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A CHARACTERIZATION OF STRONGLY SEPARABLE ALGEBRAS^{*)}

By

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§ 1. In his paper [2] T. Kanzaki introduced the notion of strongly separable algebras over a commutative ring and obtained an interesting characterization of such algebras [2, Theorem 1].

Throughout the present note, $A \ni 1$ will represent always an algebra over a commutative ring $R \ni 1$, and C the center of A .

Let P be the set of elements $\sum x_i \otimes y_i$ in $A \otimes_R A$ such that $\sum x_i x \otimes y_i = \sum x_i \otimes x y_i$ for all x in A , and let φ be the A - A -(module-) homomorphism of $A \otimes_R A$ into A defined by $\varphi(\sum x_i \otimes y_i) = \sum x_i y_i$. If $\varphi(P)$ contains 1, then A is said to be a strongly separable (R -) algebra. (The definition is somewhat different from the original one in [2]. But, as is easily seen, the two definitions are equivalent).

The purpose of the present note is to give another characterization of a strongly separable algebra A when it is R -finitely generated and projective (Theorem 2).

§ 2. A is said to be a Frobenius (resp. symmetric) algebra if A is a finitely generated, projective R -module and there exists an A -isomorphism: $A_A \cong \text{Hom}_R(A, R)_A$ (resp. ${}_A A_A \cong {}_A \text{Hom}_R(A, R)_A$), where $\text{Hom}_R(A, R)$ is regarded as an A - A -module by the following operations:

$$bfa(x) = f(axb) \quad a, b, x \in A, f \in \text{Hom}_R(A, R)$$

At first we shall quote here the following theorem which is due to Kanzaki [2, Theorem 1].

Theorem 1. *A is a strongly separable (R -) algebra if and only if A is a separable (R -) algebra and $A = C \oplus [A, A]$, as C -module, where $[A, A]$ is the C -submodule of A generated by all $xy - yx$ (x, y in A).*

Lemma 1. *Let $B \ni 1$ be a ring, and Z the center of B . If $B = Z \oplus [B, B]$ as Z -module, then $\text{Hom}_Z(B, Z)^B$ (the set of elements f in $\text{Hom}_Z(B, Z)$ such that $af = fa$ for all a in B) is a free Z -module of rank 1.*

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Proof. Let f be an element in $\text{Hom}_Z(B, Z)^B$. Then $f(xy - yx) = 0$ for all x, y in B . Since $B = Z \oplus [B, B]$ f is uniquely determined by the effect on Z , namely, by $f(1)$. Thus we see that $\text{Hom}_Z(B, Z)^B$ is Z -isomorphic to Z .

Lemma 2. *Let A be R -finitely generated and projective. If A is a strongly separable (R -) algebra, then A is a symmetric (R -) algebra.*

Proof. By Theorem 1 and Theorem 2.3 [1], A is a central separable C -algebra. Then, by Theorem 3.1 (c) [1], we have an A - A -isomorphism:

$$A \otimes_C \text{Hom}_C(A, C)^A \cong \text{Hom}_C(A, C),$$

where the left side term is regarded as an A - A -module by the operations on the first factor. Since $\text{Hom}_C(A, C)^A$ is a free C -module of rank 1 by Theorem 1 and Lemma 1, the above isomorphism induces naturally the A - A -isomorphism

$${}_A A_A \cong \text{Hom}_C(A, C).$$

Thus A is a symmetric C -algebra. By Proposition A. 4 [1], C is a commutative Frobenius whence symmetric R -algebra. Then by Theorem 2 [3] we can conclude that A is a symmetric R -algebra.

Theorem 2. *Let A be R -finitely generated and projective. Then, A is a strongly separable R -algebra if and only if there exist elements $r_1, r_2, \dots, r_n; l_1, l_2, \dots, l_n$ in A and a homomorphism h in $\text{Hom}_R(A, R)$ such that*

$$(*) \quad a = \sum r_i h(l_i a) = \sum h(ar_i) l_i, \quad h(xy) = h(yx)$$

for all a, x, y in A and that $\sum r_i l_i$ is a unit in C .

Proof. Let A be a strongly separable R -algebra. Then by Lemma 2 A is a symmetric algebra, whence by Theorem 1 [3] there exist elements $r_1, r_2, \dots, r_n; l_1, l_2, \dots, l_n$ in A and h in $\text{Hom}_R(A, R)$ such that $(*)$ hold for all a, x, y in A . When this is the case, the mapping

$$A \ni a \longrightarrow ha \in \text{Hom}_R(A, R)$$

gives an A - A -isomorphism of ${}_A A_A$ onto ${}_A \text{Hom}_R(A, R)_A$.

Now, consider the A - A -isomorphism:

$${}_A A \otimes_R A_A \longrightarrow {}_A \text{Hom}_R(\text{Hom}_R(A, R), A)_A \longrightarrow {}_A \text{Hom}_R(A, A)_A$$

defined by

$$x \otimes y \longrightarrow (f = ha \longrightarrow f(x)y = h(ax)y) \longrightarrow (a \longrightarrow f(x)y = h(ax)y) \\ x, y, a \in A, \quad f \in \text{Hom}_R(A, R).$$

Then, as is easily seen, the image of P in $\text{Hom}_R(A, A)$ is just $\text{Hom}_A(A, A) = A_r$,

the right multiplication ring of A . Since the image of $\sum r_i \otimes l_i a$ is

$$(x \rightarrow \sum h(xr_i)l_i a = xa) = a_r,$$

we see that P is the set $\{\sum r_i \otimes l_i a | a \in A\}$. Since A is a strongly separable R -algebra there exists an element a_0 in A such that $\sum r_i l_i a_0 = 1$. Noting here that $\sum r_i l_i$ is in the center of A , we see that $\sum r_i l_i$ is a unit in C . The converse part of the theorem is almost obvious.

References

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