ON DIMENSIONS OF SIMPLE RING EXTENSIONS

By

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Let A be a division ring, and B a division subring of A. If T is an intermediate ring of $A/V_A^2(B)$ then $[T: V_A^2(B)]_l=[V_A(B): V_A(T)]_r$, provided we do not distinguish between two infinite dimensions ([8, Lemma 2]). If A/B is left locally finite then so is $A/V_A^2(B)$ ([8, Theorem 1]). Moreover, if A/B is Galois and $A/V_A^2(B)$ is left locally finite then for any intermediate ring B' of A/B left finite over B there holds $[B': B]_r=[B': B]_l$ ([7, Corollary 2]).

The purpose of the present paper is to extend those results stated above to simple ring extensions. As one will see later, our extension of [7, Corollary 2] is especially satisfactory (Theorem 3).

Throughout the present paper, we use the following conventions: A will represent a ring with 1, B a unital subring of A (i.e. a subring of A containing 1), V the centralizer $V_A(B)$ of B in A, and H the double centralizer $V_A^2(B)=V_A(V)$ of B in A. Moreover, $\mathfrak{A}$ and $\mathfrak{G}$ will denote the absolute endomorphism ring $\mathrm{Hom}(A, A)$ of A and the group of all B-ring automorphisms of A, respectively. If X is a subset of A, then X_l, X_r and $\tilde{\mathfrak{X}}$ will mean the sets of all left multiplications effected by elements of X, of all the right multiplications effected by elements of X and of all inner automorphisms effected by regular elements of A contained in X, respectively.

The following easy lemma will play an essential role in our subsequent consideration.

Lemma 1. *Let A be a right Artinian ring, and $\mathfrak{B}$ a subring of $\mathfrak{A}$ such that A is $V_{\mathfrak{A}}(\mathfrak{B})\cdot\mathfrak{B}$-irreducible.* If $\mathfrak{B}A_r=\mathfrak{B}$ and A is $\mathfrak{B}$-unital (i.e. $x\mathfrak{B}\neq 0$ for every non-zero $x\in A$), then $V_{\mathfrak{A}}(\mathfrak{B})$ is an (Artinian) simple subring of A_l and $V_{\mathfrak{A}}^2(\mathfrak{B})$ is the closure of $\mathfrak{B}$ (in the finite topology).*

Proof. Noting that every $x\mathfrak{B}$ $(x\in A)$ is a right A-submodule of A and A is $\mathfrak{B}$-unital, one will easily see that A contains a minimal $\mathfrak{B}$-submodule M. Since $A=MV_{\mathfrak{A}}(\mathfrak{B})=\sum_{\alpha\in V_{\mathfrak{A}}(\mathfrak{B})}M\alpha$ and each $M\alpha$ is either $\mathfrak{B}$-isomorphic to M or 0, A is homogeneously $\mathfrak{B}$-completely reducible. Hence, $V_{\mathfrak{A}}(\mathfrak{B})$ is simple, and $\mathfrak{B}$ is dense in $V_{\mathfrak{A}}^2(\mathfrak{B})\,(\ni 1)$ by [2, Theorem VI. 2.2]. Now, the rest of our assertion will be easily seen.

The converse of Lemma 1 will be rather familiar: Let B be a direct summand of the left B-module A. If a subring $\mathfrak{B}$ of $\mathfrak{A}$ is dense in $V_{\mathfrak{A}}(B_l)$ then it is known that $B_l = V_{\mathfrak{A}}(\mathfrak{B})$[1]. In particular, if $\mathfrak{G}'A_r$ is dense in $V_{\mathfrak{A}}(B_l)$ then $J(\mathfrak{G}', A) = \{a \in A\,;\ a\sigma = a$ for every $\sigma \in \mathfrak{G}'\}$ coincides with B, where $\mathfrak{G}'$ is a group of B-ring automorphisms of A. Further, if B is a simple ring and a subring $\mathfrak{B}$ of $\mathfrak{A}$ is dense in $V_{\mathfrak{A}}(B_l)$ then it turns our that A is $V_{\mathfrak{A}}(\mathfrak{B})\cdot\mathfrak{B}$-irreducible. Accordingly, combining the above with Lemma 1, we obtain the following:

Corollary 1. *If A is a right Artinian ring, then $B' \to V_{\mathfrak{A}}(B_l')$ and $\mathfrak{B}' \to$ (1) $V_{\mathfrak{A}}(\mathfrak{B}')$ are mutually converse $1-1$ dual correspondences between unital simple subrings B' of A and closed intermediate rings $\mathfrak{B}'$ of $\mathfrak{A}/A_r$ such that A is $V_{\mathfrak{A}}(\mathfrak{B}')\cdot\mathfrak{B}'$-irreducible.*

Remark 1. In Corollary 1, $\mathfrak{B}'$ can be characterized as a closed subring of $\mathfrak{A}$ such that $\mathfrak{B}'A_r = \mathfrak{B}'$ and that A is $\mathfrak{B}'$-unital and $V_{\mathfrak{A}}(\mathfrak{B}')\cdot\mathfrak{B}'$-irreducible (Lemma 1). Moreover, in case A is a division ring, the $V_{\mathfrak{A}}(\mathfrak{B}')\cdot\mathfrak{B}'$-irreducibility of A is an easy consequence of the assumption that $\mathfrak{B}'A_r = \mathfrak{B}'$ and A is $\mathfrak{B}'$-unital. Hence, Corollary 1 contains essentially [1, Satz V, 1].

The next lemma stated without proof is [3, Lemma 1], in particular, the first assertion is an immediate consequence of Lemma 1.

Lemma 2. *Let A be a simple ring, W a unital subring of V, and let A be $B\cdot W$-A-irreducible.*

(a) *V and $V_A(W)$ are simple rings, and A is homogeneously completely reducible as B-A-module and $[A|B_l \cdot A_r] = [V|V]$*[2].

(b) *If S is a unital simple subring of B such that $[B:S]_l < \infty$, then $[V_A(S):V]_r \leqslant [B:S]_l$. If moreover A/S is left locally finite, then $[V_A(S):V]_l \leqslant [B:S]_l$.*

If A is right Artinian and $B\cdot V$-A-irreducible, then A is obviously a simple ring. In what follows, we shall often treat with a simple ring extension A/B such that A is $B\cdot V$-A-irreducible. Such an extension is, we believe, not so extraordinary. In fact, as was shown in [4] and [10], if A/B is q-Galois and left locally finite then A is $B\cdot V$-A-irreducible.

Corollary 2. *Let a simple ring A be $B\cdot V$-A-irreducible.*

(a) *If $\mathfrak{H}$ is a subset of $\mathfrak{A}$ containing $\tilde{V}$ such that $V_{\mathfrak{A}}(A_r[\mathfrak{H}]) = B_l$, then B is regular and $A_r[\mathfrak{H}]$ is dense in $V_{\mathfrak{A}}(B_l)$.*

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1) Cf. G. Azumaya, On Morita's theorems, Proceedings of a Symposium held at Hokkaido University, July 10–14, 1964 (in Japanese).

2) $[A|B_l\cdot A_r]$ means the length of the composition series of A as $B_l\cdot A_r$-module, and $[V|V]$ does the capacity of the simple ring V (= length of the composition series of V as one-sided V-module).

(b) *If* $\mathfrak{H}$ *is a subgroup of* $\mathfrak{G}$ *containing* $\tilde{V}$ *and* $J(\mathfrak{H}, A)=B$, *then* B *is regular,* $\mathfrak{H}A_r$ *is dense in* $V_{\mathfrak{A}}(B_l)$ *and* $(\mathfrak{H}|H)H_r$ *is dense in* $\mathrm{Hom}_{B_l}(H, H)$.

Proof. By Lemma 2 (a), V and H are simple rings.

(a) Since $A_r[\mathfrak{H}]\supseteq\tilde{V}A_r = V_l\cdot A_r$, A is $V_{\mathfrak{A}}(A_r[\mathfrak{H}])\cdot A_r[\mathfrak{H}]$-irreducible. Hence, the assertion is clear by Lemma 1.

(b) By the validity of (a), it suffices to prove the last assertion. Let h be an arbitrary non-zero element of H. Then, $(Bh)\mathfrak{H}H_r=eH$ with some non-zero idempotent e. Since $A=(Bh)\mathfrak{H}A_r=((Bh)\mathfrak{H}H_r)A_r=eA$, e has to be 1. Hence, H is $B_l\cdot(\mathfrak{H}|H)H_r$-irreducible. Consequently, again by Lemma 1, $(\mathfrak{H}|H)H_r$ is dense in $\mathrm{Hom}_{B_l}(H, H)$.

Obviously, [8, Lemma 2] and [8, Theorem 1] are contained in the following theorem.

Theorem 1. *Let a simple ring* A *be* $B\cdot V$-A-*irreducible.*

(a) *If* T *is an intermediate ring of* A/H *such that* A *is* T-A-*irreducible then* $[V:V_A(T)]_r=[T:H]_l$, *provided we do not distinguish between two infinite dimensions* (cf. Lemma 2 (a)).

(b) *Let* B *be simple. If* B' *is an intermediate ring of* A/B *left finite over* B *such that* A *is* B'-A-*irreducible, then* $[V_A^2(B'):H]_l=[V:V_A(B')]_r<\infty$ *and* $V_A^2(B')=H[B']$.

(c) *Let* B *be simple. If* B' *is an intermediate ring of* A/B *right finite over* B *such that* A *is* A-B'-*irreducible, then* $[V_A^2(B'):H]_r=[V:V_A(B')]_l<\infty$ *and* $V_A^2(B')=H[B']$.

(d) *Let* B *be simple. If* A/B *is left* (*or right*) *locally finite, then* A *is* h-*Galois and* (*two-sided*) *locally finite over* H *and then* A/A' *is inner Galois and* $[A':H]_r=[A':H]_l=[V:V_A(A')]_r=[V:V_A(A')]_l$ *for every simple intermediate ring* A' *of* A/H *left* (*or right*) *finite over* H.

Proof. (a) Since $V_l\cdot A_r$ is dense in $\mathrm{Hom}_{H_l}(A, A)$ by Lemma 1, one will easily see that $[(V_l|T)A_r:A_r]_r=[T:H]_l$, provided we do not distinguish between two infinite dimensions (cf. [6, Lemma 1.4]). On the other hand, by [6, Lemma 1.4], $[(V_l|T)A_r:A_r]_r=[V:V_A(T)]_r$. Combining those, it follows at once our assertion.

(b) Obviously, $H\subseteq H[B']\subseteq V_A^2(B')$ and A is $H[B']$-A-irreducible. Since $V_A(H[B'])=V_A(B')=V_A(V_A^2(B'))$ and $\infty>[B':B]_l\geqslant[V:V_A(B')]_r$ by Lemma 2 (b), (a) implies that $[V_A^2(B'):H]_l=[V:V_A(B')]_r=[H[B']:H]_l$.

(c) By Lemma 2 (b), we obtain $\infty>[B':B]_r\geqslant[V:V_A(B')]_l\geqslant[V_A^2(B'):H]_r\geqslant[H[B']:H]_r\geqslant[V:V_A(B')]_l$, namely, $[V:V_A(B')]_l=[H[B']:H]_r=[V_A^2(B'):H]_r$.

(d) By (b) (or (c)) and [3, Theorem 1], A/H is h-Galois and locally finite. Since $\tilde{V}\cdot A_r=V_l\cdot A_r$ is dense in $\mathrm{Hom}_{H_l}(A, A)$ (Lemma 1), A/A' is inner Galois by [11, Proposition 4], and then $[A':H]_r=[A':H]_l=[V:V_A(A')]_r=[V:V_A(A')]_l$ by [3, Theorem 1] or [9, Theorem 8].

Now, let A/B be a left locally finite simple ring extension. We consider the following conditions[3]:

(i) B is regular.

(ii) A is $B\cdot V$-A-irreducible.

(iii) A is $B\cdot V$-A-irreducible and $\mathfrak{G}(A', A/B)\,|\,H=\mathfrak{G}(H, A/B)$ for every $A'\in\mathfrak{R}^0/H$ left finite over H.

(iv) A is A-$B\cdot V$-irreducible.

(v) A is A-$B\cdot V$-irreducible and $\mathfrak{G}(A', A/B)\,|\,H=\mathfrak{G}(H, A/B)$ for every $A'\in\mathfrak{R}^0/H$ left finite over H.

(vi) $\mathfrak{G}(B_1, A/B)|B_2=\mathfrak{G}(B_2, A/B)$ for every $B_1\supseteq B_2$ in $\mathfrak{R}_{l.f}$.

(vii) H/B is Galois.

(viii) H/B is Galois and $[V_A^2(T):H]_l=[V:V_A(T)]_r$ for every $T\in\mathfrak{R}^0_{l.f}$.

(ix) $(T\cap H)\mathfrak{G}(T, A/B)\subseteq H$ for every $T\in\mathfrak{R}^0_{l.f}$.

In [4, Theorems 3, 4 and 5], one of the present authors has given several useful conditions those which are equivalent to the condition that A/B be q-Galois. Now, we shall add other equivalent ones to those.

Theorem 2. *Let a simple ring A be left locally finite over a simple ring B. In order that A/B be q-Galois, it is necessary and sufficient that any of the following equivalent conditions be satisfied*:

(1) (ii)+(vi)+(vii).

(2) (iv)+(vi)+(vii).

(3) (i)+(vi)+(viii).

(4) (iii)+(vii)+(ix).

(5) (v)+(vii)+(ix).

Proof. If A/B is q-Galois then (i)–(ix) are all satisfied ([4, Theorems 3, 4 and 5] and [9, Theorem 6]). Conversely, if one of the conditions (1), (2) and (3) is satisfied and if T is in $\mathfrak{R}^0_{l.f}$, then $T\cap H$ is in $\mathfrak{R}_{l.f}$ ([6, Lemma 1.6] and [5 Theorem 1.1]) and $J(\mathfrak{G}(T, A/B), T)=J(\mathfrak{G}(T, A/B), T)\cap J(\tilde{\mathfrak{V}}\,|\,T, T)=J(\mathfrak{G}(T, A/B)\,|\,H\cap T, H\cap T)=J(\mathfrak{G}(H\cap T, A/B), H\cap T)\subseteq J(\mathfrak{G}(H/B)|H\cap T, H\cap T)=B$, where $\mathfrak{G}(H/B)$ means the Galois group of H/B. Hence, A/B is q-Galois by [4, Theorems 3 and 4]. Finally, assume (4) or (5). Then, A/H is locally finite by Theorem 1 (d). If $T\in\mathfrak{R}^0_{l.f}$, then $J(\mathfrak{G}(T, A/B), T)\subseteq J(\mathfrak{G}(T[H], A/B)\,|\,T, T)=J(\mathfrak{G}(T[H], A/B)\,|\,H\cap T, H\cap T)\subseteq J(\mathfrak{G}(H/B)\,|\,H\cap T, H\cap T)=B$. Hence, A/B is q-Galois by [4, Theorem 5].

Finally, we shall extend [7, Corollary 2] to simple ring extensions. Let

3) As to notations, we follow [4] and [10].

$\mathfrak{H}$ be a (multiplicative) sub-semigroup of $\mathfrak{A}$. If $\mathfrak{H}A_r$ and $\mathfrak{H}A_l$ form subrings of $\mathfrak{A}$ (or, $A_r\mathfrak{H}\subseteq\mathfrak{H}A_r$ and $A\mathfrak{H}_l\subseteq\mathfrak{H}A_l$) and $V_{\mathfrak{A}}(\mathfrak{H})\cap A_r=B_r$ and $V_{\mathfrak{A}}(\mathfrak{H})\cap A_l=B_l$, then $\mathfrak{H}$ is called a *Galois semigroup* of A/B.

Lemma 3. *Let a simple ring A be $B\cdot V$-A-irreducible, and $\mathfrak{H}$ a Galois semigroup of A/B containing $\tilde{V}$. Let T be a B-B-submodule of A possessing a linearly independent left B-basis.*

(a) *If T is left finite over B then $(\mathfrak{H}|T)V_r$ possesses a linearly independent V_r-basis that forms at the same time a linearly independent A_r-basis of $(\mathfrak{H}|T)A_r$.*

(b) *In order that T be left finite over B, it is necessary and sufficient that $[(\mathfrak{H}|T)V_r|V_r]$ be finite.*

Proof. By Corollary 2, B is regular and $\mathfrak{H}A_r$ is dense in $V_{\mathfrak{A}}(B_l)$.

(a) The proof will be completed in the same way as in [5, Lemma 1.2 (i)]. Let $\Gamma=\{g_{pq};\ p,q=1,\cdots,u\}$ be a system of matrix units of V such that $V_V(\Gamma)$ is a division ring. If $g_p=g_{pp}$ then $A=\oplus_1^u g_pA$ and $(g_{qp})_l$ induces a B-A-isomorphism of g_pA onto g_qA. Since A is homogeneously B-A-completely reducible and $[A|B_l\cdot A_r]=[V|V]=u$ (Lemma 2 (a)), g_pA is B-A-irreducible, so that $(g_pA)_r$ is B_r-A_r-irreducible. Accordingly, $(\sigma|T)(g_pA)_r$ being B_r-A_r-homomorphic to $(g_pA)_r$ for every $\sigma\in\mathfrak{H}$, $\mathrm{Hom}_{B_l}(T,A)=\sum_{\sigma\in\mathfrak{H}}\sum_p(\sigma|T)(g_pA)_r=\oplus_1^s(\sigma_\lambda|T)(g_{p_\lambda}A)_r$ with some $\sigma_\lambda\in\mathfrak{H}$ and some g_{p_λ}, where each $(\sigma_\lambda|T)(g_{p_\lambda}A)_r$ is B_r-A_r-isomorphic to arbitrary fixed $(g_pA)_r$. Recalling here that $A=\oplus_1^u g_pA$, the last relation yields $s=u\cdot[T:B]_l$, and so $\mathfrak{T}=\sum_\lambda(\sigma_\lambda|T)(g_{p_\lambda}V)_r=\oplus_\lambda(\sigma_\lambda|T)(g_{p_\lambda}V)_r$ possesses a linearly independent V_r-basis $\{\varepsilon_1,\cdots,\varepsilon_t\}$ and $[\mathfrak{T}:V_r]_r=[T:B]_l$. Since $(\mathfrak{H}|T)A_r=\mathfrak{T}A_r$ and $[(\mathfrak{H}|T)A_r:A_r]_r=[T:B]_l$, the V_r-basis $\{\varepsilon_1,\cdots,\varepsilon_t\}$ is still a linearly independent A_r-basis of $(\mathfrak{H}|T)A_r$. Now, one will easily see that $\mathfrak{T}=\mathrm{Hom}_{B_l\cdot B_r}(T,A)=(\mathfrak{H}|T)V_r$.

(b) In virtue of (a), it remains only to prove the sufficiency. If $[(\mathfrak{H}|T)V_r|V_r]$ is finite then $(\mathfrak{H}|T)V_r$ is finite over V_r, and so $(\mathfrak{H}|T)A_r=((\mathfrak{H}|T)V_r)A_r$ is finite over A_r, too. Hence, our assertion is a consequence of the density of $\mathfrak{H}A_r$ in $V_{\mathfrak{A}}(B_l)$.

Proposition 1. *Assume that a simple ring A is $B\cdot V$-A-irreducible and A-$B\cdot V$-irreducible. Let $\mathfrak{H}$ be a Galois semigroup of A/B containing $\tilde{V}$. Let T be a B-B-submodule of A possessing a finite linearly independent left B-basis and a linearly independent right B-basis, and $\{\varepsilon_1,\cdots,\varepsilon_t\}$ a linearly independent V_r-basis of $(\mathfrak{H}|T)V_r$ that forms at the same time a linearly independent A_r-basis of $(\mathfrak{H}|T)A_r$* (cf. Lemma 3 (a)). *If $V_A(\cup_1^t T\varepsilon_i)\cap V$ contains a unital division subring U such that $[V:U]_l=[V:U]_r<\infty$, then $[T:B]_r\leqslant[T:B]_l$.*

Proof. By Corollary 2, B is regular, $\mathfrak{H}A_r$ and $\mathfrak{H}A_l$ are dense in $V_{\mathfrak{A}}(B_l)$ and $V_{\mathfrak{A}}(B_r)$, respectively. Hence, $\mathrm{Hom}_{B_l}(T, A)=(\mathfrak{H}\mid T)A_r=\oplus_1^t \varepsilon_i A_r$ ($t=[T:B]_l$). Moreover, one will easily see that $\mathfrak{T}=\oplus_1^t \varepsilon_i V_r=\mathrm{Hom}_{B_l\text{-}B_r}(T, A)\supseteq(\mathfrak{H}\mid T)V_l$. If $\{v_1, \cdots, v_m\}$ is a linearly independent left U-basis of V, then by the assumption we have $\mathfrak{T}=\sum_{i=1}^t \varepsilon_i(\sum_{k=1}^m U_r v_{kr})=\sum_{i,k}\varepsilon_i v_{kr} U_l$. Hence, $tm\geqslant[\mathfrak{T}:U_l]_r$, so that $[(\mathfrak{H}\mid T)V_l\mid V_l]$ is finite. Accordingly, by the proposition symmetric to Lemma 3, it follows $\infty>[T:B]_r=[(\mathfrak{H}\mid T)V_l:V_l]_r$. Then, noting that $[\mathfrak{T}:U_l]_r=[\mathfrak{T}\mid V_l]\cdot(m/[V\mid V])$, we readily obtain $t\cdot[V\mid V]\geqslant[\mathfrak{T}\mid V_l]\geqslant[(\mathfrak{H}\mid T)V_l\mid V_l]=[T:B]_r\cdot[V\mid V]$. We have proved therefore $[T:B]_l\geqslant[T:B]_r$.

Now, we shall prove our last theorem.

Theorem 3. *Let a simple ring A be $B\cdot V$-A-irreducible and A-$B\cdot V$-irreducible, and $\mathfrak{H}$ a Galois semigroup of A/B.*

(a) *Let T be a B-B-submodule of A possessing a linearly independent left B-basis and a linearly independent right B-basis. If A/H is left locally finite then $[T:B]_l=[T:B]_r$, provided we do not distinguish between two infinite dimensions.*

(b) *Let V be contained in B. If T is an intermediate ring of A/B then $[T:B]_l=[T:B]_r$, provided we do not distinguish between two infinite dimensions.*

Proof. One may remark first that B and H are regular by Corollary 2 and Lemma 2 (a), and assume that $\mathfrak{H}$ contains $\tilde{V}$.

(a) Since A is $H\cdot V$-A-irreducible and left locally finite over $H=V_A^2(H)$, A/H is two-sided locally finite by Theorem 1 (d). Hence, by the symmetry of our assumption, it suffices to prove that if $[T:B]_l<\infty$ then $[T:B]_r\leqslant[T:B]_l$. Now, let $\{\varepsilon_1, \cdots, \varepsilon_t\}$ be a linearly independent V_r-basis of $(\mathfrak{H}|T)V_r$ that forms at the same time a linearly independent A_r-basis of $(\mathfrak{H}|T)A_r$ (Lemma 3). By Theorem 1 (d), there holds then $\infty>[H[E, \cup_1^t T\varepsilon_i]:H]_l=[V:V_A(H[E, \cup_1^t T\varepsilon_i])]_r=[V:V_A(H[E, \cup_1^t T\varepsilon_i])]_l$ where E is a system of matrix units such that $V_A(E)$ is a division ring. Hence, $[T:B]_r\geqslant[T:B]_l$ by Proposition 1.

(b) Again by the symmetry of our assumption, it suffices to prove that if $[T:B]_l<\infty$ then $[T:B]_r\leqslant[T:B]_l$. Since $U=V_A(T)=V_B(T)\subseteq V_A(T\,\mathrm{Hom}_{A_l\text{-}B_r}(T, A))\subseteq V$ (field) and $[V:U]\leqslant[T:B]_l<\infty$ by Lemma 2, our assertion is again a consequence of Proposition 1.

As a direct consequence of Theorem 3 (a), we obtain the following, that contains evidently [7, Corollary 2].

Corollary 3. *Let a simple ring A be $B\cdot V$-A-irreducible and A-$B\cdot V$-irreducible, and T an intermediate ring of A/B. If $J(\mathfrak{G}, A)=B$ and A/H*

is left locally finite then $[T:B]_l=[T:B]_r$, *provided we do not distinguish between two infinite dimensions.*

Remark 2. Theorem 3 (a) may be regarded as an extension of [1, Folgerung zu Satz VII, 2]. However, [1] contains considerable errors: The definition of $e\mathrm{Hom}_L({}_LM, {}_LN)$ is absurd, and Satz V, 1, Hilfssatz VI, 2 and Hilfssatz VI, 4 are open to doubt.

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