THE PRINCIPLE OF LIMITING AMPLITUDE

By

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1. Introduction.

We shall be concerned with the initial value problem

$$(1.1)\qquad \left[\frac{\partial^2}{\partial t^2}-\Delta+q(x)\right]u(x,t)=f_2(x)e^{-i\sqrt{\sigma}t}\qquad (t>0),$$

$$u(x,0)=f_0(x),\qquad \frac{\partial u}{\partial t}(x,0)=f_1(x),$$

where x is a point of 3-dimensional Euclidean space $R^3=E$, Δ denotes the Laplace operator in E, σ is a positive number, and $q(x)$ is a real-valued function defined on E. The principle of limiting amplitude for the problem (1.1) consists in the following: The solution $u(x,t)$ of the problem (1.1) is such that at any point $x\in E$ we have

$$\lim_{t\to\infty}u(x,t)e^{i\sqrt{\sigma}t}=u_+(x,\sigma)\qquad (\sigma>0),$$

where $u_+(x,\sigma)$ is the solution of the equation $(-\Delta+q)u=\sigma u+f_2$ with the Sommerfeld radiation condition:

$$u_+(x,\sigma)=0(|x|^{-1}),\qquad \frac{\partial u_+}{\partial|x|}(x,\sigma)-i\sqrt{\sigma}\,u_+(x,\sigma)=o(|x|^{-1})$$

at infinity. In [1] O. A. Ladyzhenskaya gave this proof for the case that q and f_2 have compact supports, $f_0=f_1=0$, and the operator A has no eigenvalue (For the definition see §2). But in this case it should be assumed, in addition, that there exists no solution $\omega\notin L^2(E)$ of the equation $(-\Delta+q)\omega=0$ satisfying the conditions $\omega=0(|x|^{-1})$, $\dfrac{\partial\omega}{\partial x_j}=0(|x|^{-2})$ at infinity, although it is not assumed there. In [2] D. M. Èĭdus gave this proof for the case that $q\in C^1(E)$, $q=0(|x|^{-2-\alpha})\left(\alpha>\dfrac{1}{6}\right)$ at infinity, and $f_0\in\mathfrak{D}(A)$, $f_1\in\mathfrak{D}(A^{\frac{1}{2}})$, $f_2\in L^2(E)$, $f_j=0(|x|^{-3-\gamma})$ $(\gamma>0)$ $(j=0,1,2)$ at infinity, assuming that $q(x)\geqq 0$ *in* E, which quarantee the absence of not only eigenvalues of the operator A, but also the above-mentioned functions ω. In [3] K. Asano and T. Shirota constructed such a function ω for the case that $q\in C_0^\infty(E)$ and the inequality

$$-q(x) \leqq \frac{1}{4|x|^2}$$

does not hold in E. In [5], by using such a function ω the present authors constructed a solution $u(x, t)$ of (1.1) with $q \in C_0^2(E)$, $f_0 \in C_0^2(E)$, $f_1 \in C_0^1(E)$ and $f_2=0$ such that

$$u(x, t) = \omega(x) \qquad \text{for} \quad |x| \leqq t .$$

The so constructed solution $u(x, t)$ of (1.1) is special but interesting to us, because that it give a relation of the principle of limiting amplitude and characteristics. They also remarked in [5] that, in the case that q and f_j $(j=0, 1, 2)$ have compact supports and the operator A has no eigenvalue, the principle of limiting amplitude for (1.1) is valid if and only if the before-mentioned functions ω do not exist. Therefore it seems natural for us to conjecture the following: The principle of limiting amplitude for (1.1) with the condition $q=0(|x|^{-2-\alpha})$ $(\alpha>0)$ $(|x| \to \infty)$ is valid if and only if the operator A has no eigenvalue and the above-mentioned functions ω do not exist.

In the present paper we establish that the above conjecture is true for $q=0(|x|^{-3-\alpha})$ $(|x| \to \infty)$ with some differentiability conditions. In particular the condition $q=0$ $(|x|^{-3-\alpha})$ $(\alpha>0)$ $(|x| \to \infty)$ is used in proving the equality (5.5), Lemma 8, Lemma 10, and the inequality (5.20). The main results of this paper were presented in our note [5].

2. Results.

For simplicity and clearness we restrict ourselves in what follows to the problem

$$\text{(2.1)} \qquad \left[\frac{\partial^2}{\partial t^2} - \Delta + q(x)\right] u(x, t) = 0 \qquad (t>0),$$

$$u(x, 0) = 0, \qquad \frac{\partial u}{\partial t}(x, 0) = f(x).$$

Throughout the present paper $q(x)$ and $f(x)$ are assumed to satisfy the following conditions (C_1), (C_2) and (C_3):

(C_1) *$q(x)$ is a locally Hölder continuous real-valued function defined on E and behaves like $0(|x|^{-2-\alpha})$ $(\alpha>0)$ at infinity, i.e. there exist positive numbers α, C_0, R_0 such that*

$$|q(x)| \leqq C_0 |x|^{-2-\alpha} \qquad \textit{for} \quad |x| \geqq R_0 .$$

Under the condition (C_1) the operator $-\Delta + q$ defined on $C_0^\infty(E)$ has the unique self-adjoint extension in $L^2(E)$, which we denote by A. It is known that the domain $\mathfrak{D}(A)$ of A is the Sobolev space $W_2^2(E)$.

(C_2) *A has no eigenvalue.*

Under the condition (C_2) A is positive definite. By $\mathfrak{D}(A^{\frac{1}{2}})$ we denote the domain of the self-adjoint operator $A^{\frac{1}{2}}$.

(C_3) *f belongs to $\mathfrak{D}(A^{\frac{1}{2}})$ and behaves like* $0(|x|^{-3-\gamma})$ $(\gamma>0)$ *at infinity.*

In what follows we denote $\int_E \varphi(x)\psi(x)dx$ by $\langle\varphi, \psi\rangle$.

We have the following.

Theorem 1. *Assume that $\langle f, \omega\rangle=0$ for all solutions $\omega \notin L^2(E)$ of the equation $(-\Delta+q)\omega=0$ such that $\omega=0(|x|^{-1})$, $\dfrac{\partial\omega}{\partial x_j}=0(|x|^{-2})$ $(|x|\to\infty)$. Then the solution $u(t)=u(x,t)$ of the problem* (2.1) *is such that we have*

$$\lim_{t\to\infty}(u(t), \varphi)_{L^2(E)}=0 \qquad \textit{for all} \quad \varphi\in L^2(E), \tag{2.2}$$

and

$$\lim_{t\to\infty}\|u(t)\|_{L^2(K)}=0 \qquad \textit{for all compact} \quad K\subset E. \tag{2.3}$$

Furthermore, suppose that $f\in\mathfrak{D}(A)$ and $\sup_{x\in E}|\nabla q(x)|<\infty$, where $\nabla q(x)$ denotes grad $q(x)$. Then we have

$$\lim_{t\to\infty}\sup_{x\in K}|u(x,t)|=0 \qquad \textit{for all compact} \quad K\subset E. \tag{2.4}$$

The main result is the following.

Theorem 2. *Assume that $q\in C^2(E)$ and $q=0(|x|^{-3-\alpha})$, $D^\beta q=0(|x|^{-2-\alpha})$ $(|x|\to\infty)$ $(|\beta|=1,2)$. Then the solution of* (2.1) *is such that for any $\varphi\in L^2(E)$ satisfying $\varphi=0(|x|^{-2-\delta})$ $\left(\delta>\dfrac{1}{2}\right)$ $(|x|\to\infty)$ we have*

$$\lim_{t\to\infty}\langle u(t), \varphi\rangle=4\pi\langle\varphi, \omega\rangle\cdot\langle f, \omega\rangle\cdot\langle q, \omega\rangle^{-2}, \tag{2.5}$$

where ω is the one mentioned in Theorem 1.

By virtue of Theorem 2 and theorem 6 in [2] we have

Corollary. *Let $q(x)$ be as in Theorem 2. Then the solution $u(x,t)$ of* (2.1) *is such that we have*

$$\lim_{t\to\infty}\sup_{x\in K}|u(x,t)|=0 \qquad \textit{for all compact} \quad K\subset E$$

if and only if such functions ω as in Theorem 1 *do not exist.*

3. Proof of Theorem 1.

Since we can't find the Theorem in the literatures, for the sake of the completeness of description we shall give the proof.

Let us define an operator T for functions in $L^6(E)$ by

$$T\varphi(x) = -\frac{1}{4\pi}\int_E \frac{q(y)\varphi(y)}{|x-y|}dy \qquad (\varphi \in L^6(E)).$$

Lemma 1. 1) *T is a compact operator on $L^6(E)$ to $L^6(E)$ and the adjoint operator T^* of T with respect to the bilinear form $\langle,\rangle$ is a compact operator on $L^{\frac{6}{5}}(E)$ to $L^{\frac{6}{5}}(E)$ given as follows*:

$$T^*\psi(x) = -\frac{1}{4\pi}q(x)\int_E \frac{\psi(y)}{|x-y|}dy \qquad \left(\psi \in L^{\frac{6}{5}}(E)\right).$$

2) *By M, M' we denote the subspaces $\{\omega \in L^6(E); (I-T)\omega=0\}$, $\{\omega' \in L^{\frac{6}{5}}(E); (I-T^*)\omega'=0\}$ of $L^6(E)$, $L^{\frac{6}{5}}(E)$ respectively, where I denotes the identity operator. Then we have*

$$\omega = 0(|x|^{-1}), \quad \frac{\partial\omega}{\partial x_j} = 0(|x|^{-2}) \ (|x| \to \infty) \qquad \textit{for} \quad \omega \in M,$$

and

$$\omega' = 0(|x|^{-3-\alpha}) \ (|x| \to \infty) \qquad \textit{for} \quad \omega' \in M'.$$

Proof. First we shall show that T is a bounded operator on $L^6(E)$ to $L^6(E)$. In what follows we denote by C constants independent of $x \in E$. we have

$$\int_E \frac{|q(y)\varphi(y)|}{|x-y|}dy \leqq \|\varphi\|_{L^6}\left[\int_E\left(\frac{|q(y)|}{|x-y|}\right)^{\frac{6}{5}}dy\right]^{\frac{5}{6}}.$$

$\int_E\left(\frac{|q(y)|}{|x-y|}\right)^{\frac{6}{5}}dy$ can be estimated as follows:

$$\int_E\left(\frac{|q(y)|}{|x-y|}\right)^{\frac{6}{5}}dy \leqq \sup_{|y|\leqq R_0}|q(y)|^{\frac{6}{5}}\int_{|y|\leqq R_0}\frac{1}{|x-y|^{\frac{6}{5}}}dy + C_0^{\frac{6}{5}}\int_{|y|\geqq R_0}\left(\frac{1}{|x-y|\,|y|^{2+\alpha}}\right)^{\frac{6}{5}}dy$$
$$\leqq C(1+|x|)^{-\frac{6}{5}} + \rho_\alpha(x),$$

where

$$|\rho_\alpha(x)| \leqq C(1+|x|)^{-\frac{3}{5}-\frac{6}{5}\alpha} \qquad \text{for} \quad 0<\alpha<\frac{1}{2},$$
$$\leqq C(1+|x|)^{-\frac{6}{5}+\varepsilon} \qquad \text{for} \quad \alpha=\frac{1}{2},$$
$$\leqq C(1+|x|)^{-\frac{6}{5}} \qquad \text{for} \quad \alpha>\frac{1}{2},$$

and ε is an arbitrary positive number, so that for any $\alpha>0$ $[\rho_\alpha(x)]^{\frac{5}{6}} \in L^6(E)$. These estimates imply that

$$|T\varphi(x)| \leqq C\|\varphi\|_{L^6}\left[(1+|x|)^{-1}+(\rho_\alpha(x))^{\frac{5}{6}}\right], \tag{3.1}$$

and

$$\|T\varphi\|_{L^6} \leqq C\|\varphi\|_{L^6} \tag{3.2}$$

which shows the boundedness of T.

Next we shall show that T is a compact operator on $L^6(E)$ to $L^6(E)$. The inequality (3.2) implies that $\|T\varphi\|_{L^6}$ are uniformly bounded with respect to φ satisfying $\|\varphi\|_{L^6} \leqq 1$. On the other hand an argument similar to the one used in deriving (3.2) gives

$$\left\|\frac{\partial}{\partial x_j} T\varphi\right\|_{L^6} \leqq C\|\varphi\|_{L^6} \qquad (j=1,2,3),$$

which imply that $T\varphi$ are uniformly equi-continuous in $L^6(E)$ with respect to $\varphi \in L^6(E)$ satisfying $\|\varphi\|_{L^6} \leqq 1$. Thus T is compact. From this it follows by the Riesz-Schauder theory that T^* is also a compact operator on $L^{\frac{6}{5}}(E)$ to $L^{\frac{6}{5}}(E)$ and that $\dim M = \dim M' < \infty$.

For $\varphi \in L^6(E)$ and $\psi \in L^{\frac{6}{5}}(E)$, since

$$\begin{aligned}\int_{E\times E} \frac{|q(y)\varphi(y)\psi(x)|}{|x-y|} dxdy &\leqq \|\varphi\|_{L^6} \int_E |\psi(x)| \left[\int_E \left(\frac{|q(y)|}{|x-y|}\right)^{\frac{6}{5}} dy\right]^{\frac{5}{6}} dx \\ &\leqq C\|\varphi\|_{L^6}\|\psi\|_{L^{\frac{6}{5}}},\end{aligned}$$

we have

$$\begin{aligned}\langle T^*\psi, \varphi\rangle &= \langle \psi, T\varphi\rangle \\ &= \int_E \psi(x)\left[-\frac{1}{4\pi}\int_E \frac{q(y)\varphi(y)}{|x-y|} dy\right] dx \\ &= \int_E \varphi(y)\left[-\frac{1}{4\pi} q(y) \int_E \frac{\psi(x)}{|x-y|} dx\right] dy,\end{aligned}$$

which implies that

$$T^*\psi(x) = -\frac{1}{4\pi} q(x) \int_E \frac{\psi(y)}{|x-y|} dy \qquad \text{for} \quad \psi \in L^{\frac{6}{5}}(E).$$

For $\omega \in M$ we have

$$\omega(x)=-\frac{1}{4\pi}\int_E\frac{q(y)\omega(y)}{|x-y|}dy\,. \tag{3.3}$$

It follows from (3.1) and (3.3) that not only $\omega(x)$ is bounded, but also $\omega=0(|x|^{-1})$, $\frac{\partial\omega}{\partial x_j}=0(|x|^{-2})$ $(|x|\to\infty)$.

For $\omega'\in M'$ we have

$$\omega'(x)=-\frac{1}{4\pi}q(x)\int_E\frac{\omega' y)}{|x-y|}dy\,, \tag{3.4}$$

which implies that

$$\omega'(x)=\left(\frac{1}{4\pi}\right)^2 q(x)\int_E\frac{q(y)}{|x-y|}dy\int_E\frac{\omega'(z)}{|y-z|}dz\,.$$

It follows from the condition (C_1) that

$$\int_E\frac{|q(y)|}{|x-y||y-z|}dy\leqq C\frac{1}{1+|z|}\,,$$

where C is a constant independent of $x, z\in E$. This implies that

$$\int_{E\times E}\frac{|q(y)\omega'(z)|}{|x-y||y-z|}dy\,dz\leqq C\|\omega'\|_{L^{\frac{6}{5}}}\,.$$

Therefore we obtain

$$|\omega'(x)|\leqq C\|\omega'\|_{L^{\frac{6}{5}}}(1+|x|)^{-2-\alpha}\,.$$

This and (3.4) imply that $\omega'=0(|x|^{-3-\alpha})$ $(|x|\to\infty)$.

Thus we have proved Lemma 1.

Lemma 2. *Suppose that* $\varphi\in L^2(E)$, $\varphi=0(|x|^{-2-\delta})$ $\left(\delta>\frac{1}{2}\right)$ $(|x|\to\infty)$, and $\langle\varphi,\omega\rangle=0$ *for all* $\omega\in M$. *Then we have* $\varphi\in R(A^{\frac{1}{2}})$, *where* $R(A^{\frac{1}{2}})$ *denotes the range of* $A^{\frac{1}{2}}$.

Proof. Let $\omega'\in M'$ and set $\omega(x)=-\frac{1}{4\pi}\int_E\frac{\omega'(y)}{|x-y|}dy$. Then we find by the equalities (3.3) and (3.4) that $q(x)\omega(x)=\omega'(x)$ and $\omega\in M$, because that it follows from Lemma 1 that $\omega'(x)$ is continuous in E and $\omega'=0(|x|^{-3-\alpha})$ $(|x|\to\infty)$. Hence the assumption implies that $\frac{1}{4\pi}\int_E\frac{\varphi(y)}{|x-y|}dy\in L^6(E)$ and $\left\langle\frac{1}{4\pi}\int_E\frac{\varphi(y)}{|x-y|}dy,\omega'\right\rangle=0$ for all $\omega'\in M'$. Therefore by virtue of Lemma

1 and the Riesz-Schauder theory we find that there exists a solution $u\in L^6(E)$ of the equation

$$(I-T)u=\frac{1}{4\pi}\int_E\frac{\varphi(y)}{|x-y|}dy$$

which implies that $u(x)$ satisfies the equation $(-\Delta+q)u=\varphi$ with the conditions $u=0(|x|^{-\delta})$, $\frac{\partial u}{\partial x_j}=0(|x|^{-1-\delta})$ $(|x|\to\infty)$, so that $\Delta u\in L^2(E)$ and $u\in W^2_{2,\mathrm{loc}}$. Set $u_\varepsilon(x)=u(x)e^{-\varepsilon|x|^2}$ $(\varepsilon>0)$. Then we find that $u_\varepsilon\in\mathfrak{D}(A)$, $\sup_{\varepsilon\leqq\varepsilon_0}\|A^{\frac{1}{2}}u_\varepsilon\|_{L^2}<\infty$, where ε_0 is a positive number, and $\lim_{\varepsilon\to0}\|Au_\varepsilon-\varphi\|_{L^2}=0$. In fact, $\Delta u\in L^2(E)$ implies that $u_\varepsilon\in L^2(E)$ and $\Delta u_\varepsilon\in L^2(E)$, from which it follows that $u_\varepsilon\in W^2_2(E)=\mathfrak{D}(A)$. The conditions $u=0(|x|^{-\delta})$, $\frac{\partial u}{\partial x_j}=0(|x|^{-1-\delta})$ $(|x|\to\infty)$ imply that

$$\|\Delta u_\varepsilon-\Delta u\|^2_{L^2}\leqq 2\int_E|\Delta u|^2(1-e^{-\varepsilon|x|^2})^2dx+0(\varepsilon^{\delta+\frac{1}{2}})+0(\varepsilon^{\frac{3}{2}})\qquad(\varepsilon\to0),$$

from which it follows that $\|\Delta u_\varepsilon-\Delta u\|_{L^2}\to0$ as $\varepsilon\to0$. Since $qu=0(|x|^{-2-\alpha-\delta})$ $(|x|\to\infty)$, it is clear that $\|qu_\varepsilon-qu\|_{L^2}\to0$ as $\varepsilon\to0$. These show that $\|Au_\varepsilon-\varphi\|_{L^2}\to0$ as $\varepsilon\to0$. Noting that $A^{\frac{1}{2}}$ is self-adjoint and $\mathfrak{D}(A)\subset\mathfrak{D}(A^{\frac{1}{2}})$, we have

$$\|A^{\frac{1}{2}}u_\varepsilon\|^2_{L^2}\leqq 2\sum_{j=1}^{3}\int_E\left|\frac{\partial u}{\partial x_j}\right|^2dx+\int_E|q||u|^2dx+0(\varepsilon^{\delta-\frac{1}{2}})+0(\varepsilon^{\frac{3}{2}})\qquad(\varepsilon\to0)$$

which shows that $\sup_{0<\varepsilon\leqq\varepsilon_0}\|A^{\frac{1}{2}}u_\varepsilon\|_{L^2}<\infty$. Thus we have shown that $u_\varepsilon\in\mathfrak{D}(A)$, $\sup_{0<\varepsilon\leqq\varepsilon_0}\|A^{\frac{1}{2}}u_\varepsilon\|_{L^2}<\infty$ and $\|Au_\varepsilon-\varphi\|_{L^2}\to0$ as $\varepsilon\to0$. The fact $\sup_{0<\varepsilon\leqq\varepsilon_0}\|A^{\frac{1}{2}}u_\varepsilon\|_{L^2}<\infty$ implies that there exists a function $g\in L^2(E)$ and a sequence $\{\varepsilon_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty}\varepsilon_j=0$ and $A^{\frac{1}{2}}u_{\varepsilon_j}$ converges to g in $L^2(E)$ weakly. Let H be the convex hull of $\{A^{\frac{1}{2}}u_{\varepsilon_j}\}_{j=1}^\infty$. Then it follows from the fact $\lim_{\varepsilon\to0}\|Au_\varepsilon-\varphi\|_{L^2}=0$ that $\{g,\varphi\}$ belongs to the weak closure of the graph $\{H, A^{\frac{1}{2}}H\}$, so that $\{g,\varphi\}$ belongs to the strong closure of $\{H, A^{\frac{1}{2}}H\}$. This shows that $g\in\mathfrak{D}\{A^{\frac{1}{2}}\}$ and $\varphi=A^{\frac{1}{2}}g$, because that $A^{\frac{1}{2}}u_\varepsilon\in\mathfrak{D}(A^{\frac{1}{2}})$. Thus we have proved Lemma 2.

Proof of Theorem 1. Let E_λ be the resolution of the identity generated by the operator A such that $E_{\lambda+0}=E_\lambda$. Then, since $f\in\mathfrak{D}(A^{\frac{1}{2}})$ and A has no eigenvalue, the solution $u(t)$ of the initial value problem (2.1) is given by

$$u(t)=\int_0^\infty\frac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}}dE_\lambda f. \tag{3.5}$$

We decompose $u(t)$ as follows:

$$u(t)=\int_0^N \frac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}}dE_\lambda f+\int_N^\infty \frac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}}dE_\lambda f$$
$$\equiv u_1(t)+u_2(t)\,.$$

Then, since

$$\|u_2(t)\|_{L^2}^2=\int_N^\infty \frac{\sin^2\sqrt{\lambda}\,t}{\lambda}d\|E_\lambda f\|_{L^2}^2 \tag{3.6}$$
$$\leqq \frac{1}{N}\|f\|_{L^2}^2,$$

we can choose N so large that $\|u_2(t)\|_{L^2}$ becomes sufficiently small uniformly with respect to $t>0$. Let N be fixed sufficiently large. Since it follows from the assumption and Lemma 1 that $\langle f, \omega\rangle=0$ for all $\omega\in M$, we find by the condition (C_3) and Lemma 2 that there exists a function $g\in\mathfrak{D}(A^{\frac{1}{2}})$ such that

$$A^{\frac{1}{2}}g=f\,. \tag{3.7}$$

Therefore we have

$$(u_1(t),\varphi)_{L^2}=\int_0^N \sin\sqrt{\lambda}\,t d(E_\lambda g,\varphi) \qquad \text{for all} \quad \varphi\in L^2(E).$$

On the other hand it follows from condition (C_2) and theorem 6 in [4] that $(E_\lambda g,\varphi)_{L^2}$ is an absolutely continuous function of $\lambda\in[0, N]$, so that $\frac{d}{d\lambda}(E_\lambda g,\varphi)_{L^2}\in L^1(0, N)$. Consequently by virtue of Riemann-Lebesgue's theorem we obtain

$$\lim_{t\to\infty}(u_1(t),\varphi)_{L^2}=0 \qquad \text{for all} \quad \varphi\in L^2(E).$$

This and (3.6) prove (2.2).

Now we proceed to prove (2.3). It follows from the condition (C_2) that the operator A is strictly positive definite, i.e. A satisfies the inequality

$$(A\varphi,\varphi)>0 \qquad \text{for} \quad 0\neq\varphi\in\mathfrak{D}(A). \tag{3.8}$$

We define a functional $(\varphi,\phi)_1$ for functions $\varphi,\phi\in C_0^\infty(E)$ as follows:

$$(\varphi,\phi)_1=\int_E\left(\nabla\varphi(x)\overline{\nabla\phi(x)}+q(x)\varphi(x)\overline{\phi(x)}\right)dx\,. \tag{3.9}$$

Then by virtue of (3.8) we find that $C_0^\infty(E)$ is a pre-Hilbert space with the inner product $(\varphi,\phi)_1$ $(\varphi,\phi\in C_0^\infty(E))$. Let H_1 be the completion of $C_0^\infty(E)$ for the norm $\|\cdot\|_1=\sqrt{(\cdot,\cdot)_1}$ and let $\mathfrak{H}$ be the Hilbert space $H_1\times L^2(E)$ with the inner product

$$(\varphi, \psi)_{\mathfrak{H}} = (\varphi_1, \psi_1)_1 + (\varphi_2, \psi_2)_{L^2(E)} \qquad (\varphi, \psi \in \mathfrak{H}),$$

where $\varphi=[\varphi_1, \varphi_2]$, $\psi=[\psi_1, \psi_2]$ and $\varphi_1, \psi_1 \in H_1$; $\varphi_2, \psi_2 \in L^2(E)$.

Let us define an operator G_0 of $\mathfrak{H}$ to $\mathfrak{H}$ as follows:

$$G_0 = \begin{bmatrix} 0 & L \\ N & 0 \end{bmatrix},$$

where the operators $L: L^2(E) \to H_1$ and $N: H_1 \to L^2(E)$ have domains $\mathfrak{D}(L) = C_0^\infty(E) = \mathfrak{D}(N)$, and

$$L\varphi = \varphi \qquad \text{for} \quad \varphi \in \mathfrak{D}(L),$$
$$N\psi = (\Delta - q)\psi \qquad \text{for} \quad \psi \in \mathfrak{D}(N).$$

By G we denote the closure of G_0 in $\mathfrak{H}$. Then we find that the operator G generates a strongly continuous contraction semi-group on $\mathfrak{H}$. In fact, the equality

$$(G_0\varphi, \varphi)_{\mathfrak{H}} + (\varphi, G_0\varphi)_{\mathfrak{H}} = 0 \qquad \text{for} \quad \varphi \in \mathfrak{D}(G_0)$$

implies that for $\lambda > 0$ we have the inequality

$$\|(\lambda I - G)\varphi\|_{\mathfrak{H}} \geqq \lambda \|\varphi\|_{\mathfrak{H}} \qquad \text{for} \quad \varphi \in \mathfrak{D}(G_0), \tag{3.10}$$

where $\|\psi\|_{\mathfrak{H}} = \sqrt{(\psi, \psi)_{\mathfrak{H}}}$ for $\psi \in \mathfrak{H}$. Furthermore the condition (C_1) and (3.8) imply that

$$\mathfrak{D}(G) \supset \mathfrak{D}(A) \times \mathfrak{D}(A^{\frac{1}{2}}). \tag{3.11}$$

By (3.11) we find that for $\lambda > 0$

$$R(\lambda I - G) \supset \mathfrak{D}(G_0) \tag{3.12}$$

where $R(\lambda I - G)$ denotes the range of the operator $\lambda I - G$. (3.10) and (3.12) imply that for any $\lambda > 0$ the operator $\lambda I - G$ has the inverse $(\lambda I - G)^{-1}$ defined on $\mathfrak{H}$ such that $\|(\lambda I - G)^{-1}\|_{\mathfrak{H}} \leqq \lambda^{-1}$. From this it follows by the Hille-Yosida's theorem that G is the infinitesimal generator of a strongly continuous semi-group $U(t)$ on $\mathfrak{H}$ such that

$$\|U(t)\|_{\mathfrak{H}} \leqq 1 \qquad (t \geqq 0). \tag{3.13}$$

Since it follows from (3.11) that $[0, f] \in \mathfrak{D}(G)$, we can find that

$$U(t)[0, f] = \big[u(x, t),\ u_t(x, t)\big] \qquad \text{for} \quad t > 0, \tag{3.14}$$

where $u_t(x, t) = \dfrac{\partial}{\partial t} u(x, t)$. (3.13) and (3.14) imply that

$$\left\| \left[u(\cdot\,, t),\; u_t(\cdot\,, t) \right] \right\|_{\mathfrak{H}}^2 \leqq \|f\|_{L^2}^2 \qquad \text{for} \quad t>0,$$

i. e.

$$(3.15) \qquad \int_E \left[|\nabla u(x, t)|^2 + q(x)|u(x, t)|^2 + |u_t(x, t)|^2 \right] dx \leqq \|f\|_{L^2}^2 \qquad \text{for } t>0,$$

from which it follows by (3.8) and (3.9) that

$$(3.16) \qquad \|u_t(\cdot\,, t)\|_{L^2} \leqq \|f\|_{L^2} \qquad \text{for} \quad t>0.$$

On the other hand (3.5) and (3.7) imply that

$$(3.17) \qquad \|u(\cdot\,, t)\|_{L^2} \leqq \|g\|_{L^2} \qquad \text{for} \quad t>0.$$

From (3.15), (3.16) and (3.17) it follows by the boundedness of $q(x)$ that

$$(3.18) \qquad \|\nabla u(\cdot\,, t)\|_{L^2} \leqq C \qquad \text{for} \quad t>0.$$

Hereafter we denote by C contants independent of $t>0$. Thus from (2.2), (3.17) and (3.18) we obtain by the Rellich selection theorem that

$$(2.3) \qquad \lim_{t\to\infty} \|u(\cdot\,, t)\|_{L^2(K)} = 0 \qquad \text{for all compact } K\subset E.$$

Next we shall prove (2.4). It follows from (3.11), (3.13), (3.14) and the identity $GU(t)[0,f] = U(t)G[0,f]$ that

$$\int_E \left[|\nabla u_t(x ,t)|^2 + q(x)|u_t(x, t)|^2 + \left| \left(\varDelta - q(x) \right) u(x, t)\right|^2 \right] dx \leqq \| A^{\frac{1}{2}} f\|_{L^2}^2 \quad \text{for } t>0.$$

From this, (3.16) and (3.17) we find by the boundedness of $q(x)$ that we have

$$(3.19) \qquad \|\varDelta u(\cdot\,, t)\|_{L^2} \leqq C \qquad \text{for} \quad t>0,$$

$$(3.20) \qquad \|\nabla u_t(\cdot\,, t)\|_{L^2} \leqq C \qquad \text{for} \quad t>0.$$

Furthermore, suppose that $f\in \mathfrak{D}(A)$ and $\sup_{x\in E} |\nabla q(x)| < \infty$. Then by (3.11) $[f, 0] \in \mathfrak{D}(G)$. Hence. setting $v(x, t) = u_t(x, t)$ we find that

$$U(t)[f, 0] = \left[v(x, t),\; v_t(x, t) \right].$$

Therefore an argument similar to the one used in deriving (3.20) gives

$$(3.21) \qquad \|\nabla v_t(\cdot\,, t)\|_{L^2} \leqq C \qquad \text{for all} \quad t>0.$$

On the other hand we have

$$\nabla \varDelta u(x, t) = \nabla v_t(x, t) + \nabla \left(q(x)\, u(x, t) \right).$$

From this, (3.17), (3.18) and (3.21) it follows by the boundedness of $q(x)$ and $\nabla q(x)$ that

(3. 22) $$\|\nabla\Delta u(x, t)\|_{L^2}\leqq C \qquad \text{for all} \quad t>0.$$

From (2. 2), (3. 19) and (3. 22) we find by the Rellich selection theorem that

(3. 23) $$\lim_{t\to\infty}\|\Delta u(\cdot, t)\|_{L^2(K)}=0 \qquad \text{for all compact} \quad K\subset E.$$

Consequently from (2. 3) and (3. 23) we obtain by Sobolev's lemma

(2. 4) $$\lim_{t\to\infty}\sup_{x\in K}|u(x, t)|=0 \qquad \text{for all compact} \quad K\subset E.$$

Thus we have proved Theorem 1.

4. Lemmas.

In this section, we shall establish some lemmas required in proving Theorem 2.

The following lemma 3, 4 and 5 can be verified easily.

Lemma 3. 1) *Let* $a>0$, $t>0$ *be fixed. Then the integral*

$$\int_{a-i\infty}^{a+i\infty}\left|\frac{e^{\zeta t}}{\lambda+\zeta^2}\right|d\zeta$$

is a bounded continuous function of $\lambda\in(0, \infty)$.

2) *Let* $a'<0$, $t_0>0$ *be fixed. Then the integral*

$$\int_{a'-i\infty}^{a'+i\infty}\left|\frac{e^{\zeta t}}{\lambda+\zeta^2}\right|d\zeta$$

converges uniformly with respect to $t\geqq t_0$. *Furthermore we have*

(4. 1) $$\int_{a'-i\infty}^{a'+i\infty}\frac{e^{\zeta t}}{\lambda+\zeta^2}d\zeta=0 \qquad \textit{for all} \quad t>0.$$

Lemma 4. *Let* $t>0$, $a>0$ *be fixed. If* $l(\lambda)$ *is locally bounded in* $(0, \infty)$ *and belongs to* $L^1(0, \infty)$, *then the integral*

$$\int_0^\infty|l(\lambda)|d\lambda\int_{-a+iN}^{a+iN}\left|\frac{e^{\zeta t}}{\lambda+\zeta^2}\right|d\zeta$$

converges for any fixed N $(|N|>3a)$.

Lemma 5. *Let* $a>0$ *be fixed. Then*

$$u(t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{\zeta t}R(-\zeta^2)fd\zeta$$

is the solution of the problem (2. 1), *where* $R(-\zeta^2)f$ *denotes* $(A+\zeta^2)^{-1}f$.

Now we shall study an asymptotic property of functions of the form

$$\varphi(x)=\int_E \frac{v(y)}{|x-y|}dy\,.$$

We have the following one similar to Lemma 3.2 in [4].

Lemma 6. *Let* $v(x)\in L^2(E)$ *and* $v(x)=0\,(|x|^{-3-\varepsilon})$ $(0<\varepsilon<1)$ $(|x|\to\infty)$. *If* $\int_E v(x)dx=0$, *then we have*

$$\varphi(x)=0\,(|x|^{-1-\varepsilon})\ (|x|\to\infty). \tag{4.2}$$

Proof. Let R_1 be fixed so that $|v(x)|\leqq C_1|x|^{-3-\varepsilon}$ for $|x|\geqq R_1$ and let $|x|$ be so large that $\frac{1}{2}|x|=R>R_1$.

Then we have

$$\varphi(x)=\int_{|y|\leqq R}\frac{v(y)}{|x-y|}dy+\int_{|y|\geqq R}\frac{v(y)}{|x-y|}dy\equiv I+I'. \tag{4.3}$$

I' can be estimated as follows:

$$\begin{aligned}|I'|&\leqq C_1\int_{|y|\geqq R}\frac{dy}{|x-y||y|^{3+\varepsilon}}\\&\leqq C\int_R^\infty\int_0^\pi\frac{r^2\sin\theta}{r^{3+\varepsilon}(|x|^2+r^2-2|x|r\cos\theta)^{\frac{1}{2}}}d\theta dr\\&=C\int_R^\infty\int_0^\pi\frac{r^2}{r^{3+\varepsilon}}\cdot\frac{1}{|x|r}\frac{d}{d\theta}(|x|^2+r^2-2|x|r\cos\theta)^{\frac{1}{2}}d\theta dr\\&=C\frac{1}{|x|}\int_R^{|x|}r^{-1-\varepsilon}dr+C\int_{|x|}^\infty r^{-2-\varepsilon}dr\\&\leqq C|x|^{-1-\varepsilon}.\end{aligned}$$

Hereafter we denote by C constants independent of x. We proceed to estimate I. We have

$$\begin{aligned}I=&\frac{1}{|x|}\int_{|y|\leqq R}v(y)dy-\frac{1}{|x|}\int_{|y|\leqq R}\frac{|y|^2v(y)}{|x-y|(|x|+|x-y|)}dy\\&+2\int_{|y|\leqq R}\frac{\left(\frac{x}{|x|},y\right)v(y)}{|x-y|(|x|+|x-y|)}dy\equiv I_1+I_2+I_3.\end{aligned}$$

It follows from the assumption that

$$I_1=-|x|^{-1}\int_{|y|\geqq R}v(y)dy\,.$$

This implies that

$$|I_1| \leqq C_1 |x|^{-1} \int_{|y| \geqq R} \frac{dy}{|y|^{3+\varepsilon}}$$
$$\leqq C |x|^{-1-\varepsilon} .$$

I_j $(j=2,3)$ are estimated as follows:

$$|I_2| \leqq C |x|^{-3} \left[\int_{|y| \leqq R_1} |y|^2 |v(y)| dy + \int_{R_1 \leqq |y| \leqq R} |y|^2 |v(y)| dy \right]$$
$$\leqq C |x|^{-3} + C |x|^{-1-\varepsilon} ,$$
$$|I_3| \leqq C |x|^{-2} \left[\int_{|y| \leqq R_1} |y| |v(y)| dy + \int_{R_1 \leqq |y| \leqq R} |y| |v(y)| dy \right]$$
$$\leqq C |x|^{-2} + C(1-\varepsilon)^{-1} |x|^{-1-\varepsilon} .$$

These estimates and (4.3) give (4.2).

Lemma 7. *Let $\omega \in M$ and $\omega \neq 0$. Then $\langle q, \omega \rangle \neq 0$ and furthermore $dim\, M = dim\, M' = 1$, where M and M' are the ones defined in Lemma 1.*

Proof. Suppose that $\omega \in M$ and $\langle q, \omega \rangle = 0$. Then the function $\omega(x)$ satisfies

$$\omega(x) = -\frac{1}{4\pi} \int_E \frac{q(y)\,\omega(y)}{|x-y|}\, dy . \tag{3.3}$$

Since Lemma 1 implies that $\omega = 0(|x|^{-1})$ $(|x| \to \infty)$, from (3.3) it follows by Lemma 6 that $\omega = 0(|x|^{-1-\alpha})$. Applying repeatedly Lemma 6 to (3.3) we find that $\omega \in L^2(E)$. Hence $\omega \in \mathfrak{D}(A)$ and $A\omega = 0$. Therefore by the condition (C_2) it follows that $\omega = 0$. Thus $\langle q, \omega \rangle \neq 0$ for $\omega \in M$ such that $\omega \neq 0$. From this it follows that $dim\, M = 1$, which completes the proof of Lemma 7, since Lemma 1 implies that $dim\, M = dim\, M'$.

Remark. Lemma 7 will be established also in the course of the proof of Theorem 2, for the case that q satisfies the assumptions in Theorem 2.

Lemma 8. *For $\lambda > 0$ we set*

$$\theta(\lambda) = \theta(x, \lambda) = \frac{1}{2\pi i} \left(u_+(x, \lambda) - u_-(x, \lambda) \right),$$

where $u_\pm(x, \lambda) = R(\lambda \pm i0) f(x)$. Let us define the norm for functions $\varphi \in C^2(E)$ by

$$\|\varphi\|_{C^2_{3+\alpha}} = \sup_{x \in E, |\beta| \leqq 2} |D^\beta \varphi(x)| \cdot (1 + |x|^2)^{\frac{3+\alpha}{2}}$$

and let $C^2_{3+\alpha}$ be the Banach space $\{\varphi \in C^2(E);\ \|\varphi\|_{C^2_{3+\alpha}} < \infty\}$. Then the function of λ *$T_\lambda(\varphi) = \langle \theta(\lambda), \varphi \rangle$ $(\varphi \in C^2_{3+\alpha})$ is a nuclear operator on $C^2_{3+\alpha}$ to $L^1(0, \infty)$, and*

$\|T_\lambda\|_{(C^2_{3+\alpha})^*}=\|\theta(\lambda)\|_{(C^2_{3+\alpha})^*}$ *belongs to* $L^1(0,\infty)$.

Proof. We introduce the finite measure

$$d\mu=(1+|x|^2)^{-\frac{3+\alpha}{2}}dx$$

in E, where dx is the Lebesgue measure in E. Then we decompose T_λ as follows:

$$\begin{array}{ccccccc}
C^2_{3+\alpha} & \xrightarrow{J_1} & \prod\limits_{|\beta|\leqq 2} L^\infty(\mu) & \xrightarrow{I_1} & \prod\limits_{|\beta|\leqq 2} L^2(\mu) & \xrightarrow{J_2} & \prod\limits_{|\beta|\leqq 2} L^2(dx) \\
\cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\
\varphi & & \left\{(D^\beta\varphi)\cdot(1+|x|^2)^{\frac{3+\beta}{2}}\right\} & & \left\{(D^\beta\varphi)\cdot(1+|x|^2)^{\frac{3+\alpha}{2}}\right\} & & \left\{(D^\beta\varphi)\cdot(1+|x|^2)\right\}^{\frac{3+\alpha}{4}}
\end{array}$$

$$\begin{array}{cccccccccc}
\xrightarrow{J_3} & W^2_2 & \xrightarrow{I_2} & L^\infty(dx)=L^\infty(\mu) & \xrightarrow{I_3} & L^2(\mu) & \xrightarrow{J_4} & L^2(dx) & \xrightarrow{J_5} & L^1(0,\infty). \\
& \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\
& \varphi\cdot(1+|x|^2)^{\frac{3+\alpha}{4}} & & \varphi\cdot(1+|x|^2)^{\frac{3+\alpha}{4}} & & \varphi\cdot(1+|x|^2)^{\frac{3+\alpha}{4}} & & \varphi & & T_\lambda(\varphi)
\end{array}$$

Let F be the closed subspace of all vector-valued functions $\left\{(D^\beta\varphi)\cdot(1+|x|^2)^{\frac{3+\alpha}{4}}\right\}_{|\beta|\leqq 2}$ in $\prod\limits_{|\beta|\leqq 2} L^2(dx)$ whose components $(D^\beta\varphi)\cdot(1+|x|^2)^{\frac{3+\alpha}{4}}\in L^2(dx)$ $(|\beta|\leqq 2)$.

Furthermore let P_F be the projection operator of $\prod\limits_\beta L^2(dx)$ onto F and J_3' be the operator of F to W^2_2 which assigns each vector-valued function $\left\{(D^\beta\varphi)\cdot(1+|x|^2)^{\frac{3+\alpha}{4}}\right\}_\beta$ in F to the scalar-valued function $\varphi\cdot(1+|x|^2)^{\frac{3+\alpha}{4}}$ in W^2_2. Then J_3 may be regarded as the composite operator $J_3'\cdot P_F$. Hence J_3 is defined on the whole space $\prod\limits_\beta L^2(dx)$, so that by the closed graph theorem J_3 is a bounded operator on $\prod\limits_\beta L^2(dx)$ to W^2_2.

I_k $(k=1,2,3)$ are identities. Since $d\mu$ is a finite measure we find that I_1 and I_3 are semi-integral (see [8]). And Sobolev's lemma implies that I_2 is bounded. It is obvious that J_k $(k=1,2,4)$ are bounded. Thus the identity operator on $C^2_{3+\alpha}$ *to* $L^2(dx)$ is nuclear (see [8], theorem 14).

On the other hand J_5 is a bounded operator on $L^2(E)$ to $L^1(0,\infty)$. In fact, setting

$$\rho(\varphi)=\int_0^\infty\left|\frac{d}{d\lambda}(E_\lambda f,\varphi)\right|d\lambda \qquad \left(\varphi\in L^2(E)\right),$$

we find that $\rho(\varphi)$ is a semi-norm on $L^2(E)$ and satisfies the inequality

$$\rho(\varphi)\leqq\|f\|^2_{L^2}+\|\varphi\|^2_{L^2} \qquad \text{for all} \quad \varphi\in L^2(E),$$

which implies that

$$\rho(\varphi) \leqq \|f\|_{L^2}^2 + 1 \qquad \text{for} \quad \|\varphi\|_{L^2} \leqq 1 .$$

This implies that

$$\rho(\|\varphi\|_{L^2}^{-1}\varphi) \leqq \|f\|_{L^2}^2 + 1 \qquad \text{for} \quad 0 \neq \varphi \in L^2(E),$$

so that

$$\rho(\varphi) \leqq (\|f\|_{L^2}^2 + 1)\|\varphi\|_{L^2} \qquad \text{for all} \quad \varphi \in L^2(E).$$

This shows that J_5 is a bounded operator on $L^2(E)$ to $L^1(0, \infty)$, if we note that $\frac{d}{d\lambda}(E_\lambda f, \varphi) = (\theta(\lambda), \varphi)$ for all $\varphi \in L^2(E)$. Thus T_λ is a nuclear operator on $C_{3+\alpha}^2$ to $L^1(0, \infty)$, so that $\|T_\lambda\|_{(C_{3+\alpha}^2)^*} \in L^1(0, \infty)$, which completes the proof of Lemma 8.

5. Proof of Theorem 2.

Suppose that there exist functions $\omega \in M$ such that $\omega \neq 0$. Then Lemma 7 implies that $dim\ M = 1$. Therefore, taking $\omega \in M$ such that $\langle q, \omega \rangle = 1$, since $q = 0\,(|x|^{-3-\alpha})$ $(|x| \to \infty)$, we see that any $\varphi \in L^2(E)$ satisfying the condition $\varphi = 0\,(|x|^{-2-\delta})\left(\delta > \frac{1}{2}\right)$ $(|x| \to \infty)$ can be decomposed as follows:

$$\begin{aligned} \varphi &= \langle \varphi, \omega \rangle q + (\varphi - \langle \varphi, \omega \rangle q) \\ &\equiv \varphi_1 + \varphi_2 , \end{aligned}$$

while $\langle \varphi_2, \omega \rangle = 0$. An argument similar to the one used in proving Theorem 1 gives that $\lim_{t \to \infty} \langle u(t), \varphi_2 \rangle = 0$. Therefore we have only to prove

$$\lim_{t \to \infty} \langle u(t), q \rangle = 4\pi \langle f, \omega \rangle . \tag{5.1}$$

Now it follows from theorem 4 in [2], the condition (C_2) and theorem 6 in [4] that $\frac{d}{d\lambda}\langle E_\lambda f, q \rangle$ is continuous in $\lambda \in (0, \infty)$ and belongs to $L^1(0, \infty)$. Therefore by virtue of Lemma 3, Lemma 4, Lemma 5 and Fubini's theorem we obtain

$$\begin{aligned} \langle u(t), q \rangle &= \frac{1}{2\pi i}\int_0^\infty \frac{d}{d\lambda}\langle E_\lambda f, q \rangle\, d\lambda \int_{a-i\infty}^{a+i\infty} \frac{e^{\zeta t}}{\lambda + \zeta^2}\, d\zeta \\ &= \frac{1}{2\pi i}\int_0^\infty \frac{d}{d\lambda}\langle E_\lambda f, q \rangle\, d\lambda \int_{\Gamma_1 + \Gamma_2} \frac{e^{\zeta t}}{\lambda + \zeta^2}\, d\zeta + \int_{N^2}^\infty \frac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}} \frac{d}{d\lambda}\langle E_\lambda f, q \rangle\, d\lambda \end{aligned}$$

for $N > 3a$, where Γ_1 and Γ_2 are the curves $\{s - iN;\ 0 < s \leqq a\} \cup \{a + i\tau; -N <$

$\tau<N\}\cup\{s+iN;\ 0<s\leqq a\}$, $\{s+iN;\ -a\leqq s<0\}\cup\{-a+i\tau;\ -N<\tau<N\}\cup\{s-iN;\ -a\leqq s<0\}$ taken in the positive direction respectively. Then we can take N so large that $\left|\int_{N^2}^{\infty}\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\frac{d}{d\lambda}\langle E_\lambda f, q\rangle d\lambda\right|$ becomes sufficiently small uniformly with respect to $t>0$. Let N be fixed sufficiently large. Since for $\zeta\in\Gamma_2$ $Re\,\zeta<0$, Lemma 4 and Lebesgue's theorem imply that

$$\lim_{t\to\infty}\int_0^\infty\frac{d}{d\lambda}\langle E_\lambda f, q\rangle d\lambda\int_{\Gamma_2}\frac{e^{\zeta t}}{\lambda+\zeta^2}d\zeta=0\,.$$

Consequently we have only to prove

$$\lim_{t\to\infty}\int_0^\infty\frac{d}{d\lambda}\langle E_\lambda f, q\rangle d\lambda\int_{\Gamma_1}\frac{e^{\zeta t}}{\lambda+\zeta^2}d\zeta=8\pi^2 i\ \langle f,\omega\rangle\,. \tag{5.2}$$

Lemma 4 and Fubini's theorem imply that

$$\int_0^\infty\frac{d}{d\lambda}\langle E_\lambda f, q\rangle d\lambda\int_{\Gamma_1}\frac{e^{\zeta t}}{\lambda+\zeta^2}d\zeta=\int_{\Gamma_1}\langle R(-\zeta^2)f, q\rangle e^{\zeta t}d\zeta\,. \tag{5.3}$$

For $Re\,\zeta>0$ we set $u(\zeta)=R(-\zeta^2)f$. Then $u(\zeta)$ satisfies

$$u(x,\zeta)=\frac{1}{4\pi}\int_E\frac{e^{-\zeta|x-y|}}{|x-y|}f(y)\,dy-\frac{1}{4\pi}\int_E\frac{e^{-\zeta|x-y|}}{|x-y|}q(y)\,u(y,\zeta)\,dy \tag{5.4}$$

for $Re\,\zeta>0$. This and (3.4) imply that

$$\langle u(\zeta), q\rangle=-\frac{1}{\zeta}\int_{E\times E}\frac{e^{-\zeta|x-y|}}{|x-y|}f(y)\,q(x)\,\omega(x)\,dx\,dy+\zeta\langle u(\zeta), p(\zeta)\rangle \tag{5.5}$$

for $Re\,\zeta>0$, where

$$p(\zeta)=p(x,\zeta)=q(x)\int_E q(y)\,\omega(y)\,|x-y|\,dy\int_0^1 d\tau'\int_0^1\tau e^{-\zeta|x-y|\tau\tau'}\,d\tau\,. \tag{5.6}$$

In fact, setting $\omega'=q\omega$ we see that ω' satisfies (3.4) and $\int_E\omega'(x)dx=1$. (5.4) implies that .

$$\langle u(\zeta),\omega'\rangle=\frac{1}{4\pi}\int_{E\times E}\frac{e^{-\zeta|x-y|}}{|x-y|}f(y)\,\omega'(x)\,dx\,dy-\frac{1}{4\pi}\int_{E\times E}\frac{1}{|x-y|}q(y)u(y,\zeta)\times$$
$$\times\ \omega'(x)\,dx\,dy-\frac{1}{4\pi}\int_{E\times E}\frac{e^{-\zeta|x-y|}-1}{|x-y|}q(y)\,u(y,\zeta)\,\omega'(x)\,dx\,dy\,.$$

(3.4) shows that the second term in the right-hand side is equal to $\langle u(\zeta),\omega'\rangle$. This implies that

$$\int_{E\times E}\frac{e^{-\zeta|x-y|}}{|x-y|}f(y)\,\omega'(x)\,dx\,dy=-\zeta\int_{E\times E}q(y)\,u(y,\zeta)\,\omega'(x)\,dx\,dy\int_0^1 e^{-\zeta|x-y|\tau}d\tau$$

$$= -\zeta \int_{E\times E} q(y)u(y,\zeta)\omega'(x)dxdy + \zeta^2 \int_{E\times E} q(y)u(y,\zeta)\omega'(x)|x-y|dxdy \times$$
$$\times \int_0^1 d\tau' \int_0^1 \tau e^{-\zeta|x-y|\tau\tau'} d\tau ,$$

which gives (5.5). Furthermore (3.3), (5.3), (5.4), (5.5) and (5.6) imply that

$$(5.7) \quad \int_0^\infty \frac{d}{d\lambda}\langle E_\lambda f, q\rangle d\lambda \int_{\Gamma_1} \frac{e^{\zeta t}}{\lambda+\zeta^2} d\zeta$$
$$= 4\pi\langle f, \omega\rangle \int_{\Gamma_1} \frac{e^{\zeta t}}{\zeta} d\zeta + \int_{\Gamma_1} e^{\zeta t} F(\zeta) d\zeta + \int_{\Gamma_1} \zeta e^{\zeta t} \langle R(-\zeta^2)f, T_\zeta^{*3} p(\zeta)\rangle d\zeta ,$$

where

$$F(\zeta) = \int_{E\times E} f(y)q(x)\omega(x)dxdy \int_0^1 e^{-\zeta|x-y|\tau} d\tau + \zeta \sum_{j=0}^{2} \langle T_\zeta^j \phi_\zeta, p(\zeta)\rangle ,$$

$$\phi_\zeta(x) = \frac{1}{4\pi}\int_E \frac{e^{-\zeta|x-y|}}{|x-y|} f(y)dy ,$$

$$T_\zeta\phi(x) = -\frac{1}{4\pi}\int_E \frac{e^{-\zeta|x-y|}}{|x-y|} q(y)\phi(y)dy ,$$

$$T_\zeta^*\phi(x) = -\frac{1}{4\pi} q(x) \int_E \frac{e^{-\zeta|x-y|}}{|x-y|} \phi(y)dy ,$$

$$T^0\phi(x) = \phi(x) , \qquad T^j\phi(x) = T(T^{j-1}\phi)(x) \qquad (j=1,2,3).$$

First we shall show that

$$(5.8) \qquad \lim_{t\to\infty} 4\pi\langle f, \omega\rangle \int_{\Gamma_1} \frac{e^{\zeta t}}{\zeta} d\zeta = 8\pi^2 i \langle f, \omega\rangle .$$

We have

$$\int_{\Gamma_1} \frac{e^{\zeta t}}{\zeta} d\zeta = \operatorname{Res} \frac{e^{\zeta t}}{\zeta}\Big|_{\zeta=0} - \int_{\Gamma_2} \frac{e^{\zeta t}}{\zeta} d\zeta .$$

Lebesgue's theorem implies that $\lim_{t\to\infty} \int_{\Gamma_2} \frac{e^{\zeta t}}{\zeta} d\zeta = 0$, which gives (5.8).

Next we shall show that

$$(5.9) \qquad \lim_{t\to\infty} \int_{\Gamma_1} e^{\zeta t} F(\zeta) d\zeta = 0 .$$

The conditions (C_3) and $q = 0(|x|^{-3-\alpha})$ $(|x|\to\infty)$ imply that $F(\zeta)$ is holomorphic in $\{\zeta;\ \mathrm{Re}\,\zeta>0\}$ and is continuous in $\{\zeta;\ \mathrm{Re}\,\zeta\geqq 0\}$. From this it follows by Lebesgue's theorem that

$$\int_{\Gamma_1} e^{\zeta t} F(\zeta) d\zeta = i \int_{-N}^{N} e^{ist} F(is)\, ds\,,$$

from which we obtain (5.9) by Riemann-Lebesgue's theorem.

Thus the proof of Theorem 2 will be complete, if we prove the following

Lemma 9. *Let $q(x)$ be as in Theorem 2. Then*

$$\lim_{t\to\infty} \int_{\Gamma_1} \zeta e^{\zeta t} \langle R(-\zeta^2) f, p_3(\zeta)\rangle d\zeta = 0\,, \tag{5.10}$$

where $p_3(\zeta) = p_3(x, \zeta) = T_\zeta^{*3} p(x, \zeta)$.

To prove Lemma 9 we shall establish the following.

Lemma 10. 1) *For fixed $\lambda>0$ $\langle\theta(\lambda), p_3(\zeta)\rangle$ is a holomorphic function of $\zeta \in \{\mathrm{Re}\,\zeta>0\}$, where $\theta(\lambda)$ is the one introduced in Lemma 8.*

2) *Let $q(x)$ be as in Theorem 2. Then we have*

$$\sup_{x\in E, |\beta|\leq 2} |D_x^\beta p_3(x, \zeta)|(1+|x|^2)^{\frac{3+\alpha}{2}} \leqq C_K \qquad \textit{for } \mathrm{Re}\,\zeta\geqq 0,\ |\zeta|\leqq K,$$

where K is a positive number and C_K is a constant independent of $x\in E$, $\zeta\in\{\mathrm{Re}\,\zeta\geqq 0,\ |\zeta|\leqq K\}$.

Proof of Lemma 10. 1) It follows from the condition (C_3) and theorem 4 in [2] that for $\lambda>0$ $\theta(x, \lambda)$ is a continuous function of $x\in E$ and behaves like $0(|x|^{-1})$ $(|x|\to\infty)$. Therefore, since $q=0(|x|^{-2-\alpha})$ $(|x|\to\infty)$, we see easily that for $\lambda>0$ $\langle\theta(\lambda), p_3(\zeta)\rangle$ is a holomorphic function of $\zeta\in\{\mathrm{Re}\,\zeta>0\}$, which proves 1).

We proceed to prove 2). The condition $q=0(|x|^{-3-\alpha})$ $(|x|\to\infty)$ and (5.6) imply that for $\mathrm{Re}\,\zeta\geqq 0$

$$\begin{aligned} |p(x, \zeta)| &\leqq |q(x)|\Big[|x|\int_E |q(y)\,\omega(y)|dy + \int_E |y||q(y)\,\omega(y)|dy\Big] \\ &\leqq C(1+|x|)^{-2-\alpha}\,. \end{aligned} \tag{5.11}$$

Here and in what follows we denote various constants independent of $\zeta(\mathrm{Re}\,\zeta\geqq 0)$ by C. (5.11) and the equality

$$T_\zeta^* p(x, \zeta) = -\frac{1}{4\pi} q(x) \int_E \frac{e^{-\zeta|x-y|}}{|x-y|} p(y, \zeta)\, dy$$

imply that

$$|T_\zeta^* p(x, \zeta)| \leqq C(1+|x|)^{-3-\alpha} \qquad \text{for } \mathrm{Re}\,\zeta\geqq 0 \tag{5.12}$$

In the same way we obtain

$$|T_\zeta^{*j} p(x, \zeta)| \leqq C(1+|x|)^{-4-\alpha} \qquad \text{for } \mathrm{Re}\,\zeta\geqq 0 \ \ (j=2, 3). \tag{5.13}$$

The condition $D^\beta q = 0(|x|^{-2-\alpha})$ $(|x|\to\infty)$ $(|\beta|=1,2)$, (5.12) and (5.13) imply that

$$|D_x^\beta p_3(x,\zeta)| \leqq C_K(1+|x|)^{-3-\alpha} \qquad \text{for } \operatorname{Re}\zeta\geqq 0,\ |\zeta|\leqq K\ \ (|\beta|\leqq 2),$$

which proves 2) of Lemma 10.

Proof of Lemma 9. It follows from 2) of Lemma 10 and Lemma 8 that there exists a function $g(\lambda)\in L^1(0,\infty)$ such that

$$|\langle\theta(\lambda), p_3(\zeta)\rangle| \leqq g(\lambda) \qquad \text{for } \operatorname{Re}\zeta\geqq 0,\ |\zeta|\leqq K. \tag{5.14}$$

This, Lemma 4, 1) of Lemma 10 and theorem 4 in [2] imply that

$$\begin{aligned}\int_{\Gamma_1}\zeta e^{\zeta t}\langle R(-\zeta^2)f, p_3(\zeta)\rangle\,d\zeta &= \int_{\Gamma_1}\zeta e^{\zeta t}\,d\zeta\int_0^\infty \frac{\langle\theta(\lambda), p_3(\zeta)\rangle}{\lambda+\zeta^2}\,d\lambda\\ &= \lim_{\varepsilon\to 0}\int_0^\infty d\lambda\int_{\Gamma_\varepsilon}\zeta e^{\zeta t}\frac{\langle\theta(\lambda), p_3(\zeta)\rangle}{\lambda+\zeta^2}\,d\zeta\,,\end{aligned}$$

where Γ_ε is the path obtained replacing a by ε in Γ_1. Therefore, by virtue of (5.14), Lemma 4, 2) of Lemma 10, theorem 4 in [2], Lebesgue's theorem and Riemann-Lebesgue's theorem we find that we have only to prove

$$\lim_{t\to\infty}\lim_{\varepsilon\to 0}\int_0^{4N^2} d\lambda\int_{\varepsilon-iN}^{\varepsilon+iN}\frac{\langle\theta(\lambda), p_3(\zeta)\rangle}{\lambda+\zeta^2}\,\zeta e^{\zeta t}\,d\zeta = 0\,. \tag{5.15}$$

Now we shall prove

$$\lim_{t\to\infty}\lim_{\varepsilon\to 0}\int_0^N d\lambda\int_{-N}^N e^{(\varepsilon+is)t}\,\frac{(\lambda-s)\langle\lambda\theta(\lambda^2), p_3(\varepsilon+is)\rangle}{(\lambda-s)^2+\varepsilon^2}\,ds = 0\,. \tag{5.16}$$

Set $\rho = t-(|x-y|+|y-z|+|z-u|+|u-v|\tau\tau')$. Then by virtue of Fubini's theorem, for fixed $t>0$ and fixed $\varepsilon>0$ we obtain

$$\int_{-N}^N e^{(\varepsilon+is)t}\,\frac{(\lambda-s)\langle\lambda\theta(\lambda^2), p_3(\varepsilon+is)\rangle}{(\lambda-s)^2+\varepsilon^2}\,ds = \left(\frac{1}{4\pi}\right)^3 e^{\varepsilon t}\int_E \lambda\theta(x,\lambda^2)\varphi_{\varepsilon,t}(x,\lambda)\,dx, \tag{5.17}$$

where

$$\begin{aligned}\varphi_{\varepsilon,t}(x,\lambda) = q(x)\int_E\frac{q(y)}{|x-y|}\,dy\int_E\frac{q(z)}{|y-z|}\,dz\int_E\frac{q(u)}{|z-u|}\,du\int_E|u-v|\,q(v)\,\times\\ \times\,\omega(v)\,dv\int_0^1 d\tau'\int_0^1\tau e^{-\varepsilon(t-\rho)}\,d\tau\int_{-N}^N\frac{(s-\lambda)\,e^{i\rho s}}{(s-\lambda)^2+\varepsilon^2}\,ds\,.\end{aligned} \tag{5.18}$$

First we shall show that

$$\lim_{\varepsilon\to 0}\int_0^N d\lambda\int_{-N}^N e^{(\varepsilon+is)t}\,\frac{(\lambda-s)\langle\lambda\theta(\lambda^2), p_3(\varepsilon+is)\rangle}{(\lambda-s)^2+\varepsilon^2}\,ds \tag{5.19}$$

$$=\left(\frac{1}{4\pi}\right)^3\int_0^N d\lambda\int_E \lambda\theta(x,\lambda^2)\lim_{\varepsilon\to 0}\varphi_{\varepsilon,t}(x,\lambda)dx\,.$$

Let $t>0$ be fixed. Then we find that there exists a constant C such that

$$\sup_{x\in E,|\beta|\leqq 2}|D^\beta\varphi_{\varepsilon,t}(x,\lambda)|(1+|x|^2)^{\frac{3+\alpha}{2}}\leqq C\left(1+\log\frac{N+\lambda}{N-\lambda}\right) \tag{5.20}$$

for all $\lambda<N$ and all $\varepsilon\leqq\varepsilon_0$, where ε_0 is a positive number. In fact, since $s\cos s$ is a odd function, for $\lambda<N$ we have

$$\begin{aligned}\int_{-N}^N\frac{(s-\lambda)e^{i\rho s}}{(s-\lambda)^2+\varepsilon^2}ds &= e^{i\rho\lambda}\left[\int_{(-N-\lambda)\rho}^{(N-\lambda)\rho}\frac{s\cos s}{s^2+\varepsilon^2\rho^2}ds+i\int_{(-N-\lambda)\rho}^{(N-\lambda)\rho}\frac{s\sin s}{s^2+\varepsilon^2\rho^2}ds\right]\\ &= e^{i\rho\lambda}\left[\int_{(-N-\lambda)\rho}^{(\lambda-N)\rho}\frac{\cos s}{s}ds-\varepsilon^2\rho^2\int_{(-N-\lambda)\rho}^{(\lambda-N)\rho}\frac{\cos s}{s(s^2+\varepsilon^2\rho^2)}ds\right.\\ &\quad\left.+i\left[\int_{(-N-\lambda)\rho}^{(N-\lambda)\rho}\frac{\sin s}{s}ds-i\varepsilon^2\rho^2\int_{(-N-\lambda)\rho}^{(N-\lambda)\rho}\frac{\sin s}{s(s^2+\varepsilon^2\rho^2)}ds\right]\right.\end{aligned} \tag{5.21}$$

which implies that

$$\left|\int_{-N}^N\frac{(s-\lambda)e^{i\rho s}}{(s-\lambda)^2+\varepsilon^2}ds\right|\leqq C'\left(1+\log\frac{N+\lambda}{N-\lambda}\right), \tag{5.22}$$

where C' is a constant independent of ε,λ,ρ. Since $q=0(|x|^{-3-\alpha})$, $D^\beta q=0(|x|^{-2-\alpha})$ $(|x|\to\infty)$ $(|\beta|=1,2)$ and $t-\rho\geqq 0$, (5.18) and (5.22) imply (5.20). On the other hand it follows from theorem 4 in [2] that for each function φ satisfying $|\varphi(x)|\leqq C(1+|x|^2)^{-\frac{2+\alpha}{2}}$, $\int_E\lambda\theta(x,\lambda^2)\varphi(x)dx$ is a continuous function of $\lambda\in(0,\infty)$. Consequently by virtue of (5.17), (5.20), Lemma 8 and Lebesgue's theorem we obtain (5.19).

Next we shall show that

$$\lim_{t\to\infty}\int_0^N d\lambda\int_E\lambda\theta(x,\lambda^2)\lim_{\varepsilon\to 0}\varphi_{\varepsilon,t}(x,\lambda)\,d\lambda=0\,. \tag{5.23}$$

From (5.18), (5.21) and (5.22) it follows by Lebesgue's theorem that for fixed $\lambda<N$ we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\varphi_{\varepsilon,t}(x,\lambda) &= q(x)\int_E\frac{q(y)}{|x-y|}dy\int_E\frac{q(z)}{|y-z|}dz\int_E\frac{q(u)}{|z-u|}du\int_E|u-v|\times\\ &\times q(v)\,\omega(v)\,dv\int_0^1 d\tau'\int_0^1\tau e^{i\lambda\rho}d\tau\left[\int_{(-N-\lambda)\rho}^{(\lambda-N)\rho}\frac{\cos s}{s}ds+i\pi\right.\\ &\left.+i\left\{\int_{(-N-\lambda)\rho}^{(N-\lambda)\rho}\frac{\sin s}{s}ds-\pi\right\}\right]\equiv J_{1,t}(x,\lambda)+J_{2,t}(x,\lambda)+J_{3,t}(x,\lambda)\,.\end{aligned} \tag{5.24}$$

Set

$$\varphi_2(x, \lambda) = e^{-i\lambda t} J_{2,t}(x, \lambda).$$

Then, since $\rho = t - (|x-y| + |y-z| + |z-u| + |u-v|\tau\tau')$ and $q = 0(|x|^{-3-\alpha})$, $D^\beta q = 0(|x|^{-2-\alpha})(|x| \to \infty)(|\beta| = 1, 2)$, we find that

$$\sup_{x \in E, |\beta| \leqq 2} |D^\beta \varphi_2(x, \lambda)| (1 + |x|^2)^{\frac{3+\alpha}{2}} \leqq C$$

for all $\lambda < N$, where C is a constant independent of λ. Therefore by virtue of Lemma 8 and Riemann-Lebesgue's theorem we obtain

$$(5.25) \qquad \lim_{t\to\infty} \int_0^N d\lambda \int_E \lambda\theta(x, \lambda^2) J_{2,t}(x, \lambda)\, dx$$
$$= \lim_{t\to\infty} \int_0^N e^{i\lambda t}\, d\lambda \int_E \lambda\theta(x, \lambda^2) \varphi_2(x, \lambda) dx = 0.$$

Let $\rho - t$ be fixed. Then $\rho \to \infty$ as $t \to \infty$. This implies that for fixed $\lambda < N$ and fixed $\rho - t$

$$\lim_{t\to\infty} \int_{(-N-\lambda)\rho}^{(\lambda-N)\rho} \frac{\cos s}{s} ds = \lim_{t\to\infty} \int_{(-N-\lambda)\rho}^{(N-\lambda)\rho} \frac{\sin s}{s} ds - \pi = 0 .$$

Consequently an argument similar to the one used in proving (5.19) gives that

$$(5.26) \qquad \lim_{t\to\infty} \int_0^N d\lambda \int_E \lambda\theta(x, \lambda^2) J_{k,t}(x, \lambda) dx = 0 \qquad (k = 1, 3).$$

Thus (5.24), (5.25) and (5.26) imply (5.23). Therefore (5.19) and (5.23) prove (5.16).

An argument similar to the one used in proving (5.16) gives that

$$\lim_{t\to\infty} \lim_{\varepsilon\to 0} \int_N^{2N} d\lambda \int_{-N}^N e^{(\varepsilon+is)t} \frac{(\lambda - s)\langle \lambda\theta(\lambda^2), p_3(\varepsilon + is)\rangle}{(\lambda - s)^2 + \varepsilon^2} ds = 0 ,$$

$$\lim_{t\to\infty} \lim_{\varepsilon\to 0} \int_0^{2N} d\lambda \int_{-N}^N e^{(\varepsilon+is)t} \frac{\varepsilon \langle \lambda\theta(\lambda^2), p_3(\varepsilon + is)\rangle}{(\lambda - s)^2 + \varepsilon^2} ds = 0 .$$

These and (5.16) imply that

$$\lim_{t\to\infty} \lim_{\varepsilon\to 0} \int_0^{2N} d\lambda \int_{\varepsilon-iN}^{\varepsilon+iN} e^{\zeta t} \frac{\langle \lambda\theta(\lambda^2), p_3(\zeta)\rangle}{\lambda + i\zeta} d\zeta = 0 .$$

In the same way we obtain

$$\lim_{t\to\infty} \lim_{\varepsilon\to 0} \int_0^{2N} d\lambda \int_{\varepsilon-iN}^{\varepsilon+iN} e^{\zeta t} \frac{\langle \lambda\theta(\lambda^2), p_3(\zeta)\rangle}{\lambda - i\zeta} d\zeta = 0 .$$

Thus we have proved (5. 15), which completes the proof of Lemma 9.

Remark. If the condition (C_2) is not assumed, then, as O. A. Ladyzhenskaya pointed out in [1], it is seen that in general the principle of limiting amplitude is not valid. Let E_λ be the resolution of the identity generated by the operator A. Furthermore let $-\mu_n$ $(\mu_n>0)$ $(n=1,2,\cdots,k)$ be the negative eigenvalues of the operator A and $\varphi_n(n=1,2,\cdots,k)$ be the eigenfunctions associated with the eigenvalues $-\mu_n$, and $\psi_n(n\geqq 1)$ be the eigenfunctions of A associated with the eigenvalue zero. Then the solution $u(t)$ of (2.1) is represented as follows:

$$u(t)=\int_{0+}^{\infty}\frac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}}\,dE_\lambda f+t\sum_{n=1}^{\infty}(f,\psi_n)_{L^2}\psi_n+\sum_{n=1}^{k}\frac{e^{\sqrt{\mu_n}\,t}-e^{-\sqrt{\mu_n}\,t}}{2\sqrt{\mu_n}}(f,\varphi_n)_{L^2}\varphi_n\,.$$

The first term in the right-hand side behaves like $o(t)$ at infinity. Therefore Theorem 2 implies that our conjecture is true in the case that $q=0\,(|x|^{-3-\alpha})$ at infinity.

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