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RELATIONS BETWEEN TWO MARTIN TOPOLOGIES ON A RIEMANN SURFACE

By

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Let R be a Riemann surface. Let G be a domain in R with relative boundary ∂G of positive capacity. Let $U(z)$ be a positive superharmonic function in G such that the Dirichlet integral $D(\min(M, U(z))) < \infty$ for every M . Let D be a compact domain in G . Let ${}_D U^M(z)$ be the lower envelope of superharmonic functions $\{U_n(z)\}$ such that $U_n(z) \geq \min(M, U(z))$ on $D + \partial G$ except a set of capacity zero, $U_n(z)$ is harmonic in $G - D$ and $U_n(z)$ has M.D.I. (minimal Dirichlet integral) $\leq D(\min(M, U(z))) < \infty$ over $G - D$ with the same value as $U_n(z)$ on $\partial G + \partial D$. Then ${}_D U^M(z)$ is uniquely determined. Put ${}_D U(z) = \lim_{M \rightarrow \infty} {}_D U^M(z)$. If for any compact domain D ${}_D U(z) = U(z)$ or ${}_D U(z) \leq U(z)$, we call $U(z)$ a full harmonic (F.H.) or a full superharmonic (F.S.H.) in G respectively. If $U(z)$ is an F.S.H. in G and $U(z) = 0$ on ∂G except a set of capacity zero, $U(z)$ is called an F_0 .S.H. in G . Let $U(z)$ be an F.S.H. in G . Then ${}_D U(z) \uparrow$ as $D \uparrow$. For a non compact domain D , put $U(z) = \lim_{n \rightarrow \infty} U(z)$, where $\{G_n\}$ is an exhaustion of G with compact relative boundary ∂G_n ($n = 0, 1, 2, \dots$).

$\mathcal{M}(U(z))$ of an F_0 .S.H. $U(z)$ in G . Let D be a domain in G . Suppose there exists at least one C_1 -function $V(z)$ in $G - D$ such that $V(z) = 1$ on D , $= 0$ on ∂G except a set of capacity zero and $D(V(z)) < \infty$. Let $\omega(D, z, G)$ be a harmonic function in $G - D$ such that $\omega(D, z, G) = 1$ on D , $= 0$ on ∂G except a set of capacity zero and $\omega(D, z, G)$ has M.D.I. over $G - D$. We call $\omega(D, z, G)$ a C.P. (capacitary potential) of D . Let $U(z)$ be an F_0 .S.H. in G . Then $D_{Cg_M}(U(z)) = MD(\omega(g_M, z, G)) \uparrow$ as $M \rightarrow 0$,¹⁾ where $g_M = E[z: U(z) > M]$.

Put $\mathcal{M}(U(z)) = \lim_{M \rightarrow 0} \frac{1}{2\pi} D_{Cg_M}(U(z))$.

$\mathcal{M}(U(z))$ of an F.S.H. $U(z)$ in G . For any compact domain D in G , if we can define functions $U_n(z)$ such that $U_n(z)$ is superharmonic in G , $U_n(z)$ is harmonic in $G - D$, $U_n(z) \geq \min(M, U(z))$ on D , $U_n(z) = 0$ on ∂G except a set of capacity zero and $U_n(z)$ has M.D.I. over $G - D$. Let ${}^0 U^M(z)$ be the lower envelope of $\{U_n(z)\}$. Put ${}_D U(z) = \lim_{M \rightarrow \infty} {}^0 U^M(z)$ (clearly ${}_D U(z) \leq {}_D U(z)$).

1) Z. Kuramochi: Superharmonic functions in a domain of a Riemann surface. Nagoya Math. J., to appear.

Since D is compact, ${}^0D U(z) = 0$ on ∂G except a set of capacity zero. For a non compact domain D , ${}^0D U(z)$ is defined as ${}_D U(z)$. For $U(z)$, put $\mathfrak{M}^f(U(z)) = \lim_{n \rightarrow \infty} \mathfrak{M}^f({}^0G_n U(z))$, where $\{G_n\}$ is an exhaustion of G with compact relative boundary ∂G_n .

Let $\{R_n\}$ with compact relative boundary ∂R_n ($n=0, 1, 2, \dots$) Let $U(z)$ be an F.S.H. in $R - R_0$ such that $U(z) = 0$ on ∂R_0 . Consider $R - R_0$ as G . Then ${}_D U(z)$ is defined. In this case we say that ${}_D U(z)$ is defined relative to $R - R_0$. It is clear that the mapping $U(z) \rightarrow {}_D U(z)$ depends on the domain (G or $R - R_0$) in which ${}_D U(z)$ is defined. In the following we use ${}_D U(z)$ relative to $R - R_0$ which will be denoted by ${}^R_D U(z)$ to distinguish from ${}_D U(z)$ (relative to G). We understand ${}_D U(z)$ (without R on D) means ${}_D U(z)$ of $U(z)$ relative to G .

Martin topologies on $R - R_0$ and on a subdomain $G \subset (R - R_0)$. Let $N(z, p)$ be an N -Green's function of G such that $N(z, p)$ is positively harmonic in $G - p$, $N(z, p) = 0$ on ∂G except a set of capacity zero, $N(z, p)$ has a logarithmic singularity at p and $N(z, p)$ has M.D.I. (where Dirichlet integral is taken with respect to $N(z, p) + \log |z - p|$ in a neighbourhood of p). We suppose N -Martin topology is defined on $G + B$ using $N(z, p)$, s and the distance between p_1 and p_2 is given as

$$\delta(p_1, p_2) = \sup_{z \in D} \left| \frac{N(z, p_1)}{1 + N(z, p_1)} - \frac{N(z, p_2)}{1 + N(z, p_2)} \right|,$$

where D is a fixed compact domain and B is the set of the ideal boundary. Let $L(z, p)$ be an N -Green's function of $R - R_0$ with pole at p . Then also N -Martin topology is introduced on $R - R_0 + B^L$ with metric:

$$\delta(p_1, p_2) = \sup_{z \in R_1} \left| \frac{L(z, p_1)}{1 + L(z, p_1)} - \frac{L(z, p_2)}{1 + L(z, p_2)} \right|,$$

where B^L is the set of the ideal boundary points.

In the following for simplicity we call above two topologies L and N -topologies.

Let $p \in R - R_0 + B_1^L$ (B_1^L is the set of minimal boundary points of $R - R_0$). If ${}_{CG}^R L(z, p) < L(z, p)$ (CG is thin at p), we denote by $p \overset{L}{\in} G$. Then

Theorem 1. Suppose $p \in R - R_0 + B_1^L$ and $p \overset{L}{\in} G$. Then $U(z, p) = L(z, p) - N(z, p)$ is an F.S.H. in G with $D(\min(M, U(z))) \leq 2\pi M$, whence $\mathfrak{M}^f(U(z, p)) \leq 1$.

Proof. $N(z, p) : p \in R - R_0 + B^L$ is continuous on ∂G except p . Hence ${}_{CG}^R L(z, p) = L(z, p)$ on ∂G and $U(z, p) = 0$ on ∂G except a set of capacity zero.

Case 1. $p \in G$. In this case, clearly $U(z, p) = N(z, p)$ and $D(\min(M, U(z, p))) \leq 2\pi M$.

Case 2. $p \in \partial G$. Put $G_n = G + v_n(p)$, Then $CG_n \uparrow CG$ and ${}_{CG_n}^R L(z, p) \uparrow {}_{CG}^R L(z, p)$

as $n \rightarrow \infty$, where $v_n(p) = E \left[z : \text{dist}(z, p) < \frac{1}{n} \right]$. By $p \in G_n$, we have

$$D(\min(M, U(z, p))) \leq \lim_n D(\min(M, L(z, p) - c_{G_n}^R L(z, p))) \leq 2\pi M$$

Case 3. $p \in B_1^L - B_s^L$. In this case it was proved²⁾ $D(\min(M, U(z, p))) \leq 2\pi M$, where B_s^L is the set of singular points, i.e. set of point p such that $\omega(p, z, R - R_0) > 0$ and B_1^L is the set of minimal boundary points of $R - R_0$.

Case 4. $p \in B_s^L$. It was proved only $D(U(z, p)) < \infty$ but as case 3 it can be proved $D(\min(M, U(z, p))) \leq 2\pi M$.

Hence ${}_p^R U(z, p)$ can be defined. Now $c_{G+D}^R({}_p^R U(z, p)) = c_G^R L(z, p)$ by $CG + D) \subset CG$ and $c_{G+D}^R L(z, p) \leq L(z, p)$. Hence ${}_p U(z, p) = c_{G+D}^R(L(z, p) - c_G^R L(z, p)) = c_{G+D}^R L(z, p) - c_G^R L(z, p)$ ³⁾ $\leq L(z, p) - c_G^R L(z, p) = U(z, p)$. By $D(\min(M, U(z, p))) \leq 2\pi M$ we have at once $\mathfrak{M}^V(U(z, p)) \leq 1$. Thus $U(z, p)$ is an F₀.S.H. in G with $\mathfrak{M}^V(U(z, p)) \leq 1$.

Lemma 1. 1). Let $p_i \in R - R_0$ and $p_i \xrightarrow{L} p \in R - R_0 + B^L$ (p_i tends to p relative to L -topology). Then $L(z, p) - \lim_i c_G^R L(z, p_i) \leq L(z, p) - c_G^R L(z, p)$.

2). Let $p_i \xrightarrow{L} p^\alpha \in R - R_0 + B_1^L$ and $p_o \xrightarrow{M} p^\beta \in G + B : p_i \in G$. Then

$$N(z, p^\beta) = (1 - a)(L(z, p^\alpha) - c_G^R L(z, p^\alpha)) : 1 \geq a \geq 0.$$

Proof of 1). For any $\varepsilon > 0$ we can find a number n_0 such that $c_G^R L(z, p^\alpha) \leq c_{G \cap R_n}^R L(z, p^\alpha) + \varepsilon$ for $n \geq n_0$. Since $L(z, p_i) \rightarrow L(z, p^\alpha)$ on $CG \cap R_n$, $\lim_i c_G^R L(z, p_i) \geq \lim_i c_{G \cap R_n}^R L(z, p_i) \geq c_G^R L(z, p^\alpha) - \varepsilon$. Let $\varepsilon \rightarrow 0$. Then we have (1).

Proof of 2). $L(z, p_i) - c_G^R L(z, p_i) = N(z, p_i)$ in G for $p_i \in G$. By the assumption $\lim_i L(z, p_i)$ and $\lim_i N(z, p_i)$ exist, whence $\lim_i c_G^R L(z, p_i)$ exists. We denote this limit by $U(z)$. Let μ be a canonical mass distribution⁴⁾ of $U(z)$ on $R - R_0 + B_1^L$. Assume μ has a positive mass in $\text{int}(G \cap Cv_n(p^\alpha))$ ($\text{int } G$ means the interior of G relative to L -topology and $v_n(p^\alpha)$ is a neighbourhood of p^α relative to L -topology). Then we can find a number n_0 such that G_{n_0} has a positive mass on $\bar{G}_{n_0} \cap Cv_n(p^\alpha)$, where $G_n = E \left[z \in R - R_0 + B^L : \text{dist}(z, CG) > \frac{1}{n} \right]$. Since $\text{dist}(CG + v_{n+i}(p^\alpha), G_{n_0} - v_n(p^\alpha)) > 0$,

2) Z. Kuramochi: Correspondence of boundaries of Riemann surfaces. Journ. Fac. Sci. Hokkaido Uni., XVII (1963). See page 101.

3) If $p \in G$, $U(z, p) = N(z, p)$, we suppose $p \in B^G$. Then $L(z, p)$ is harmonic in $R - R_0$, whence $\sup L(z, p) < \infty$ on a compact domain D and it is clear ${}_p U(z) = c_{G+D}^R(L(z, p) - c_G^R L(z, p))$. If D is non compact, consider $D \cap G_n$ and let $n \rightarrow \infty$.

4) Z. Kuramochi: Potentials on Riemann surfaces. Journ. Fac. Sci. Hokkaido Univ., XVI (1962).

$$c_{G+v_n+\mathfrak{z}(p^a)}^R U(z) < U(z).^{5)}$$

Hence by $c_{G+v_n+\mathfrak{z}(p^a)} L(z, p^a) = L(z, p^a)$ (for $p^a \in R - R + B_1^L$) we have

$$\begin{aligned} N(z, p^a) &= L(z, p^a) - U(z) > c_{G+v_n+\mathfrak{z}(p^a)}^R L(z, p^a) - c_{G+v_n+\mathfrak{z}(p^a)}^R U(z) \\ &= c_{G+v_n+\mathfrak{z}(p^a)}^R (L(z, p^a) - U(z)) = v_{n+\mathfrak{z}(p^a)} (L(z, p^a) - U(z)). \end{aligned} \quad (1)$$

On the other hand, $L(z, p^a) - U(z) = N(z, p^a)$ is an F₀.S.H. in G , whence

$$v_{n+\mathfrak{z}(p^a)} (L(z, p^a) - U(z)) \leq L(z, p^a) - U(z). \quad (2)$$

(1) contradicts (2). Hence $\mu = 0$ on $Cv_n(p) \cap \text{int } G$. Let $n \rightarrow \infty$. Then $\mu = 0$ except on $p + CG$. put $V(z) = \int L(z, p) d\mu'(p)$, where μ' is the restriction of μ on CG . Let a be the mass of μ at p . Then $1 \geq a \geq 0$, $c_G^R V(z) = V(z)$ and $U(z) = V(z) + aL(z, p^a)$. Now $V(z) = (1-a)L(z, p^a)$ on ∂G except a set of capacity zero. Hence $V(z) = c_G^R V(z) = (1-a)c_G^R L(z, p^a)$. Thus $U(z) = (1-a)c_G^R L(z, p^a) + aL(z, p^a)$ and

$$N(z, p^a) = L(z, p^a) - \lim_i c_G^R L(z, p_i) = (1-a) (L(z, p^a) - c_G^R L(z, p^a)).$$

We denote by $\overset{L}{B}(G)$ the set of points p such that $p \in R - R_0 + B_1^L$, $p \in B$ and $p \in \overset{L}{G}$. Clearly $\overset{L}{B}(G)$ is an F_σ set relative to L -topology by the upper semi-continuity of $L(z, p) - c_G^R L(z, p)$ and if $p \in \partial G$, $p \in \overset{L}{B}(G)$ if and only if p is an irregular point for the Dirichlet problem in G by Lemma 1. (2).

Lemma 2. Let $p_i \in \overset{L}{B}(G) + G$ and $p_1 \neq p_2$. Then $L(z, p_1) - c_G^R L(z, p_1) \neq L(z, p_2) - c_G^R L(z, p_2)$.

Assume $L(z, p_1) - c_G^R L(z, p_1) = L(z, p_2) - c_G^R L(z, p_2) = U(z)$. Let n be a number such that $\text{dist}(v_n(p_1), v_n(p_2)) > 0$, where $v_n(p_i)$ is a neighbourhood of p_i relative to L -topology. Now $p_2 \in v_n(p_2)$ imply

$$(G \cap v_n(p_2)) \overset{L}{\ni} p_2.^{6)}$$

Let $V_n = G - v_n(p_1)$. Then $V_n \supset (G \cap v_n(p_2)) \overset{L}{\ni} p_2$. Whence

$$c_{V_n}^R L(z, p) < L(z, p).$$

By $CV_n \supset CG$ we have $c_{V_n}^R (c_G^R L(z, p_i)) = c_G^R L(z, p_i) : i = 1, 2$. Now $c_{V_n}^R L(z, p) \downarrow$ as $n \rightarrow \infty$ by $CV_n \downarrow$. Hence there exist a point z_0 in V_{n_0} , a number n_0 and a const. $\delta > 0$ such that $c_{V_n}^R L(z_0, p_2) < L(z_0, p_2) - \delta$ for $n \geq n_0$. Hence

$$\begin{aligned} c_{V_n}^R (U(z_0)) &= c_{V_n}^R L(z_0, p_2) - c_G^R L(z_0, p_2) < c_G^R L(z_0, p_2) - c_G^R L(z_0, p_2) - \delta \\ &= U(z_0) - \delta : n \geq n_0. \end{aligned} \quad (3)$$

5) See page 60 of 4).

6) See page 99 of 2).

By $CV_n + (v_n(p_1) \cap CG) \supset v_n(p_1)$, we have

$${}^R_{CV_n}L(z, p_1) + {}_{v_n(p_1) \cap CG}{}^RL(z, p_1) \geq {}_{v_n(p_1)}{}^RL(z, p_1) = L(z, p_1).$$

We proved if a domain $\Omega \in p$, $\lim_{n \rightarrow \infty} {}_{v_n(p) \cap CG}{}^RL(z, p) = 0$.⁷⁾ Hence for any $\varepsilon > 0$ there exists a number n' such that ${}^R_{CV_n}L(z_0, p_1) \geq L(z_0, p) - \varepsilon$ for $n \geq n'$. Hence

$$\begin{aligned} {}^R_{CV_n}U(z_0) &= {}^R_{CV_n}L(z_0, p_1) - {}^R_{CV_n}({}_{CG}{}^RL(z_0, p_1)) = {}^R_{CV_n}L(z_0, p_1) - {}^RL(z_0, p_1) \\ &\geq L(z_0, p) - {}^RL(z_0, p_1) - \varepsilon = U(z_0) - \varepsilon, \quad \text{for } n \geq n'. \end{aligned} \quad (4)$$

By (3) and (4) $U(z_0) - \delta \geq U(z_0) - \varepsilon$. This is a contradiction. Hence $L(z, p_1) - {}^RL(z, p_1) \neq L(z, p_2) - {}^RL(z, p_2)$.

Let p^α be a point in $G + \bar{B}(G)$. If there exists a sequence $\{p_i\}$ such that $p_i \xrightarrow{L} p^\alpha$ and $p_i \xrightarrow{M} p^\beta \in G + B$, we say that p^β lies on p^α . We denote the set of points p lying on p^α by $\mathfrak{p}(p^\alpha)$. Then

Lemma 3. *Let $p^\alpha \in G + \bar{B}(G)$. Then $\mathfrak{p}(p^\alpha)$ contains only one point p^β of $G + B_1$ and $L(z, p^\alpha) - {}^RL(z, p) = N(z, p^\beta)$, where B_1 is the set of minimal boundary points of G relative to N -topology. We denote such p^β by $f(p^\alpha)$.*

Let $p_i \xrightarrow{L} p^\alpha$ and $p_i \xrightarrow{M} p^\beta$. Then by Lemma 1.2) $N(z, p^\beta) = (1 - a_\beta)(L(z, p^\alpha) - {}^RL(z, p^\alpha))$. Hence any function $N(z, p^\beta)$ corresponding to p^α is a submultiple of a fixed function and there exists at most one minimal or inner point $p^{\beta'}$ of $G + B_1$ in $\mathfrak{p}(p^\alpha)$ such that $\mathfrak{M}(p^{\beta'}) = 1$ ($\mathfrak{M}(p^{\beta'}) = \mathfrak{M}(N(z, p^{\beta'})) = 1$ is a necessary condition for $p^{\beta'}$ to be minimal).⁸⁾ Let $p^\alpha \in G + \bar{B}(G)$ and $v_n(p^\alpha)$ be a neighbourhood of p^α relative to L -topology and $\bar{v}_n(p^\alpha)$ be the closure of $v_n(p^\alpha)$ relative by M -topology. Then by $p \in G + \bar{B}(G)$ $L(z, p^\alpha) - {}^RL(z, p^\alpha) = \delta_\beta N(z, p^\beta)$: $\delta_\beta = \frac{1}{1 - a_\beta}$ and by ${}_{v_n(p^\alpha)}L(z, p^\alpha) = L(z, p)$ and $CG + v_n(p^\alpha) \supset CG$ we have

$$\begin{aligned} \delta_\beta N(z, p^\beta) &= L(z, p^\alpha) - {}^RL(z, p^\alpha) = {}_{CG + v_n(p^\alpha)}(L(z, p^\alpha) - {}^RL(z, p^\alpha)) \\ &= \delta_{\bar{v}_n(p^\alpha)} N(z, p^\beta). \end{aligned}$$

Let $n \rightarrow \infty$. Then $N(z, p^\beta) = {}_F N(z, p^\beta) > 0$, where $F = \bigcap_{n > 0} \bar{v}_n(p^\alpha)$ is a M -closed set, whence $N(z, p^\beta)$ is representable by a canonical mass distribution on F .⁹⁾ This implies $\mathfrak{p}(p^\alpha)$ contains at least one point in $G + B_1$. Thus $\mathfrak{p}(p^\alpha)$ contains only one point p^* in $G + B_1$ and $(1 - a^*)(L(z, p^\alpha) - {}^RL(z, p^\alpha)) = N(z, p^*)$. On the other hand, $\mathfrak{M}(L(z, p) - {}^RL(z, p)) \leq 1$ by Theorem 1 and $\mathfrak{M}(N(z, p^*)) = 1$.

7) See 6).

8) See Lemma 4 of 1).

9) See 5).

Hence $a^* = 0$ and $L(z, p^a) - c_G^R L(z, p^a) = N(z, p^*)$.

Theorem 2. Let p^β be a point in $G + B_1$. Let $f^{-1}(p^\beta)$ be the set of points p in $R - R_0 + B^L$ (not only in $G + \overset{L}{B}(G)$) such that $L(z, p) - c_G^R L(z, p) = N(z, p^\beta)$. Then $f^{-1}(p^\beta)$ consists of only one point $p \in G + \overset{L}{B}(G)$. Hence the mapping $f(p^a) : p^a \in G + \overset{L}{B}(G)$ is one-to-one manner between $G + \overset{L}{B}(G)$ and $G + B_1$ and further $f^{-1}(p^\beta)$ is a continuous function of p^β in $G + B_1$, but $f(p^a)$ is not necessarily continuous in $G + \overset{L}{B}(G)$.

Let $p \in f^{-1}(p^\beta)$. Then $L(z, p) - c_G^R L(z, p)$ is minimal in G and is equal to $N(z, p^\beta) : p \in G + B_1$. There exists a canonical distribution $\mu(p^a)$ on $R - R_0 + B_1^L$ such that $L(z, p) = \int L(z, p^a) d\mu(p^a)$. Hence

$$N(z, p^\beta) = L(z, p) - c_G^R L(z, p) = \int (L(z, p^a) - c_G^R L(z, p^a)) d\mu(p^a).^{10)}$$

Now by Lemma 3 $L(z, p^a) - c_G^R L(z, p^a) = N(z, q)$ is minimal in G , where $p^a \in G + \overset{L}{B}(G)$ and $q = f(p^a)$. Clearly $L(z, p^a) - c_G^R L(z, p^a) = 0$ for $p^a \notin G + \overset{L}{B}(G)$. Since $N(z, p^\beta)$ is minimal $\mu(p^a)$ must be a point mass a at $p' \in R - R_0 + B_1^L$ and clearly $p' \in G + \overset{L}{B}(G)$. Hence $N(z, p^\beta) = a(L(z, p') - c_G^R L(z, p')) : a > 0$. But $\mathfrak{M}'(N(z, p^\beta)) = 1$ and $\mathfrak{M}(L(z, p^a) - c_G^R L(z, p)) \leq 1$ by Theorem 1, hence $a = 1$ and $N(z, p^\beta) = L(z, p') - c_G^R L(z, p') : p' \in G + \overset{L}{B}(G)$.

Suppose there exist two points p_1 and p_2 in $G + \overset{L}{B}(G)$ such that $L(z, p_i) - c_G^R L(z, p_i) = N(z, p^\beta) : i = 1, 2$. Then by Lemma 2 $p_1 = p_2$. Thus $f^{-1}(p^\beta)$ is uniquely determined and $f^{-1}(p^\beta) \in G + \overset{L}{B}(G)$.

We show $f^{-1}(p^\beta)$ is continuous in $G + B_1$. Let $p_i^\beta \in G + B_1$ and $p_i^\beta \xrightarrow{M} p^\beta \in G + B_1$ as $i \rightarrow \infty$ and let $p_i^a = f^{-1}(p_i^\beta)$. Then $\{p_i^a\}$ has at least one limiting point p in $R - R_0 + B^L$, since $R - R_1 + B^L$ is compact. Let $\{p_j^a\}$ be a subsequence of $\{p_i^a\}$ such that $p_j^a \rightarrow p$ and $p_j^\beta \rightarrow p^\beta : p_j^\beta = f(p_j^a)$. Then $\lim_j L(z, p_j^a) = L(z, p)$, $\lim_j N(z, p_j^\beta) = N(z, p^\beta)$ and $\lim_j c_G^R L(z, p_j^a)$ exists, i.e. $L(z, p) - \lim_j c_G^R L(z, p_j^a) = N(z, p^\beta)$. Let $p' = f^{-1}(p^\beta)$. Then $L(z, p') - c_G^R L(z, p') = N(z, p^\beta)$ and $p' \in G + \overset{L}{B}(G)$. By $\lim_j c_G^R L(z, p_j^a) \geq c_G^R(\lim_j L(z, p_j^a)) = c_G^R L(z, p)$, we have

$$L(z, p) - c_G^R L(z, p) \geq L(z, p) - \lim_j c_G^R L(z, p_j^a) = N(z, p^\beta).$$

Let $\mu(q)$ be a canonical mass distribution of $L(z, p)$ on $R - R_0 + B_1^L$. Then

$$L(z, p) = \int L(z, q) d\mu(q) \text{ and } \int d\mu(q) = 1 \text{ by } \mathfrak{M}'(L(z, p)) = \frac{1}{2\pi} \int_{\partial R_0} \frac{\partial}{\partial n} L(z, p) ds$$

10) Because $\int c_G^R L(z, p) d\mu(p) = c_G^R(\int L(z, p) d\mu(p))$. See Theorem 1 of 1).

$= \int d\mu(p) = 1$. Now

$$L(z, p) - c_G^R L(z, p) = \int (L(z, q) - c_G^R L(z, q)) d\mu(q) = \int N(z, q^\beta) \delta(q) d\mu(q),$$

where $\delta(q) = 1$ or 0 according as $q \in G + \bar{B}(G)$ or not and $q^\beta = f(q) \subset G + B_1$. Hence $\mathfrak{W}'(L(z, p) - c_G^R L(z, p)) = \int \delta(q) d\mu(q)$ by Theorem 6.¹¹⁾ On the other hand, by $N(z, p^\beta) \leq L(z, p) - c_G^R L(z, p)$, $\mathfrak{W}'(N(z, p^\beta)) = 1 \leq \mathfrak{W}'(L(z, p) - c_G^R L(z, p)) \leq 1$ by Theorem 1. Hence $\delta(q) = 1$ if $\mu(q) > 0$ and $\int d\mu(q) = 1 = \mathfrak{W}'(L(z, p) - c_G^R L(z, p))$. Both $L(z, p) - c_G^R L(z, p)$ and $N(z, p^\beta)$ are F₀.S.H.s in G . Let $V_M = E[z : L(z, p) - c_G^R L(z, p) > M]$ and $V'_M = E[z : N(z, p) > M]$. Then $V_M \supset V'_M$. $\mathfrak{W}'(N(z, p^\beta)) = \frac{MD(\omega(V'_M, z, G))}{2\pi} = 1$ for any M , since $N(z, p^\beta)$ is minimal and $\lim_{V'_M} N(z, p^\beta) = N(z, p^\beta)$. Also $1 = \mathfrak{W}'(L(z, p) - c_G^R L(z, p)) \geq \frac{MD(\omega(V_M, z, G))}{2\pi} \geq \frac{MD(\omega(V'_M, z, G))}{2\pi} = 1$, because $MD(\omega(V_M, z, G)) \uparrow$ as $M \rightarrow 0$. Hence $V_M \supset V'_M$ and $\omega(V_M, z, G) = \omega(V'_M, z, G)$ for any M . This implies $L(z, p) - c_M^R L(z, p) = N(z, p^\beta)$ and $p = f^{-1}(p^\beta) = p' \in G + \bar{B}(G)$. Since any subsequence $p_j^a \rightarrow p'$, $\{p_i^a\}$ converges to $f^{-1}(p_\beta)$ as $p_i^a \rightarrow p^\beta$.

We show $f(p^a)$ is not necessarily continuous. Let $R - R_0$ be $E[0 < |z| < 1] = \Omega$, and F be a closed set on the real axis such that $z_0 = 0$ is an irregular point for the Dirichlet problem of $G = \Omega - F$, where $F = \sum_{K=0}^{\infty} F_K$ and F_K is a segment. Then $L(z, p)$ of Ω and $N(z, p)$ of G are Green's functions $G(z, p)$ and $G'(z, p)$ of Ω and G respectively. Then by Lemma 3 there exists a sequence $\{p_i\}$ such that $G(z, p_i)$ converges to a function $G'(z, p^\beta)$ with $\mathfrak{W}'(G'(z, p^\beta)) = 1$ and $p_i \rightarrow z_0$. Hence $p^\beta = f(z_0)$. Let p_0 be a fixed point in G . Let q_i be a point such that q_i is so near F_i that $G'(p_0, q_i) \leq \frac{1}{i}$. Then $\lim_i G(z, q_i) = 0$. For any i we can find $G'(z, p'_i)$ such that p'_i lies on a curve connecting p_i and q_i and that $G(p_0, p'_i) \rightarrow a(G'(p_0, p^\beta))$ as $i \rightarrow \infty$, where $0 < a < 1$. Also we choose a subsequence $\{p'_j\}$ from $\{p'_i\}$ so that $p'_i \rightarrow z_0$ (relative to L -topology) and $G'(z, p'_j) \rightarrow p^{\beta'}$ (relative to M -topology): $p^{\beta'} \neq p^\beta$. Then since $p'_j \in G'_j$, $p'_j = f(p'_j)$ and $p'_j \xrightarrow{L} z_0$ but $f(p'_j) \xrightarrow{M} p^{\beta'} \neq p^\beta = f(z_0)$. Hence $f(p)$ is not continuous at z_0 .

We call the *harmonic dimension* of $p \in (\partial G + B)$ relative to G and $R - R_0$ the number of linearly independent F₀.S.H.s with finite $\mathfrak{W}'$ in G and $R - R_0$ which are harmonic in G and $R - R_0$ respectively. Then by Lemma 1 we have the following

11) If μ is canonical, $\mathfrak{W}'(U(z)) = \int d\mu(p)$. See Theorem 6 of 1).

Corollary. *Harmonic dimension of p relative to $R-R_0$ is equal to that of p relative to G .*

Applications to extremisations. Let $U(z)$ be an F_0 .S.H. in $R-R_0$ with $\mathfrak{M}'(U(z)) < \infty$. Then there exists a canonical distribution μ such that $U(z) = \int L(z, p) d\mu(p)^{12)}$ and $\int d\mu(p) = \mathfrak{M}'(U(z))$. Put $V(z) = U(z) - {}_{CG}^R U(z)$. Then $V(z) = \int (L(z, p) - {}_{CG}^R L(z, p)) d\mu(p) = \int N(z, q) \delta(p) d\mu(p)$, where $q = f(p)$ and $\delta(p) = 1$ or 0 according as $p \in G + \overset{L}{B}(G)$ or not. Hence $V(z)$ is an F.S.H. in G with $\mathfrak{M}'(V(z)) \leq \int \delta(p) d\mu(p) < \infty$ and $U(z) - V(z) = {}_{CG} U(z)$ is full harmonic in G . We denote $V(z)$ by ${}_{in\ ex} U(z)$. Let $V'(z)$ be an F.S.H. in G with $\mathfrak{M}'(V'(z)) < \infty$. Then $V(z)$ is a potential such that $V(z) = \int_{G+B_1} N(z, q) d\mu(q)^{13)}$ and $\int d\mu(q) = \mathfrak{M}'(V'(z))$. Put $U'(z) = \int L(z, p) d\mu(q)$, where $p = f^{-1}(q)$. Then $U'(z)$ is an F_0 .S.H. in $R-R_0$ with $\mathfrak{M}'(U'(z)) \leq \int d\mu(q)$ and $U'(z) - V(z)$ is full harmonic in G . We denote $U'(z)$ by ${}_{ex} V'(z)$. Then $U'(z) - {}_{CG} U'(z) = V'(z)$.

Let $\{G_n\}$ be an exhaustion of G with compact relative boundary ∂G_n . Since ${}_{G_n} V'(z)$ is full harmonic in $G - \bar{G}_n$, the solution of Neumann's problem (to obtain an F_0 .S.H. $W(z)$ in $R-R_0$ such that $W(z) - {}_{G_n} V'(z)$ is full harmonic in G_{n+i+j} and $W(z)$ is full harmonic in $G - G_{n+i}$) can be obtained by smooting process by $\text{dist}(\partial G_{n+i}, \partial G_{n+i+j}) > 0$ for a given singularity of ${}_{G_n} V'(z)$ in $\bar{G}_n$ and its solution is unique. It is evident that this solution coincides with ${}_{ex}({}_{G_n} V'(z))$. Clearly ${}_{ex}({}_{G_n} V'(z)) \uparrow$ as $n \rightarrow \infty$. On the other hand, $f^{-1}(p) : p \in G + B_1$ is continuous, we have ${}_{ex}(V'(z)) = \lim_{n \rightarrow \infty} ({}_{ex} {}_{G_n} V'(z))$. Hence ${}_{ex} V'(z)$ is the least F_0 .S.H. in $R-R_0$ such that ${}_{ex} V'(z) - V(z)$ is full harmonic in G . We have easily the following

Theorem 3. 1). Let $U(z)$ be an F_0 .S.H. in $R-R_0$ with $\mathfrak{M}'(U(z)) < \infty$. Then ${}_{ex}({}_{in\ ex} U(z)) \leq U(z)$ and ${}_{ex}({}_{in\ ex} U(z)) = U(z)$ if and only if the canonical distribution of $U(z)$ has no mass on CG .

2). Let $V(z)$ be an F.S.H. in G with $\mathfrak{M}'(V(z)) < \infty$. Then

$${}_{in\ ex}({}_{ex} V(z)) = V(z).$$

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12) See 4).

13) See 1).