NOTE ON q-GALOIS EXTENSIONS OF SIMPLE RINGS

By

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In the previous paper [2], the notions of q-Galois extensions and h-q-Galois extensions of simple rings were introduced with a view to unifying all the earlier works cited at the end of [2] under the assumption that the simple ring extension in question is left locally finite and h-q-Galois. Recently, in his paper [1], T. Nagahara has obtained some useful q-Galois conditions and proved that any left locally finite q-Galois extension is necessarily h-q-Galois ([1, Th. 8]).

The purpose of this note is to present a direct proof to the last. In fact, the sharpening of [2, Th. 1] together with the modification of the proof of [2, Cor. 1] enables us to complete our proof. It will be noteworthy that all the results are obtained without the assumption that V is simple.

As to notations and terminologies used in this note, we follow the previous paper [2]. Moreover, as in [1], by $\mathfrak{R}$ we denote the set of all regular intermediate rings of A/B, and we shall use the following notations: $\mathfrak{R}^0=\{A'\in\mathfrak{R};\ [A'|A']=[A|A]\}$, $\mathfrak{R}_{l.f.}=\{A'\in\mathfrak{R};\ [A':B]_l<\infty\}$, $\mathfrak{R}^0_{l.f.}=\mathfrak{R}^0\cap\mathfrak{R}_{l.f.}$, and for any subset S of A we set $\mathfrak{R}/S=\{A'\in\mathfrak{R};\ A'\supseteq S\}$, $\mathfrak{R}^0/S=\mathfrak{R}^0\cap\mathfrak{R}/S$, $\mathfrak{R}_{l.f.}/S=\mathfrak{R}_{l.f.}\cap\mathfrak{R}/S$ and $\mathfrak{R}^0_{l.f.}/S=\mathfrak{R}^0_{l.f.}\cap\mathfrak{R}/S$.

At first, to our end, [2, Th. 1] (that was useful in [1], too) has to be improved as follows:

Theorem 1. *Assume that (a simple ring) A is left locally finite over (a unital simple subring) B and $\mathrm{Hom}_{B_l}(T, A)=\mathfrak{G}(T, A/B)A_r$ for each $T\in\mathfrak{R}^0_{l.f.}$.*

(a) *If B' is a simple intermediate ring of A/B with $[B':B]_l<\infty$, then for any finite subset F of A there exists $T'\in\mathfrak{R}^0_{l.f.}/B'[F]$ such that each intermediate ring A' of A/T' is B'–A'-completely reducible.*

(b) *If B' is in $\mathfrak{R}_{l.f.}$, then for any finite subset F of A there exists $T'\in\mathfrak{R}^0_{l.f.}/B'[F]$ such that each intermediate ring A' of A/T' is homogeneously B'–A'-completely reducible with $[A'|B'_l\cdot A'_r]=[V_A(B')|V_A(B')]$ (and $[V_{A'}(B):V_{A'}(B')]_r<\infty$).*

Proof. (a) Let M be a minimal B'–A-submodule of A such that the composition series of M as A-module is of the shortest length among minimal B'–A-submodules of A. Then, $M=eA$ with a non-zero idempotent e. In virtue of [2, Lemma 1], we can find $T^*\in\mathfrak{R}^0_{l.f.}/B'[e, F]$ with $[eT^*|T^*]=[M|A]$.

One may remark here that eT^* is a B'–T^*-submodule of T^*, for $B'e \subseteq B'eA \cap T^* = eA \cap T^* = eT^*$. Since $\mathrm{Hom}_{B_l}(T^*, A) = \mathfrak{G}(T^*, A/B)A_r = \sum_1^s \tau_i A_r$ with some $\tau_i \in \mathfrak{G}(T^*, A/B)$, there exists $T' \in \mathfrak{R}^0_{l.f.}/T^*$ such that $\mathrm{Hom}_{B_l}(T^*, A') = \mathfrak{G}(T^*, A'/B)A'_r$ for each intermediate ring A' of A/T'. By the above remark $B'eT^* = eT^*$ and [2, Lemma 1], one will easily see that $M' = eA'$ is a (minimal) B'–A'-submodule of A' such that the length of the composition series of M' as A'-module coincides with $[M|A] = [eT^*|T^*]$ (and so $[M'|A'] \leqslant [N'|A']$ for each non-zero B'–A'-submodule N' of A'). Since $\mathrm{Hom}_{B_l}(T^*, A')$ is T^*_r–A'_r-completely reducible by [2, Lemma 2 (a)], the T^*_r-A'_r-submodule $\mathrm{Hom}_{B'_l}(T^*, A') = \oplus_1^t \mathfrak{M}_j$ with some irreducible $\mathfrak{M}_j$'s. By [2, Lemma 2 (b)], $\mathfrak{M}_j = \sigma_j u_{jl} A'_r$ with some $\sigma_j \in \mathfrak{G}(T^*, A'/B)$ and non-zero $u_j \in V_{A'}(B)$. Since $\mathfrak{M}_j$ is contained in $\mathrm{Hom}_{B'_l}(T^*, A')$ and $B'e \subseteq eT^* \subseteq T^*$, each $M_j = (B'e)\mathfrak{M}_j$ is a B'–A'-submodule of A'. Further, there holds $M_j = u_j \cdot (B'e)\sigma_j \cdot A' = u_j \cdot (B'eT^*)\sigma_j \cdot A' = u_j \cdot (eT^*)\sigma_j \cdot A' = u_j \cdot e\sigma_j \cdot A'$, whence it follows $[M_j | A'] = [u_j \cdot e\sigma_j \cdot A' \,|\, A'] \leqslant [e\sigma_j \cdot A' | A'] \leqslant [e\sigma_j \cdot T^*\sigma_j | T^*\sigma_j] = [eT^*|T^*] = [M'|A']$ by [2, Lemma 1]. Recalling here that $[M'|A']$ is the least, we see that each M_j is either 0 or B'–A'-irreducible. Finally, noting that A' is $B'_l \cdot \mathrm{Hom}_{B'_l}(A', A')$-irreducible, there holds $A' = e(B'_l \cdot \mathrm{Hom}_{B'_l}(A', A')) = (B'e)\,\mathrm{Hom}_{B'_l}(T^*, A') = (B'e)\sum_1^t \mathfrak{M}_j = \sum_1^t M_j$, which proves evidently the complete reducibility of A' as B'–A'-module.

(b) Let $V' = V_A(B') = \sum U' g'_{pq}$, where $\Gamma' = \{g'_{pq}\text{'s}\}$ is a system of matrix units and $U' = V_{V'}(\Gamma')$ a division ring. Let $M = eA$ be chosen as in (a), and T^* a member of $\mathfrak{R}^0_{l.f.}/B'[e, F, \Gamma']$ such that $[eT^*|T^*] = [M|A]$. Then, there exists $T' \in \mathfrak{R}^0_{l.f.}/T^*$ such that $\mathrm{Hom}_{B_l}(T^*, A') = \mathfrak{G}(T^*, A'/B)A'_r$ for each intermedidate ring A' of A/T'. Since $\mathrm{Hom}_{B'_l \cdot A'_r}(A', A')$ coincides with the simple ring $(V_{A'}(B'))_l$, the proof of (a) enables us to see that A' is homogeneously B'–A'-completely reducible and $[A'|B'_l \cdot A'_r] = [V_{A'}(B') | V_{A'}(B')] = [V'|V']$. Accordingly, A' is $B' \cdot V_{A'}(B')$–A'-irreducible, and hence $[V_{A'}(B) : V_{A'}(B')]_r \leqslant [B' : B]_l < \infty$ by [2, Prop. 1 (b)].

Lemma 1. *Assume that A/B is left locally finite and* $\mathrm{Hom}_{B_l}(T, A) = \mathfrak{G}(T, A/B)A_r$ *for each* $T \in \mathfrak{R}^0_{l.f.}$.

(a) *If ρ is a B-ring homomorphism of an intermediate ring A_1 of A/B with $[A_1 : B]_l < \infty$ into A such that A is $A_1\rho$–A-irreducible, then ρ is contained in $\mathfrak{G}(A_0, A/B) | A_1$ for each $A_0 \in \mathfrak{R}^0_{l.f.}/A_1$.*

(b) *If B' is in $\mathfrak{R}_{l.f.}$ then $\mathfrak{G}(T, A/B)|B' \subseteq \mathfrak{G}(B', A/B)$ for each $T \in \mathfrak{R}^0_{l.f.}/B'$, and so* $\mathrm{Hom}_{B_l}(B', A) = \mathfrak{G}(B', A/B)\,A_r$.

Proof. The proof of (a) will be obvious by that of [2, Lemma 4]. In what follows, we shall prove (b). Let $V' = V_A(B') = \sum U' g'_{pq}$, where $\Gamma' = \{g'_{pq}\text{'s}\}$ is a system of matrix units and $U' = V_{V'}(\Gamma')$ a division ring. We set $\bar{B} = B'[\Gamma']$

and $\bar{T}=T[\Gamma']$ Now, let σ be an arbitrary element of $\mathfrak{G}(T, A/B)$. By (a), $\sigma=\bar{\sigma}|T$ for some $\bar{\sigma}\in\mathfrak{G}(\bar{T}, A/B)$. Obviously, $V_A(B'\sigma)=V_A(B'\bar{\sigma})$ cotains $\Gamma'\bar{\sigma}$ as a system of matrix units and the centralizer of $\Gamma'\bar{\sigma}$ in $V_A(B'\sigma)$ coincides with $V_A(\bar{B}\bar{\sigma})$. If a is an arbitrary non-zero element of $V_A(\bar{B}\bar{\sigma})$ and $T^*=(\bar{T}\bar{\sigma})[a]$ $(\in\mathfrak{R}^{\circ}_{l.f.})$, then $\bar{\sigma}^{-1}=\tau^*|\bar{T}\bar{\sigma}$ for some $\tau^*\in\mathfrak{G}(T^*, A/B)$ again by (a). For any $x\in\bar{B}$, the equality $x\bar{\sigma}\cdot a=a\cdot x\bar{\sigma}$ yields $x\cdot a\tau^*=a\tau^*\cdot x$. Hence, $a\tau^*$ is contained in the division ring $V_A(\bar{B})=U'$. Accordingly, $a\tau^*$ is a regular element of $T^*\tau^*$, and so a is a regular element. We have proved thus $V_A(\bar{B}\bar{\sigma})$ is a division ring, which implies that $V_A(B'\sigma)=\sum V_A(\bar{B}\bar{\sigma})\cdot g'_{pq}\bar{\sigma}$ is a simple ring.

Theorem 2 ([1, Th. 2]). *Assume that A/B is left locally finite and* $\mathrm{Hom}_{B_l}(T, A)=\mathfrak{G}(T, A/B)A_r$ *for each* $T\in\mathfrak{R}^{\circ}_{l.f.}$. *If* $B_1\supseteq B_2$ *are in* $\mathfrak{R}_{l.f.}$ *then* $\mathfrak{G}(B_1, A/B)|B_2=\mathfrak{G}(B_2, A/B)$.

Proof. By Lemma 1 (b), the argument in the proof of [2, Th. 3] applies to obtain $\mathfrak{G}(B_2, A/B)\subseteq\mathfrak{G}(B_1, A/B)|B_2$. If T is an arbitrary member of $\mathfrak{R}^{\circ}_{l.f.}/B_1$, it follows obviously $\mathfrak{G}(B_1, A/B)\subseteq\mathfrak{G}(T, A/B)|B_1$. Accordingly, again by Lemma 1 (b), we obtain $\mathfrak{G}(B_1, A/B)|B_2\subseteq(\mathfrak{G}(T, A/B)|B_1)|B_2=\mathfrak{G}(T, A/B)|B_2 \subseteq\mathfrak{G}(B_2, A/B)$, which completes our proof.

In [2], A/B was defined to be q-Galois if (1) B is regular, (2) $\mathrm{Hom}_{B_l}(B', A)=\mathfrak{G}(B', A/B)A_r$ for each $B'\in\mathfrak{R}_{l.f.}$ and (3) $\mathfrak{G}(B_1, A/B)|B_2\subseteq\mathfrak{G}(B_2, A/B)$ for each $B_1\supseteq B_2$ in $\mathfrak{R}_{l.f.}$. However, Th. 2 implies that A/B is left locally finite and q-Galois if and only if B is regular, A/B is left locally finite and $\mathrm{Hom}_{B_l}(T, A)=\mathfrak{G}(T, A/B)A_r$ for each $T\in\mathfrak{R}^{\circ}_{l.f.}$. This fact has been pointed out in [1]. Moreover, [1, Th. 8] asserts that if A/B is left locally finite and q-Galois then so is A/B' for each $B'\in\mathfrak{R}_{l.f.}$, namely, A/B is h-q-Galois. This fact is obviously contained in the following theorem.

Theorem 3. *Assume that A/B is left locally finite and* $\mathrm{Hom}_{B_l}(T, A)=\mathfrak{G}(T, A/B)A_r$ *for each* $T\in\mathfrak{R}^{\circ}_{l.f.}$. *If B' is in* $\mathfrak{R}_{l.f.}$ *then* $\mathrm{Hom}_{B'_l}(B'', A)=\mathfrak{G}(B'', A/B')A_r$ *for each* $B''\in\mathfrak{R}_{l.f.}/B'$.

Proof. Let T be an arbitrary member of $\mathfrak{R}^{\circ}_{l.f.}/B'$. Evidently, $\mathrm{Hom}_{B'_l}(T, A)$ is a T_r–A_r-submodule of $\mathrm{Hom}_{B_l}(T, A)=\mathfrak{G}(T, A/B)A_r$ and then $\mathrm{Hom}_{B'_l}(T, A)=\bigoplus_1^s\sigma_i u_{il}A_r$ with some $\sigma_i\in\mathfrak{G}(T, A/B)$ and non-zero $u_i\in V$ [2, Lemma 2 (a), (b)]). By Th. 1 (b), there exists $T'\in\mathfrak{R}^{\circ}_{l.f.}/T$ such that each intermediate ring A' of A/T' is homogeneously B'–A'-completely reducible with $[A'|B'_l\cdot A'_r]=[V_A(B')|V_A(B')]$ and $\mathrm{Hom}_{B_l}(T, A')=\mathfrak{G}(T, A'/B)A'_r$. Now, let σu_l be an arbitrary $\sigma_i u_{il}$. By Th. 2, there exists $\sigma^*\in\mathfrak{G}(T', A/B)$ such that $\sigma=\sigma^*|T$. If $A''=(T'\sigma^*)[T', u](\in\mathfrak{R}^{\circ}_{l.f.}/T')$, then $\sigma^{*-1}=\tau''|T'\sigma^*$ with some $\tau''\in\mathfrak{G}(A'', A/B)$ again by Th. 2. We set here $A'=A''\tau''$ $(\in\mathfrak{R}^{\circ}_{l.f.}/T')$ and $\sigma'=\tau''^{-1}$. Let M_0 be a minimal B'–A'-submodule of A' such that $N_0=M_0\sigma u_l$ is non-zero (and so B'–

A''-irreducible). Obviously, $A' = M_0 \oplus M_1 \oplus \cdots \oplus M_t$ and $A'' = N_0 \oplus N_1 \oplus \cdots \oplus N_t$ with some B'–A'-isomorphic M_i's and B'–A''-isomorphic N_i's, where $t+1 = [V_A(B') | V_A(B')]$. Then, we can easily find such a B'-isomorphism ν of A' onto A'' that $a'_r \nu = \nu (a' \sigma')_r$ for each $a' \in A'$. As $\nu^{-1} \sigma' u_l$ is contained in $\mathrm{Hom}_{B'_l \cdot A''_r}(A'', A'') = (V_{A''}(B'))_l$ (simple), $\sigma' u_l = \nu v_{1l} + \cdots + \nu v_{ml}$ with some regular elements v_j's in $V_{A''}(B')$. Noting that T is in $\mathfrak{R}^0_{l.f.}$, one will easily see that every $(\nu v_{jl} | T) A''_r$ is a T_r–A''_r-irreducible submodule of $\mathrm{Hom}_{B'_l}(T, A'') \subseteq \mathrm{Hom}_{B_l}(T, A'') = \mathfrak{G}(T, A''/B) A''_r$. It follows therefore $(\nu v_{jl} | T) A''_r = \tau w_{jl} A''_r$ with some $\tau \in \mathfrak{G}(T, A''/B)$ and non-zero $w_j \in V_{A''}(B)$ ([2, Lemma 2 (b)]). We have then $A'' = v_j A'' = v_j \cdot A' \nu = v_j \cdot (T \cdot A') \nu = v_j \cdot T \nu \cdot A'' = T (\nu v_{jl} | T) A''_r = w_j \cdot T \tau \cdot A'' = w_j A''$, whence it follows that w_j is a regular element of $V_{A''}(B)$. Hence, $\tau \widetilde{w}_j$ is contained in $\mathfrak{G}(T, A''/B) \cap \mathrm{Hom}_{B'_l}(T, A''/B')$. We have proved thus $\sigma u_l = \sigma' u_l | T$ is contained in $\mathfrak{G}(T, A/B') A_r$, and so $\mathrm{Hom}_{B'_l}(T, A) = \mathfrak{G}(T, A/B') A_r$. Then, our assertion is a consequence of Lemma 1 (b).

Now, [2, Cor. 1] can be restated as follows:

Corollary 1. *Let A be left locally finite over B, and $\mathfrak{H}$ a subset of $\mathfrak{G}$ with $\mathfrak{H}\tilde{V} = \mathfrak{H}$. If $\mathfrak{H} A_r$ is dense in* $\mathrm{Hom}_{B_l}(A, A)$ *then $\mathfrak{G}(B', A/B) = \mathfrak{H} | B'$, $\mathfrak{H}(B') A_r$ is dense in* $\mathrm{Hom}_{B'_l}(A, A)$ *and $J(\mathfrak{H}(B'), A) = B'$ for each $B' \in \mathfrak{R}_{l.f.}$.*

Proof. If T is in $\mathfrak{R}^0_{l.f.}$, then $\mathfrak{G}(T, A/B) \subseteq \mathrm{Hom}_{B_l}(T, A) = (\mathfrak{H} | T) A_r$ yields $\mathrm{Hom}_{B_l}(T, A) = \mathfrak{G}(T, A/B) A_r$ and $\mathfrak{G}(T, A/B) = \mathfrak{H} | T$ ([2, Lemma 2 (c)]). Accordingly, for any $A' \in \mathfrak{R}^0_{l.f.}/B'$ there holds $\mathfrak{G}(B', A/B) = \mathfrak{G}(A', A/B) | B' = (\mathfrak{H} | A') | B' = \mathfrak{H} | B'$ (Th. 2). Now, our assertion will be easy by Ths. 2, 3 and [2, Remark 1].

Let B' be an intermediate ring of A/B such that A is B'–A-irreducible and $[B' : B]_l < \infty$. Then $A = B' \mathfrak{l}' A = \sum_{a \in A} \mathfrak{l}' a$ for an arbitrary minimal left ideal $\mathfrak{l}'$ of B', which means that A is homogeneously B'-completely reducible. Hence, B' is also homogeneously B'-completely reducible, and so B' is contained in $\mathfrak{R}_{l.f.}$. Combining this fact with [4, Lemma 1.1], we readily see that *if A is left locally finite over B and B–A-irreducible then every intermediate ring of A/B is in $\mathfrak{R}$.* The first assertion of the following corollary is a direct consequence of Th. 1 (b), [2, Remark 4] and the last remark. Moreover, Th. 3 and [2, Remark 1] yield the second one.

Corollary 2 (cf. [1, Ths. 10, 7 and Lemma 16]). *Assume that A/B is left locally finite and* $\mathrm{Hom}_{B_l}(T, A) = \mathfrak{G}(T, A/B) A_r$ *for each $T \in \mathfrak{R}^0_{l.f.}$.*

(a) *If V is a division ring then every intermediate ring of A/B is simple, and conversely.*

(b) *If B' is in $\mathfrak{R}_{l.f.}$ then $J(\mathfrak{G}(B'', A/B'), B'') = B'$ for each $B'' \in \mathfrak{R}_{l.f.}/B$. In particular, if B' is an intermediate ring of A/B such that A is B'–A-irreducible and $[B' : B]_l < \infty$ then $J(\mathfrak{G}(B'', A/B'), B'') = B'$ for each $B'' \in \mathfrak{R}_{l.f.}/B'$.*

Patterning after the proof of [3, Th. 1], one will readily see that [3, Th. 1] is still valid under the same assumption as in Th. 3. Moreover, taking the validity of Th. 3 into the mind, it will be obvious that the proof of [3, Th. 2] is efficient even under the same assumption as in Th. 3. Thus, we obtain the following:

Theorem 4. *Assume that* A/B *is left locally finite and* $\mathrm{Hom}_{B_l}(T, A)=\mathfrak{G}(T, A/B)A_r$ *for each* $T\in\mathfrak{R}^0_{l.f.}$. *Let* B' *be a simple intermediate ring of* A/B, *and* δ *a derivation of* B' *into* A *vanishing on* B. *If* $[B':B]_l<\infty$ *or* B' *is f-regular then* δ *can be extended to an inner derivation of* A.

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