n$  satisfying*

$$x < n \leq 2x, \quad n \equiv l \pmod{k}, \quad V(n) \leq 2,$$

where  $V(n)$  denotes the total number of prime divisors of  $n$ . In particular, if we write  $a(k, l)$  for the least positive integer  $n (> 1)$  satisfying

$$n \equiv l \pmod{k}, \quad V(n) \leq 2,$$

then we have

$$a(k, l) < c_2 k^{3.5}$$

with some absolute constant  $c_2 > 0$ .

It is of some interest to compare our results presented above, though they are not the best possible, with a recent result of T. Tatzuzaawa [5] on the existence of a prime number  $p$  satisfying  $x < p \leq 2x$ ,  $p \equiv l \pmod{k}$  and a celebrated theorem of Yu. V. Linnik concerning the upper bound for the least prime  $p \equiv l \pmod{k}$  (cf. [3: X]).

Our proof of the theorem is based essentially upon the general sieve methods due to A. Selberg. The deepest result which we shall refer to is:

$$\pi(x) = \text{li } x + O\left(x \exp(-c_3(\log x)^{1/2})\right)$$

with a positive constant  $c_3$ , where  $\pi(x)$  denotes, as usual, the number of primes not exceeding  $x$  (in fact, a slightly weaker result will suffice for our purpose). Apart from this, the proof is entirely elementary.

*Notations.* Throughout in the following,  $k$  represents a fixed positive

integer,  $l$  an integer with  $0 \leq l \leq k-1$ ,  $(k, l)=1$ . The letters  $p, q$  are used to denote prime numbers and,  $d, m, n, r$  to denote positive integers. The functions  $\mu(n)$  and  $\varphi(n)$  are Möbius' and Euler's functions, respectively. The function  $g(n)$  is defined as follows:  $g(1)=1$  and for  $n>1$   $g(n)$ =the greatest prime divisor of  $n$ .

$s, t, u, v, w, x, y, z$  will be used to denote real numbers, constant or variable.  $c$  represents positive constants, not depending on  $k$  and  $l$ , which are not necessarily the same in each occurrence; the constants implied in the symbol  $O$  are either absolute or else uniform in  $k$  and  $l$ .

**2. Preliminaries.** There needs the following lemma for later calculations:

**Lemma 1.** *We have*

$$\sum_{p \leq t} \frac{1}{p} = \log \log t + c_1 + O\left(\frac{1}{\log t}\right),$$

where  $c_1$  is a constant;

$$\sum_{p \leq t} \frac{\log p}{p} = \log t + O(1);$$

$$\prod_{p \leq t} \left(1 - \frac{1}{p}\right)^{-1} = e^C \log t + O(1),$$

where  $C$  is Euler's constant; and

$$\varphi(m) > c \frac{m}{\log \log 3m}.$$

These results are well known. For a proof see [3: I, Theorems 3.1, 4.1 and 5.1].

Let  $M \geq 0$ ,  $N \geq 2$  be arbitrary but fixed integers and put

$$y = 2k(N+1), \quad w = y^{\frac{1}{2}-\varepsilon},$$

where  $0 < \varepsilon < \frac{1}{4}$ : we shall fix  $\varepsilon = \frac{1}{7}$  later on. Further we put

$$z = y^{\frac{1}{\alpha}}, \quad z_1 = y^{\frac{1}{\beta}}, \quad z_2 = y^{\frac{1}{\gamma}},$$

where  $\alpha, \beta, \gamma$  are fixed real numbers satisfying

$$10 \geq \gamma \geq 4 \geq \alpha > 2 \geq \beta > 1.$$

First we wish to evaluate from below the number  $S_1$  of those integers of the form  $kn+l$  ( $M < n \leq M+N$ ) which are not divisible by any prime  $p \leq z$ . Applying the 'lower' sieve of A. Selberg (see [2] and [7]), we find that

$$S_1 \geq (1-Q)N - R_1,$$

where

$$Q = \sum_{\substack{p \leq z \\ (p,k)=1}} \frac{1}{pZ_p} \quad \text{with} \quad Z_p = \sum_{\substack{n \leq w/\sqrt{p} \\ q(n) < p \\ (n,k)=1}} \frac{\mu^2(n)}{\varphi(n)}$$

and

$$R_1 = O\left(w^2 \sum_{p \leq z} \frac{1}{pZ_p^2}\right).$$

It will be shown later that

$$Z_p > c \frac{\varphi(k)}{k} \log p \quad \text{for all } p \leq z,$$

and so we have, by Lemma 1,

$$R_1 = O\left(w^2 (\log \log 3k)^2\right).$$

We put

$$H_p = \prod_{\substack{q < p \\ (q,k)=1}} \left(1 - \frac{1}{q}\right)^{-1} \quad (p \leq z).$$

Then it is easily verified that

$$1-Q = \prod_{\substack{p \leq z \\ (p,k)=1}} \left(1 - \frac{1}{p}\right) - \sum_{\substack{p \leq z \\ (p,k)=1}} \frac{H_p - Z_p}{pH_pZ_p}.$$

**Lemma 2.** *We have*

$$\begin{aligned} S_1 &\geq \frac{kN}{\varphi(k)} \prod_{p \leq z} \left(1 - \frac{1}{p}\right) - N \sum_{\substack{p \leq z \\ (p,k)=1}} \frac{H_p - Z_p}{pH_pZ_p} \\ &\quad + O\left(\frac{N(\log \log 3k)^3}{z \log z}\right) + O\left(w^2 (\log \log 3k)^2\right). \end{aligned}$$

*Proof.* We have only to prove that

$$\prod_{\substack{p \leq z \\ (p,k)=1}} \left(1 - \frac{1}{p}\right) = \frac{k}{\varphi(k)} \prod_{p \leq z} \left(1 - \frac{1}{p}\right) + O\left(\frac{(\log \log 3k)^3}{z \log z}\right)$$

or

$$(1) \quad \prod_{\substack{p \leq z \\ p|k}} \left(1 - \frac{1}{p}\right)^{-1} = \frac{k}{\varphi(k)} + O\left(\frac{(\log \log 3k)^3}{z}\right).$$

Now we have

$$\begin{aligned} 0 &\leq \prod_{\substack{p \leq z \\ p|k}} \left(1 - \frac{1}{p}\right) - \frac{\varphi(k)}{k} = \prod_{\substack{p \leq z \\ p|k}} \left(1 - \frac{1}{p}\right) - \prod_{p|k} \left(1 - \frac{1}{p}\right) \\ &\leq \sum_{\substack{p > z \\ p|k}} \sum_{\substack{d|k \\ d \equiv 0 \pmod{p}}} \frac{\mu^2(d)}{d} = \sum_{\substack{p > z \\ p|k}} \frac{1}{p} \sum_{\substack{d|k/p \\ (d, p)=1}} \frac{\mu^2(d)}{d} \\ &= O\left(\frac{1}{z} \frac{\log k}{\log z} \log \log 3k\right) = O\left(\frac{\log \log 3k}{z}\right), \end{aligned}$$

from which follows (1) at once.

Let  $q$  be a prime number in the interval  $z < q \leq z_1$  with  $(q, k) = 1$ . We next evaluate from above the number  $S(q)$  of those integers  $kn + l$  ( $M < n \leq M + N$ ) which are not divisible by any prime  $p \leq z$  and are divisible by the prime  $q$ . Applying the 'upper' sieve of A. Selberg (see the Appendix below), we find that

$$S(q) \leq \frac{N}{q W_q} + R(q),$$

where

$$W_q = \sum_{\substack{n \leq z^\alpha \\ q(n) \leq z \\ (n, k) = 1}} \frac{\mu^2(n)}{\varphi(n)}$$

with

$$a = \frac{\alpha}{2} \left(1 - 2\varepsilon - \frac{\log q}{\log y}\right)$$

and

$$R(q) = O\left(\frac{z^{2a}}{W_q^2}\right) = O\left(\frac{w^2}{q W_q^2}\right).$$

Now, let  $r \geq 1$  be a fixed integer and let  $S_2$  denote the number of those integers of the form  $kn + l$  ( $M < n \leq M + N$ ) which are not divisible by any prime  $p \leq z$  and are divisible by at least  $r + 1$  distinct primes  $q$  in the interval  $z < q \leq z_1$  with  $(q, k) = 1$ . Clearly  $S_2$  is not greater than

$$\frac{1}{r+1} \sum_{\substack{z < q \leq z_1 \\ (q, k)=1}} S(q).$$

Hence :

**Lemma 3.** *We have*

$$S_2 \leq \frac{N}{r+1} \sum_{\substack{z < q \leq z_1 \\ (q, k)=1}} \frac{1}{q W_q} + O\left(\frac{w^2(\log \log 3k)^2}{\log^2 y}\right).$$

*Proof.* It will later be shown that

$$W_q > c \frac{\varphi(k)}{k} \log y \quad \text{for } z < q \leq z_1.$$

It follows that

$$\begin{aligned} \frac{1}{r+1} \sum_{\substack{z < q \leq z_1 \\ (q, k)=1}} R(q) &= O\left(\frac{w^2(\log \log 3k)^2}{\log^2 y} \sum_{z < q \leq z_1} \frac{1}{q}\right) \\ &= O\left(\frac{w^2(\log \log 3k)^2}{\log^2 y}\right), \end{aligned}$$

since

$$\sum_{z < q \leq z_1} \frac{1}{q} = \log \frac{\alpha}{\beta} + O(1) = O(1).$$

**3. Some lemmas.** Here we collect some auxiliary results which will be needed in the next two sections.

**Lemma 4.** *We have*

$$\sum_{d|m} \frac{\mu^2(d) \log d}{d} = O\left((\log \log 3m)^2\right).$$

*Proof.* The left-hand side is equal to

$$\begin{aligned} \sum_{d|m} \frac{\mu^2(d)}{d} \sum_{p|d} \log p &= \sum_{p|m} \log p \sum_{\substack{d|m \\ d \equiv 0 \pmod{p}}} \frac{\mu^2(d)}{d} \\ &= \sum_{p|m} \frac{\log p}{p} \sum_{\substack{d|m/p \\ (d, p)=1}} \frac{\mu^2(d)}{d}, \end{aligned}$$

where we have

$$\sum_{\substack{d|m/p \\ (d, p)=1}} \frac{\mu^2(d)}{d} \leq \sum_{d|m} \frac{1}{d} = O(\log \log 3m)$$

and

$$\begin{aligned}\sum_{p|m} \frac{\log p}{p} &= \sum_{\substack{p|m \\ p \leq \log m}} \frac{\log p}{p} + O(1) \\ &= O(\log \log 3m).\end{aligned}$$

This proves Lemma 4.

**Lemma 5.** *We have*

$$\sum_{\substack{n \leq t \\ (n, m)=1}} \frac{\mu^2(n)}{\varphi(n)} = \frac{\varphi(m)}{m} \log t + O(\log \log 3m).$$

*Proof.* H. N. Shapiro and J. Warga [4: Appendix I] have proved that

$$\sum_{\substack{n \leq t \\ (n, m)=1}} \frac{\mu^2(n)}{n} = \frac{\varphi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right) \log t + O(\log \log 3m).$$

Using this inequality we obtain

$$\begin{aligned}\sum_{\substack{n \leq t \\ (n, m)=1}} \frac{\mu^2(n)}{\varphi(n)} &= \sum_{\substack{n \leq t \\ (n, m)=1}} \frac{\mu^2(n)}{n} \prod_{p|n} \left(1 + \frac{1}{p-1}\right) \\ &= \sum_{\substack{n \leq t \\ (n, m)=1}} \frac{\mu^2(n)}{n} \sum_{d|n} \frac{1}{\varphi(d)} \\ &= \sum_{\substack{d \leq t \\ (d, m)=1}} \frac{\mu^2(d)}{d\varphi(d)} \sum_{\substack{n \leq t/d \\ (n, dm)=1}} \frac{\mu^2(n)}{n} \\ &= \sum_{\substack{d \leq t \\ (d, m)=1}} \frac{\mu^2(d)}{d\varphi(d)} \left( \frac{\varphi(dm)}{dm} \prod_{p|dm} \left(1 - \frac{1}{p^2}\right) \log \frac{t}{d} + O(\log \log 3dm) \right) \\ &= \frac{\varphi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right) \sum_{\substack{d \leq t \\ (d, m)=1}} \frac{\mu^2(d)}{d^2} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} \log \frac{t}{d} \\ &\quad + O\left( \sum_{d \leq t} \frac{\mu^2(d)}{d\varphi(d)} \log \log 3dm \right) \\ &= \frac{\varphi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right) \sum_{\substack{d=1 \\ (d, m)=1}}^{\infty} \frac{\mu^2(d)}{d^2} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} \log t \\ &\quad + O\left( \sum_{d > t} \frac{\mu^2(d)}{d^2} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} \log d \right) \\ &\quad + O\left( \sum_{d \leq t} \frac{\mu^2(d)}{d\varphi(d)} \log \log 3dm \right)\end{aligned}$$

$$= \frac{\varphi(m)}{m} \log t + O(\log \log 3m),$$

since

$$\sum_{\substack{d=1 \\ (d,m)=1}}^{\infty} \frac{\mu^2(d)}{d^2} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} = \prod_{p \nmid m} \left(1 + \frac{1}{p^2 - 1}\right) = \prod_{p \nmid m} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

Now, for  $u > 0$ ,  $v \geq 2$ , let  $G(u, v)$  denote the number of positive integers  $n \leq u$  with  $g(n) \leq v$ .

Define the function  $\rho(s)$  by the following properties:

$$(2) \quad \begin{cases} \rho(s) = 0 & (s < 0); & \rho(s) = 1 & (0 \leq s \leq 1); \\ s\rho'(s) = -\rho(s-1) & (s > 1); & \rho(s) \text{ continuous} & \text{for } s > 0. \end{cases}$$

Then the following result has been proved by N. G. de Bruijn [1]:

**Lemma 6.** *Let  $u > 0$ ,  $v \geq 2$ , and put  $t = (\log u)/\log v$ . Then we have*

$$G(u, v) = O(ue^{-ct})$$

and, more precisely,

$$G(u, v) = u\rho(t) \left(1 + O\left(\frac{\log(2+t)}{\log v}\right)\right) + O(1) + O(ut^2 P(v)),$$

where  $P(v)$  is a function satisfying

$$P(v) \downarrow 0 \quad (v \rightarrow \infty), \quad P(v) > (\log v)/v \quad (v \geq 2),$$

$$|\pi(v) - \text{li } v| < vP(v)/\log v \quad (v \geq 2).$$

As to the function  $\rho(s)$  itself, it is not difficult to prove the following result, which is known as a lemma of N. C. Ankeny:

**Lemma 7.** *For  $s_1 \geq s_2 \geq 1$  we have*

$$\rho(s_1) \leq \rho(s_2) e^{-(s_1 - s_2)},$$

so that

$$\int_s^\infty \rho(t) dt \leq \rho(s) \quad (s \geq 1).$$

For a proof of this result see [8].

**4. Evaluation of  $S_1$ .** We are now going to find an explicit lower bound for  $S_1$  on the basis of Lemma 2.



First we have to evaluate  $Z_p$  and  $H_p - Z_p$  for  $p \leq z$ . To accomplish this we distinguish three cases on the magnitude of the prime  $p$ .

It is clear that

$$T_p \stackrel{\text{def}}{=} \sum_{\substack{n > w/\sqrt{p} \\ g(n) \leq p \\ (n, k) = 1}} \frac{1}{n} \geq H_p - Z_p \geq 0.$$

*Case 1:*  $2 \leq p \leq \exp(\log y)^{\frac{2}{3}}$ . By partial summation we get

$$T_p \leq \sum_{\substack{n > w/\sqrt{p} \\ g(n) \leq p}} \frac{1}{n} = \sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} + O(y^{-c_5}),$$

where  $c_5 = \frac{1}{2} - \varepsilon - \frac{1}{2\alpha}$ . By Lemma 6 we have

$$\begin{aligned} \sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} &= O\left(\sum_{n > w/\sqrt{p}} n^{-(1+c/\log p)}\right) \\ &= O\left((\log y)^{\frac{2}{3}} \exp(-c(\log y)^{\frac{1}{3}})\right). \end{aligned}$$

It follows that

$$H_p - Z_p = O\left(\frac{1}{\log^2 y}\right), \quad Z_p > c \frac{\varphi(k)}{k} \log p,$$

since, by Lemma 1,

$$H_p = \prod_{\substack{q \leq p \\ (q, k) = 1}} \left(1 - \frac{1}{q}\right)^{-1} \geq \frac{\varphi(k)}{k} e^c \log p + O(1).$$

*Case 2:*  $\exp(\log y)^{\frac{2}{3}} < p \leq z_2$ . We have

$$\begin{aligned} T_p &= \sum_{\substack{n > w/\sqrt{p} \\ g(n) \leq p}} \frac{1}{n} \sum_{d|(n, k)} \mu(d) \\ &= \sum_{\substack{d|k \\ g(d) \leq p}} \frac{\mu(d)}{d} \sum_{\substack{n > w/d\sqrt{p} \\ g(n) \leq p}} \frac{1}{n} \\ &= \sum_{\substack{d|k \\ g(d) \leq p}} \frac{\mu(d)}{d} \sum_{\substack{n > w/\sqrt{p} \\ g(n) \leq p}} \frac{1}{n} \\ &\quad + \sum_{\substack{d|k \\ g(d) \leq p}} \frac{\mu(d)}{d} \sum_{\substack{w/\sqrt{p} \leq n < w/d\sqrt{p} \\ g(n) \leq p}} \frac{1}{n} \end{aligned}$$

$$= \prod_{\substack{q|k \\ q \leq p}} \left(1 - \frac{1}{q}\right) \sum_{\substack{n > w/\sqrt{p} \\ q(n) \leq p}} \frac{1}{n} + O((\log \log 3k)^2),$$

since we have, by Lemma 4,

$$\begin{aligned} \sum_{\substack{d|k \\ q(d) \leq p}} \frac{\mu(d)}{d} \sum_{\substack{w/\sqrt{p} \leq n < w/d\sqrt{p} \\ q(n) \leq p}} \frac{1}{n} &= O\left(\sum_{d|k} \frac{\mu^2(d) \log d}{d}\right) \\ &= O((\log \log 3k)^2). \end{aligned}$$

Now, by partial summation, we have

$$\sum_{\substack{n > w/\sqrt{p} \\ q(n) \leq p}} \frac{1}{n} = \sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} + O(y^{-c_s}).$$

Here, by Lemma 6, we find that

$$\begin{aligned} \sum_{n > \exp(\log y)^2} \frac{G(n, p)}{n^2} &= O\left(\sum_{n > \exp(\log y)^2} n^{-(1+e/\log p)}\right) \\ &= O(\log y \exp(-c \log y)) \\ &= O((\log y) y^{-c}), \end{aligned}$$

so that

$$\sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} = \sum_{\exp(\log y)^2 \leq n < w/\sqrt{p}} \frac{G(n, p)}{n^2} + O\left(\frac{1}{\log^2 y}\right).$$

Let us write  $I$  for the interval  $w/\sqrt{p} < n \leq \exp(\log y)^2$ . Then, by making use of the result in Lemma 6, we obtain

$$\begin{aligned} \sum_{n \in I} \frac{G(n, p)}{n^2} &= \sum_{n \in I} \frac{1}{n} \rho\left(\frac{\log n}{\log p}\right) \left(1 + O\left(\frac{\log \log y}{\log p}\right)\right) \\ &\quad + O\left(\sum_{n \in I} \frac{1}{n}\right) + O\left(\sum_{n \in I} \frac{1}{n} \left(\frac{\log n}{\log p}\right)^2 P(p)\right). \end{aligned}$$

It is easily verified that

$$\sum_{n \in I} \frac{1}{n} \rho\left(\frac{\log n}{\log p}\right) = \log p \int_{t_p}^{\infty} \rho(t) dt + O(y^{-c_s}),$$

where  $t_p = (\log w/\sqrt{p})/\log p$ ;

$$\sum_{n \in I} \frac{1}{n} = O(y^{-c_s}); \quad \sum_{n \in I} \frac{1}{n} \left( \frac{\log n}{\log p} \right)^2 P(p) = O\left( \frac{1}{\log^2 y} \right),$$

where we have taken  $P(v) = c \exp(-c(\log v)^{\frac{1}{2}})$ .

We thus have

$$T_p = \log p \int_{t_p}^{\infty} \rho(t) dt \left( 1 + O\left( \frac{\log \log y}{\log p} \right) \right) + O\left( \frac{1}{\log^2 y} \right).$$

Hence

$$\begin{aligned} H_p - Z_p &\leq \prod_{\substack{q|k \\ q \leq p}} \left( 1 - \frac{1}{q} \right) \log p \int_{t_p}^{\infty} \rho(t) dt \left( 1 + O\left( \frac{\log \log y}{\log p} \right) \right) \\ &\quad + O\left( (\log \log 3k)^2 \right), \\ Z_p &\geq \prod_{\substack{q|k \\ q \leq p}} \left( 1 - \frac{1}{q} \right) \left( e^c - \int_{t_p}^{\infty} \rho(t) dt \right) \log p + O(\log \log y) \\ &\quad + O\left( (\log \log 3k)^2 \right). \end{aligned}$$

*Case 3:*  $z_2 < p \leq z$ . Put  $t_p = (\log w / \sqrt{p}) / \log p$ , as before. If  $0 < t_p \leq 1$  then we have

$$\begin{aligned} Z_p &= \sum_{\substack{n \leq w/\sqrt{p} \\ (n, k)=1}} \frac{\mu^2(n)}{\varphi(n)} \\ &= \frac{\varphi(k)}{k} \log \frac{w}{\sqrt{p}} + O(\log \log 3k) \\ &= \frac{\varphi(k)}{k} t_p \log p + O(\log \log 3k), \\ H_p - Z_p &= \frac{\varphi(k)}{k} (e^c - t_p) \log p + O(\log \log 3k), \end{aligned}$$

by Lemma 5. If  $t_p > 1$  then

$$\begin{aligned} Z_p &\geq \sum_{\substack{n \leq w/\sqrt{p} \\ (n, k)=1}} \frac{\mu^2(n)}{\varphi(n)} - \sum_{\substack{p \leq q \leq w/\sqrt{p} \\ (q, k)=1}} \sum_{\substack{n \leq w/\sqrt{p} \\ n \equiv 0 \pmod{q} \\ (n, k)=1}} \frac{\mu^2(n)}{\varphi(n)} \\ &= \sum_{\substack{n \leq w/\sqrt{p} \\ (n, k)=1}} \frac{\mu^2(n)}{\varphi(n)} - \sum_{\substack{p \leq q \leq w/\sqrt{p} \\ (q, k)=1}} \frac{1}{\varphi(q)} \sum_{\substack{n \leq w/q\sqrt{p} \\ (n, qk)=1}} \frac{\mu^2(n)}{\varphi(n)}, \end{aligned}$$

where, again by Lemma 5,

$$\begin{aligned}
& \sum_{\substack{p \leq q \leq w/\sqrt{p} \\ (q, k)=1}} \frac{1}{\varphi(q)} \sum_{\substack{n \leq w/q\sqrt{p} \\ (n, qk)=1}} \frac{\mu^2(n)}{\varphi(n)} \\
&= \sum_{\substack{p \leq q \leq w/\sqrt{p} \\ (q, k)=1}} \frac{1}{\varphi(q)} \left( \frac{\varphi(qk)}{qk} \log \frac{w}{q\sqrt{p}} + O(\log \log 3qk) \right) \\
&= \frac{\varphi(k)}{k} \sum_{\substack{p \leq q \leq w/\sqrt{p} \\ (q, k)=1}} \frac{1}{q} \log \frac{w}{q\sqrt{p}} + O\left( \sum_{p \leq q \leq w/\sqrt{p}} \frac{\log \log 3qk}{\varphi(q)} \right) \\
&= \frac{\varphi(k)}{k} \left( \log \frac{w}{\sqrt{p}} \log \frac{\log \frac{w}{\sqrt{p}}}{\log p} - \log \frac{w}{\sqrt{p}} + \log p \right) + O(\log \log y),
\end{aligned}$$

and hence

$$Z_p \geq \frac{\varphi(k)}{k} (2t_p - 1 - t_p \log t_p) \log p + O(\log \log y).$$

Therefore

$$H_p - Z_p \leq \frac{\varphi(k)}{k} (e^c - (2t_p - 1 - t_p \log t_p)) \log p + O(\log \log y).$$

Here we have, as in the proof of Lemma 2,

$$\begin{aligned}
H_p &= \prod_{\substack{q < p \\ q|k}} \left(1 - \frac{1}{q}\right) \prod_{q < p} \left(1 - \frac{1}{q}\right)^{-1} \\
&= \left( \frac{\varphi(k)}{k} + O\left(\frac{\log \log 3k}{p}\right) \right) (e^c \log p + O(1)) \\
&= \frac{\varphi(k)}{k} e^c \log p + O(\log \log 3k).
\end{aligned}$$

We are now in position to be able to evaluate the sum

$$\sum_{\substack{p \leq z \\ (p, k)=1}} \frac{H_p - Z_p}{p H_p Z_p}.$$

Define :

$$A(t) = \begin{cases} \frac{e^c - t}{t} & (0 < t \leq 1), \\ \frac{e^c - (2t - 1 - t \log t)}{2t - 1 - t \log t} & (1 < t < e^c), \end{cases}$$

where  $t=e^c$  is the unique solution of

$$2t-1-t \log t = 0, \quad t > 1,$$

so that  $1.8 < c_6 < 1.9$ ; and

$$B(t) = \frac{\int_t^\infty \rho(s) ds}{e^c - \int_t^\infty \rho(s) ds} \quad \left(t > \frac{1}{4}\right).$$

Let us put, for the sake of brevity,

$$z_3 = \exp(\log y)^{2/3}.$$

Then we have

$$\begin{aligned} \sum_{\substack{2 \leq p \leq z_3 \\ (p, k)=1}} \frac{H_p - Z_p}{p H_p Z_p} &= O\left(\frac{k}{\varphi(k)} \frac{\log \log 3k}{\log^2 y} \sum_p \frac{1}{p \log^2 p}\right) \\ &= O\left(\frac{k}{\varphi(k)} \frac{\log \log 3k}{\log^2 y}\right), \end{aligned}$$

and, noticing that

$$\prod_{\substack{q|k \\ q \leq p}} \left(1 - \frac{1}{q}\right)^{-1} \leq \frac{k}{\varphi(k)}$$

for every  $p$ ,

$$\begin{aligned} \sum_{\substack{z_3 < p \leq z_2 \\ (p, k)=1}} \frac{H_p - Z_p}{p H_p Z_p} &\leq \frac{k}{\varphi(k)} e^{-c} \sum_{z_3 < p \leq z_2} \frac{B(t_p)}{p \log p} \\ &\quad + O\left(\frac{k}{\varphi(k)} \frac{(\log \log 3k)^3}{\log^{4/3} y}\right) + O\left(\frac{k}{\varphi(k)} \frac{\log \log y}{\log^{4/3} y}\right), \end{aligned}$$

where we have used the inequality

$$\sum_{z_3 < p \leq z_2} \frac{1}{p \log^2 p} = O\left(\sum_{\pi(z_3) < n \leq \pi(z_2)} \frac{1}{n \log^3 n}\right) = O\left(\frac{1}{\log^2 z_3}\right).$$

We now assume that  $\gamma, 4 \leq \gamma \leq 10$ , be integral. Write  $J_\nu$  for the interval  $y^{1/\nu+1} < p \leq y^{1/\nu}$  ( $\nu \geq \gamma$ ). Then, since the function  $B(t)$  is monotone decreasing,

$$\sum_{z_3 < p \leq z_2} \frac{B(t_p)}{p \log p} = \sum_{\gamma \leq \nu < c(\log y)^{\frac{1}{3}}} \sum_{p \in J_\nu} \frac{B(t_p)}{p \log p}$$

$$\begin{aligned}
&\leq \sum_{r \leq \nu < c(\log y)^{\frac{1}{3}}} \left( \sum_{p \in J_r} \frac{1}{p} \right) \max_{p \in J_r} \frac{B(t_p)}{\log p} \\
&\leq \sum_{r \leq \nu < c(\log y)^{\frac{1}{3}}} \log \frac{\nu+1}{\nu} \frac{\nu+1}{\log y} B\left(\left(\frac{1}{2} - \varepsilon\right)\nu - \frac{1}{2}\right) \\
&\quad + O\left(\sum_{r \leq \nu < c(\log y)^{\frac{1}{3}}} \frac{\nu}{\log^2 y}\right) \\
&\leq \frac{1}{\log y} \sum_{\nu=r}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} B\left(\left(\frac{1}{2} - \varepsilon\right)\nu - \frac{1}{2}\right) + O\left(\frac{1}{\log^{4/3} y}\right).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\sum_{\substack{z_3 < p \leq z_2 \\ (p, k) = 1}} \frac{H_p - Z_p}{p H_p Z_p} &\leq \frac{k}{\varphi(k)} \frac{e^{-c}}{\log y} \sum_{\nu=r}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} B\left(\left(\frac{1}{2} - \varepsilon\right)\nu - \frac{1}{2}\right) \\
&\quad + O\left(\frac{k}{\varphi(k)} \frac{(\log \log 3k)^3}{\log^{4/3} y}\right) + O\left(\frac{k}{\varphi(k)} \frac{\log \log y}{\log^{4/3} y}\right).
\end{aligned}$$

We have similarly

$$\begin{aligned}
\sum_{\substack{z_2 < p \leq z \\ (p, k) = 1}} \frac{H_p - Z_p}{p H_p Z_p} &\leq \frac{k}{\varphi(k)} e^{-c} \sum_{z_2 < p \leq z} \frac{A(t_p)}{p \log p} \\
&\quad + O\left(\frac{k}{\varphi(k)} \frac{(\log \log 3k) \log \log y}{\log^2 y}\right).
\end{aligned}$$

Put

$$n = [\log^{1/2} y], \quad u_j = \alpha + \frac{r - \alpha}{n} j \quad (j \geq 0),$$

and write  $K_j$  for the interval

$$y^{1/u_{j+1}} < p \leq y^{1/u_j} \quad (0 \leq j \leq n-1).$$

Now, the function  $A(t)$  is continuous, monotone decreasing in the interval  $0 < t \leq e$  and monotone increasing in the interval  $e < t < e^{e^e}$ . Thus, if we denote by  $m$  the integer for which

$$\left(\frac{1}{2} - \varepsilon\right) u_m - \frac{1}{2} \leq e < \left(\frac{1}{2} - \varepsilon\right) u_{m+1} - \frac{1}{2},$$

then

$$\begin{aligned}
\sum_{z_2 < p \leq z} \frac{A(t_p)}{p \log p} &= \sum_{j=0}^{n-1} \sum_{p \in K_j} \frac{A(t_p)}{p \log p} \\
&\leq \sum_{j=0}^{n-1} \left( \sum_{p \in K_j} \frac{1}{p} \right) \max_{p \in K_j} \frac{A(t_p)}{\log p} \\
&\leq \sum_{j=0}^{n-1} \log \frac{u_{j+1}}{u_j} \frac{u_{j+1}}{\log y} A\left(\left(\frac{1}{2} - \varepsilon\right)u_j - \frac{1}{2}\right) \\
&\quad + \log \frac{u_{m+1}}{u_m} \frac{u_{m+1}}{\log y} \max\left(A\left(\left(\frac{1}{2} - \varepsilon\right)u_m - \frac{1}{2}\right), A\left(\left(\frac{1}{2} - \varepsilon\right)u_{m+1} - \frac{1}{2}\right)\right) \\
&\quad + \sum_{j=m+1}^{n-1} \log \frac{u_{j+1}}{u_j} \frac{u_{j+1}}{\log y} A\left(\left(\frac{1}{2} - \varepsilon\right)u_{j+1} - \frac{1}{2}\right) + O\left(\sum_{j=0}^{n-1} \frac{u_j}{\log^2 y}\right) \\
&= \frac{1}{\log y} \int_{\alpha}^r A\left(\left(\frac{1}{2} - \varepsilon\right)u - \frac{1}{2}\right) du + O\left(\frac{1}{\log^{3/2} y}\right),
\end{aligned}$$

where it should be noticed that we have uniformly

$$u_{j+1} \log \frac{u_{j+1}}{u_j} = \frac{r - \alpha}{n} + O\left(\frac{1}{n^2}\right) \quad (0 \leq j \leq n-1).$$

Hence

$$\begin{aligned}
\sum_{\substack{z_2 < p \leq z \\ (p, k) = 1}} \frac{H_p - Z_p}{p H_p Z_p} &\leq \frac{k}{\varphi(k)} \frac{e^{-c}}{\log y} \int_{\alpha}^r A\left(\left(\frac{1}{2} - \varepsilon\right)u - \frac{1}{2}\right) du \\
&\quad + O\left(\frac{k}{\varphi(k)} \frac{(\log \log y) \log \log 3k}{\log^2 y}\right) + O\left(\frac{k}{\varphi(k)} \frac{1}{\log^{3/2} y}\right).
\end{aligned}$$

Collecting these results, we thus obtain, via Lemma 2, the following

**Lemma 8.** *We have*

$$\begin{aligned}
S_1 &\geq \frac{kN}{\varphi(k)} \frac{e^{-c}}{\log y} \left( \alpha - \int_{\alpha}^r A\left(\left(\frac{1}{2} - \varepsilon\right)u - \frac{1}{2}\right) du \right. \\
&\quad \left. - \sum_{\nu=r}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} B\left(\left(\frac{1}{2} - \varepsilon\right)\nu - \frac{1}{2}\right) \right) \\
&\quad + O\left(\frac{kN}{\varphi(k)} \frac{(\log \log 3k)^3}{\log^{4/3} y}\right) + O\left(\frac{kN}{\varphi(k)} \frac{\log \log y}{\log^{4/3} y}\right) \\
&\quad + O\left(\frac{N(\log \log 3k)^3}{y^{1/4} \log y}\right) + O\left(y^{1-2\varepsilon} (\log \log 3k)^2\right).
\end{aligned}$$

**5. Evaluation of  $S_2$ .** By virtue of Lemma 3, our present task is only to estimate the quantity

$$\sum_{\substack{z < q \leq z_1 \\ (q, k) = 1}} \frac{1}{q W_q}.$$

We set

$$C(t) = \begin{cases} \frac{\alpha}{a} & (0 < a \leq 1), \\ \frac{\alpha}{2a - 1 - a \log a} & (1 < a \leq 2), \end{cases}$$

where

$$a = \frac{\alpha}{2} \left( 1 - 2\varepsilon - \frac{1}{t} \right).$$

Then, it is not difficult to verify, by Lemma 5, that, with  $t = t_q = (\log y)/\log q$

$$\begin{aligned} W_q &= \sum_{\substack{n \leq z^a \\ q(n) \leq z \\ (n, k) = 1}} \frac{\mu^2(n)}{\varphi(n)} \\ &\geq \frac{\varphi(k)}{k} \frac{\log y}{C(t_q)} + O(\log \log 3k) \quad (z < q \leq z_1), \end{aligned}$$

and consequently

$$\sum_{\substack{z < q \leq z_1 \\ (q, k) = 1}} \frac{1}{q W_q} \leq \frac{k}{\varphi(k)} \frac{1}{\log y} \sum_{z < q \leq z_1} \frac{C(t_q)}{q} + O\left(\frac{k}{\varphi(k)} \frac{(\log \log 3k)^2}{\log^2 y}\right).$$

Put

$$n = [\log^{1/2} y], \quad u_j = \beta + \frac{\alpha - \beta}{n} j \quad (0 \leq j \leq n),$$

and write  $L_j$  for the interval

$$y^{1/u_{j+1}} < q \leq y^{1/u_j} \quad (0 \leq j \leq n-1).$$

Then we have

$$\begin{aligned} \sum_{z < q \leq z_1} \frac{C(t_q)}{q} &= \sum_{j=0}^{n-1} \sum_{q \in L_j} \frac{C(t_q)}{q} \leq \sum_{j=0}^{n-1} \left( \sum_{q \in L_j} \frac{1}{q} \right) \max_{q \in L_j} C(t_q) \\ &\leq \sum_{j=0}^{n-1} \left( \log \frac{u_{j+1}}{u_j} + O\left(\frac{1}{\log y}\right) \right) C(u_j) = \int_{\beta}^{\alpha} C(u) \frac{du}{u} + O\left(\frac{1}{\log^{1/2} y}\right), \end{aligned}$$



since the function  $C(t)$  is continuous and decreases monotonously for  $t > (1-2\varepsilon)^{-1}$  and since we have uniformly

$$\log \frac{u_{j+1}}{u_j} = \frac{\alpha - \beta}{n} \frac{1}{u_j} + O\left(\frac{1}{n^2}\right) \quad (0 \leq j \leq n-1).$$

We thus have proved the following

**Lemma 9.** *We have*

$$\begin{aligned} S_2 \leq & \frac{1}{r+1} \frac{kN}{\varphi(k)} \frac{1}{\log y} \int_{\beta}^{\alpha} \frac{C(u)}{u} du \\ & + O\left(\frac{kN}{\varphi(k)} \frac{(\log \log 3k)^2}{\log^2 y}\right) + O\left(\frac{kN}{\varphi(k)} \frac{1}{\log^{3/2} y}\right) + O\left(\frac{y^{1-2\varepsilon} (\log \log 3k)^2}{\log^2 y}\right). \end{aligned}$$

**6. Numerical computations.** We need the following easy lemma, a part of which has already been used in the proof of Lemmas 8 and 9.

**Lemma 10.** *The function*

$$f(s) = \frac{1}{2s-1-s \log s} \quad (1 < s < e^e)$$

*is positive, convex, and monotone decreasing for  $1 < s \leq e$  and monotone increasing for  $e < s < e^e$ .*

Putting  $f_1(s) = (f(s))^{-1}$ , we see that  $f_1(s) > 0$ ,  $f_1'(s) = 1 - \log s$  and  $f_1''(s) = -1/s$ , and the result follows at once.

We now choose  $\varepsilon = \frac{1}{7}$  and take

$$\alpha = 4, \quad \beta = 2, \quad \text{and} \quad r = 10.$$

Our aim in this section is to compute numerically two integrals and a sum appearing in Lemmas 8 and 9.

(i) Computation of

$$\int_{\alpha}^{\gamma} A\left(\left(\frac{1}{2} - \varepsilon\right)u - \frac{1}{2}\right) du = \int_4^{10} A\left(\frac{5}{14}u - \frac{1}{2}\right) du.$$

The integral is equal to

$$(3) \quad \int_4^{4.2} A\left(\frac{5}{14}u - \frac{1}{2}\right) du + \int_{4.2}^{10} A\left(\frac{5}{14}u - \frac{1}{2}\right) du,$$

where the first integral is found to be

$$\begin{aligned}
&= e^c \int_4^{4.2} \left( \frac{5}{14}u - \frac{1}{2} \right)^{-1} du - \int_4^{4.2} du \\
&= e^c \frac{14}{5} \log \frac{14}{13} - 0.2 < 0.1696,
\end{aligned}$$

while the second is

$$= e^c \int_{4.2}^{10} F(u) du - \int_{4.2}^{10} du$$

with  $F(u) = f(s(u))$ , where  $f(s)$  is the function defined in Lemma 10 and  $s(u) = \frac{5}{14}u - \frac{1}{2}$ . To estimate the integral of  $F(u)$  over  $(4.2, 10)$  we proceed as follows.

We find :

$$\begin{aligned}
F(4.2) &= 1.0000; & F(4.5) &< 0.9080; \\
F(5) &< 0.8011; & F(6) &< 0.6803; \\
F(7) &< 0.6197; & F(8) &< 0.5907; \\
F(9) &< 0.5820; & F(10) &< 0.5896.
\end{aligned}$$

By Lemma 10, the function  $F(u)$  is convex for  $4.2 \leq u \leq 10$ . Hence

$$\begin{aligned}
\int_{4.2}^{10} F(u) du &\leq \frac{3}{20} (F(4.2) + F(4.5)) + \frac{1}{4} (F(4.5) + F(5)) \\
&\quad + \frac{1}{2} (F(5) + F(10)) + (F(6) + F(7) + F(8) + F(9)) \\
&< 3.8817,
\end{aligned}$$

and the second integral in (3) is less than

$$3.8817 e^c - 5.8 < 1.1137.$$

Thus we have

$$\int_4^{10} A \left( \frac{5}{14}u - \frac{1}{2} \right) du < 0.1696 + 1.1137 = 1.2833.$$

(ii) Computation of

$$\begin{aligned}
&\sum_{\nu=7}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} B \left( \left( \frac{1}{2} - \varepsilon \right) \nu - \frac{1}{2} \right) \\
&= \sum_{\nu=10}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} B \left( \frac{5}{14} \nu - \frac{1}{2} \right).
\end{aligned}$$

By the definition (2), the function  $\rho(s)$  is positive and monotone decreasing for  $s > 0$ , and moreover

$$\rho(s) = 1 - \log s \quad \text{for } 1 \leq s \leq 2.$$

Put  $s(\nu) = \frac{5}{14}\nu - \frac{1}{2}$ . Then we have  $s(10) = \frac{43}{14} > 3$  and

$$\rho(s(10)) \leq \rho(3) \leq \rho(2)e^{-1} = (1 - \log 2)e^{-1} < 0.1129,$$

by Lemma 7. Now, using Lemma 7 again, we find that for  $\nu \geq 10$

$$B(s(\nu)) \leq \frac{\rho(s(\nu))}{e^c - \rho(s(\nu))} \leq \frac{\rho(s(10))}{e^c - \rho(s(10))} e^{-\frac{5}{14}(\nu-10)}.$$

Since  $(\nu+1)\log((\nu+1)/\nu)$  decreases monotonously as  $\nu \rightarrow \infty$ , we thus obtain

$$\begin{aligned} & \sum_{\nu=10}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} B(s(\nu)) \\ & \leq 11 \log \frac{11}{10} \frac{\rho(s(10))}{e^c - \rho(s(10))} \frac{1}{1 - e^{-5/14}} < 0.2366. \end{aligned}$$

(iii) Computation of

$$\int_{\beta}^{\alpha} \frac{C(u)}{u} du = \int_2^4 \frac{C(u)}{u} du.$$

For  $2 \leq u \leq 4$  we have

$$\frac{3}{7} \leq a = 2 \left( \frac{5}{7} - \frac{1}{u} \right) \leq \frac{13}{14}.$$

Hence

$$\begin{aligned} \int_2^4 \frac{C(u)}{u} du &= 2 \int_2^4 \left( \frac{5}{7}u - 1 \right)^{-1} du \\ &= \frac{14}{5} \log \frac{13}{3} < 4.1058. \end{aligned}$$

**7. Proof of the theorem.** Let  $1 \leq k < x$ ,  $0 \leq l \leq k-1$ ,  $(k, l) = 1$ . Take

$$M = \left[ \frac{x-l}{k} \right], \quad N = \left[ \frac{x}{k} \right],$$

and put

$$y = 2k(N+1), \quad z = y^{1/4}, \quad z_1 = y^{1/2}, \quad w = y^{5/14}.$$

Then it is clear that  $y > 2x$  and that  $M < n \leq M+N$  implies  $x < kn+l \leq 2x$ .

By  $D(x; k, l)$  we denote the number of those integers of the form  $kn+l$  ( $M < n \leq M+N$ ) which are divisible by no primes  $p \leq z$ , by at most two primes  $q$  in  $z < q \leq z_1$ , and by no integers of the form  $q^2$ ,  $q$  being a prime in  $z < q \leq z_1$ : clearly such an integer  $kn+l$  ( $M < n \leq M+N$ ), if it exists, has at most two prime factors, i.e.  $V(kn+l) \leq 2$ .

In order to estimate  $D(x; k, l)$  from below, we apply Lemma 8 and Lemma 9 with  $r=2$ . Let us note that we have from the data in §6

$$e^{-c} \left( 4 - \int_4^{10} A(s(u)) du - \sum_{\nu=10}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} B(s(\nu)) \right) > e^{-c} (4 - 1.2833 - 0.2366) > 1.3923$$

and

$$\frac{1}{3} \int_2^4 \frac{C(u)}{u} du < \frac{4.1058}{3} = 1.3686.$$

Now, the number  $R_2$  of those integers  $kn+l$  ( $M < n \leq M+N$ ) which are not divisible by any prime  $p \leq z$  and are divisible by some integer  $q^2$  with  $q$  in  $z < q \leq z_1$  does not exceed

$$\sum_{z < q \leq z_1} \left( \frac{N}{q^2} + 1 \right) = O\left(\frac{N}{z}\right) + O(z_1).$$

We find, therefore, that

$$\begin{aligned} D(x; k, l) &\geq S_1 - S_2 - R_2 \geq (1.3923 - 1.3686) \frac{kN}{\varphi(k)} \frac{1}{\log y} \\ &+ O\left(\frac{kN}{\varphi(k)} \frac{(\log \log 3k)^3}{\log^{4/3} y}\right) + O\left(\frac{kN}{\varphi(k)} \frac{\log \log y}{\log^{4/3} y}\right) \\ &+ O\left(\frac{N(\log \log 3k)^3}{y^{1/4} \log y}\right) + O(y^{5/7} (\log \log 3k)^2) + O\left(\frac{N}{y^{1/4}}\right). \end{aligned}$$

Since  $N = \frac{x}{k} + O(1)$ ,  $2x < y \leq 4x$ , it follows that

$$\begin{aligned} D(x; k, l) &\geq 0.0237 \frac{1}{\varphi(k)} \frac{x}{\log x} \\ &+ O\left(\frac{1}{\varphi(k)} \frac{x(\log \log 3k)^3}{\log^{4/3} x}\right) + O\left(\frac{1}{\varphi(k)} \frac{x \log \log x}{\log^{4/3} x}\right) \end{aligned}$$

$$+ O\left(\frac{1}{k} \frac{x^{3/4} (\log \log 3k)^3}{\log x}\right) + O\left(x^{5/7} (\log \log 3k)^2\right) + O\left(\frac{1}{k} x^{3/4}\right).$$

Let  $c_7 > 3.5$  be a fixed number. If  $x \geq k^{c_7}$  and  $k$  is sufficiently large, then all the error terms on the right-hand side of the above inequality for  $D(x; k, l)$  are of negligible order of magnitude, with respect to the leading term. Thus, for all large enough  $k$ ,  $x \geq k^{c_7}$  implies that

$$D(x; k, l) > 0.0236 \frac{1}{\varphi(k)} \frac{x}{\log x} > 1.$$

Hence, by continuity argument, we conclude that there is a (finite) natural number  $k_0$  such that, if  $k \geq k_0$  and  $x \geq k^{3.5}$  then we have  $D(x; k, l) > 0$ . Therefore there exists an absolute constant  $c_1 > 0$  such that

$$D(x; k, l) > 0 \quad \text{for all } x \geq c_1 k^{3.5}, \quad k \geq 1.$$

• This completes the proof of our theorem.

## Appendix

### ON THE 'UPPER' SIEVE OF A. SELBERG

Here we aim at generalizing the results obtained in [6].

Let  $N > 1$  and let  $a_1, a_2, \dots, a_N$  be rational integers not necessarily different from each other. Let  $S$  denote the number of those integers  $a_j$  ( $1 \leq j \leq N$ ) which are not divisible by any prime number  $p \leq z$ , where  $z \geq 2$ . Suppose that for every positive integer  $d$

$$S_a \stackrel{\text{def}}{=} \sum_{\substack{n \leq N \\ a_n \equiv 0 \pmod{d}}} 1 = \frac{\omega(d)}{d} N + R(d),$$

where  $R(d)$  is the error term for  $S_a$  and  $\omega(d)$  is a multiplicative function of  $d$ . We put

$$f(d) = \frac{d}{\omega(d)}$$

and suppose that  $f(d) > 1$  for all  $d > 1$ .

Let  $w$  be an arbitrary but fixed real number such that  $w \geq 2$ . We define for positive integers  $m$  and  $d$

$$f_1(m) = \sum_{n|m} \mu(n) f\left(\frac{m}{n}\right),$$

$$W(d) = \sum_{\substack{r \leq w/d \\ (r, d)=1}} \varepsilon_z(r) \frac{\mu^2(r)}{f_1(r)}, \quad W = W(1),$$

$$\lambda(d) = \varepsilon_z(d) \mu(d) \prod_{p|d} \left(1 - \frac{1}{f(p)}\right)^{-1} \cdot \frac{W(d)}{W},$$

where  $\varepsilon_z(n)=0$  or 1 according as  $n$  has or has not a prime factor  $> z$ . Then we have, since  $\lambda(1)=1$ ,

$$S \leq \sum_{n \leq N} \left( \sum_{\substack{d \leq w \\ d|a_n}} \lambda(d) \right)^2 = \sum_{d \leq w^2} \left( \sum_{\substack{d_1, d_2 \leq w \\ \{d_1, d_2\} = d}} \lambda(d_1) \lambda(d_2) \right) \frac{N}{f(d)} \\ + \sum_{d_1, d_2 \leq w} |\lambda(d_1) \lambda(d_2) R(\{d_1, d_2\})|,$$

where  $\{d_1, d_2\}$  denotes the least common multiple of  $d_1$  and  $d_2$ .

Now

$$\begin{aligned} & \sum_{d \leq w^2} \left( \sum_{\substack{d_1, d_2 \leq w \\ \{d_1, d_2\} = d}} \lambda(d_1) \lambda(d_2) \right) \frac{1}{f(d)} \\ &= \sum_{r \leq w} f_1(r) \left( \sum_{\substack{d \leq w \\ d \equiv 0 \pmod{r}}} \frac{\lambda(d)}{f(d)} \right)^2 \\ &= \frac{1}{W^2} \sum_{r \leq w} f_1(r) \left( \sum_{\substack{d \leq w \\ d \equiv 0 \pmod{r}}} \varepsilon_z(d) \mu(d) \frac{1}{f_1(d)} \sum_{\substack{m \leq w/d \\ (m, d)=1}} \varepsilon_z(m) \frac{\mu^2(m)}{f_1(m)} \right)^2 \\ &= \frac{1}{W^2} \sum_{r \leq w} f_1(r) \left( \varepsilon_z(r) \frac{\mu(r)}{f_1(r)} \sum_{\substack{n \leq w/r \\ (n, r)=1}} \varepsilon_z(n) \frac{\mu^2(n)}{f_1(n)} \sum_{d|n} \mu(d) \right)^2 \\ &= \frac{1}{W^2} \sum_{r \leq w} \varepsilon_z(r) \frac{\mu^2(r)}{f_1(r)} = \frac{1}{W}. \end{aligned}$$

We thus have proved the following

**Theorem.** *Under the notations and conditions described above we have*

$$S \leq \frac{N}{W} + R$$

with

$$R = \sum_{d_1, d_2 \leq w} |\lambda(d_1) \lambda(d_2) R(\{d_1, d_2\})|.$$

This is a generalization of [3: II, Theorem 3.1].

To evaluate the remainder term  $R$  explicitly, let us suppose that for all positive integers  $d, d_1, d_2$

$$|R(d)| \leq B\omega(d), \quad \omega(\{d_1, d_2\}) \leq \omega(d_1)\omega(d_2),$$

where  $B > 0$  is a constant independent of  $d$ . These conditions imply

$$R \leq B \left( \sum_{d \leq w} \lambda(d) \omega(d) \right)^2.$$

Then, it is not difficult to show that we have, in general,

$$R = O\left(w^2(\log \log w)^2\right),$$

and, in the special case where  $\omega(p) \leq 1$  for all primes  $p \leq z$ ,

$$R = O\left(\frac{w^2}{W^2}\right),$$

where the constants implied in the symbol  $O$  depend only on the constant  $B$ .

The proof of these estimates of the remainder term  $R$  can easily be carried out just in the same way as in [6], and we shall omit the details (cf. also [7]).

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