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ON A REPRESENTATION OF A CERTAIN LINEAR TRANSFORMATION ON VECTOR LATTICES

By

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As an example of conditionally σ-complete vector lattices[1] we have various function spaces consisting in real valued measurable functions on a measure space Ω. On a function space, by a given measurable transformation $\omega \to \omega^t$ on Ω and a fixed measurable function $f_0(\omega)$ on Ω we can define, by the below $(*)$, a linear transformation T which has a suitably restricted domain and preserves an orthogonality,

$$(*) \qquad (Tf)(\omega) = f_0(\omega) f(\omega^t)$$

In this paper, on a conditionally σ-complete vector lattice we shall consider a certain linear transformation which preserves an orthogonality and give a representation of it, which may be considered as an abstract from of the above $(*)$. This is also a generalization of a representation of a *dilatator* which is defined by Nakano and investigated in detail in his book [1].

Briefly we shall state some fundamental facts about the theory of representations of conditionally σ-complete vector lattices, which is necessary for the description of our results. (cf. [1] Chap. I, III, VII)[2]. Let R be a conditionally σ-complete vector lattice having a order-complete element[3] e. For any element p of R we can define a projection operator[4] $[p]$; $[p]a = \bigcup_{n=1}^{\infty}(a^+ \frown n|p|) - \bigcup_{n=1}^{\infty}(a^- \frown n|p|)$[5], the set P of all projection operators $[p]$ is a σ-complete Boolean lattice[6]. The set E of all maximal ideals[7] $\mathfrak{p}$ of P is a totally disconnected compact Hausdorff space with the topology such that all of $U_{[p]} = \{\mathfrak{p} | \mathfrak{p} \ni [p]\}$

1) A vector lattice in which every upper bounded countable subset has a supremum, that is, *a continuous semi-ordered linear space* in Nakano's terminology, cf. [1] Chap. I §6.

2) We does not necessarily use notations and terminology in [1].

3) e is positive and $e \cap a = 0$ implies $a = 0$.

4) $[p]$ is called *projector* in [1], cf. Chap. I §7.

5) $a^+ = a \cup 0$, $a^- = (-a)^+$ and $|a| = a^+ + a^-$.

6) Cf. [1] Chap. I §8.

7) A family $\mathfrak{p}$ of projection operators is called *ideal*, if $\mathfrak{p} \neq 0$, $\mathfrak{p} \ni [p]$, $[p] \leqq [q]$ implies $\mathfrak{p} \ni [q]$ and $\mathfrak{p} \ni [p]$, $[q]$ implies $\mathfrak{p} \ni [p][q]$. An ideal is called to be *maximal*, if there is no other ideal including $\mathfrak{p}$.

constitutes an open base, namely E is a Stone's representation space[8] of P. Let $C^\infty(E)$ be all of almost finite[9] continuous functions on E, then in natural way $C^\infty(E)$ becomes a conditionally σ-complete vector lattice[10]. The representation[11] of R is as follows: There exists an isomorphism from R into $C^\infty(E)$, $a \to \varphi_a$, such that $\varphi_e = 1$ and all of representations of $a \in R$, denoted by $C_R(E)$, is a order-convex[12] subspace of $C^\infty(E)$.

Let $\mathfrak{p} \to \mathfrak{p}^t$ be a continuous transformation from some compact open subset U_0 of E to E and satisfy the following condition;

(1) *for any dense F_σ-open set U of E the inverse image U^{t-1} of U is also dense in U_0,*

then we see easily for any almost finite continuous function $\varphi(\mathfrak{p})$ on E $\varphi(\mathfrak{p}^t)$, putting $\varphi(\mathfrak{p}^t) = 0$ for $\mathfrak{p} \notin U_0$, is also an almyst finite continuous function on E. Furthermore, by composing the above point transformation and the multiplication by a fixed $\varphi_0 \in C^\infty(E)$ we have linear transformation $\hat{T}$ on $C^\infty(E)$, namely

$$(\hat{T}\varphi)(\mathfrak{p}) = \varphi_0(\mathfrak{p})\varphi(\mathfrak{p}^t) .$$

Naturally $\hat{T}$ induces a linear transformation T on R, but T has the restricted domain D;

$$D = \left\{ a \mid \varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t) \in C_R(E) \right\} ,$$

and T is represented in $C_R(E)$ as follows

$$(**) \qquad \varphi_{Ta}(\mathfrak{p}) = \varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t) \qquad (a \in D) .$$

Especially, if φ_0 belongs to $C_R(E)$, then T has the following properties;

(2) *D contains $[p]e$ for any projection operator $[p]$,*

(3) *T preserves an orthogonality, that is, $a \perp b$ for $a, b \in D$ implies $Ta \perp Tb$.*

(4) *T is a closed transformation with respect to the uniform convergence[13] in order and the orthogonal sum,*

In detail (4) means that if either $\{a_n\}$ and $\{Ta_n\}$ converge uniformly in order

8) E is called the *proper space* of R, cf. [1] Chap. III §15, §16.

9) An extended real valued function which is finite valued on a dense subset is called to be *almost finite.*

10) Cf. [1] Chap. VII §41 Theorems 41.1–41.3.

11) Cf. [1] Chap. III the second spectral theory.

12) $C_R(E) \ni \varphi$, $|\varphi| \geqq |\psi|$ implies $C_R(E) \ni \psi$.

13) A sequence $\{a_n\}$ is said to be *uniformly convergent* in order to a limit a if there exist a positive element l and a sequnce of positive numbers ε_n converging to 0 such that $|a_n - a| \leqq \varepsilon_n l$ $(n = 1, 2, \cdots)$.

to a and b respectively or $\{a_n\}$ is an orthogonal sequence and $a=\sum_{n=1}^{\infty} a_n$ and $b=\sum_{n=1}^{\infty} Ta_n$, then a belongs to D and $Ta=b$.

(2) and (3) are almost evident. (4) is as follows. Let $\chi_{[p]}$ be a characteristic function of $U_{[p]}$ and $[p]^t$ be defined by $U_{[p]^t}=\{\mathfrak{p}|\mathfrak{p}^t\in U_{[p]}\}$. If $\{a_n\}$ is an orthogonal sequence in D and $a=\sum_{n=1}^{\infty} a_n$ and $b=\sum_{n=1}^{\infty} Ta_n$, then $\chi_{[Ta_n]}(\mathfrak{p})\varphi_b(\mathfrak{p})=\varphi_{Ta_n}(\mathfrak{p})=\varphi_0(\mathfrak{p})\varphi_{a_n}(\mathfrak{p}^t)=\varphi_0(\mathfrak{p})\chi_{[a_n]^t}(\mathfrak{p}^t)\varphi_a(\mathfrak{p}^t)=\chi_{[a_n]^t}(\mathfrak{p})\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$ on E. According to (1), $[a]=\bigcup_{n=1}^{\infty}[a_n]$, implies $[a]^t=\bigcup_{n=1}^{\infty}[a_n]^t$, so $\sum_{n=1}^{\infty}\chi_{[a_n]^t}(\mathfrak{p})=\chi_{[a]^t}(\mathfrak{p})$ on some dense open set of E[14]. Therefore we have $\varphi_b(\mathfrak{p})=\sum_{n=1}^{\infty}\chi_{[Ta_n]}(\mathfrak{p})\varphi_b(\mathfrak{p})=\sum_{n=1}^{\infty}\chi_{[a_n]^t}(\mathfrak{p})\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)=\chi_{[a]^t}(\mathfrak{p})\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)=\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$ on some dense open set of E, that is, $a\in D$ and $Ta=b$. Next if $\{a_n\}$ and $\{Ta_n\}$ converge uniformly in order to a and b respectively, especially we assume $|a_n-a|\leqq\frac{1}{n}e$ $(n=1,2,\cdots)$, then $|\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)|\leqq\frac{1}{n}|\varphi_0(\mathfrak{p})|+|\varphi_0(\mathfrak{p})\varphi_{a_n}(\mathfrak{p}^t)|=\varphi_{\frac{1}{n}|Te|+|Ta_n|}(\mathfrak{p})$, as $C_R(E)$ is order-convex in $C^\infty(E)$, we can find $c\in R$ such that $\varphi_c(\mathfrak{p})=\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$ on E, that is, a belongs to D and $Ta=c$. Therefore we see $|Ta_n-Ta|\leqq\frac{1}{n}|Te|$ $(n=1,2,\cdots)$, namely $b=Ta$. For general case $|a_n-a|\leqq\frac{1}{n}l$ $(n=1,2,\cdots)$ for some positive element l, we can find an orthogonal sequence $\{[p_n]\}$ of projection operators such that $\bigcup_{n=1}^{\infty}[p_n]=[e]$ and all $[p_n]l$ are bounded with respect to e, that is, $0\leqq[p_n]l\leqq r_n e$ for some positive number r_n. As $T[p_\nu]a_n=[p_\nu]^tTa_n$ $(n=1,2,\cdots)$, applying the above argument we see $[p_\nu]a\in D$ and $T[p_\nu]a=[p_\nu]^tb$ for all $\nu=1,2,\cdots$. Again, by the closedness of T with respect to orthogonal sum it follows $a\in D$ and $Ta=b$.

Conversely we can obtain the following:

Theorem. *Let T be a linear transformation on R with domain D and satisfy the above three conditions* (2), (3) *and* (4), *then we can find a continuous transformation $\mathfrak{p}\to\mathfrak{p}^t$ from some compact open subset U_0 of E to E and an almost finite continuous function φ_0 on E, and T can be represented by the form* (**). *Namely,*

i) *the transformation t satisfies* (1),

ii) *a belongs to D if and only if $\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$ belongs to $C_R(E)$,*

iii) $\varphi_{Ta}(\mathfrak{p})=\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$ *for all* $a\in D$.

14) Cf. [1] Chap. III §16 Theorem 16.4.

Proof. At first T induces a transformation $[p]\to[p]^t$ on P defined by

$$[p]^t = \left[T[p]e\right]$$

This mapping t is a homomorphism from P to itself as a Boolean lattice. Because, if $[p]$ and $[q]$ are mutually orthogonal, then by (2) $[p]^t$ and $[q]^t$ are also mutually orthogonal and hence $([p]\cup[q])^t=([p]+[q])^t=[p]^t+[q]^t=[p]^t\cup[q]^t$. Therefore, shown easily, t is a homomorphism, that is, $([p]\cup[q])^t=[p]^t\cup[q]^t$ and $([p][q])^t=[p]^t[q]^t$ for all $[p]$ and $[q]$. Furthermore, if $[p]=\bigcup_{n=1}^{\infty}[p_n]$ then $[p]^t=\bigcup_{n=1}^{\infty}[p_n]^t$. Because, assuming $\{[p_n]\}$ to be mutually orthogonal without loss of generality, $[p]e=\sum_{n=1}^{\infty}[p_n]e$ and $b=\sum_{n=1}^{\infty}T[p_n]e=\sum_{n=1}^{\infty}[p_n]^tTe$ imply $[b]=\bigcup_{n=1}^{\infty}[p_n]^t$ and $b=T[p]e$ by the assumption (4), so we see $\bigcup_{n=1}^{\infty}[p_n]^t=[b]=[T[p]e]=[p]^t$. Next, this homomorphism induces a continuous transformation $\mathfrak{p}\to\mathfrak{p}^t$ from $U_{[Te]}$ to E defined by

$$\mathfrak{p}^t=\left\{[p]\mid[p]^t\in\mathfrak{p}\right\}\qquad(\mathfrak{p}\in U_{[Te]}).$$

By the definition $\mathfrak{p}\in U_{[p]^t}$ if and only if $\mathfrak{p}^t\in U_{[p]}$, so the inverse image of $U_{[p]}$ by t is $U_{[p]^t}$. As $[p]=\bigcup_{n=1}^{\infty}[p_n]$ implies $[p]^t=\bigcup_{n=1}^{\infty}[p_n]^t$, $U_{[p]}=\left(\bigcup_{n=1}^{\infty}U_{[p_n]}\right)^-$ and $U_{[p]^t}=\left(\bigcup_{n=1}^{\infty}U_{[p_n]^t}\right)^-$, and $\bigcup_{n=1}^{\infty}U_{[p_n]^t}$ is the inverse image of $\bigcup_{n=1}^{\infty}U_{[p_n]}$ hence t satisfies the condition (1).

For an element $a=\sum_{i=1}^{n}\xi_i[p_i]e$, $[p_i]$ $(1\leqq i\leqq n)$ are mutually orthogonal, evidently a belongs to D and $Ta=\sum_{i=1}^{n}\xi_iT[p_i]e=\sum_{i=1}^{n}\xi_i[p_i]^tTe$, so it follows $\varphi_{Ta}(\mathfrak{p})=\left(\sum_{i=1}^{n}\xi_i\chi_{[p_i]^t}(\mathfrak{p})\right)\varphi_{Te}(\mathfrak{p})=\varphi_a(\mathfrak{p}^t)\varphi_{Te}(\mathfrak{p})$. If a is a bounded element with respect to e, $|a|\leqq re$ for some $r>0$, then $\varphi_a(\mathfrak{p}^t)\varphi_{Te}(\mathfrak{p})=\varphi_c(\mathfrak{p})$ for some c by the order-convexity of $C_R(E)$ in $C^\infty(E)$, and we can find elements a_n with the above type such that $|a_n-a|\leqq\frac{1}{n}e$ $(n=1,2,\cdots)$, therefore $|Ta_n-c|\leqq\frac{1}{n}|Te|$ $(n=1,2,\cdots)$ and (4) implies $a\in D$ and $Ta=c$. For any $a\in D$ we can find $[p_n]$ $(n=1,2,\cdots)$ which are mutually orthogonal and $\bigcup_{n=1}^{\infty}[p_n]=[e]$ and $[p_n]a$ is a bounded element with respect to e for all $n=1,2,\cdots$, then $\sum_{n=1}^{\infty}[p_n]a=a$, and $\sum_{\nu=1}^{n}|T[p_\nu]a|=|\sum_{\nu=1}^{n}T[p_\nu]a|\leqq|Ta|$ for all $n=1,2,\cdots$ implies that $\sum_{\nu=1}^{n}T[p_\nu]a$ converges to some b in order; $b=\sum_{n=1}^{\infty}T[p_n]a$. So it follows $Ta=b$ from (4) and $\varphi_b(\mathfrak{p})=$

$\sum_{n=1}^{\infty}\varphi_{T[p_n]a}(\mathfrak{p})=\sum_{n=1}^{\infty}\varphi_{Te}(\mathfrak{p})\chi_{[p_n]^t}(\mathfrak{p})\varphi_a(\mathfrak{p}^t)=\varphi_{Te}(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$. Finally if $\varphi_{Te}(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$ belongs to $C_R(E)$ and we put $\varphi_b(\mathfrak{p})=\varphi_{Te}(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$, then similarly to the above it can be proved from (4) $a\in D$ and $b=Ta$. The uniqueness of such representation of T as type (**) is almost evident.

Reference

[1] H. NAKANO: Modern spectral theory, Tokyo Mathematical Book Series, Vol. 2 (1950).

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