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QUASI-PROJECTIVE MODULES, PERFECT MODULES, AND A THEOREM FOR MODULAR LATTICES

By

Yôichi MIYASHITA

Contents

§ 0. **Introduction.** Throughout the present paper, R is a ring with 1, and M a unital R-left module. "Submodule" and "homomorphism" will mean "R-submodule" and "R-homomorphism", respectively. For any R-submodule A of M, we denote by $\nu(M\to M/A)$ and $\iota(A\to M)$ the projection of M onto M/A and the injection of A into M, respectively. M is called R-perfect if for any pair of submodules A, B of M with $A+B=M$ there exists a submodule B_0 of B that is minimal with respect to the property that $A+B_0=M$. In this case, B_0 is called a d-complement of A (in M). A submodule D of M is said to be d-dense in M, if $D+X\neq M$ for all proper submodule X. Then, $\mathfrak{R}({}_RM)$ (radical of M, i.e. the meet of all maximal submodules of M) = the sum of all d-dense submodules (Prop. 1.4). From this point of view, we should like to have another look at radicals of modules (Th. 2.11, Th. 2.12 and Th. 4.3). M is called R-quasi-projective if for any submodule B of M, $\mathrm{Hom}({}_RM, {}_RM/B) = \mathrm{Hom}({}_RM, {}_RM)\cdot\nu(M\to M/B)$ holds. If M is R-quasi-projective and A, B are d-complements of each other then $M=A\oplus B$. By the light of this fact, we shall prove the following: If ${}_RP$ is an R-projective module, then the following conditions for P are equivalent. (i) P is R-perfect. (ii) Every homomorphic image of P has a projective cover. (iii) $\mathfrak{R}({}_RP)$ is d-dense in P, and P is a direct sum of sum-irreducible submodules, where a sum-irreducible module is a module such that the sum of its two proper submodules is always proper (Th. 3.3 and Th. 3.7).

R is called a semi-perfect ring, if ${}_RR$ is perfect, and R is called h-central if R is quasi-projective as an R-R-module. As a generalization of Wedderburn's structure theorem for semi-simple rings, we shall obtain the following: R is a semi-perfect h-central ring if and only if R is a direct sum of a finite number of finite dimensional matrix rings over sum-irreducible h-central rings (Th. 4.5).

In §5, we shall generalize the uniqueness theorem of maximal independent sets of uniform submodules which was given in the preceding paper [8], namely, Th. 5.9 is valid for any modular lattice with 0. In virtue of this generalization, we can present another proof to Azumaya's generalization of Krull-Remak-Schmidt's theorem.

§ 1. Preliminary results and perfect modules.

A submodule D of M is said to be (R-) *d-dense* in M, if $D+X \neq M$ for any proper submodule X of M. If D is d-dense in $M \neq 0$, then $D \neq M$. Evidently $\{0\}$ is d-dense in M.

Proposition 1.1. (1) *If A and B are d-dense submodules of M, then so is $A+B$.* (2) *Let A and B be submodules of M with $A \supseteqq B$. If B is d-dense in A then B is d-dense in M.* (3) *Let φ be a homomorphism from M to an R-left module N. If B is a d-dense submodule of M, then so is $B\varphi$ in N.*

Proof. (1) will be easily seen. (2) If $B+X=M$ for some submodule X, then $A=A \cap (B+X)=B+(A \cap X)$. Since B is d-dense in A, we have $A \cap X=A$, that is, $A \subseteqq X$. Therefore, as $B \subseteqq A \subseteqq X$, $M=B+X=X$. (3) By (2), we may assume that $M\varphi=N$. If $B\varphi+Y=N$ for some submodule $Y \subseteqq N$, then $M=B+Y\varphi^{-1}$, where $Y\varphi^{-1}=\{m \in M;\ m\varphi \in Y\}$. Since B is d-dense in M, we have $M=Y\varphi^{-1}$, and therefore $N=M\varphi=(Y\varphi^{-1})\varphi=Y$.

The sum of all d-dense submodules of M is called the *radical* of ${}_RM$, and denoted by $\mathfrak{R}({}_RM)$. The following is a direct consequence from Prop. 1.1.

Proposition 1.2. (1) $\mathfrak{R}({}_RM) \cdot \mathrm{Hom}({}_RM, {}_RM) \subseteqq \mathfrak{R}({}_RM)$. (2) *Every finitely generated submodule of $\mathfrak{R}({}_RM)$ is d-dense in M.* (3) *For any submodule A of M, $\mathfrak{R}({}_RA) \subseteqq \mathfrak{R}({}_RM)$.*

Let $M=A+B$, where A and B are submodules of M. If B is minimal with respect to the property $A+B=M$, then B is called a *d-complement* of A (in M). A submodule B is called a *d-complemented submodule* of M if B is a d-complement of some A.

Proposition 1.3. *Let $M=A+B$, where A and B are submodules of M. If B is a d-complement of A then $A \cap B$ is d-dense in B, and conversely. If B is a d-complement of A, $D \cap B$ is d-dense in B for any d-dense submodule D of M, and $\mathfrak{R}({}_RB)$ coincides with $\mathfrak{R}({}_RM) \cap B$.*

Proof. Let B be a d-complement of A. If $(A\cap B)+X=B$ for some submodule X of B, then $M=A+B=A+X$. The minimality of B implies $X=B$. Hence $A\cap B$ is d-dense in B. Conversely, assume that $A\cap B$ is d-dense in B. Let $A+B_0=M$ for some submodule B_0 of B. Then $B=B\cap(A+B_0)=(A\cap B)+B_0$. Since $A\cap B$ is d-dense in B, we have $B=B_0$. Hence B is a d-complement of A. Let D be a d-dense submodule of M, and let $B=(D\cap B)+X$ for some submodule X. Then $M=A+B=A+(D\cap B)+X$. Since $D\cap B(\subseteqq D)$ is d-dense in M, we have $M=A+X$. Then, the minimality of B implies $X=B$. Hence $D\cap B$ is d-dense in B. By Prop. 1.2, $\mathfrak{R}({}_RB)\subseteqq\mathfrak{R}({}_RM)\cap B$. Conversely, for any $x\in\mathfrak{R}({}_RM)\cap B$, Rx is d-dense in M, and therefore $Rx=Rx\cap B$ is d-dense in B.

Corollary. *If* $M=\sum_{\lambda\in\Lambda}\oplus M_\lambda$ *with submodules* M_λ *then* $\mathfrak{R}({}_RM)=\sum_{\lambda\in\Lambda}\oplus\mathfrak{R}({}_RM_\lambda)$.

Proof. Since $\mathfrak{R}({}_RM)\cdot\mathrm{Hom}({}_RM,{}_RM)\subseteqq\mathfrak{R}({}_RM)$, we have $\mathfrak{R}({}_RM)=\sum_{\lambda\in\Lambda}\oplus(\mathfrak{R}({}_RM)\cap M_\lambda)$. Since any direct summand is a d-complemented submodule, we have $\mathfrak{R}({}_RM)\cap M_\lambda=\mathfrak{R}({}_RM_\lambda)$ by Prop. 1.3, and so $\mathfrak{R}({}_RM)=\sum_{\lambda\in\Lambda}\oplus\mathfrak{R}({}_RM_\lambda)$.

Proposition 1.4. $\mathfrak{R}({}_RM)\neq M$ *if and only if* M *has a maximal submodule. When this is the case,* $\mathfrak{R}({}_RM)$ *coincides with the meet of all maximal submodules of* M.

Proof. Assume $\mathfrak{R}({}_RM)\neq M$. Evidently every d-dense submodule of M is contained in all maximal submodules of M. Let x be any element of M with $x\notin\mathfrak{R}({}_RM)$, and $Rx+X=M$ for some proper submodule X. By Zorn's lemma, X is contained in some maximal submodule X_1 with $x\notin X_1$. Hence $\mathfrak{R}({}_RM)$ coincides with the meet of all maximal submodules of M. Conversely, we assume that M has a maximal submodule A. If $x\in M$ and $x\notin A$, then $Rx+A=M$, and therefore Rx is not d-dense in M. Hence $\mathfrak{R}({}_RM)\neq M$, by Prop. 1.2 (2).

Corollary. $\mathfrak{R}({}_RM/\mathfrak{R}({}_RM))=0$.

Remark. (1) The dual of Prop. 1.4 is also true: The meet of all dense submodules (cf. [8]) of M coincides with the sum of all minimal R-submodules of M. (2) For a ring $R(\ni 1)$, $\mathfrak{R}({}_RR)$ coincides with the Jacobson radical of R.

The following lemma is evident.

Lemma 1.5. *Let* A, B, C *be submodules of* M *such that* $A+C=B+C$ *and* $A\cap C=B\cap C$. *If* $A\subseteqq B$ *then* $A=B$.

Proposition 1.6. *If a submodule* A $(\subseteqq M)$ *satisfies the descending chain condition for its submodules, then for any submodule* B *of* M *with* $A+B=M$ *there exists a submodule* B_0 *of* B *that is minimal with respect to the property* $A+B_0=M$.

Proof. This is evident from Lemma 1.5.

Theorem 1.7. *If a submodule A of M satisfies the descending chain condition for submodules and $A \cap \mathfrak{R}({}_RM)=0$, then A is a direct sum of a finite number of minimal submodules, and a direct summand of M.*

Proof. Let B be a d-complement of A (Prop. 1.6). Then, since $A \cap B \subseteqq \mathfrak{R}({}_RB) \subseteqq \mathfrak{R}({}_RM)$ by Prop. 1.3, we have $A \cap B \subseteqq A \cap \mathfrak{R}({}_RM)=0$, and so $M=A \oplus B$. Since A satisfies the descending chain condition for submodules and $\mathfrak{R}({}_RA)=\mathfrak{R}({}_RM) \cap A=0$ (Prop. 1.3), the above shows that every submodule of A is a direct summand of A. Hence, as is well known, A is completely reducible.

M is called (R-) *perfect*, if for any pair of submodules A, B of M with $A+B=M$, there exists a submodule B_0 of B that is minimal with respect to $M=A+B_0$.

Proposition 1.8. *If a submodule B of M is a d-complement of A, then for any d-dense submodule D of M, $(B+D)/D$ is a d-complement of $(A+D)/D$ in M/D. Therefore, if M is perfect, then so is M/D.*

Proof. Let X be a submodule with $B+D \supseteqq X \supseteqq D$ and $(A+D)/D+X/D =M/D$. Then, as $X=X \cap (B+D)=(X \cap B)+D$, we have $M=A+D+X= A+D+(X \cap B)$, and so $A+(X \cap B)=M$, because D is d-dense in M. Therefore, $X \cap B=B$ by the minimality of B, that is, $X \supseteqq B$. Hence $X=B+D$, that is, $X/D=(B+D)/D$, which means that $(B+D)/D$ is a d-complement of $(A+D)/D$ in M/D.

Proposition 1.9. *Let B be a d-complement of A in M. If M is perfect, then so is B. In particular, every direct summand of a perfect module is perfect.*

Proof. Let $B=X+Y$, where X and Y are submodules of B. Then $M=A+B=A+X+Y$. Let Y_0 be a d-complement of $A+X$ in M such that $Y_0 \subseteqq Y$. As $M=A+(X+Y_0)$, the minimality of B implies that $X+Y_0=B$. If Y_1 is a submodule of B with $X+Y_1=B$ and $Y_1 \subseteqq Y_0$, then $M=A+X+Y_1$. Since Y_0 is a d-complement of $A+X$ in M, we have $Y_1=Y_0$. Hence Y_0 is a d-complement of X in B such that $Y_0 \subseteqq Y$. Thus B is perfect.

Proposition 1.10. *Let A be a submodule of M, D a d-dense submodule of M with $D \subseteqq A$. Then, A is d-dense in M if and only if A/D is d-dense in M/D.*

Proof. If A is d-dense in M, A/D is d-dense in M/D by Prop. 1.1 (3). Conversely, assume that A/D is d-dense in M/D. If $A+B=M$ for some submodule B, then $A/D+(B+D)/D=M/D$, and so $(B+D)/D=M/D$ by assumption. Hence $B+D=M$. Since D is d-dense in M, we have $B=M$.

Corollary. *If D is a d-dense submodule of M, then $\mathfrak{R}({}_RM/D)=\mathfrak{R}({}_RM)/D$.*

Proposition 1.11. *Let A be a submodule of M, B a d-complement of A (in M). Then $\mathfrak{R}({}_RM)=(\mathfrak{R}({}_RM)\cap A)+(\mathfrak{R}({}_RM)\cap B)$.*

Proof. Put $\mathfrak{R}({}_RM)=N$. Then, $A\cap B$ is d-dense in B and $\mathfrak{R}({}_RB)=N\cap B$ by Prop. 1.3. Therefore, by Cor. to Prop. 1.10, $\mathfrak{R}({}_RB/(A\cap B))=(N\cap B)/(A\cap B)$. Now, $B/(A\cap B)\cong(A+B)/A=M/A$ canonically, and $\mathfrak{R}({}_RB/(A\cap B))=(N\cap B)/(A\cap B)\leftarrow\mathfrak{R}({}_RM/A)=((N\cap B)+A)/A\subseteqq(N+A)/A\subseteqq\mathfrak{R}({}_RM/A)$ under the above isomorphism. Hence $((N\cap B)+A)/A=(N+A)/A$, that is, $(N\cap B)+A=N+A$. Thus we have $N=(N\cap A)+(N\cap B)$.

Theorem 1.12. *If M is perfect, then so is M/A for any submodule A of M.*

Proof. If B is a d-complement of A, then B is perfect, and $A\cap B$ is d-dense in B by Prop. 1.9 and Prop. 1.3. Therefore, by Prop. 1.8, $B/(A\cap B)\cong M/A$ is perfect.

Proposition 1.13. *If M is perfect and $\mathfrak{R}({}_RM)=0$, then M is R-completely reducible.*

Proof. For any submodule A, we take a d-complement B of A. Then $A\cap B\subseteqq\mathfrak{R}({}_RB)\subseteqq\mathfrak{R}({}_RM)=0$ by Prop. 1.3, and so $M=A\oplus B$. Hence, every submodule of M is an R-direct summand of M, and hence, as is well known, M is completely reducible.

Proposition 1.14. *If $M\neq 0$ is R-projective, then $M\supsetneqq\mathfrak{R}({}_RM)=\mathfrak{R}(R)\cdot M\supseteqq\mathfrak{R}(M_K)$, where $K=\mathrm{Hom}({}_RM, {}_RM)$.* (See Bass [3; p. 474]).

Proof. The first important assertion $M\neq\mathfrak{R}({}_RM)$ is proved in Bass [3]. For an R-free module $F={}_RR^{(\Lambda)}$, we have $\mathfrak{R}({}_RF)=\mathfrak{R}(R)^{(\Lambda)}=\mathfrak{R}(R)\cdot F$ by Cor. to Prop. 1.3. As is well known, $\mathrm{Hom}({}_RF, {}_RF)=(R)_\Lambda$, the set of all raw-finite matrices over R. We set $T=(R)_\Lambda$, and take an idempotent $e\in T$. Then, $\mathfrak{R}({}_RFe)=\mathfrak{R}({}_RF)\cap Fe=\mathfrak{R}(R)\cdot Fe$ by Prop. 1.3, and as is easily seen, $\mathrm{Hom}({}_RFe, {}_RFe)=eTe$. Since F_T is T-projective[1], a symmetric argument to the above yields $\mathfrak{R}(F_T)=F\cdot\mathfrak{R}(T)$. As is well known, $\mathfrak{R}(T)\subseteqq(\mathfrak{R}(R))_\Lambda$[2], and so $\mathfrak{R}(F_T)=F\cdot\mathfrak{R}(T)\subseteqq F\cdot(\mathfrak{R}(R))_\Lambda\subseteqq\mathfrak{R}({}_RF)$. Len X be any eTe-d-dense submodule of Fe, and let $X\cdot T+Y=F$ for some T-submodule Y of F. Then $Fe=X\cdot Te+Ye=X+Ye$, and so $Ye=Fe\supseteqq X$, because X is eTe-d-dense in Fe. Hence $Y\supseteqq X\cdot T$, and we have $Y=X\cdot T+Y=F$. This implies that $X\subseteqq\mathfrak{R}(F_T)$. Thus

1) The proof will proceed as that of the case $\#\Lambda<\aleph_0$.

2) Cf. [10; p. 113].

we have $\mathfrak{R}(Fe_{eTe}) \subseteqq \mathfrak{R}(F_T)$. Therefore $\mathfrak{R}(Fe_{eTe}) \subseteqq \mathfrak{R}(F_T)e \subseteqq \mathfrak{R}({}_RF)e = \mathfrak{R}({}_RFe)$. Noting that the projective module M can be written as Fe with some F and e, the proof is now complete.

Proposition 1.15. *If every proper submodule of M contained in a maximal submodule, then $\mathfrak{R}({}_RM)$ is R-d-dense in M. In particular, if M is R-finitely generated, then $\mathfrak{R}({}_RM)$ is R-d-dense in M.*

Proof. If $\mathfrak{R}({}_RM)+X=M$ for some proper submodule X of M then, by assumption, X is contained in a maximal submodule X_1, so that $\mathfrak{R}({}_RM)+X_1 =M$. On the other hand, $\mathfrak{R}({}_RM) \subseteqq X_1$ by Prop. 1.14, and we have a contradiction $X_1=M$.

§ 2. Quasi-projective modules.

M is called *R-quasi-projective* if $\mathrm{Hom}({}_RM, {}_RM/A) = \mathrm{Hom}({}_RM, {}_RM)\cdot \nu(M \to M/A)$ for any submodule A of M, or equivalently, $\mathrm{Hom}({}_RM, {}_RN) = \mathrm{Hom}({}_RM, {}_RM)\varphi$ for any R-epimorphism $\varphi : M \to N$, where $\nu(M \to M/A)$ is the projection of M on M/A. M is called *R-quasi-injective* if $\mathrm{Hom}({}_RA, {}_RM) = \iota(A \to M)\cdot \mathrm{Hom}({}_RM, {}_RM)$ for any submodule A of M, where $\iota(A \to M)$ is the injection of A into M.

Proposition 2.1. *The following are equivalent:* (i) *M is R-quasi-projective.* (ii) *For any submodules A, B of M and any epimorphism $\varphi : A \twoheadrightarrow M/B$*[3]*, there exists a homomorphism $\psi : M \to A$ with $\psi\varphi = \nu(M \to M/B)$.*

Proof. (i)$\Rightarrow$(ii) $\varphi : A \twoheadrightarrow M/B$ induces an isomorphism $\bar{\varphi} : A/\mathrm{Ker}\,\varphi \cong M/B$. Further, we have a homomorphism $\varphi^* = \nu(M \to M/B)\bar{\varphi}^{-1} : M \to M/\mathrm{Ker}\,\varphi$. Then, by assumption, there exists a homomorphism $\psi : M \to M$ with $\varphi^* = \psi \cdot \nu(M \to M/\mathrm{Ker}\,\varphi)$. Then $\psi\varphi = \nu(M \to M/B)$. (ii)$\Rightarrow$(i) Let φ be an epimorphism from M to B/A, where A and B are submodules of M with $A \subseteqq B$. φ induces an isomorphism $\rho : B/A \cong M/\mathrm{Ker}\,\varphi$. By assumption there exists a homomorphism $\psi : M \to B$ with $\psi \cdot \nu(M \to M/A)\rho = \nu(M \to M/\mathrm{Ker}\,\varphi)$. Then $\psi \cdot \nu(M \to M/A) = \varphi$.

Proposition 2.2. *The following are equivalent:* (i) *M is R-quasi-injective.* (ii) *For any submodules A, B of M and any monomorphism $\varphi : A \rightarrowtail M/B$*[4] *there exists a homomorphism $\psi : M/B \to M$ with $\varphi\psi = 1_A$.*

Proof. (i)$\Rightarrow$(ii) Let $\varphi : A \cong C/B$, where B and C are submodules of M with $B \subseteqq C$. Then we have a homomorphism $\varphi^* = \nu(C \to C/B)\varphi^{-1} : C \to M$. By assumption, there exists a homomorphism $\psi : M \to M$ with $\psi | C$ (restriction to $C) = \varphi^*$. As $B\psi = 0$, ψ induces a homomorphism $\bar{\psi} : M/B \to M$. Then, $\varphi\bar{\psi} = 1_A$.

3) The symbol "$\twoheadrightarrow$" means an epimorphism.

4) The symbol "$\rightarrowtail$" means a monomorphism.

(ii)$\Rightarrow$(i) Let $\varphi: A \to M$, where A is a submodule of M. Then, φ induces a monomorphism $\sigma: A\varphi \rightarrowtail M/\mathrm{Ker}\,\varphi$. By assumption, there exists a homomorphism $\psi: M/\mathrm{Ker}\,\varphi \to M$ with $\sigma\psi = 1_{A\varphi}$. We put $\varphi^* = \nu(M \to M/\mathrm{Ker}\,\varphi)\psi$, then $\varphi^*: M \to M$ and $\varphi^*|A = \varphi$.

Theorem 2.3. *Let M be R-quasi-projective. If A and B are d-complements of each other, then $M = A \oplus B$.*

Proof. Since $\nu(M \to M/A)|B: B \to M/A$ is an epimorphism, by Prop. 2.1 there exists a homomorphism $\sigma: M \to B$ with $\sigma(\nu(M \to M/A)|B = \nu(M \to M/A)$. Then $M\sigma + A = M$, $M\sigma \subseteqq B$, and therefore the minimality of B implies that $B = M\sigma = A\sigma + B\sigma$. Since $m\sigma + A = m + A$ for all $m \in M$, we have $A\sigma \subseteqq A$. Therefore $M = A + A\sigma + B\sigma = A + B\sigma$, $B\sigma \subseteqq B$. Again, by the minimality of B, we have $B\sigma = B$, that is, $B + \mathrm{Ker}\,\sigma = M$. Since $\mathrm{Ker}\,\sigma \subseteqq A$, the minimality of A implies that $\mathrm{Ker}\,\sigma = A$. From this fact, we know that $\nu(M \to M/A)|B$ is $1-1$, that is, $A \cap B = 0$.

Remark. As is easily seen from the above proof, if M is R-quasi-projective and B is a d-complemented submodule of M then there exists a homomorphism $\sigma: M \to M$ with $M\sigma = B\sigma = B$. Further, if B satisfies the ascending chain condition for R-submodules then $B \cap \mathrm{Ker}\,\sigma = 0$, that is, $M = B \oplus \mathrm{Ker}\,\sigma$. In particular, if M is R-finitely generated and projective, and R is left Noetherian, then every d-complemented submodule of M is an R-direct summand of M.

Theorem 2.4. *Let M be R-quasi-projective.*

(1) *Every R-direct summand of M is R-quasi-projective.*

(2) *If M_0 is an R-K-submodule of M, then M/M_0 is R-quasi-projective, where $K = End({}_RM)$ acting on the right.*

(3) *M is Q-quasi-projective for any intermediate ring Q between R_0 and* $\mathrm{End}(M_K)$, *where R_0 is the image of R in* $\mathrm{End}(M)$ *acting on the left.*

Proof. (1) Let ρ be an idempotent in K, and let φ be an epimorphism from $M\rho$ to an R-module B. Then $M(\rho\varphi) = B$. Therefore $\mathrm{Hom}({}_RM, {}_RB) = \mathrm{Hom}({}_RM, {}_RM)\rho\varphi$, and so $\mathrm{Hom}({}_RM\rho, {}_RB) = \iota(M\rho \to M)\cdot\mathrm{Hom}({}_RM, {}_RB) = \mathrm{Hom}({}_RM\rho, {}_RM\rho)\varphi$. Hence $M\rho$ is R-quasi-projective. (2) Let M_0 be an R-K-submodule of M, and σ an epimorphism from M/M_0 to an R-module B. Then $\nu(M \to M/M_0)\sigma$ is onto. Therefore, for any homomorphism $\varphi: M/M_0 \to B$, there exists a homomorphism $\psi: M \to M$ with $\psi\cdot\nu(M \to M/M_0)\sigma = \nu(M \to M/M_0)\varphi$. If we define $\psi^*: M/M_0 \to M/M_0$ by $(m + M_0)\psi^* = m\psi + M_0$ $(m \in M)$, then $\psi^*\sigma = \varphi$. (3) Noting that $\mathrm{Hom}({}_QM, {}_QM) = \mathrm{Hom}({}_RM, {}_RM)$, (3) will be easily seen.

Remark. Every completely reducible module is quasi-injective and quasi-projective.

Proposition 2.5. *If M is R-perfect, then the following are equivalent:* (i) *M is R-quasi-projective.* (ii) *For each d-dense submodule D of M,* $\mathrm{Hom}({}_RM, {}_RM/D) = \mathrm{Hom}({}_RM, {}_RM)\cdot\nu(M\to M/D)$.

Proof (i)$\Rightarrow$(ii) is trivial. (ii)$\Rightarrow$(i) For any submodule A of M, we take a d-complement B of A. Then $A\cap B$ is d-dense (in B, and so) in M (Prop. 1.3). Let φ be arbitrary homomorphism from M to M/A, and let α be the isomorphism: $B/(A\cap B)\cong M/A$ given by $(b+(A\cap B))\alpha = b+A$ $(b\in B)$. Then $\varphi\alpha^{-1}$: $M\to M/(A\cap B)$. Since $A\cap B$ is d-dense in M, there exists a homomorphism ψ: $M\to M$ with $\psi\cdot\nu(M\to M/(A\cap B)) = \varphi\alpha^{-1}$. Then $\psi\cdot\nu(M\to M/A) = \psi\cdot\nu(M\to M/(A\cap B))\alpha = \varphi$.

The following fact was found by Johnson and Wong [6]: M is R-quasi-injective if and only if $M\cdot\mathrm{Hom}({}_RM^\wedge, {}_RM^\wedge)\subseteqq M$, where $M^\wedge$ is the injective envelope of M. By making use of this fact we can prove the following:

Proposition 2.6. *Let n be a natural number. If M is R-quasi-injective then so is M^n, and conversely.*

Proof. Evidently, $(M^n)^\wedge = (M^\wedge)^n$ and $\mathrm{Hom}({}_R(M^\wedge)^n, {}_R(M^\wedge)^n) = (\mathrm{Hom}({}_RM^\wedge, {}_RM^\wedge))_n$, and therefore $M^n\cdot(\mathrm{Hom}({}_RM^\wedge, {}_RM^\wedge))_n \subseteqq (M\cdot\mathrm{Hom}({}_RM^\wedge, {}_RM^\wedge))^n \subseteqq M^n$. Hence M^n is R-quasi-injective. Conversely, if M^n is R-quasi-injective, then M $(\cong(M,0,\cdots,0))$ is R-quasi-injectiv by [5; Cor. 2.2] or [8; Th. 4.3].

The next is the dual of Johnson and Wong [6; 1.1. Th.].

Theorem 2.7. *Let ${}_RP$ be an R-projective module, and D an R-d-dense submodule of P. Then the following are equivalent:*

(i) *P/D is R-quasi-projective.*

(ii) $D\cdot\mathrm{Hom}({}_RP, {}_RP)\subseteqq D$.

Proof. (ii)$\Rightarrow$(i) is a direct consequence of Th. 2.4 (2). (i)$\Rightarrow$(ii) Assume that there exists a homomorphism φ: $P\to P$ with $D\varphi\not\subseteqq D$. Then φ induces a homomorphism $\bar\varphi$: $P/D\to P/(D\varphi+D)$. If σ: $P/D\twoheadrightarrow P/(D\varphi+D)$ is defined by $(x+D)\sigma = x+(D\varphi+D)$, by assumption there exists a homomorphism ψ: $P/D\to P/D$ with $\psi\sigma=\bar\varphi$. Furthermore, since P is projective, there exists a homomorphism ρ: $P\to P$ with $x\rho+D=(x+D)\psi$ for all $x\in P$. Then $x\rho+(D\varphi+D) = x\varphi+(D\varphi+D)$, that is, $x(\varphi-\rho)\in D\varphi+D$ for all $x\in P$. Since $D\varphi\not\subseteqq D$ and $D\rho\subseteqq D$, we have $D(\varphi-\rho)\not\subseteqq D$, and so $D(\varphi-\rho)^{-1}\neq P$. Therefore $D(\varphi-\rho)^{-1}+D\neq P$. But this is impossible. In fact, if a is an element of P not contained in $D(\varphi-\rho)^{-1}+D$, and if $a(\varphi-\rho) = d\varphi+d'$ $(d, d'\in D)$, then $(a-d)(\varphi-\rho) = d'+d\rho\in D$, that is, $a-d\in D(\varphi-\rho)^{-1}$, whence it follows $a\in D(\varphi-\rho)^{-1}+D$.

Corollary. *Let ${}_RP$ be an R-projective module, D a d-dense submodule of P, and n a natural number. Then the following are equivalent:* (i) *P/D is R-quasi-projective.* (ii) *$(P/D)^n$ is R-quasi-projective.*

Proof. (ii)$\Rightarrow$(i) is a consequence of Th. 2.4 (1). (i)$\Rightarrow$(ii) $\mathrm{Hom}({}_RP^n, {}_RP^n)=(\mathrm{Hom}({}_RP, {}_RP))_n$. By Prop. 1.1, D^n is R-d-dense in P^n, and $D^n\cdot(\mathrm{Hom}({}_RP, {}_RP))_n \subseteqq (D\cdot \mathrm{Hom}({}_RP, {}_RP))^n \subseteqq D^n$ by Th. 2.7. Hence P^n/D^n ($\cong (P/D)^n$) is R-quasi-projective by Th. 2.4 (2).

Let N be a T-right module, where T is a ring with 1 ($\neq 0$). Then $N^n=\{(x_1, \cdots, x_n); x_i\in N\}$ may be regarded naturally as a $(T)_n$-right module. We set $S=\mathrm{Hom}(N_T, N_T)$ (acting on the left). Then N^n is as usual an S-left, $(T)_n$-right module, and, to be easily verified, the correspondence $X\to X^n$ is an order isomorphism between the T-submodules of N and the $(T)_n$-submodules of N^n. Now, we shall prove the following:

Theorem 2.8. (1) *N is T-quasi-projective if and only if N^n is $(T)_n$-quasi-projective.*

(2) *N is T-quasi-injective if and only if N^n is $(T)_n$-quasi-injective.*

Proof. (1) Suppose that N is T-quasi-projective. If X is a T-submodule of N, then $(N/X)^n{}_{(T)_n}\cong N^n/X^n{}_{(T)_n}$. Hence we may consider $(N/X)^n{}_{(T)_n}$ instead of $N^n/X^n{}_{(T)_n}$. Let Φ be arbitrary $(T)_n$-homomorphism from N^n to $(N/X)^n$. Then, noting that $(\Phi x)e_{ij}=\Phi(xe_{ij})$ for each x in N^n and each matrix unit e_{ij} of $(T)_n$, we can easily see that there exists a T-homomorphism φ from N to N/X such that $\Phi(x_1, \cdots, x_n)=(\varphi x_1, \cdots, \varphi x_n)$ ($(x_1, \cdots, x_n)\in N^n$). By assumption, $\varphi=s\cdot\nu(N\to N/X)$ for some $s\in S$. Since $\mathrm{Hom}(N^n{}_{(T)_n}, N^n{}_{(T)_n})=S$, the last implies that N^n is $(T)_n$-quasi-projective. Next, suppose that N^n is $(T)_n$-quasi-projective. If X is a T-submodule of N and φ a T-homomorphism from N to N/X, then φ induces a $(T)_n$-homomorphism Φ: $N^n{}_{(T)_n}\to(N/X)^n$. By assumption, there exists an element s of S ($\cong \mathrm{Hom}(N^n{}_{(T)_n}, N^n{}_{(T)_n})$) with $\Phi(x_1, \cdots, x_n)=(sx_1+X_1, \cdots, sx_n+X)$ for all $(x_1, \cdots, x_n)\in N^n$. Then $\varphi x_1=sx_1+X$ for all $x_1\in N$. (2) The proof of this part is quite symmetric to the above. Therefore we omit it.

Corollary. *Let T be a ring with* 1 ($\neq 0$). *Then T_T is injective if and only* if *$(T)_{n(T)_n}$ is injective.*

Proof. One may remark here a ring is injective if and only if it is quasi-injective. T_T is quasi-injective if and only if $T^n{}_{(T)_n}$ is quasi-injective by the above theorem. And this is equivalent to that $(T)_{n(T)_n}$ is quasi-injective by Prop. 2.6.

A ring R is called *h-central*, if R is quasi-projective as an R-R-module.

Proposition 2.9. *R is an h-central ring if and only if for any ideal I of R, $Z(R/I)=(Z(R)+I)/I$, where $Z(*)$ is the center of $*$. If R is R-R-perfect, then R is h-central if and only if $Z(R/I)=(Z(R)+I)/I$ for each ideal I of R contained in $R({}_RR_R)$.*

Proof. Evidently, $(Z(R)+I)/I\subseteqq Z(R/I)$. Suppose that R is h-central, and let $a+I$ be any element of $Z(R/I)$. If $\varphi: R \rightarrow R/I$ is an R-R-homomorphism defined by $x\varphi = xa + \mathrm{I}$, there exists an element c of $Z(R)$ with $xa+I=xc+I$ for all $x \in R$. Then $a+I=c+I$. Hence $(Z(R)+I)/I=Z(R/I)$. Conversely, we assume that $(Z(R)+I)/I=Z(R/I)$ for each ideal I of R. Let φ be any R-R-homomorphism from R to R/I. Then $I\varphi = I(1\varphi)=0$, so that φ induces an R-R-homomorphism $\bar{\varphi}: R/I \rightarrow R/I$. Then there exists an element $c+I$ of $Z(R/I)$ with $x\varphi = xc+I$ for all $x \in R$. By assumption, we may assume that $c \in Z(R)$. Then $x\varphi = (xc)\cdot \nu(R \rightarrow R/I)$ for all x in R. Hence R is h-central. The second half is evident from the above proof and Th. 2.5.

Corollary. *Let n be a natural number. If R is h-central then so is $(R)_n$, and conversely.*

Proof. As is easily seen, for any ideal $\bar{I}$ of $(R)_n$ there exists an ideal I of R with $(I)_n = \bar{I}$. Then there exists a canonical ring isomorphism from $(R/I)_n$ to $(R)_n/(I)_n$, and $((Z(R)+I)/I)_n \leftrightarrow (Z(R)+I)_n/(I)_n = ((Z(R))_n + (I)_n)/(I)_n = (Z((R)_n) + (I)_n)/(I)_n$, $Z((R/I)_n) = (Z(R/I))_n \leftarrow Z((R)_n/(I)_n)$ under the above isomorphism. Therefore $(Z((R)_n)+(I)_n)/(I)_n = Z((R)_n/(I)_n)$ if and only if $((Z(R)+I)/I)_n = (Z(R/I))_n$, or equivalently, $(Z(R)+I)/I = Z(R/I)$.

We set $J=\{\alpha \in K = \mathrm{Hom}({}_RM, {}_RM);\ M\alpha$ is R-d-dense in $M\}$, that is an ideal of K by Prop. 1.1.

Proposition 2.10. *If M is R-quasi-projective, then $J \subseteqq \mathfrak{R}(K)$.*

Proof. Let $\gamma \in J, A$ a left ideal of K with $K\gamma + A = K$. Then, there exists an element α of A with $K\gamma + K\alpha = K$. Therefore, $M\gamma + M\alpha = M$, and so $M\alpha = M$, because $M\gamma$ is R-d-dense in M. By Prop. 2.1, there exists an element β of K with $\beta\alpha = 1$. Thus we have $K\alpha = K$, and so $A=K$. Hence $K\gamma$ is d-dense in ${}_KK$, that is, $\gamma \in \mathfrak{R}(K)$.

Theorem 2.11. *If M is R-projective and $\mathfrak{R}({}_RM)$ is R-d-dense in M, then $J=\mathfrak{R}(K)$.*

Proof. By Prop. 1.13, $\mathfrak{R}({}_RM) \supseteqq \mathfrak{R}(M_K)$. Since $\mathfrak{R}(M_K) \supseteqq M \cdot \mathfrak{R}(K)$ by Prop. 1.1, we have $\mathfrak{R}(K) \subseteqq J$. On the hand, $J \subseteqq \mathfrak{R}(K)$ by Prop. 2.10, and hence we have $J = \mathfrak{R}(K)$.

Remark. In Th. 2.11, $\mathrm{End}({}_RM/\mathfrak{R}({}_RM))$ is naturally isomorphic to $K/\mathfrak{R}(K)$.

The following is the dual of Faith and Utumi [5; Th. 3.1].

Theorem 2.12. *If M is R-quasi-projective and R-perfect, then $J=\mathfrak{R}(K)$ and K/J is a regular ring.*

Proof. By Prop. 2.10, $J \subseteqq \mathfrak{R}(K)$. Since a regular ring is semi-simple, it suffices to prove that K/J is a regular ring. Take any element γ of K not contained in J, and let A be a d-complement of $M\gamma$. Then $M\gamma \cap A$ is d-dense in A by Prop. 1.3. For the epimorphism $\gamma\cdot\nu(M\to M/A)$: $M\to M/A$, there exists a homomorphism ε : $M\to M$ with $\varepsilon\gamma\cdot\nu(M\to M/A)=\nu(M\to M/A)$ (Prop. 2. 1.). Then, $(\gamma-\gamma\varepsilon\gamma)\cdot\nu(M\to M/A)=\gamma(1-\varepsilon\gamma)\cdot\nu(M\to M/A)=0$, and hence $M(\gamma-\gamma\varepsilon\gamma)\subseteqq A\cap M\gamma$. As $A\cap M\gamma$ is d-dense in M, so is $M(\gamma-\gamma\varepsilon\gamma)$. Hence $\gamma-\gamma\varepsilon\gamma\in J$, as desired.

Remark. Further, if $\mathfrak{R}({}_RM)$ is R-d-dense in M, then $M/\mathfrak{R}({}_RM)$ is R-completely reducible, and $\mathrm{End}({}_RM/\mathfrak{R}({}_RM))\cong K/\mathfrak{R}(K)$ naturally.

§ 3. Perfect projective modules.

The projective cover of ${}_RM$ is an epimorphism φ: ${}_RP\twoheadrightarrow M$ from an R-projective module ${}_RP$ to M such that $\mathrm{Ker}\,\varphi$ is R-d-dense in P. The following proposition (the uniqueness of projective cover) is well known.

Proposition 3.1. *Let φ : $P\twoheadrightarrow M$ and φ' : $P'\twoheadrightarrow M$ be projective covers of M. If σ is a homomorphism from P to P' with $\sigma\varphi'=\varphi$, then σ is an isomorphism.*

Proposition 3.2 (Shukla [11]). *If M has a projective cover, and φ is an epimorphism from an R-module ${}_RL$ to M with $S\varphi=M$, where S is an R-submodule of L, then exists an R-submodule S_0 of S that is minimal with respect to the property $S_0\varphi=M$.*

Proof. We may assume that $L=S$. Let ψ : $P\to M$ be a projective cover of M, and let σ be a homomorphism from P to L with $\sigma\varphi=\psi$. Then $P\sigma$ is a desired one.

Theorem 3.3. *Let ${}_RP$ be an R-projective module. The following are equivalent*:

(i) *Every homomorphic image of P has a projective cover.*

(ii) *P is R-perfect.*

Proof. (i)$\Rightarrow$(ii) Let A, B be submodules of P with $A+B=P$. By assumption, P/A has a projective cover. As $B\twoheadrightarrow(B+A)/A=P/A$, by Prop. 3.2 there exists a submodule B_0 of B that is minimal with respect to the property $B_0+A=P$. Hence P is perfect. (ii)$\Rightarrow$(i) Let A be any submodule of M, and let B be a d-complement of A in M. Then, by Th. 2.3, B is a direct summand of P, so that B is projective. If $\varphi=\nu(P\to P/A)|B$, then $B\varphi=P/A$ and $\mathrm{Ker}\,\varphi=A\cap B$ is d-dense in B by Prop. 1.3. Hence φ : $B\twoheadrightarrow P/A$ is a projective cover of P/A.

Proposition 3.4. *If an R-left module P is perfect and projective, then $\mathfrak{R}({}_RP)$ is R-d-dense in P.*

Proof. We may assume $P\neq 0$, then $\mathfrak{R}({}_RP)\neq P$ by Prop. 1.14. If $\mathfrak{R}({}_RP)$

is not d-dense in P, $\mathfrak{R}({}_RP)$ contains a non-zero direct summand X of P by Th. 2.3. Since X is a direct summand, X is projective, and so $\mathfrak{R}({}_RX)\neq X$ by Prop. 1.14. On the other hand, Prop. 1.3 yields a contradiction $\mathfrak{R}({}_RX)=\mathfrak{R}({}_RP)\cap X=X$.

$M\neq 0$ is called *R-sum-irreducible*, if for any pair of proper submodules A, B of M, $A+B$ is also a proper submodule. The following is evident.

Proposition 3.5. *Let $\mathfrak{R}({}_RM)\neq M$. Then M is R-sum-irreducible if and only if $\mathfrak{R}({}_RM)$ is maximal in M. When it is the case, $\mathfrak{R}({}_RM)$ is the unique maximal submodule of M, and $M=Rx$ for each $x\in M$ not contained in $\mathfrak{R}({}_RM)$.*

In this paper, a ring T $(\ni 1)$ is called a *sum-irreducible ring*, if ${}_TT$ is sum-irreducible. As is well known, ${}_TT$ is sum-irreducible if and only if T_T is sum-irreducible.

Theorem 3.6. *Let M be R-quasi-projective. Then the following are equivalent:*

(i) *M is R-sum-irreducible.*

(ii) *$K=\mathrm{Hom}\,({}_RM,{}_RM)$ is a sum-irreducible ring.*

Proof. (i)$\Rightarrow$(ii) Let $\mathfrak{l}+\mathfrak{l}'=K$, where $\mathfrak{l}$ and $\mathfrak{l}'$ are left ideals of K. Then, $K\alpha+K\alpha'=K$ for some $\alpha\in\mathfrak{l}$, $\alpha'\in\mathfrak{l}'$. Then $M=M\alpha+M\alpha'$, and so $M=M\alpha$ or $M=M\alpha'$. If $M=M\alpha$, there exists an element β of K with $\beta\alpha=1$, so that $\mathfrak{l}\supseteqq K\alpha=K$. Hence K is a sum-irreducible ring. (ii)$\Rightarrow$(i) Let A and B be submodules of M with $M=A+B$. Then $M/(A\cap B)=A/(A\cap B)\oplus B/(A\cap B)$. Let $\bar{\omega}$ be the projection of $M/(A\cap B)$ to $A/(A\cap B)$. Then, by assumption we can easily obtain ω in K such that $(A\cap B)\omega\subseteqq(A\cap B)$ and $(x+(A\cap B))\bar{\omega}=x\omega+(A\cap B)$ $(x\in M)$. Since $K=K\omega+K(1-\omega)$, $K=K\omega$ or $K=K(1-\omega)$. Therefore, $M=M\omega$ or $M=M(1-\omega)$. As $M\omega\subseteqq A$ and $M(1-\omega)\subseteqq B$, we have eventally $M=A$ or $M=B$.

Theorem 3.7. *If an R-left module $P\neq 0$ is projective, the following are equivalent:*

(i) *P is R-perfect.*

(ii) *$\mathfrak{R}({}_RP)$ is R-d-dense in P, and P is a direct sum of R-sum-irreducible submodules.*

Proof. (i)$\Rightarrow$(ii) We put $\mathfrak{R}({}_RP)=N$. Then, N is R-d-dense in P by Prop. 3.4, and P/N is R-completely reducible by Th. 1.12 and Prop. 1.13. Let $P/N=\sum_{\gamma\in\Gamma}\oplus A_\gamma/N$, where each A_γ/N is minimal. As $P\twoheadrightarrow P/N\twoheadrightarrow A_\gamma/N$, each A_γ/N has a projective cover $\varphi_\gamma: P_\gamma\twoheadrightarrow A_\gamma/N$ by Th. 3.3. Since $\mathrm{Ker}\,\varphi_\gamma$ is d-dense in P_γ and $P_\gamma/\mathrm{Ker}\,\varphi_\gamma$ $(\cong A_\gamma/N)$ is minimal, P_γ is R-sum-irreducible. Let P' be the external direct sum of $\{P_\gamma;\ \gamma\in\Gamma\}$, and $\varphi: P'\to P/N$ the homomorphism defined by $((x_\gamma))\varphi = \sum x_\gamma \varphi_\gamma$ $((x_\gamma) \ni P')$. Then, P' being projective, there exists a homomorphism $\tau: P' \to P$ with $\tau \cdot \nu(P \to P/N) = \varphi$. Since N is d-dense in P, τ is an epimorphism. Let $\{\gamma_1, \cdots, \gamma_n\}$ be any finite subset of Γ, and let $P'' = \{(x_\gamma) \in P';\ x_\gamma = 0$ for each $\gamma \notin \{\gamma_1, \cdots, \gamma_n\}\}$. We put $\sum_i \oplus (A_{\gamma_i}/N) = B$.

Then, we have a commutative diagram
$$\begin{array}{ccc} P'' & \xrightarrow{\varphi} & B \\ \tau \downarrow & & \uparrow \pi \\ P & \xrightarrow{\phi} & P/N \end{array}$$
where $\phi = \nu(P \to P/N)$, and π is the projection of P/N to B. Since $\varphi | P'': P'' \to B$ is a projective cover of B and $(P''\tau)\phi\pi = B$, by Prop. 3.2 there exists a submodule P_0 of $P''\tau$ that is minimal with respect to the property $P_0 + \mathrm{Ker}\,\phi\pi = P$. Then, by Th. 2.3, P_0 is an R-direct summand of P, so that P_0 is projective. Since $P_0 \subseteqq P''\tau$, we have $(P_0\tau^{-1} \cap P'')\varphi = (P_0\tau^{-1} \cap P'')\tau\phi\pi = P_0\phi\pi = B$. Recalling here that $\varphi | P''$ is a projective cover of B, it follows $P_0\tau^{-1} \cap P'' = P''$, that is, $P''\tau \subseteqq P_0$. Thus we have $P''\tau = P_0$. Since $\mathrm{Ker}\,\varphi \cap P''$ is d-dense in P'', $(\mathrm{Ker}\,\varphi \cap P'')\tau = \mathrm{Ker}\,\phi\pi \cap P_0$ is d-dense in P_0. Hence $\phi\pi | P_0$: $P_0 \to B$ is a projective cover of B. Therefore, Prop. 3.1 yields that $\tau | P''$ is an isomorphism. Thus we conclude that τ is an isomorphism. (ii)$\Rightarrow$(i) Let $P = \sum_{\gamma \in \Gamma} \oplus P_\gamma$, where each P_γ is sum-irreducible. Then, $P/N = \sum_{\gamma \in \Gamma} \oplus (P_\gamma + N)/N$, and $(P_\gamma + N)/N \cong P_\gamma/(P_\gamma \cap N) = P_\gamma/\mathfrak{R}({}_RP_\gamma)$ is R-minimal. Now, take an arbitrary proper submodule A of P. Since P/N $(\twoheadrightarrow P/(A+N))$ is R-completely reducible, there exists an R-isomorphism $\varphi: (P'+N)/N = \sum_{\gamma \in \Gamma'} \oplus (P_\gamma + N)/N \cong P/(A+N)$, where Γ' is a subset of Γ and $P' = \sum_{\gamma \in \Gamma'} \oplus P_\gamma$. Let $\sigma: P' \twoheadrightarrow (P'+N)/N$ and $\rho: P/A \twoheadrightarrow P/(A+N)$ be canonical homomorphisms. Then, there exists a homomorphism $\phi: P' \to P/A$ with $\phi\rho = \sigma\varphi$, because P' is R-projective. Since N is d-dense in P, $\mathrm{Ker}\,\rho = (A+N)/A$ is d-dense in P/A. As $P'\phi\rho = P'\sigma\varphi = P/(A+N)$, we have $P'\phi + \mathrm{Ker}\,\rho = P/A$, so that $P'\phi = P/A$. Since $\mathrm{Ker}\,\phi$ is contained in $\mathrm{Ker}\,\sigma\varphi = \mathrm{Ker}\,\sigma = P' \cap N$ that is d-dense in P' (Prop. 1.3), $\phi: P' \twoheadrightarrow P/A$ is a projective cover of P/A. Hence every homomorphic image of P has a projective cover and therefore P is R-perfect by Th. 2.3.

Corollary. *Let ${}_RP$ be an R-projective module, and let $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$. Then P is R-perfect if and only if every P_i is R-perfect.*

In the proof (i)$\Rightarrow$(ii) of Th. 3.7, one may remark that the minimality of A_γ/N is not needed to prove that τ is an isomorphism. We obtained therefore the following:

Proposition 3.8. *Let ${}_RP$ be projective and perfect, and let $P/N = \sum_{\lambda \in \Lambda} \oplus A_\lambda/N$, where $N = \mathfrak{R}({}_RP)$. If $\varphi_\lambda: P_\lambda \to A/_\lambda N$ $(\lambda \in \Lambda)$ is a projective cover of A_λ/N, and P' the external direct sum of $\{P_\lambda;\ \lambda \in \Lambda\}$, then any homo-*

morphism $\alpha: P' \to P$ *with* $\alpha \cdot \nu(P \to P/N) = \varphi$ *is an isomorphism, where* φ: $P' \to P/N$ *is defined by* $((x_\lambda))\varphi = \sum x_\lambda \varphi_\lambda$ $((x_\lambda) \in P')$.

Theorem 3.9. *Let* ${}_RP$ *be projective and perfect,* $\{Q_\gamma;\ \gamma \in \Gamma\}$ *a set of projective submodules such that each* $Q_\gamma \cap \mathfrak{R}({}_RP)$ *is R-d-dense in* Q_γ. *Let* $P = \sum_{\lambda \in \Lambda} \oplus P_\lambda$, *where each* P_λ *is sum-irreducible. If* $\{Q_\gamma;\ \gamma \in \Gamma\}$ *is independent modulo* $\mathfrak{R}({}_RP)$, $\{Q_\gamma: \gamma \in \Gamma\}$ *is independent and* $P = (\sum_{\lambda \in \Lambda'} \oplus P_\lambda) \oplus (\sum_{\gamma \in \Gamma} \oplus Q_\gamma)$ *for some subset* Λ' *of* Λ.

Proof. We put $N = \mathfrak{R}({}_RP)$, and for any element x of P and any submodule A of P we denote $x+N$ and $(A+N)/N$ by $\bar{x}$ and $\bar{A}$, respectively. Since $\bar{P} = \sum_{\lambda \in \Lambda} \oplus \bar{P}_\lambda$ is completely reducible, as is easily seen, $\bar{P} = (\sum_{\lambda \in \Lambda'} \oplus \bar{P}_\lambda) \oplus (\sum_{\gamma \in \Gamma} \oplus \bar{Q}_\gamma)$ for some subset Λ' of Λ. Then $P = \sum_{\lambda \in \Lambda'} P_\lambda + \sum_{\gamma \in \Gamma} Q_\gamma + N$, and so $P = \sum_{\lambda \in \Lambda'} P_\lambda + \sum_{\gamma \in \Gamma} Q_\gamma$ (Prop. 3.4). Now, $\rho_\lambda: P_\lambda \to \bar{P}_\lambda$ and $\sigma_\lambda: Q_\gamma \to \bar{Q}_\gamma$ are projective covers of $\bar{P}_\lambda$ and $\bar{Q}_\gamma$, respectively. Let P' and Q' be the external direct sum of P_λ's $(\lambda \in \Lambda')$ and Q_γ's $(\gamma \in \Gamma)$, respectively. If $\alpha: (P', Q') \to P$ and $\varphi': (P', Q') \to \bar{P}$ are respectively defined by $((p_\lambda), (q_\gamma))\alpha = \sum p_\lambda + \sum q_\gamma$ and $((p_\lambda), (q_\gamma))\varphi' = \sum \bar{p}_\lambda + \sum \bar{q}_\gamma$, then the following diagram is commutative:

$$\begin{tikzcd} (P', Q') \arrow[r, "\varphi'"] \arrow[d, "\alpha"'] & \bar{P} \\ P \arrow[ur, "\nu(P \to P/N)"'] & \end{tikzcd}$$

By Prop. 3.8, α is then an isomorphism, and so $P = (\sum_{\lambda \in \Lambda'} \oplus P_\lambda) \oplus (\sum_{\gamma \in \Gamma} \oplus Q_\gamma)$, as desired.

For any non-zero idemoptent e of $K = \mathrm{Hom}({}_RM, {}_RM)$, $\mathrm{Hom}({}_KKe, {}_KKe) \cong eKe \cong \mathrm{Hom}(eK_K, eK_K)$. Consequently, Ke is a sum-irreducible left ideal if and only if eK is a sum-irreducible right ideal (Th. 3.6). A ring T $(\ni 1)$ is called a *semi-perfect ring* (Bass [3]), if T is a direct sum of sum-irreducible left ideals. By the above remark, this is equivalent to that T is a direct sum of sum-irreducible right ideals.

Theorem 3.10. *The following are equivalent*:

(i) *M is R-finitely generated, projective and perfect.*

(ii) *M is a direct sum of a finite number of R-projective, sum-irreducible submodules.*

(iii) *M is R-finitely generated and projective, and K is a semi-perfect ring.*

Proof. The equivalence (i)$\Longleftrightarrow$(ii) will be easily seen by Th. 3.7 and Prop. 3.5 (and Prop. 1.14). Next, let M be R-finitely generated and projective. For an idempotent e of K, $\mathrm{End}({}_RMe) \cong eKe \cong \mathrm{End}({}_KKe)$. Therefore, Me is R-sum-irreducible if and only if Ke is a sum-irreducible left ideal of K (Th. 3.6). From this, as is easily seen, M is a direct sum of a finite number of sum-irreducible submodules if and only if K is a direct sum of (a finite number of) sum-irreducible left ideals. Accordingly, Th. 3.7 proves at once (i)$\Longleftrightarrow$(iii).

Proposition 3.11. *If P is a perfect R-projective module, then $P/\mathfrak{a}P$ is a perfect $R/\mathfrak{a}$-projective module for any ideal $\mathfrak{a}$ of R.*

Proof. As is well known, $P/\mathfrak{a}P$ is $R/\mathfrak{a}$-projective. Since $P/\mathfrak{a}P$ *is* R-perfect (Th. 1.12), $P/\mathfrak{a}P$ is $R/\mathfrak{a}$-perfect.

Proposition 3.12. *If $P_R \neq 0$ is projective and perfect, then $\mathfrak{R}({}_RP)$ is the unique maximal submodule of P that contains no non-zero R-direct summand of P.*

Proof. We set $N=\mathfrak{R}({}_RP)$. Then, since N is R-d-dense in P, N does not contain a non-zero R-direct summand. If a non-zero submodule X of P is not contained in N, there exists a proper submodule Y of P with $P=X+Y$. By Th. 2.3, X contains a non-zero R-direct summand of P.

An element c of R is called a root element of R if cR does not contain a non-zero idempotent (see Azumaya [1]). We denote the set of all root elements of R by $\mathfrak{c}$. Azumaya called R a strongly semi-primary ring if $\mathfrak{c}$ is an ideal and $R/\mathfrak{c}$ satisfies the descending chain condition for right ideals, and proved the following ([10]): For a ring R $(\ni 1)$, the following are equivalent: (i) R is strongly semi-primary. (ii) $\mathfrak{c}$ is an ideal and R is a direct sum of directly indecomposable right ideals. (iii) There are mutually orthogonal idempotents $e_1, \cdots, e_n$ such that $\sum_i e_i = 1$ and each e_iRe_i is a completely primary ring (or a sum-irreducible ring).

By Th. 3.6, (iii) is equivalent to that R is a semi-perfect ring, namely, a strongly semi-primary ring is nothing but a semi-perfect ring.

Proposition 3.13. *Let D be a d-dense submodule of M.* (1) *M has a projective cover if and only if so does M/D.* (2) *Every homomorphic image of M has a projective cover if and only if so does every homomorphic image of M/D.*

Proof. (1) Let $\varphi: P\rightarrow M$ be a projective cover of M. If $D\varphi^{-1}+X=P$ for some R-submodule X of P, then $D+X\varphi=M$, and so $X\varphi=M$, that is, $X+\operatorname{Ker}\varphi=P$. Since $\operatorname{Ker}\varphi$ is d-dense in P, we have $X=P$, whence it follows that $D\varphi^{-1}$ is d-dense in P. Hence $\varphi\cdot\nu(M\rightarrow M/D)$: $P\twoheadrightarrow M/D$ is a projective cover of M/D. Conversely, if $\psi: Q\twoheadrightarrow M/D$ is a projective cover of M/D, there exists an R-homomorphism $\sigma: Q\rightarrow M$ with $\sigma\cdot\nu(M\rightarrow M/D)=\psi$. Then, $Q\sigma+D=M$, and therefore $Q\sigma=M$. As $\operatorname{Ker}\sigma\subseteqq\operatorname{Ker}\psi$, $\operatorname{Ker}\sigma$ is d-dense in Q. Hence, σ is a projective cover of M. (2) For any submodule A of M we have a canonical homomorphism $\sigma_A: M/A\rightarrow M/(A+D)$, and $\operatorname{Ker}\sigma_A=(A+D)/A$

is d-dense in M/A. Therefore, by (1), M/A has a projective cover if and only if $M/(A+D)\cong(M/D)/((D+A)/D)$ has a projective cover. Hence, every homomorphic image of M has a projective cover if and only if every homomorphic image of M/D has a projective cover.

§ 4. Semi-perfect h-central rings.

Proposition 4.1. (1) *Let* $\mathfrak{l}$ *be a left ideal of* R. *If* $\mathfrak{l}'$ *and* $\mathfrak{l}''$ *are d-complements of* ${}_R\mathfrak{l}$, *then* $\mathfrak{l}'=\mathfrak{l}''\cdot\mathfrak{l}'$. (2) *Let* $\mathfrak{a}$ *be an ideal of* R. *If* $\mathfrak{b}$ *is a d-complement of* ${}_R\mathfrak{a}_R$, *then* $\mathfrak{b}\mathfrak{c}=\mathfrak{b}$ *for any ideal* $\mathfrak{c}$ *with* $\mathfrak{a}+\mathfrak{c}=R$, *and hence* $\mathfrak{b}$ *is the unique d-complement of* $\mathfrak{a}$.

Proof. (1) Since $\mathfrak{l}''\subseteqq\mathfrak{l}''\cdot R=\mathfrak{l}''\cdot\mathfrak{l}+\mathfrak{l}''\cdot\mathfrak{l}'$, we have $R=\mathfrak{l}+\mathfrak{l}''=\mathfrak{l}+\mathfrak{l}''\cdot\mathfrak{l}'$, and the minimality of $\mathfrak{l}'$ implies $\mathfrak{l}'=\mathfrak{l}''\cdot\mathfrak{l}'$. (2) Let $R=\mathfrak{a}+\mathfrak{c}$ for some ideal $\mathfrak{c}$ of R. Then, as $\mathfrak{b}=\mathfrak{b}R=\mathfrak{b}\mathfrak{a}+\mathfrak{b}\mathfrak{c}$, we have $R=\mathfrak{a}+\mathfrak{b}=\mathfrak{a}+\mathfrak{b}\mathfrak{c}$, and the minimality of $\mathfrak{b}$ implies $\mathfrak{b}=\mathfrak{b}\mathfrak{c}$.

Theorem 4.2. (1) *Let* $R=\mathfrak{b}_1+\cdots+\mathfrak{b}_n$ *be a sum of sum-irreducible ideals* $\mathfrak{b}_i$ $(i=1,\cdots,n)$, *and* $\mathfrak{a}$ *an ideal of* R. *Then,* $\sum_{\mathfrak{b}_i\nsubseteq\mathfrak{a}}\mathfrak{b}_i$ *is the unique d-complement of* ${}_R\mathfrak{a}_R$, *and* R *is a direct sum of directly indecomposable ideals.*

(2) *R is R-R-perfect if and only if R is a sum of a finite number of sum-irreducible ideals.*

Proof. (1) Evidently $R=\mathfrak{a}+\sum_{\mathfrak{b}_i\nsubseteq\mathfrak{a}}\mathfrak{b}_i$. If $R=\mathfrak{a}+\mathfrak{b}$ for some ideal $\mathfrak{b}$, then $\mathfrak{b}_i=\mathfrak{b}_i\cdot\mathfrak{a}+\mathfrak{b}_i\cdot\mathfrak{b}$, and so $\mathfrak{b}_i=\mathfrak{b}_i\cdot\mathfrak{a}\subseteqq\mathfrak{a}$ or $\mathfrak{b}_i=\mathfrak{b}_i\cdot\mathfrak{b}\subseteqq\mathfrak{b}$. Hence $\mathfrak{b}\supseteqq\sum_{\mathfrak{b}_i\nsubseteq\mathfrak{a}}\mathfrak{b}_i$, which means that $\sum_{\mathfrak{b}_i\nsubseteq\mathfrak{a}}\mathfrak{b}_i$ is the unique d-complement of ${}_R\mathfrak{a}_R$. Since any R-R-direct summand of R is an R-R-d-complemented submodule, the latter is evident. (2) The "if" part is contained in (1). Assume that R is R-R-perfect. We put $\mathfrak{R}({}_RR_R)=I$. Then R/I is R-R-perfect and $\mathfrak{R}({}_RR/I)_R)=0$ (Th. 1.12). Hence R/I is R-R-completely reducible by Prop. 1.13. Therefore there are ideals $\mathfrak{a}_1,\cdots,\mathfrak{a}_n$ of R such that $R/I=\sum_i\oplus\mathfrak{a}_i/I$ and $\mathfrak{a}_i/I$ is R-R-minimal. Then, $R=\mathfrak{a}_1+\cdots+\mathfrak{a}_n$, and evidently this is an irredundant sum. Let $\mathfrak{b}_i$ be an R-R-d-complement of $\mathfrak{a}_1+\cdots+\check{\mathfrak{a}}_i+\cdots+\mathfrak{a}_n$ with $\mathfrak{b}_i\subseteqq\mathfrak{a}_i$. If $\mathfrak{b}_i\subseteqq I$, then $R=\mathfrak{a}_1+\cdots+\check{\mathfrak{a}}_i+\cdots+\mathfrak{a}_n+I$, and so $R=\mathfrak{a}_1+\cdots+\check{\mathfrak{a}}_i+\cdots+\mathfrak{a}_n$ (Prop. 1.15), a contradiction. Hence, $\mathfrak{b}_i+I=\mathfrak{a}_i$ by the minimolity of $\mathfrak{a}_i/I$, and so $R=\mathfrak{b}_1+\cdots+\mathfrak{b}_n+I$, whence it follows $R=\mathfrak{b}_1+\cdots+\mathfrak{b}_n$. Since $\mathfrak{b}_i$ is an R-R-complemented submodule, $R({}_R\mathfrak{b}_{iR})=\mathfrak{b}_i\cap I$ by Prop. 1.3, and hence $\mathfrak{b}_i/\mathfrak{R}({}_R\mathfrak{b}_{iR})$ is R-R-isomorphic to the minimal $\mathfrak{a}_i/I$. We have proved thus each $\mathfrak{b}_i$ is a sum-irreducible ideal (Prop. 3.5).

Theorem 4.3. *Let R be an h-central ring. Then the following are equivalent:*

(i) *R is R-R-perfect.*

(ii) *R is a direct sum of a finite number of sum-irreducible ideals.*

(iii) *The center of R is a direct sum of sum-irreducible ideals.*

Proof. The equivalence (ii)$\Longleftrightarrow$(iii) is easy by Th. 2.4 and Th. 3.6. (ii)$\Rightarrow$(i) is trivial by Th. 4.2 (2). (i)$\Rightarrow$(ii) By Th. 4.2, R is an irredundant sum of some sum-irreducible ideals $\mathfrak{b}_i$ $(i=1,\cdots,n)$. Since $\mathfrak{b}_i$ and $\mathfrak{b}_1+\cdots+\check{\mathfrak{b}}_i+\cdots+\mathfrak{b}_n$ are R-R-complements of each other (Th. 4.2), $\mathfrak{b}_i\cap(\mathfrak{b}_1+\cdots+\check{\mathfrak{b}}_i+\cdots+\mathfrak{b}_n)=0$ by Th. 2.3. Hence $R=\mathfrak{b}_1\oplus\cdots\oplus\mathfrak{b}_n$, as desired.

Proposition 4.4. *Let R be a semi-perfect ring, $\mathfrak{a}$ an ideal of R. If $\mathfrak{l}$ and $\mathfrak{r}$ are respective d-complements of ${}_R\mathfrak{a}$ and $\mathfrak{a}_R$, then $\mathfrak{l}\mathfrak{r}=\mathfrak{l}R=R\mathfrak{r}$, which coincides with the unique d-complement of ${}_R\mathfrak{a}_R$. Therefore, R is perfect as an R-R-module.*

Proof. As $\mathfrak{r}\subseteqq R\mathfrak{r}=\mathfrak{a}\mathfrak{r}+\mathfrak{l}\mathfrak{r}$, we have $R=\mathfrak{a}+\mathfrak{l}\mathfrak{r}$. Let $\mathfrak{b}$ be an ideal with $R=\mathfrak{a}+\mathfrak{b}$. Then $\mathfrak{r}R=\mathfrak{r}\mathfrak{a}+\mathfrak{r}\mathfrak{b}$, and so $R=\mathfrak{a}+\mathfrak{r}\mathfrak{b}$. The minimality of $\mathfrak{r}$ implies $\mathfrak{r}=\mathfrak{r}\mathfrak{b}\subseteqq\mathfrak{b}$, so that $R\mathfrak{r}\subseteqq\mathfrak{b}$. Similarly we have $\mathfrak{l}R\subseteqq\mathfrak{b}$, in particular, if we set $\mathfrak{b}=\mathfrak{l}\mathfrak{r}$, then $\mathfrak{l}R=\mathfrak{l}\mathfrak{r}$. Thus we know that $\mathfrak{l}\mathfrak{r}=R\mathfrak{r}=\mathfrak{l}R$ and this is the unique d-complement of ${}_R\mathfrak{a}_R$.

Proposition 4.5. *If R is a semi-perfect ring, then $\mathfrak{R}(R)=\mathfrak{R}({}_RR_R)$.*

Proof. Since $R/\mathfrak{R}(R)$ is R-R-completely reducible, we have $\mathfrak{R}({}_RR_R)\subseteqq\mathfrak{R}(R)$, and evidently $\mathfrak{R}(R)\subseteqq\mathfrak{R}({}_RR_R)$. Hence $\mathfrak{R}(R)=\mathfrak{R}({}_RR_R)$.

Theorem 4.6. *The following conditions for R are equivalent*:

(i) *R is a semi-perfect h-central ring.*

(ii) *There are sum-irreducible h-central rings $R_1,\cdots,R_r$ and natural numbers $m_1,\cdots,m_r$ such that $R\cong(R_1)_{m_1}\oplus\cdots\oplus(R_r)_{m_r}$ (external direct sum) as rings.*

Proof. (ii)$\Rightarrow$(i) Since each R_i is a sum-irreducible h-central ring, $(R_i)_{m_i}$ $(\cong\mathrm{End}({}_{R_i}R_i^{m_i}))$ is a semi-perfect h-central ring by Cor. to Prop. 2.9 and Th. 3.10. Hence, as is easily seen, R is a semi-perfect h-central ring. (i)$\Rightarrow$(ii) By Th. 4.3, $R=\mathfrak{a}_1\oplus\cdots+\mathfrak{a}_n$ with some sum-irreducible ideals $\mathfrak{a}_i$. Since each $\mathfrak{a}_i$ is evidently a semi-perfect h-central ring, we may assume that R is R-R-sum-irreducible. Let $R=\mathfrak{l}_1\oplus\cdots\oplus\mathfrak{l}_s$, where each $\mathfrak{l}_i$ is a sum-irreducible left ideal (Th. 3.7). We put $N=\mathfrak{R}(R)$. Then $R/N=\sum_i\oplus(\mathfrak{l}_i+N)/N$ and $\mathfrak{l}_i/(\mathfrak{l}_i\cap N)=\mathfrak{l}_i/\mathfrak{R}({}_R\mathfrak{l}_i)$ is minimal (Prop. 1.3). Since R/N is R-R-minimal, $\mathfrak{l}_1/\mathfrak{R}({}_R\mathfrak{l}_1)\cong\mathfrak{l}_i/\mathfrak{R}({}_R\mathfrak{l}_i)$ for all i. Hence $\mathfrak{l}_1\cong\mathfrak{l}_i$ for all i, because $\mathfrak{l}_i\twoheadrightarrow\mathfrak{l}_i/\mathfrak{R}({}_R\mathfrak{l}_i)$ is a projective cover of $\mathfrak{l}_i/\mathfrak{R}({}_R\mathfrak{l}_i)$ (Prop. 3.1). Thus $R\cong(\mathrm{End}({}_R\mathfrak{l}_i))_s$. Since $\mathfrak{l}_i$ is sum-irreducible and R is h-central, $\mathrm{End}({}_R\mathfrak{l}_i)$ is a sum-irreducible h-central ring by Th. 3.6 and Cor. to Prop. 2.9.

Proposition 4.7. *Let R be a semi-perfect ring, and $R=\mathfrak{b}_1+\cdots+\mathfrak{b}_n$ the irredundant sum of sum-irreducible ideals $\mathfrak{b}_i$ $(i=1,\cdots,n)$ of R. Then, $\mathfrak{b}_1=ReR$ for any non-zero idempotent e in $\mathfrak{b}_1+\mathfrak{R}(R)$.*

Proof. We put $\mathfrak{R}(R)$ $(=\mathfrak{R}(_RR_R))=N$. Since $Re\subseteqq\mathfrak{b}_1+N$, we have $Re=\mathfrak{b}_1e+Ne$, so that $Re=\mathfrak{b}_1e\subseteqq\mathfrak{b}_1$, because Ne is R-d-dense in Re (Prop. 1.1). Hence $ReR\subseteqq\mathfrak{b}_1$. Since $\mathfrak{b}_1$ is the unique R-R-d-complement of $\mathfrak{b}_2+\cdots+\mathfrak{b}_n$, there holds $\mathfrak{R}(_R\mathfrak{b}_{1R})=\mathfrak{b}_1\cap N$ (Prop. 1.3), and $\mathfrak{b}_1/(\mathfrak{b}_1\cap N)\cong(\mathfrak{b}_1+N)/N$ is R-R-minimal (Prop. 3.5). As $e\in N$, the minimality of $(\mathfrak{b}_1+N)/N$ implies $(ReR+N)/N=(\mathfrak{b}_1+N)/N$, or equivalently, $ReR+N=\mathfrak{b}_1+N$. Hence $\mathfrak{b}_1=ReR+(\mathfrak{b}_1\cap N)$, so that $\mathfrak{b}_1=ReR$, because $\mathfrak{b}_1\cap N$ is R-R-d-dense in $\mathfrak{b}_1$ (Prop. 1.3).

§ 5. A theorem for modular lattices.

In this section, we shall restate the fundamental theorem which was given in the preceding paper[8]. The theorem is valid for any modular lattice with 0. Throughout the section, $\mathfrak{L}$ will means a modular lattice with 0, and we shall use conveniently "$+$" instead of "$\vee$".

A non-empty subset $\mathfrak{S}$ is called *independent*, if each finite subset of $\mathfrak{S}$ is independent, namely, $a_1\wedge(a_2+\cdots+a_n)=0$ for any different $a_1,\cdots,a_n$ in $\mathfrak{S}$.

Proposition 5.1. *If $\{a_2,\cdots,a_n\}$ $(n\geqq3)$ is an independent subset of $\mathfrak{L}$, and $a_1\wedge(a_2+\cdots+a_n)=0$ for $a_1\in\mathfrak{L}$, then $(a_1+a_2)\wedge(a_3+\cdots+a_n)=0$.*

Proof. $a_1\wedge(a_2+\cdots+a_n)=0$ yields $(a_2+\cdots+a_n)\wedge(a_1+a_2)=a_2$, and so $0=a_2\wedge(a_3+\cdots+a_n)=(a_2+\cdots+a_n)\wedge(a_1+a_2)\wedge(a_3+\cdots+a_n)=(a_1+a_2)\wedge(a_3+\cdots+a_n)$.

Proposition 5.2. *Let $\{a_1,\cdots,a_n\}$ be an independent subset of $\mathfrak{L}$, and a an element of $\mathfrak{L}$. If $a\wedge(a_1+\cdots+a_n)=0$, then $\{a,a_1,\cdots,a_n\}$ is independent (and conversely).*

Proof. We may assume that $n\geqq2$. Since $(a+a_1)\wedge(a_2+\cdots+a_n)=0$ by Prop. 5.1, we have $(a+a_1)\wedge(a+a_2+\cdots+a_n)=a$. Therefore $a_1\wedge(a+a_2+\cdots+a_n)=a_1\wedge(a+a_1)\wedge(a+a_2+\cdots+a_n)=a_1\wedge a=0$.

Proposition 5.3. *Let $\{a_\lambda;\lambda\in\Lambda\}$ be an independent finite subset of $\mathfrak{L}$, and let Λ_1,Λ_2 be non-empty subsets of Λ. If $\Lambda_1\cap\Lambda_2=\emptyset$,* then $\sum_{\lambda\in\Lambda_1}a_\lambda\wedge\sum_{\lambda\in\Lambda_2}a_\lambda=0$. *If $\Lambda_1\cap\Lambda_2\neq\emptyset$, then $\sum_{\lambda\in\Lambda_1}a_\lambda\wedge\sum_{\gamma\in\Lambda_2}a_\lambda=\sum_{\lambda\in\Lambda_1\cap\Lambda_2}a_\lambda$.*

Proof. Let $\{a_\lambda;\lambda\in\Lambda\}=\{a_1,\cdots,a_n\}$. Then, by Prop. 5.1, $a_1\wedge(a_2+\cdots+a_n)=0$ yields $(a_1+a_2)\wedge(a_3+\cdots+a_n)=0$, so that $\{a_1+a_2,a_3,\cdots,a_n\}$ is independent by Prop. 5.2. Repeating the same procedure, we know that $(a_1+\cdots+a_i)\wedge(a_{i+1}+\cdots+a_n)=0$ for all i. Thus we obtain the first assertion. Next we assume that $\Lambda_1\cap\Lambda_2\neq\emptyset$. Then $\sum_{\lambda\in\Lambda_1-\Lambda_2}a_\lambda\wedge\sum_{\lambda\in\Lambda_2}a_\lambda=0$ and $\sum_{\lambda\in\Lambda_1\cap\Lambda_2}a_\lambda\leqslant\sum_{\lambda\in\Lambda_2}a_\lambda$. Therefore $\sum_{\lambda\in\Lambda_2}a_\lambda\wedge\sum_{\lambda\in\Lambda_1}a_\lambda=\sum_{\lambda\in\Lambda_2}a_\lambda\wedge(\sum_{\lambda\in\Lambda_1\cap\Lambda_2}a_\lambda+\sum_{\lambda\in\Lambda_1-\Lambda_2}a_\lambda)=\sum_{\lambda\in\Lambda_1\cap\Lambda_2}a_\lambda+$

$(\sum_{\lambda\in\Lambda_2}a_\lambda\wedge\sum_{\lambda\in\Lambda_1-\Lambda_2}a_\lambda)=\sum_{\lambda\in\Lambda_1\cap\Lambda_2}a_\lambda$, as desired.

Proposition 5.4. *Let $\{a\}\cup\{a_\lambda;\lambda\in\Lambda\}$ be an independent subset of $\mathfrak{L}$. If $\{b_\gamma;\gamma\in\Gamma\}$ is an independent subset of $\mathfrak{L}$ such that $b_\gamma\leqslant a$ for all $\gamma\in\Gamma$, then $\{a_\lambda;\lambda\in\Lambda\}\cup\{b_\gamma;\gamma\in\Gamma\}$ is independent.*

Proof. The proof will be easy, and may be omitted.

Let a, b, c be elements of $\mathfrak{L}$. If $a\wedge c=b\wedge c=0$ and $a\wedge(b+c)\neq 0$, then $b\wedge(a+c)\neq 0$ (Prop. 5.2). In this case, we write $a\overset{c}{\sim}b$. Evidently, if $a, b\in L$ and $0\neq a\leqslant b$, then $a\overset{0}{\sim}b$. Let x, y be in $\mathfrak{L}$. If there exists a subset $\{c_1,\cdots,c_n, x_1,\cdots,x_{n-1}\}$ with $x\overset{c_1}{\sim}x_1\overset{c_2}{\sim}x_2\overset{c_3}{\sim}\cdots\sim x_{n-1}\overset{c_n}{\sim}y$, we write $x\sim y$. Obviously, $\sim$ is an equivalence relation.

A non-zero element u is called *uniform*, if $x\wedge y\neq 0$ for all non-zero elements x, y of $\mathfrak{L}$ with $x, y\leqslant u$. In the rest of this section, we assume that $\mathfrak{L}$ contains at least one uniform element.

Proposition 5.5. *Let u, v be uniform elements of $\mathfrak{L}$ with $u\wedge v\neq 0$, and x an element of $\mathfrak{L}$. If $x\wedge u=0$ then $x\wedge v=0$ (and conversely).*

Proof. If $x\wedge v\neq 0$, then $0\neq(x\wedge v)\wedge(u\wedge v)=x\wedge v\wedge u$, and so $x\wedge u\neq 0$.

Corollary. *Let $\{u_\lambda;\lambda\in\Lambda\}$ and $\{v_\lambda;\lambda\in\Lambda\}$ be subsets of uniform elements of $\mathfrak{L}$ such that $u_\lambda\wedge v_\lambda\neq 0$ for all $\lambda\in\Lambda$. If $\{u_\lambda;\lambda\in\Lambda\}$ is independent, then so is $\{v_\lambda;\lambda\in\Lambda\}$.*

Proof. By the validity of Prop. 5.2 and Prop. 5.5, the proof will proceed in the same way as that of [8; Prop. 1.1] does.

Proposition 5.6. *Let $\{a_1,\cdots,a_n\}$ $(n\geqq 2)$ be an independent subset of $\mathfrak{L}$, and a an element of $\mathfrak{L}$. If $a\wedge(a_1+\cdots+a_n)\neq 0$ and $a\wedge(a_2+\cdots+a_n)=0$, then $a\wedge(a_1+c)\overset{c}{\sim}(a+c)\wedge a_1$, where $c=a_2+\cdots+a_n$.*

Proof. We set $a'=a\wedge(a_1+c)$. Evidently $a'\wedge c=(a_1+c)\wedge a\wedge c=0$ and $((a+c)\wedge a_1)\wedge c=a_1\wedge c=0$. Moreover, $((a+c)\wedge a_1)+c=(a+c)\wedge(a_1+c)$ and $a'\wedge(a+c)\wedge(a_1+c)=a'\neq 0$.

Corollary. *Under the assumption of* Prop. 5.6, *if a and a_1 are uniform, then $a\overset{c}{\sim}a_1$.*

Let $\mathfrak{U}$ be the set of all uniform elements of $\mathfrak{L}$. A non-empty subset $\mathfrak{U}_0$ of $\mathfrak{U}$ is called a (*)-subset of $\mathfrak{U}$, if $\mathfrak{U}_0$ satisfies the following condition:

(*) if $v\in\mathfrak{U}_0$ and $0\neq u\leqslant v$, then $u\in\mathfrak{U}_0$.

Evidently, if $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are (*)-subsets with $\mathfrak{U}_1\cap\mathfrak{U}_2\neq\emptyset$, then so is $\mathfrak{U}_1\cap\mathfrak{U}_2$.

Proposition 5.7. *Let $\mathfrak{U}_0$ be a (*)-subset of $\mathfrak{U}$, $\{u_\lambda:\lambda\in\Lambda\}$ an independent* *subset of* $\mathfrak{U}_0$, *and* u *an element of* $\mathfrak{U}$ *such that* $\{u\} \cup \{u_\lambda; \lambda \in \Lambda\}$ *is dependent. Then, there exists a unique minimal finite subset* $\{u_1, \cdots, u_n\}$ *of* $\{u_\lambda; \lambda \in \Lambda\}$ *such that* $u \wedge (u_1 + \cdots + u_n) \neq 0$. *For such* u_i, *there holds* $u \frown u_i$ $(i = 1, \cdots, n)$. *Accordingly, if* $\{u_\lambda; \lambda \in \Lambda\}$ *is a maximal independent subset of* $\mathfrak{U}_0$, *then* $\{u_\lambda; u_\lambda \in \mathfrak{U}_u\}$ *is a maximal independent* subset of $\mathfrak{U}_0 \cap \mathfrak{U}_u$, where $\mathfrak{U}_u = \{v \in \mathfrak{U}; u \frown v\}$[5].

Proof. This is a direct consequence from Prop. 5.3 and Cor. to Prop. 5.6.

Proposition 5.8. *If* $\{u_\lambda; \lambda \in \Lambda\}$ *and* $\{v_\gamma; \gamma \in \Gamma\}$ *are maximal independent subsets of a* (*)-*subset* $\mathfrak{U}_0$, *then* $\sharp\Lambda = \sharp\Gamma$, *where* $\sharp\Lambda$ *means the cardinal number of* Λ.

Proof. We shall distinguish here between two cases.

Case 1. $\sharp\Lambda < \aleph_0$ or $\sharp\Gamma < \aleph_0$. We may assume $\sharp\Gamma \leqq \sharp\Lambda$. We set $\{v_\gamma; \gamma \in \Gamma\} = \{v_1, \cdots, v_s\}$, and take $\lambda_0 \in \Lambda$. If $\{v_i\} \cup \{u_\lambda; \lambda \neq \lambda_0\}$ is dependent for all i, then for each v_i there exists a unique minimal finite subset $\{u_{ij}\text{'s}\}$ of $\{u_\lambda; \lambda \neq \lambda_0\}$ with $v_i' = v_i \wedge \sum_i u_{ij} \neq 0$. Evidently $\{v_1', \cdots, v_n'\}$ is independent, and further, as $u_{\lambda_0} \wedge (v_1' + \cdots + v_s') \leqslant u_{\lambda_0} \wedge \sum_{ij} u_{ij} = 0$, $\{u_{\lambda_0}\} \cup \{v_1', \cdots, v_s'\}$ is independent (Prop. 5.2), so that $\{u_{\lambda_0}\} \cup \{v_1, \cdots, v_s\}$ is independent (Cor. to Prop. 5.5), a contradiction. Hence, for some v_k, $\{v_k\} \cup \{u_\lambda; \lambda \neq \lambda_0\}$ is independent. Now, $u_{\lambda_0}' = u_{\lambda_0} \wedge (v_k + u_{\lambda_1} + \cdots + u_{\lambda_n}) \neq 0$ for some $\{u_{\lambda_1}, \cdots, u_{\lambda_n}\} \subseteqq \{u_\lambda; \lambda \neq \lambda_0\}$. We assume that $\{c, v_k\} \cup \{u_\lambda; \lambda \neq \lambda_0\}$ is independent for some $c \in \mathfrak{U}_0$. Then, since $\{u_\lambda; \lambda \in \Lambda\}$ is a maximal independent subset of $\mathfrak{U}_0$, $c \wedge (u_{\lambda_0}' + u_{\mu_1} + \cdots + u_{\mu_m}) \neq 0$ for some $\{u_{\mu_1}, \cdots, u_{\mu_m}\} \subseteqq \{u_\lambda; \lambda \neq \lambda_0\}$. On the other hand, $c \wedge (u_{\lambda_0}' + u_{\mu_1} + \cdots + u_{\mu_m}) \leqslant c \wedge ((v_k + u_{\lambda_1} + \cdots + u_{\lambda_n}) + u_{\mu_1} + \cdots + u_{\mu_m}) = 0$, a contradiction. Hence $\{v_k\} \cup \{u_\lambda; \lambda \neq \lambda_0\}$ is a maximal independent subset of $\mathfrak{U}_0$. Continuing the same argument step by step, we obtain eventually $s = \sharp\Lambda$.

Case 2. $\sharp\Lambda \geqq \aleph^0$ and $\sharp\Gamma \geqq \aleph_0$. For each $v \in \{v_\gamma; \gamma \in \Gamma\}$, there exists a unique minimal finite subset $\{u_1, \cdots, u_n\}$ of $\{u_\lambda; \lambda \in \Lambda\}$ such that $v \wedge (u_1 + \cdots + u_n) \neq 0$. We shall prove that $\cup_v \{u_1, \cdots, u_n\} = \{u_\lambda; \lambda \in \Lambda\}$. For each $u \in \{u_\lambda; \lambda \in \Lambda\}$, there exists a unique minimal finite subset $\{v_1, \cdots, v_m\}$ of $\{v_\gamma; \gamma \in \Gamma\}$ such that $u \wedge (v_1 + \cdots + v_m) \neq 0$. Now, if $\{u_{ij}\text{'s}\}$ is the unique minimal finite subset of $\{u_\lambda; \lambda \in \Lambda\}$ with $v_i' = v_i \wedge \sum_j u_{ij} \neq 0$ $(i = 1, \cdots, m)$, then $u \wedge (v_1' + \cdots + v_m') \neq 0$ by Cor. to Prop. 5.5. Therefore, we have $u \wedge \sum_{ij} u_{ij} \neq 0$, which implies $u \in \{u_{ij}; i, j\}$. Thus we have $\cup_v \{u_1, \cdots, u_n\} = \{u_\lambda; \lambda \in \Lambda\}$. Hence $\sharp\Lambda \leqq \sharp\Gamma \cdot \aleph_0 = \sharp\Gamma$. And, we have symmetrically $\sharp\Gamma \leqq \sharp\Lambda$. Hence $\sharp\Lambda = \sharp\Gamma$.

Combining Prop. 5.7 with Prop. 5.8, we obtain the following fundamental theorem.

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5) This is evidently a (*)-subset of $\mathfrak{U}$.

Theorem 5.9. *Let $\mathfrak{U}_0$ be a (*)-subset of $\mathfrak{U}$. If $\{u_\lambda; \lambda \in \Lambda\}$ and $\{v_\gamma; \gamma \in \Gamma\}$ are maximal independent subsets of $\mathfrak{U}_0$, then there exists a 1–1 mapping f from Λ onto Γ such that $u_\lambda \sim v_{f(\lambda)}$ for all $\lambda \in \Lambda$.*

Corresponding to [8; Prop. 1.9], we obtain the following whose proof is easy by Prop. 5.7.

Proposition 5.10. *Let $\{\mathfrak{U}_\lambda; \lambda \in \Lambda\}$ be a family of (*)-subsets of $\mathfrak{U}$ such that if $\lambda \neq \lambda'$ then $u \nsim u'$ for all $u \in \mathfrak{U}_\lambda$, $u' \in \mathfrak{U}_{\lambda'}$, and let $\{u_{\lambda\gamma}; \gamma \in \Gamma_\lambda\}$ be a maximal independent subset of $\mathfrak{U}_\lambda$. Then $\bigcup_{\lambda \in \Lambda}\{u_{\lambda\gamma}; \gamma \in \Gamma_\lambda\}$ is a maximal independent subset of $\bigcup_{\lambda \in \Lambda}\mathfrak{U}_\lambda$.*

The set of all R-submodules of M forms a modular lattice $\mathfrak{L}(_RM)$ with respect to dual order[6]. A subset $\mathfrak{B}$ of R-submodules is called *d-independent* if $B_1 + \bigcap_{i \neq 1} B_i = M$ for each subset $\{B_1, \cdots, B_n\}$ of $\mathfrak{B}$ $(n \geqq 2)$. A submodule B $(\neq M)$ is called *d-uniform*, if M/B is R-sum-irreducible. Evidently, every maximal submodule is d-uniform.

Let B, C be d-uniform submodules of M. If $M/B' \cong M/C'$ for some proper submodules B', C' of M with $B \subseteqq B'$ and $C \subseteqq C'$ we write $B \approx C$ (d-similar)[7]. As is easily seen, this is an equivalence relation concerning the d-uniform submodules of M.

Let A, B, C be submodules with $A+C=B+C=M$ and $A+(B \cap C) \neq M$. Then, $C \twoheadrightarrow (A+C)/(A+(B \cap C)) = M/(A+(B \cap C))$, and therefore $C/((A \cap C)+(B \cap C)) \cong M/(A+(B \cap C))$. Similarly, we have $C/((A \cap C)+(B \cap C)) \cong M/((A \cap C)+B)$. Hence $M/(A+(B \cap C)) \cong M/((A \cap C)+B)$. If moreover A and B are d-uniform, then $A \approx B$. Thus, Th. 5.9 yields at once the following:

Theorem 5.11. *Let $\{A_\lambda; \lambda \in \Lambda\}$ and $\{B_\gamma; \gamma \in \Gamma\}$ be maximal d-independent subsets of the d-uniform submodules of M. Then, there exists a 1–1 mapping f from Λ onto Γ with $M/A_\lambda \approx M/B_{f(\lambda)}$ for all $\lambda \in \Lambda$. Accordingly we may set* d-dim $_RM = \sharp\Lambda$.

Theorem 5.12. *Let $\{A_\lambda; \lambda \in \Lambda'\}$ and $\{B_\gamma; \gamma \in \Gamma'\}$ be maximal d-independent subsets of the maximal submodules of M. Then, there exists a 1–1 mapping g from Λ' onto Γ' with $M/A_\lambda \cong M/B_{g(\lambda)}$ for all $\lambda \in \Lambda'$. Accordingly we may set* max-dim $_RM = \sharp\Lambda'$.

Proof. Let $\mathfrak{L} = \mathfrak{L}(_RM)$ the modular lattice of all submodules of M defined above, $\mathfrak{U}$ the set of all d-uniform submodules of M, and $\mathfrak{U}_0$ the set of all maximal submodules of M. Then Th. 5.9 applies to obtain our assersion.

Evidently, max-dim $_RM \leqq$ d-dim $_RM$. If every proper submodule of M is contained in a maximal submodule, then max-dim $_RM =$ d-dim $_RM$ by Cor. to

6) For R-submodules A, B of M, $A \geqslant B$ if and only if $A \subseteqq B$.

7) "d-similar" is the dual concept of "similar" defined by [8].

Prop. 5.5.

Remark. (1) We consider R as an R-R-module. Then, the set of all maximal ideals is d-independent, because maximal ideals are prime ideals. Hence (max-dim ${}_RR_R=$) d-dim ${}_RR_R=$ the cardinal number of the set of all maximal ideals of R. (2) Let $M=A_1\oplus\cdots\oplus A_n$, where each A_i is R-sum-irreducible. Put $B_i=A_1\oplus\cdots\oplus\check{A}_i\oplus\cdots\oplus A_n$. Then $\{B_1,\cdots,B_n\}$ is a maximal d-independent subset of the d-uniform submodules of M. Hence, in this case, d-dim ${}_RM=n$. (3) In Th. 5.12, if every proper submodule of M is contained in a maximal submodule, then $\bigcap_{\lambda\in\Lambda'}A_\lambda=\mathfrak{R}({}_RM)$. Because, if $\bigcap_{\lambda\in\Lambda'}A_\lambda\supsetneqq\mathfrak{R}({}_RM)$, then $A+\bigcap_{\lambda\in\Lambda'}A_\lambda=M$ for some maximal submodule A of M. Then, $\{A\}\cup\{A_\lambda;\lambda\in\Lambda'\}$ is d-independent, a contradiction.

Proposition 5.13. *If D is a d-dense submodule of M, then* d-dim ${}_RM=$ d-dim ${}_RM/D$ *and* max-dim ${}_RM=$ max-dim${}_RM/D$.

Proof. Let $\{A_\lambda;\lambda\in\Lambda\}$ be a maximal d-independent subset of the set of d-uniform (resp. maximal) submodules of M. If $A'_\lambda=A_\lambda+D$, then $M/A_\lambda\twoheadrightarrow M/A'_\lambda\cong(M/D)/(A'_\lambda/D)$, and so $\{A'_\lambda/D;\lambda\in\Lambda\}$ is a d-independent subset of the set of d-uniform (resp. maximal) submodules of M/D. Let X be a submodule of M such that $X\supseteqq D$ and $(M/D)/(X/D)$ ($\cong M/X$) is R-sum-irreducible (resp. minimal). Then, since $\{A'_\lambda;\lambda\in\Lambda\}$ is a maximal d-independent subset of the set of d-uniform (resp. maximal) submodules (Cor. to Prop. 5.5), $X+(A'_1\cap\cdots\cap A'_n)\neq M$ for some finite subset $\{A'_1,\cdots,A'_n\}$ of $\{A'_\lambda;\lambda\in\Lambda\}$, and so $X/D+\bigcap_iA'_i/D\neq M/D$. Hence $\{A'_\lambda/D;\lambda\in\Lambda\{$ is a maximal d-independent subset of the set of d-uniform (resp. maximal) submodules of M/D.

Proposition 5.14. *Assume that every proper submodule of M is contained in a maximal submodule of M. Then,* d-dim ${}_RM<\aleph_0$ *if and only if $M/\mathfrak{R}({}_RM)$ is a direct sum of a finite number of R-minimal submodules (or equivalently, if $M/\mathfrak{R}({}_RM)$ satisfies the descending chain condition for R-submodules* (Prop. 1.13)). *When it is the case, M is R-finitely generated.*

Proof. As $\mathfrak{R}({}_RM)$ is R-d-dense in M by Prop. 1.14, we obtain max-dim ${}_RM=$max-dim ${}_RM/\mathfrak{R}({}_RM)$ by Prop. 5.13. If $M/\mathfrak{R}({}_RM)$ is R-finitely generated then M is R-finitely generated (and conversely). Hence it suffices to prove that if d-dim ${}_RM<\aleph_0$ and $\mathfrak{R}({}_RM)=0$ then M is a direct sum of a finite number of minimal submodules. Let $\{A_1,\cdots,A_n\}$ be a maximal d-independent subset of maximal submodules. Then $A_1\cap\cdots\cap A_n=0$ (Remark (3) to Th. 5.10). As is well known, this implies that M is a direct sum of a finite number of minimal submodules, so that M is R-finitely generated.

Corollary. *If* d-dim ${}_RR<\aleph_0$, *then* d-dim ${}_RR=$ d-dim $R_R=$ *the length of* *an R-composition series of* ${}_R R/\mathfrak{R}(R)$ (or $R/\mathfrak{R}(R)_R$).

Finally, as an application of Th. 5.9, we can derive Azumaya's generalization of Krull-Remak-Schmidt's theorem [2]. In what follows, we shall give only the facts which are needed to see the last.

(1) Let e and f be idempotents of $K=\mathrm{Hom}\,({}_RM, {}_RM)$. Then $\mathrm{Hom}\,({}_RMe, {}_RMf)\cong eKf\cong \mathrm{Hom}\,(fK_K, eK_K)$, canonically. Therefore, $Me\cong Mf$ as R-modules if and only if $fK\cong eK$ as right ideals.

(2) Let e and f be idempotents of K different from 1 such that eK and fK are d-uniform right ideals of K. Then, $eK\approx fK$ (d-similar) if and only if $K/eK\cong K/fK$, or equivalently $(1-e)K\cong(1-f)K$. For $(1-e)K(\cong K/eK)$ and $(1-f)K(\cong K/fK)$ are projective (Prop. 3.1).

(3) We put $\mathfrak{L}=$ the modular lattice $\mathfrak{L}(K_K)$ of all right ideals of K with respect to dual order, $\mathfrak{U}=$ the set of all d-uniform right ideals of K and $\mathfrak{U}_0=$ the set of all d-uniform right ideals which contain d-uniform direct summands of K. Then $\mathfrak{U}_0$ is a (*)-subset of $\mathfrak{U}$.

(4) Let $M=\sum_{\lambda\in\Lambda}\oplus M_\lambda$, where each $\mathrm{Hom}\,({}_RM_\lambda, {}_RM_\lambda)$ is a sum-irreducible ring (a completely primary ring in [2]), and let e_λ be the projection to M_λ. Then $\{(1-e_\lambda)K;\, \lambda\in\Lambda\}$ is a maximal d-independent subset of the set of all d-uniform right ideals which contain d-uniform direct summands of K. In fact, the d-independence of $\{(1-e_\lambda)K;\, \lambda\in\Lambda\}$ will be easily seen. To prove the maximality, we take any idempotent f of K such that fK is d-uniform. As $1-f\neq0$, $M(1-f)\neq0$. Let x be arbitrary non-zero element of $M(1-f)$. Then there exists a finite subset $\{M_{\lambda_1}, \cdots, M_{\lambda_n}\}$ of $\{M_\lambda;\, \lambda\in\Lambda\}$ with $x\in M_{\lambda_1}+\cdots+M_{\lambda_n}$. Since $xf=0$ and $xk=0$ for each $k\in\bigcap(1-e_{\lambda_i})K$, $fK+\bigcap(1-e_{\lambda_i})K\neq K$.

§. 6. Lemma on radical and perfectness.

Let $\mathfrak{L}$ be a lattice with 0. $\mathfrak{L}$ is called *perfect* if, for any pair of elements x, y of $\mathfrak{L}$ with $x\wedge y=0$ there exists an element y' of $\mathfrak{L}$ that is maximal with respect to the property $y\leqslant y'$ and $x\wedge y'=0$. A non-zero element d of $\mathfrak{L}$ is said to be dense in $\mathfrak{L}$, if $d\wedge y\neq0$ for each non-zero y in $\mathfrak{L}$.

Lemma 6.1. *Let $\mathfrak{L}, \mathfrak{T}$ be lattices with 0, and φ be an order preserving mapping from $\mathfrak{L}$ to $\mathfrak{T}$, ϕ an order preserving mapping from $\mathfrak{T}$ to $\mathfrak{L}$. We assume that the pair (φ, ϕ) satisfies the following conditions*:

(α) $a\wedge b=0$ $(a, b\in\mathfrak{L})\Rightarrow a\varphi\wedge b\varphi=0$.

(β) $x\wedge y=0$ $(x, y\in\mathfrak{T})\Rightarrow x\phi\wedge y\phi=0$.

(γ) $a\leqslant a\varphi\phi$ *and* $x\leqslant x\phi\varphi$ *for any* $a\in\mathfrak{L}$, $x\in\mathfrak{T}$.

Then there hold the following:

(1) *$\mathfrak{L}$ is perfect if and only if $\mathfrak{T}$ is perfect.*

(2) *If d is a dense element of $\mathfrak{L}$, then $d\varphi$ is a dense element of $\mathfrak{T}$, and*

conversely.

Proof. (1) Let $\mathfrak{T}$ be perfect, and let a, b be elements of $\mathfrak{L}$ with $a\wedge b=0$. Then $a\varphi\wedge b\varphi=0$ by assumption. If y is an element that is maximal with respect to the property $b\varphi\leqslant y$ and $a\varphi\wedge y=0$, then $a\varphi\phi\wedge y\phi=0$, and so $a\wedge y\phi=0$, because $a\leqslant a\varphi\phi$. If b' is an element of $\mathfrak{L}$ with $y\phi\leqslant b'$ and $a\wedge b'=0$, then $a\varphi\wedge b'\varphi=0$. As $y\leqslant y\phi\varphi\leqslant b'\varphi$, we have $y=b'\varphi$, and so $b'\leqslant b'\varphi\phi=y\phi$. Hence $y\phi$ is maximal with respect to the property $b\leqslant y\phi$ and $a\wedge y\phi=0$. Thus $\mathfrak{L}$ is perfect. (2) If $d\varphi\wedge x=0$ $(x\in\mathfrak{T})$ then $d\varphi\phi\wedge x\phi=0$. As $d\leqslant d\varphi\phi$, $d\wedge x\phi=0$, and so $x\phi=0$. Since $x\leqslant x\phi\varphi=0\phi=0$, we have $x=0$. Hence $d\varphi$ is dense in $\mathfrak{T}$. Conversely, assume that $d\varphi$ is dense in $\mathfrak{T}$. If $d\wedge a=0$ $(a\in\mathfrak{L})$ then $d\varphi\wedge a\varphi=0$, and so $a\varphi=0$. As $a\leqslant a\varphi\phi=0$, we have $a=0$. Thus d is dense in $\mathfrak{L}$.

Let $\mathfrak{L}, \mathfrak{T}$ be complete lattices. By $\mathrm{s}(\mathfrak{L})$ (resp. $\mathrm{s}(\mathfrak{T})$), we denote the meet of all dense elements of $\mathfrak{L}$ (resp. $\mathfrak{T}$).

Lemma 6.2. *Let $\mathfrak{L}, \mathfrak{T}$ be complete lattices, and let φ be a mapping from $\mathfrak{L}$ to $\mathfrak{T}$, and ϕ a mapping from $\mathfrak{T}$ to $\mathfrak{L}$ satisfing the following conditions:*

(α') $0\varphi=0$ *and* $(\bigcap_{\gamma\in\Gamma}a_\gamma)\varphi=\bigcap_{\gamma\in\Gamma}(a_\gamma\varphi)$ *for each subset* $\{a_\gamma;\gamma\in\Gamma\}$ *of* $\mathfrak{L}$.

(β') $0\phi=0$ *and* $(\bigcap_{\lambda\in\Lambda}x_\lambda)\phi=\bigcap_{\lambda\in\Lambda}(x_\lambda\phi)$ *for each subset* $\{x_\lambda;\lambda\in\Lambda\}$ *of* $\mathfrak{T}$.

(γ') $a=a\varphi\phi$ *and* $x\leqslant x\phi\varphi$ *for each* $a\in\mathfrak{L}$, $x\in\mathfrak{T}$.

Then there hold the following:

(1) $\mathrm{s}(\mathfrak{L})=\mathrm{s}(\mathfrak{T})\phi$.

(2) *Let* $\mathrm{s}(\mathfrak{L})$ *be dense in* $\mathfrak{L}$. *If* $x\phi\geqslant \mathrm{s}(\mathfrak{L})$ $(x\in\mathfrak{T})$ *then* $\mathrm{s}(\mathfrak{T})\leqslant x$.

Proof. $\mathrm{s}(\mathfrak{T})\leqslant\mathrm{s}(\mathfrak{L})\varphi$ and $\mathrm{s}(\mathfrak{L})\leqslant\mathrm{s}(\mathfrak{T})\phi$ by Lemma 6.1. Since $\varphi\phi=1$, $\mathrm{s}(\mathfrak{T})\leqslant\mathrm{s}(\mathfrak{L})\varphi$ implies that $\mathrm{s}(\mathfrak{T})\phi\leqslant\mathrm{s}(\mathfrak{L})$, which proves (1). If $\mathrm{s}(\mathfrak{L})$ is dense in $\mathfrak{L}$, and $x\phi\geqslant\mathrm{s}(\mathfrak{L})$, then $x\phi$ is dense in $\mathfrak{T}$, so that x is dense in $\mathfrak{T}$ by Lemma 6.1 (2). Hence $x\geqslant\mathrm{s}(\mathfrak{T})$.

Now, let N be a unital R-left module. We set $L=\mathrm{End}\,({}_RN)$, which acts on the right. Then, $A=\mathrm{Hom}\,({}_RM, {}_RN)$ is a K-left, L-right module, and $B=\mathrm{Hom}\,({}_RN, {}_RM)$ is an L-left, K-right module, where $K=\mathrm{End}\,({}_RM)$. In what follows, our attention will be restricted to the important case $A\cdot B=K$. We consider then the modular lattices $\mathfrak{L}(K_K)$, $\mathfrak{L}({}_KK)$, $\mathfrak{L}(A_L)$, $\mathfrak{L}({}_KA)$, $\mathfrak{L}(B_K)$, $\mathfrak{L}({}_LB)$, $\mathfrak{L}(B\cdot A_L)$ and $\mathfrak{L}({}_LB\cdot A)$ Among them, we can define several pairs of mappings (φ, ϕ) satisfing the conditions (α'), (β'), (γ') in Lemma 6.2:

(1) For $\mathfrak{L}(K_K)$ and $(\mathfrak{L}A_L)$, we define $X\varphi=X\cdot A$ and $Y\phi=Y\cdot B$ $(X\in\mathfrak{L}(K_K)$, $Y\in\mathfrak{L}(A_L))$.

(2) For $\mathfrak{L}({}_KK)$ and $\mathfrak{L}({}_LB)$, we define $X\varphi=B\cdot X$ and $Y\phi=A\cdot Y(X\in\mathfrak{L}({}_KK)$, $Y\in\mathfrak{L}({}_LB))$.

(3) For $\mathfrak{L}({}_KA)$ and $\mathfrak{L}({}_LB\cdot A)$, we define $X\varphi=B\cdot X$ and $Y\phi=A\cdot Y(X\in$

$\mathfrak{L}({}_KA)$, $Y \in \mathfrak{L}({}_LB \cdot A))$.

(4) For $\mathfrak{L}(B_K)$ and $\mathfrak{L}(B \cdot A_L)$, we define $X\varphi = X \cdot A$ and $Y\psi = Y \cdot B(X \in \mathfrak{L}(B_K)$, $Y \in \mathfrak{L}(B \cdot A_L))$.

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