LOCALLY FINITE OUTER GALOIS THEORY

By

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Introduction.

This paper is the continuation of the preceding paper [22]. In §1 and §2, locally finite (outer) Galois extensions are treated. The main results are parallel to those of the finite case. In these studies, Nagahara [12] is our guide. Further several results for finite Galois extensions are added (Th. 1.18). In §3, we give a normal basis theorem for a finite Galois extension.

§1. As to notations and terminologies we follow [22]. Let A be a ring with 1 ($\neq 0$), C the center of A, G a (finite or infinite) group of automorphisms of $A, B=A^G=\{x\in A\,;\ \sigma(x)=x$ for all σ in $G\}$, and $\hat{G}$ the group of all B-automorphisms of A. $\hat{G}$ is then a topological group in finite topology (cf. Jacobson [7]). We denote the closure of G in $\hat{G}$ by G^*. $\varDelta$ means the trivial crossed product of A with G: $\varDelta=\sum_{\sigma\in G}\oplus Au_\sigma$, $u_\sigma u_\tau=u_{\sigma\tau}$ $(\sigma,\tau\in G)$, $u_\sigma x=\sigma(x)u_\sigma$ $(x\in A)$. Then there is a canonical ring homomorphism j from $\varDelta$ to $\mathrm{End}\,(A_B)$ defined by $j(\sum_\sigma x_\sigma u_\sigma)(y)=\sum_\sigma x_\sigma\cdot\sigma(y)$ $(\sum_\sigma x_\sigma u_\sigma\in\varDelta,\ y\in A)$. For any intermediate ring T of A/B, $G^T=\{\sigma\in G\,;\ \sigma|T=1_T\}$ is a subgroup of G, where $\sigma|T$ means the restriction of σ to T. We call it a *fixed subgroup* of G. For any subgroup H of G, $A^H=\{x\in A\,;\ \sigma(x)=x$ for all σ in $H\}$ is an intermediate ring of A/B. We call it a *fixed subring* of A (with respect to G). Then, as is well known, the set of all fixed subgroups of G and the set of all fixed subrings of A are anti-order-isomorphic in the usual sense of Galois theory. A subring T of A is called a *G-invariant* subring of A if $\sigma(T)=T$ for all σ in G (or equivalently, $\sigma(T)\subseteq T$ for all σ in G). Let N be a fixed subgroup of G. Then, A^N is G-invariant if and only if N is a normal subgroup of G: $N\lhd G$. Let T be an intermediate ring of A/B, and put $H=G^T$. Then, for σ,τ in G, $\sigma|T=\tau|T$ if and only if $\sigma H=\tau H$. Let H and K be subgroups of G such that $H\supseteq K$ and $(H:K)<\infty$, and let $H=\sigma_1K\cup\cdots\cup\sigma_rK$ be the left coset decomposition. For any x in A^K we put $t_{H:K}(x)=\sum_i\sigma_i(x)$. Then $t_{H:K}$ is an A^H-A^H-homomorphism from A^K to A^H, and is independent of the choice of $\sigma_1,\cdots,\sigma_r$. If $K=1$, we write simply t_H instead of $t_{H:1}$.

Here we present several fundamental facts, which are essential throughout the present study. Let ${}_TM_U$ and ${}_TN_U$ be T-left, U-right modules. If ${}_TM_U$ is isomorphic to a direct summand of ${}_TN_U^r$ for some natural number r, then we write ${}_TM_U|{}_TN_U$, where ${}_TN_U^r$ means the direct sum of r copies of ${}_TN_U$. If ${}_TM_U|{}_TN_U$ and ${}_TN_U|{}_TM_U$ we write ${}_TM_U\sim{}_TN_U$ (similar) (cf. Morita [21]). To be easily seen, ${}_TM_U|{}_TN_U$ if and only if there are $T-U$-homomorphisms $f_1,\cdots,f_r$ in $\mathrm{Hom}({}_TM_U, {}_TN_U)$ and $g_1,\cdots,g_r$ in $\mathrm{Hom}({}_TN_U, {}_TM_U)$ such that $\sum_i f_ig_i=$ the identity of M, or equivalently, $\mathrm{Hom}({}_TM_U, {}_TN_U)\cdot\mathrm{Hom}({}_TN_U, {}_TM_U)=\mathrm{Hom}({}_TM_U, {}_TM_U)$, where homomorphisms act on the right side.

Let T be a ring with 1, M a unital T-left module, and $T^*=\mathrm{End}({}_TM)$.

S. 1. If ${}_TT|{}_TM$ then $M_{T^*}|T^*_{T^*}$ (i.e. M_{T^*} is finitely generated and projective) and $T=\mathrm{End}(M_{T^*})$. (Morita)

S. 2. If ${}_TM|{}_TT$ then $T^*_{T^*}|M_{T^*}$. (Morita)

S. 3. Let T be commutative. If ${}_TM|{}_TT$ and ${}_TM$ is faithful, then ${}_TT|{}_TM$. (Auslander-Buchsbaum-Goldman)

S. 4. Let $\bar{T}$ be an extension ring of T. If ${}_TT|{}_T\bar{T}$ then ${}_TT$ is a direct summand of ${}_T\bar{T}$ (and conversely). (Müller)

S. 5. Let $\bar{T}$ be an extension ring of T. If ${}_TT_T|{}_T\bar{T}_T$ then ${}_TT_T$ is a direct summand of ${}_T\bar{T}_T$. (The proof is similar to the one of S.4.)

In [22], A/B was called a *G-Galois extension* if G is finite and there are elements $a_1,\cdots,a_n$; $a_1^*,\cdots,a_n^*$ in A such that $\sum_i a_i\cdot\sigma(a_i^*)=\delta_{1,\sigma}$ $(\sigma\in G)$. In this paper, A/B is called a *finite G-Galois extension* if A/B is G-Galois and $t_G(c)=1$ for some c in A. Then, the following are equivalent:

(a) A/B is finite G-Galois.

(b) G is finite, $A_B\sim B_B$ and j: $\Delta\simeq\mathrm{End}(A_B)$.

(c) G is finite and ${}_\Delta\Delta\sim{}_\Delta A$.

(Cf. S.1, S.2, [6] and [21]).

A/B is called a *locally finite G-Galois extension* if there are fixed normal subgroups N_λ $(\lambda\in\Lambda)$ of G which satisfy the following conditions: (1) $(G:N_\lambda)<\infty$, and A^{N_λ}/B is a finite G/N_λ-Galois extension. (2) $A=\cup_\lambda A^{N_\lambda}$, and $\{A^{N_\lambda};\lambda\in\Lambda\}$ is a directed set with respect to the inclusion relation (abbre. $A=\cup_\lambda A^{N_\lambda}$ is a *directed union*). Then we call $A=\cup_\lambda A^{N_\lambda}$ a representation of the locally finite G-Galois extension A/B. If $V_A(B)=C$, an extension A/B is said to be *outer*.

Now we shall prove first the following

Proposition 1.1. *Let $G=G^*$ (i.e. G is closed in $\hat{G}$). Then the following are equivalent*:

(i) *$\{\sigma(x);\sigma\in G\}$ is finite for any x in A.*

(ii) *G is compact.*

(iii) *Every directed union of fixed subrings of A with respect to G is also a fixed subring of A with respect to G, and $\cap H=1$, where H ranges* *over all fixed subgroups of G such that $(G:H)<\infty$.*

Proof. (i)$\Longrightarrow$(ii) If we put $\prod_{x\in A}\{\sigma(x);\sigma\in G\}=D$, then $G\subseteq D$ and D is compact. Therefore it is sufficient to prove that G is closed in D. Let ρ be any element of the closure of G in D. Then, as is easily seen, ρ is a B-ring isomorphism from A into A. Let a be in A, and put $F=\{\sigma(a);\sigma\in G\}$. Then, by assumption, F is a finite subset of A, so that there is an element τ in G such that $\rho|F=\tau|F$. Then, in particular, $\rho(\tau^{-1}(a))=\tau(\tau^{-1}(a))=a$. Thus ρ is a B-automorphism of A. Hence the closure of G in D is contained in $\hat{G}$. Since G is closed in $\hat{G}$, G is closed in D, as desired. (ii)$\Longrightarrow$(iii) For any x in G, we put $H_x=\{\sigma\in G;\sigma(x)=x\}$. Then H_x is open in G, and therefore σH_x is open in G for any σ in G. Then, since G is compact, we have $(G:H_x)<\infty$. Evidently $\bigcap_{x\in A}H_x=1$. This proves the second assertion. Let $(A\neq)$ $T=\bigcup_{\lambda\in\Lambda}T_\lambda$ be a directed union of fixed subrings of A with respect to G, and let $K_\lambda=G^{T_\lambda}$. Then each K_λ is a closed subgroup of G, and $A^{K_\lambda}=T_\lambda$. Let a be an element of $A-T$, and put $U=\{\sigma\in G;\sigma(a)=a\}$. Then U is open in G, so that each $K_\lambda-U$ is closed in G. Since $a\notin T_\lambda$ and $A^{K_\lambda}=T_\lambda$, we have $K_\lambda-U\neq\emptyset$. For any finite subset $\{\lambda_1,\cdots,\lambda_n\}$ of Λ, there is an element λ_0 of Λ such that $T_{\lambda_0}\supseteq\bigcup_i T_{\lambda_i}$. Then $K_{\lambda_0}\subseteq\bigcap_i K_{\lambda_i}$, and so $\emptyset\neq K_{\lambda_0}-U\subseteq\bigcap_i K_{\lambda_i}-U=\bigcap_i(K_{\lambda_i}-U)$. Thus $\{K_\lambda-U;\lambda\in\Lambda\}$ has finite intersection property. Since G is compact, we have $\bigcap_\lambda(K_\lambda-U)\neq\emptyset$. If ρ is in $\bigcap_\lambda(K_\lambda-U)$ then $\rho\in G^T$ and $\rho(a)\neq a$. Therefore $a\not\subseteq A^K$, where $K=G^T$. Thus $A^K=T$. Hence T is a fixed subring of A with respect to G. (iii)$\Longrightarrow$(i) Let H and K be fixed subgroups of G such that $(G:H)<\infty$ and $(G:K)<\infty$. Then $H\cap K$ is also a fixed subgroup of G with $(G:H\cap K)<\infty$. Therefore $\bigcup A^H$ is a directed union of fixed subrings of A, where H ranges over all fixed subgroups of G with $(G:H)<\infty$. Then, by assumption, $\bigcup A^H$ is a fixed scbring of A with respect to G. Since $\bigcap H=1$, we have $A=\bigcup A^H$. For any x in A, there is an A^H such that $x\in A^H$. Therefore if we put $L=\{\sigma\in G;\sigma(x)=x\}$ then $(G:L)<\infty$. This implies that $\{\sigma(x);\sigma\in G\}$ is finite.

Remark. For any x in A, $\{\sigma(x);\sigma\in G\}=\{\sigma(x);\sigma\in G^*\}$.

Proposition 1.2. *Let N be a fixed normal subgroup of G such that $(G:N)<\infty$ and A^N/B is finite G/N-Galois, and G_1 a subgroup of G^* containing G. Then A^N/B is finite G_1/N_1-Galois, where $N_1=\{\sigma\in G_1;\sigma|A^N=1_{A^N}\}$.*

Proof. Put $T=A^N$. Evidently $A^{N_1}=T$. Since G is dense in G_1 and T_B is finitely generated, there holds $G|T=G_1|T$. Therefore T is G_1-invariant, $N_1\triangleleft G_1$, and $(G_1:N_1)<\infty$. There are elements $a_1,\cdots,a_n$; $a_1^*,\cdots,a_n^*$ in T such that $\sum_i a_i\cdot\sigma(a_i^*)=\delta_{N,\sigma}$ for all σ in G. If τ is in G_1-N_1 then $\tau|T=\rho|T$ for some ρ in $G-N$, and $\sum_i a_i\cdot\tau(a_i^*)=\sum_i a_i\cdot\rho(a_i^*)=0$. Thus $\sum_i a_i\cdot\sigma(a_i^*)=\delta_{N_1,\sigma}$ for σ in G_1.

Corollary. *Let A/B be locally finite G-Galois, and G_1 a subgroup of G^* containing G. Then A/B is locally finite G_1-Galois.*

Proposition 1.3. *Let H_λ $(\lambda\in\Lambda)$ be fixed subgroups of G such that $A=\bigcup_{\lambda\in\Lambda}A^{H_\lambda}$ is a directed union.*

(1) *If H is a subgroup of G such that $(G:H)<\infty$ then $A^H\subseteq A^{H_\lambda}$ for some λ in Λ.*

(2) *If K is a subgroup of G such that $(K:1)<\infty$ then $K\cap H_\mu=1$ for some μ in Λ.*

Proof. (1) Let $[H_\lambda\cup H]$ be the subgroupof G generated by $H_\lambda\cup H$. Since $G\supseteq[H_\lambda\cup H]\supseteq H$, we have $(G:[H_\lambda\cup H])\leqslant(G:H)$ for all λ in Λ. Let $(G:[H_{\lambda_0}\cup H])$ be maximum. We shall prove that $A^H\subseteq A^{H_{\lambda_0}}$. For any H_λ there is an H_μ such that $A^{H_\mu}\supseteq A^{H_\lambda}\cup A^{H_{\lambda_0}}$. Then $H_\mu\subseteq H_\lambda\cap H_{\lambda_0}$, and so $[H_\mu\cup H]\subseteq[H_\lambda\cup H]\cap[H_{\lambda_0}\cup H]$. Since $(G:[H_{\lambda_0}\cup H])$ is maximum, we have $([H_\lambda\cup H]\supseteq)$ $[H_\mu\cup H]=[H_{\lambda_0}\cup H]$. Hence $[H_{\lambda_0}\cup H]\subseteq[H_\lambda\cup H]$ for all λ in Λ. Then $A^H=\bigcup_\lambda(A^{H_\lambda}\cap A^H)=\bigcup_\lambda A^{[H_\lambda\cup H]}=A^{[H_{\lambda_0}\cup H]}=A^{H_{\lambda_0}}\cap A^H$, which means $A^H\subseteq A^{H_{\lambda_0}}$. (2) Since $A=\bigcup_\lambda A^{H_\lambda}$, we have $1=G^A=\bigcap_\lambda H_\lambda$. Let $K=\{\sigma_1=1, \sigma_2,\cdots,\sigma_r\}$. Then, for any σ_i $(i\neq1)$, there is an H_{λ_i} such that $\sigma_i\notin H_{\lambda_i}$. By assumption there is a μ such that $H_\mu\subseteq\bigcap_{i=1,\cdots,r}H_{\lambda_i}$. Then $H\cap H_\mu\subseteq H\cap(\bigcap_{i=,\cdots,r}H_{\lambda_i})=1$.

Remark. Let A/B be locally finite G-Galois, and $A=\bigcup_{\lambda\in\Lambda}A^{N_\lambda}$ its representation. If G is finite then $A=A^{N_\lambda}$ for some λ.

Proposition 1.4. *Let T be an intermediate ring of A/B such that $G|T$ is finite, and let $H=G^T$, and $G=\sigma_1H\cup\cdots\cup\sigma_rH$ a left coset decomposition of G. If there are elements $t_1,\cdots,t_n$; $t_1^*,\cdots,t_n^*$ in T such that $\sum_i t_i\cdot\sigma(t_i^*)=\delta_{H,\sigma}$ for all σ in G, then there hold the following.*

(1) *$T=A^H$, and T_B is finitely generated and projective.*

(2) *j^*: $A(\sum_k u_{\sigma_k})T=\sum_k Au_{\sigma_k}\simeq\mathrm{Hom}(T_B, A_B)$, where $j^*(\sum_k x_k u_{\sigma_k})(t)=\sum_k x_k\cdot\sigma_k(t)$, and this induces the $B-T$-isomorphism $({}_BT_T\simeq)$ $(\sum_k u_{\sigma_k})T\simeq$ $\mathrm{Hom}(T_B, B_B)$.*

(3) *The following are equivalent*: (i) $B_B|T_B$. (ii) ${}_BB|_BT$. (iii) $t_{G:H}(c)=1$ *for some c in T.*

Proof. (1) $t_{G:H}$ is a $B-B$-homomorphism from A^H to B. For any y in A^H, $T\ni\sum_i t_i\cdot t_{G:H}(t_i^*y)=\sum_i t_i\sum_k\sigma_k(t_i^*y)=\sum_k\sum_i t_i\cdot\sigma_k(t_i^*)\sigma_k(y)=y$. Hence $A^H=T$, and T_B is finitely generated and projective (cf. [3]). (2) $j^{*^{-1}}$ is the mapping such that $j^{*^{-1}}(f)=\sum_i f(t_i)(\sum_k u_{\sigma_k})t_i^*$ $(f\in\mathrm{Hom}(T_B, A_B))$. The second part will be easily seen. (3) The equivalence (i)$\Longleftrightarrow$(iii) is easy from (2).

Therefore (i) and (ii) are equivalent, because the situation is right-left symmetric.

Proposition 1.5. *Let A/B be locally finite G-Galois. Then there hold the following*:

(1) *G^* is compact.*

(2) *By j, Δ is isomorphic to a dense subring of* $\mathrm{Hom}(A_B, A_B)$.

(3) *A subgroup H of G is a closed subgroup of G if and only if H is a fixed subgroup of G.*

Proof. Let $A=\bigcup_\lambda A^{N_\lambda}$ be a representation of the locally finite G-Galois extension A/B. (1) If x is in A then $x\in A^{N_\nu}$ for some ν in Λ. Then $(G:N_\nu)<\infty$ implies that $(\{\sigma(x);\ \sigma\in G\}=)\ \{\sigma(x);\ \sigma\in G^*\}$ is finite. Hence, by Prop. 1.1, G^* is compact. (2) By Prop. 1.4 (2), $\mathrm{Im}\, j$ is dense in $\mathrm{Hom}(A_B, A_B)$. Therefore it suffices to prove that j is $1-1$. Let $\sigma_1,\cdots,\sigma_r$ be different elements in G. Then there is a finite subset F of A such that $\sigma_i|F\neq\sigma_k|F$ provided $i\neq k$. From this fact and Prop. 1.4, we can easily see that j is $1-1$. (3) Evidently, a fixed subgroup is a closed subgroup. Let H be any subgroup of G, and put $H'=G^T$, where $T=A^H$. Then $T=A^{H'}$. It suffices to prove that H is dense in H'. To prove this, we take any finite subset F of A. Then $F\subseteq A^{N_\mu}$ for some N_μ. Put $N=N_\mu$. Then, by finite Galois theory, we obtain $(G/N)^{T_1}=HN/N$ and $(A/N)^{T_1'}=H'N/N$, where $T_1=A^{HN}$ and $T_1'=A^{H'N}$ (cf. [22; Prop. 2.2]). Since $A^{HN}=A^H\cap A^N=A^{H'}\cap A^N=A^{H'N}$, we have $HN/N=H'N/N$, that is, $HN=H'N$. Hence $H|A^N=H'|A^N$, and so $H|F=H'F$. Since F is arbitrary, this implies that H is dense in H'. This completes the proof.

Theorem 1.6. *Let A/B be locally finite G-Galois, $G=G^*$, and H a subgroup of G, and let A' be an indecomposable extension ring of B such that $\mathrm{V}_{A'}(B)=\mathrm{V}_{A'}(A')$. Assume that there is a B-ring homomorphism g from A to A'. Then, for any B-ring homomorphism f from A^H to A', there is an element σ in G such that $f=g\sigma|A^H$.*

Proof. Let $A=\bigcup_{\lambda\in\Lambda} A^{N_\lambda}$ be a representation. For each N_λ, there is an element σ in G such that $f|A^{HN_\lambda}=g\sigma|A^{HN_\lambda}$ ([22; Th. 4.1]). For each λ, we put $K_\lambda=\{\sigma\in G\,;\ f|A^{HN_\lambda}=g\sigma|A^{HN_\lambda}\}$. Then $K_\lambda\neq\emptyset$, and $\{K_\lambda;\ \lambda\in\Lambda\}$ has finite intersection property. Let τ be in the closure of K_λ in G. Since $(A^{N_\lambda})_B$ is finitely generated, $\tau|A^{N_\lambda}=\alpha|A^{N_\lambda}$ for some α in K_λ. Then $\tau|A^{HN_\lambda}=\alpha|A^{HN_\lambda}$, and so $f|A^{HN_\lambda}=g\alpha|A^{HN_\lambda}=g\tau|A^{HN_\lambda}$. Hence $\tau\in K_\lambda$, and therefore K_λ is closed in G. Since G is compact (Prop. 1.5), we have $\bigcap_\lambda K_\lambda\neq\emptyset$. If ρ is in $\bigcap_\lambda K_\lambda$, then $f|A^{HN_\lambda}=g\rho|A^{HN_\lambda}$ for all λ in Λ. Since $A^H=\bigcup_\lambda A^{HN_\lambda}$, we know $f=g\rho|A^H$.

The following theorem will follow at once from Th. 1.6 and Cor. to Prop. 1.2.

Theorem 1.7. *Let A/B be locally finite outer G-Galois, and A an* *indecomposvble ring. Then $G^*=\hat{G}$, that is, G is dense in $\hat{G}$.*

Proposition 1.8. *Let A/B be locally finite G-Galois, and $G=G^*$* (cf. Cor. to Prop. 1.2). *Then there hold the following.*

(1) *For an intermediate ring T of A/B the following are equivalent.* (i) *$T=A^H$ for some subgroup H of G.* (ii) *There are subgroups H_γ $(\gamma\in\Gamma)$ of G such that $T=\cup_\gamma A^{H_\gamma}$, $(G:H_\gamma)<\infty$ and $\{A^{H_\gamma};\gamma\in\Gamma\}$ is a directed set with respect to the inclusion relation.*

(2) *If H is a subgroup of G such that $(G:H)<\infty$ then $(A^H)_B$ is finitely generated.*

Proof. Let $A=\cup_{\lambda\in\Lambda}A^{N_\lambda}$ be a representation of the locally finite G-Galois extension A/B. (1) (i)$\Longrightarrow$(ii) $T=A^H=\cup_\lambda(A^H\cap A^{N_\lambda})=\cup_\lambda A^{HN_\lambda}$ is a directed union, and $(G:HN_\lambda)<\infty$. (ii)$\Longrightarrow$(i) follows from Prop. 1.1. (2) By Prop. 1.3, $A^H\subseteq A^{N_\nu}$ for some ν in Λ. Then, $A^H=A^{HN_\nu}$ is a fixed subring of the finite G/N_ν-Galois extension A^{N_ν}/B, and therefore $(A^H)_{A^H}|(A^{N_\nu})_{A^H}$ (cf. [22; §2. p. 118]). Since $(A^{N_\nu})_B$ is finitely generated, $(A^H)_B$ is finitely generated.

Let T be an intermediate ring of A/B, and S a subset of A. T is called a *G-separable cover* of S if T satisfies the following conditions:

(1) T/B is a separable extension, and $T\supseteq S$.

(2) $G|T$ is finite.

(3) $G|T$ is strongly distinct (i.e. if $\sigma|T\neq\tau|T$ for σ, τ in G then $\sigma|T$ and $\tau|T$ are strongly distinct).

Theorem 1.9. *Let A/B be locally finite outer G-Galois, and T an intermediate ring of A/B. Then the following are equivalent:*

(i) *$T=A^H$ for some subgroup H of G such that $(G:H)<\infty$.*

(ii) *T/B is a separable extension, T_B is finitely generated, and $G|T$ is strongly distinct.*

(iii) *T is a G-separable cover of B.*

Proof. Let $A=\cup_{\lambda\in\Lambda}A^{N_\lambda}$ be a representation. (i)$\Longrightarrow$(ii) By Prop. 1.3, $T=A^H\subseteq A^{N_\nu}$ for some ν in Λ. Then T is a fixed subring of the finite G/N_ν-Galois extension A^{N_ν}/B. Then, by [19; Prop. 3.4], T/B is a separable extension. By Prop. 1.8 (2) (cf. Cor. to Prop. 1.2), T_B is finitely generated. By [22; Th. 2.6], $G|T$ is strongly distinct. (ii)$\Longrightarrow$(iii) This follows from the fact that $\{\sigma(x);\sigma\in G\}$ is finite for any x in A. (iii)$\Longrightarrow$(i) Let $\{(t_i, t_i^*); i=1,\cdots,n\}$ be a (B, T)-projective coordinate system of T/B. Then, by [22; Prop. 1.2], $\sum_i t_i\cdot\sigma(t_i^*)=\delta_{H,\sigma}$ for σ in G, where $H=G^T$. $\#(G|T)<\infty$ implies $(G:H)<\infty$. By Prop. 1.4, $A^H=T$.

Combining Th. 1.9 with Prop. 1.8, we obtain the following theorem (cf. [12; Th. 3], [28; Theorem]).

Theorem 1.10. *Let A/B be locally finite outer G-Galois, and $G=G^*$. Then, for an intermediate ring T of A/B, the following are equivalent.*

(i) *$T=A^H$ for some subgroup H of G.*

(ii) *For any finite subset F of T there is an intermediate ring T_0 of T/B such that $T_0 \supseteq F$, T_0/B is separable, T_{0_B} is finitely generated, and $G|T_0$ is strongly distinct.*

(iii) *Any finite subset of T has a G-separable cover which is contained in T.*

Next we shall proceed to the characterization of locally finite outer Galois extensions.

Proposition 1.11. *Let $V_A(B)=C$, T a G-separable cover of B, and $\{(t_i, t_i^*);\ i=1,\cdots,n\}$ a (B, T)-projective coordinate system for T/B, and put $H=G^T$. Then there hold the following.*

(1) *$\sum_i t_i\cdot\sigma(t_i^*)=\delta_{H,\sigma}$ for all σ in G.*

(2) *$A^H=T$, $(G:H)<\infty$, and T/B is a projective Frobenius extension.*

(3) *Let K be a subgroup of G containing H. Then, $\sum_i t_{K:H}(t_i)\sigma(t_i^*)=\delta_{K,\sigma}$ for all σ in G, T is (B, A^K)-projective, T/A^K is a projective Frobenius extension, and $G|A^K$ is strongly distinct. Further the following are equivalent. (α) $(A^K)_{A^K}|T_{A^K}$. (β) ${}_{(A^K)}(A^K)|_{(A^K)}T$. ($\gamma$) $t_{K:H}(c)=1$ for some c in T.*

Proof. (1) follows from [22; Prop. 1.2], and (2) is obvious by (1) and Prop. 1.4. (3) It will be easily seen that $\sum_i t_{K:H}(t_i)\sigma(t_i^*)=\delta_{K,\sigma}$ for all σ in G. Since $\sum_i tt_i\otimes t_i^*=\sum_i t_i\otimes t_i^* t$ ($\in T\otimes_B T$) for t in T, $\sum_i y\cdot t_{K:H}(t_i)\otimes t_i^*=\sum_i t_{K:H}(t_i)\otimes t_i^* y$ ($\in A^K\otimes_B T$) for all y in A^K. Hence the mapping $x\to\sum_i t_{K:H}(t_i)\otimes t_i^* x$ from T to $A^K\otimes_B T$ is an A^K-A^K-homomorphism. Since $\sum_i t_{K:H}(t_i)t_i^* x=x$, it follows that T is (B, A^K)-projective. Let $\rho|A^K\neq\tau|A^K$ for ρ, τ in G. Then $\tau^{-1}\rho\notin K$, and so $0=\tau(\sum_i t_{K:H}(t_i)\tau^{-1}\rho(t_i^*))=\sum_i\tau(t_{K:H}(t_i))\rho(t_i^*)$. Thus, by [22; Prop. 1.1], $\rho|A^K$ and $\tau|A^K$ are strongly distinct. If we set $G=K$ in Prop. 1.4, the remainder follows from Prop. 1.4.

Theorem 1.12. *Let $V_A(B)=C$. Then the following statements are equivalent.*

(i) *A/B is locally finite (outer) G-Galois.*

(ii) *For any finite subset F of A there is a G-invariant G-separable cover T of F such that ${}_BB|_BT$.*

(iii) *For any finite subset F of A there is a G-separable cover T of F which satisfies the following: If T_0 is an intermediate ring of T/B such that (α) T is (B, T_0)-projective, (β) T/T_0 is a projective Frobenius extension, (γ) $G|T_0$ is strongly distinct, then ${}_{T_0}T_0|_{T_0}T$.*

(iv) *For any finite subset F of A there is a G-separable cover T of F*

which satisfies the following: If T_0 is an intermediate ring of T/B such that (α) *T is* (B, T_0)*-projective,* (β) *T/T_0 is a projective Frobenius extension,* (γ) *$G|T_0$ is strongly distinct,* (δ) *T_0 is a G-invarinat fixed subring* (*with respect to G*), *then ${}_{T_0}T_0|_{T_0}T$.*

Proof. (i)$\Longrightarrow$(ii), (iii) Let $A=\cup_{\lambda\in\Lambda}A^{N_\lambda}$ be a representation of the locally finite G-Galois extension A/B. Then any finite subset F of A is contained in some A^{N_μ} ($\mu\in\Lambda$). By [22; Th. 1.5], A^{N_μ} is a G-invariant G-separable cover of F such that ${}_BB|_BA^{N_\mu}$. Let T_0 be an intermediate ring of A^{N_μ}/B such that A^{N_μ} is (B, T_0)-projective and that $G|T_0$ is strongly distinct. Then, by [22; Th. 2.6], T_0 is a fixed subring of the finite outer G/N_μ-Galois extension A^{N_μ}/B, whence ${}_{T_0}T_0|_{T_0}T$ by [22; §2. p. 118]. (ii)$\Longrightarrow$(i) Let F be a finite subset of A, and T a G-invariant G-separable cover of F such that ${}_BB|_BT$. If we put $N=G^T$, then $A^N=T$, $N\lhd G$ and $(G:N)<\infty$ (Prop. 1.11). By Prop. 1.11, A^N/B is a finite G/N-Galois extension. Noting that $(A^N)_B$ is finitely generated, A/B is a locally finite G-Galois extension. (iii)$\Longrightarrow$(iv) is trivial. (iv)$\Longrightarrow$(i) Let T_1 be a separable cover of an element x in A. Put $G^{T_1}=H_1$. Then $\#(G|T_1)<\infty$ implies $(G:H_1)<\infty$ and $\#\{\sigma(x);\ \sigma\in G\}<\infty$. Thus any finite subset of A is contained in a G-invariant finite subset of A. Let F be a G-invariant finite subset of A, and T a G-separable cover of F as that in (iv), and let $\{(t_i, t_i^*);\ i=1,\cdots,n\}$ be a (B, T)-projective coordinate system of T/B, and $H=G^T$. Then, by Prop. 1.11, $A^H=T$, $(G:H)<\infty$, and $\sum_i t_i\cdot\sigma(t_i^*)=\delta_{H,\sigma}$ for all σ in G. Set $N=G^F$. Then $H\subseteq N\lhd G$, and $F\subseteq A^N\subseteq A^H=T$. By Prop. 1.11, T is (B, A^N)-projective, T/A^N is a projecttive Frobenius extension, and $G|A^N$ is strongly distinct. Then, by the assumption for T, ${}_{(A^N)}(A^N)|_{(A^N)}T$, so that $t_{N:H}(c)=1$ for some c in T (Prop. 1.11 (3)). Put $t_i'=t_{N:H}(t_i)$ and $t_i^{*\prime}=t_{N:H}(t_i^*c)$. Then, t_i', $t_i^{*\prime}\in A^N$, and $\sum_i t_i'\cdot\sigma(t_i^*)=\delta_{N,\sigma}$ for all σ in G (Prop. 1.11 (3)). Further, as is easily seen, $\sum_i t_i'\cdot\sigma(t_i^{*\prime})=\delta_{N,\sigma}$ for all σ in G. Since ${}_BB|_BT$ (Prop. 1.11 (3)), we have ${}_BB|_BA^N$. Thus A^N/B is a finite G/N-Galois extension. Noting that $(A^N)_B$ is finitely generated, we conclude that A/B is a locally finite G-Galois extension.

Proposition 1.13. *Let $A^*\supseteq T\supseteq B^*$ be rings such that A^* is* (B^*, T)*-projective, A' an extension ring of B^* such that $V_{A'}(B^*)=V_{A'}(A')$, and $f_1,\cdots,f_s$ B^*-ring homomorphisms from A^* to A' such that $f_i|T$ and $f_k|T$* ($i\neq k$) *are strongly distinct. If $(B^*)^r_{B^*}\twoheadrightarrow T_{B^*}$ then $(A')^r_{A'}\twoheadrightarrow(A')^s_{A'}$.*

Proof. Let $\{(t_i, a_i^*);i=1,\cdots,n\}$ be a (B^*, T)-projective coordinate system for A^*. Then, by [22; Prop. 1.2], $\sum_i f_h(t_i)f_k(a_i^*)=\delta_{h,k}$ for all h, k. Let φ be a A'-right homomorphism from $T\otimes_{B^*}A'$ to $(A')^s_{A'}$ defined by $\varphi(t\otimes a')=(f_1(t)a',\cdots,f_s(t)a')$. Since $\sum_i f_h(t_i)f_k(a_i^*)=\delta_{h,k}$, φ is an epimorphism. $(B^*)^r_{B^*}\twoheadrightarrow T_{B^*}$ implies that $(A')^r_{A'}\twoheadrightarrow T\otimes_{B^*}A'_{A'}$. Hence we have $(A')^r_{A'}\twoheadrightarrow(A')^s_{A'}$, as desired.

Concerning Prop. 1.13, we consider the following condition.

Condition (F): If ${}_AA^r \twoheadrightarrow {}_AA^s$ for positive integers r, s, then $r \geqslant s$.

Remark. Let ${}_AA^r \twoheadrightarrow {}_AA^s$ for positive integers r, s. Then, since ${}_AA^s$ is projective, ${}_AA^s$ is isomorphic to an A-direct summand of ${}_AA^r$.

(1) If ${}_AA$ is finite dimensional, then $r \cdot \dim {}_AA \geqslant s \cdot \dim {}_AA$, and so $r \geqslant s$ (cf. [11]).

(2) Assume that there is a proper ideal $\mathfrak{A}$ of A such that ${}_AA/\mathfrak{A}$ is finite dimensional. Then, since ${}_AA^r/\mathfrak{A}^r \twoheadrightarrow {}_AA^s/\mathfrak{A}^s$, the above (1) yields $r \geqslant s$, because ${}_AA^r/\mathfrak{A}^r \simeq {}_A(A/\mathfrak{A})^r$ and ${}_AA^s/\mathfrak{A}^s \simeq {}_A(A/\mathfrak{A})^s$.

(3) If A is commutative, then $r \geqslant s$ by (2).

Proposition 1.14. *Let $V_A(B)=C$, and A an indecomposable ring satisfying* (F), *and let T be an intermediate ring of A/B, and S a subset of A. Then the following are equivalent*:

(i) *T is a G-separable cover of S.*

(ii) *$T \supseteq S$, T/B is a separable extension, and T_B is finitely generated.*

Proof. (i)$\Longrightarrow$(ii) is evident by Prop. 1.11. (ii)$\Longrightarrow$(i) By [22; Lemma 2.7], A is (B, T)-projective. Then, by Prop. 1.13, we have $\#(G|T) < \infty$, and hence T is a G-separable cover of S.

If A is commutative, then A satisfies (F). Therefore, by Th. 1.12, S. 3 and Prop. 1.14, we have the following

Theorem 1.15 (Nagahara [12]). *Let A be an indecomposable commutative ring. Then the following are equivalent.*

(i) *A/B is locally finite G-Galois.*

(ii) *For any finite subset F of A there is an intermediate ring T of A/B such that (α) T/B is a separable extension, anf T_B is finitely generated, (β) $T \supseteq F$.*

Proposition 1.16. *Let A/B be locally finite G-Galois, and H a subgroup of G. Then $G|A^H$ is strongly distinct.*

Proof. Let σ, τ be in G, and e a central idempotent of A such that $\sigma(x)e=\tau(x)e$ for all x in A^H. Let $A=\cup_{\lambda\in\Lambda}A^{N_\lambda}$ be a representation of the locally finite G-Galois extension A/B. We may assume that $e \in A^{N_\lambda}$ for all λ in Λ. Suppose that $\sigma|A^H \neq \tau|A^H$. Since $A^H=\cup_{\lambda\in\Lambda}A^{N_\lambda H}$, $\sigma|A^{N_\mu H} \neq \tau|A^{N_\mu H}$ for some μ in Λ. Then, by [22; Prop. 2.4], $(G/N_\mu)|A^{N_\mu H}$ is strongly distinct. Therefore we have $e=0$. Thus $G|A^H$ is strongly distinct.

Theorem 1.17. *Let A/B be locally finite outer G-Galois, and T an intermediate ring of A/B. Then the following are equivalent.*

(i) *$T=A^H$ for some subgroup H of G, and A_T is finitely generated.*

(ii) *$T=A^H$ for some subgroup H of G such that $(H:1)<\infty$.*

(iii) *A/T is a projective Frobenius extension, $\mathrm{Hom}(A_T, A_T)\subseteq\Delta$, and $G|T$ is strongly distinct.*

When any of the above conditions is satisfied A/A^H is finite H-Galois.

Proof. Let $A=\bigcup_{\lambda\in\Lambda}A^{N_\lambda}$ be a representation of the locally finite outer G-Galois extension A/B. (i)$\Longrightarrow$(ii) Let $A=x_1T+\cdots+x_rT$. Then $x_1,\cdots,x_r\in A^{N_\mu}$ for some $\mu\in\Lambda$, so that $A=A^{N_\mu}\cdot T=A^{N_\mu}\cdot A^H$. Hence $N_\mu\cap H=1$. Since $(G:N_\mu)<\infty$ we have $(H:1)<\infty$. (ii)$\Longrightarrow$(iii) By Prop. 1.3, $H\cap N_\mu=1$ for some $\mu\in\Lambda$. There are elements $a_1,\cdots,a_n$; $a_1^*,\cdots,a_n^*$ in A^{N_μ} such that $\sum_i a_i\cdot\sigma(a_i^*)=\delta_{N_\mu,\sigma}$ for all σ in G. Then $\sum_i a_i\cdot\sigma(a_i^*)=\delta_{1,\sigma}$ for all σ in H. Hence A/A^H is H-Galois. Therefore A/A^H is a projective Frobenius extension (cf. [22; p. 121]), and $\mathrm{Hom}(A_T, A_T)=\sum_{\tau\in H}Au_\tau\subseteq\Delta$. By Prop. 1.16, $G|T$ is strongly distinct. (iii)$\Longrightarrow$(i) Let $h=\sum_{\tau\in H}a_\tau u_\tau$ be a Frobenius homomorphism of A/T, where H is a finite subset of G and $a_\tau\neq 0$ for all τ in H. Then, since $th=ht$ for all t in T, we have $ta_\tau=a_\tau\cdot\tau(t)$ for all t in T, in particular, $ba_\tau=a_\tau b$ for all b in B. Hence $a_\tau\in V_A(B)=C$ for all τ in H. There are elements r_i, l_i in A such that $x=\sum_i h(xr_i)l_i=\sum_i r_i\cdot h(l_ix)$ for all x in A (cf. [27]). Then $u_1=\sum_i r_ihl_i=\sum_i r_i\sum_{\tau\in H}a_\tau\cdot\tau(l_i)u_\tau=\sum_{\tau\in H}\sum_i r_i\cdot a_\tau\cdot\tau(l_i)u_\tau$, and so $1=\sum_i r_ia_1l_i=a_1\sum_i r_il_i$. Thus a_1 is an invertible element in C, and $a_1^{-1}=\sum_i r_il_i$. Since H is finite there is an N_μ such that $\tau|A^{N_\mu}\neq\rho|A^{N_\mu}$ provided $\tau\neq\rho$ $(\tau,\rho\in H)$. Since A^{N_μ}/B is finite G/N_μ-Galois, there are elements d_k, e_k in A^{N_μ} such that $\sum_k d_k\cdot\sigma(e_k)=\delta_{N_\mu,\sigma}$ for all σ in G. Put $\Delta_0=\mathrm{Hom}(A_T, A_T)$. Then $\Delta_0=AhA$, and $\Delta_0\ni\sum_k\tau(d_k)he_k=\sum_{\sigma\in H}\sum_k\tau(d_k)a_\sigma\cdot\sigma(e_k)u_\sigma=a_\tau u_\tau$ for τ in H. Thus $\Delta_0=AhA=\sum_{\tau\in H}\oplus Aa_\tau u_\tau$. Since A/T is a projective Frobenius extension with Frobenius homomorphism h, ${}_AA\otimes_TA_A\simeq{}_A\Delta_{0A}$ by the correspondence $x\otimes y\rightarrow xhy$. Let φ be the A-left homomorphism from Δ to Δ_0 defined by $\varphi(\sum_\sigma x_\sigma u_\sigma)=\sum_{\tau\in H}x_\tau a_\tau u_\tau$, and ϕ the A-left homomorphism from Δ_0 to Δ defined by $\phi(xhy)=\sum_i x\cdot h(yr_i)vl_i$, where $v=\sum_{\tau\in H}u_\tau$. Then, as $h(tr_i)a_\tau=\tau h(yr_i)a_\tau$ $(\tau\in H)$, $\varphi\phi=1$. Since $a_\tau u_\tau=\sum_k\tau(d_k)he_k$, we have $\phi(a_\tau u_\tau)=\sum_k\sum_i\tau(d_k)h(e_kr_i)vl_i=\sum_{\nu,\rho\in H}\sum_k\sum_i\tau(d_k)a_\rho\cdot\rho(e_k)\rho(r_i)u_\nu l_i=\sum_{\nu\in H}\sum_i a_\tau\cdot\tau(r_i)u_\nu l_i$, and so $\varphi\phi(a_\tau u_\tau)=\sum_{\nu\in H}\sum_i a_\tau\cdot a_\nu\cdot\tau(r_i)\times\nu(l_i)u_\nu$. On the other hand, $\varphi\phi(a_\tau u_\tau)=a_\tau u_\tau$, and hence $a_\tau^2\cdot\tau(\sum_i r_il_i)=a_\tau$ for all τ in H. Since $a_1^{-1}=\sum_i r_il_i$, we have $a_\tau^2\cdot=a_\tau\cdot\tau(a_1)$. Noting that $\tau(a_1)$ is an invertible element of C, $Aa_\tau a_\tau=Aa_\tau\cdot\tau(a_1)=Aa_\tau$, and so $A=Aa_\tau+\mathrm{Ann}_A(a_\tau)$, where $\mathrm{Ann}_A(a_\tau)=\{x\in A\,;\,xa_\tau=0\}$. If $xa_\tau\in\mathrm{Ann}_A(a_\tau)$, then $0=xa_\tau^2=xa_\tau\cdot\tau(a_\tau)$, so that $xa_\tau=0$. Thus $A=Aa_\tau\oplus\mathrm{Ann}_A(a_\tau)$. Therefore Aa_τ is written as Ag_τ with a central idempotent g_τ of A. Since $Aa_\tau u_\tau\subseteq\Delta_0$, we have $g_\tau u_\tau\in\Delta_0$, and so $g_\tau\cdot t=g_\tau\tau(t)$ for all t in T. Consequently, $\Delta_0=\sum_{\tau\in H}\oplus Au_\tau$ and $H=G^T$. Hence End $({}_{\Delta_0}A)=(A^H)_r$ the right multiplications of elements of A. Since $a_\tau u_\tau\in\Delta_0=\mathrm{End}(A_T)$, we have $a_\tau u_\tau\in\mathrm{End}(A_{(A^H)})$. Noting that a_τ is in C, we can easily seen that $a_\tau u_\tau \in \mathrm{Hom}({}_{(A^H)}A_{(A^H)}, {}_{(A^H)}A_{(A^H)})$. Thus $h=\sum_{\tau\in H}a_\tau u_\tau \in \mathrm{Hom}({}_{(A^H)}A_{(A^H)}, {}_{(A^H)}A_{(A^H)})$. Then, by [27; Cor. 1], A/A^H is also a projective Frobenius extension with a Frobenius homomorphism h. Since $(H:1)<\infty$, there is an N_λ such that $H\cap N_\lambda=1$ (Prop. 1.3 (2)). Then $A^{HN_\lambda}\subseteq A^{N_\lambda}$, and $H\simeq HN_\lambda/N_\lambda$ canonically. Therefore there is an element c in A^{N_λ} such that $t_H(c)=1$ (cf. [22; §2. p. 118]), which implies $(A^H)_{(A^H)}|A_{(A^H)}$, because the A^H-right homomorphism $x\to t_H(cx)$ $(x\in A)$ from A to A^H splits. Therefore there is an element d in A such that $h(d)=1$. Then, for any x in A^H, $T\ni h(dx)=h(d)x=x$. Thus we obtain $T=A^H$. as desired.

Theorem 1.18. *Let A/B be finite outer G-Galois, and T an intermediate ring of A/B. Then the following are equivalent.*

(i) *$T=A^H$ for some subgroup H of G.*

(ii) *A/T is a projective Frobenius extension, and $G|T$ is strongly distinct.*

(iii) *T/B is a separable extension, and $G|T$ is strongly distinct.*

Proof. (i)$\Longleftrightarrow$(ii) is evident from Th. 1.17. (i)$\Longrightarrow$(iii) follows from [22; Th. 2.6] and [19; Prop. 3.4]. (iii)$\Longrightarrow$(i) follows from [22; Th. 2.6 and Lemma 2.7].

§2. Heredity of locally finite Galois extensions.

Let A_0 be a G^*-invariant subring of A such that the mapping $\sigma\to\sigma|A_0$ $(\sigma\in G^*)$ is one-to-one and such that A_0/A_0^G is a locally finite G-Galois extension, and let G^* be compact (as an automorphism group of A). Put $B_0=A_0^G$, and let $A_0=\bigcup_{\lambda\in\Lambda}A_0^{N_\lambda}$ be a representation of the locally finite G-Galois extension A_0/B_0. Then G/N_λ may be considered as a finite group of automorphisms of A^{N_λ}. And, by [22; Th. 5.1 and §2. p. 118], $A^{N_\lambda}=A_0^{N_\lambda}\otimes_{B_0}B$, A^{N_λ}/B is finite G/N_λ-Galois. Since $\bigcup_\lambda A^{N_\lambda}$ is a directed union, the compactness of G^* implies that $\bigcup_\lambda A^{N_\lambda}(\supseteq A_0)$ is a fixed subring of A with respectto G^* (Prop. 1.1), so that $A=\bigcup_\lambda A^{N_\lambda}$, because $\sigma\to\sigma|A_0$ $(\sigma\in G^*)$ is 1–1. Thus A/B is locally finite G-Galois. Let H be any subgroup of G. Then, $A^H=\bigcup_\lambda(A^H\cap A^{N_\lambda})=\bigcup_\lambda A^{HN_\lambda}$. By [22; Th. 5.1], $A^{HN_\lambda}=(A_0^{N_\lambda})^{HN_\lambda/N_\lambda}\otimes_{B_0}B=A_0^{HN_\lambda}\otimes_{B_0}B$. Hence $A^H=\bigcup_\lambda(A_0^{HN_\lambda}\cdot B)=A_0^H\cdot B$, and $A_0^H\otimes_{B_0}B\twoheadrightarrow A_0^H\cdot B=A^H$ canonically. Since the isomorphism $A_0^{HN_\lambda}\otimes_{B_0}B\simeq A_0^{HN_\lambda}\cdot B(\subseteq A^H)$ may be considered as $A_0^{HN_\lambda}\otimes_{B_0}B\to A_0^H\otimes_{B_0}B\to A^H$, we know $A^H=A_0^H\otimes_{B_0}B$. Symmetrically we obtain $A^H=B\otimes_{B_0}A_0^H$. Next we consider the set of all A_0G-left submodules of A and the set of all B_0-left submodules of B. Let $\bar{X}$ be any A_0G-left submodule of A. Then $\bar{X}\cap A^{N_\lambda}$ is an $A_0^{N_\lambda}(G/N_\lambda)$-left submodule of A^{N_λ}. Therefore, by [22; Th. 5.1], we have $\bar{X}\cap A^{N_\lambda}=A_0^{N_\lambda}((\bar{X}\cap A^{N_\lambda})\cap B)=A_0^{N_\lambda}\otimes_{B_0}(\bar{X}\cap B)$, so that $\bar{X}=\bigcup_\lambda(\bar{X}\cap A^{N_\lambda})=\bigcup_\lambda(A_0^{N_\lambda}(\bar{X}\cap B))=A_0(\bar{X}\cap B)$. Since $A_0^{N_\lambda}\otimes_{B_0}(\bar{X}\cap B)\simeq A_0^{N_\lambda}(\bar{X}$

$\cap B)\subseteq \overline{X}$ for all λ, we have $\overline{X}=A_0\otimes_{B_0}(\overline{X}\cap B)$. Evidently $\overline{X}\cap B$ is a B_0-left submodule of B. Let X be any B_0-left submodule of B. Then, as is easily seen, A_0X is an A_0G-left submodule of A, and $A_0X=\cup_\lambda A_0^{N_\lambda}\cdot X$. By [22; Th. 5.1], $A_0^{N_\lambda}\cdot X\cap B=X$ for all λ in Λ, so that $A_0X\cap B=\cup_\lambda(A_0^{N_\lambda}X\cap B)=X$. If $\overline{Y}$ is a G-invariant intermediate ring of A/A_0, then $\overline{Y}\cap B$ is an intermediate ring of B/B_0, and $\overline{Y}=A_0(\overline{Y}\cap B)$. Symmetrically we have $\overline{Y}=(\overline{Y}\cap B)A_0$. If Y is an intremediate ring of B/B_0 such that $A_0Y=YA_0$, then A_0Y is a G-invariant intermediate ring of A/A_0. Since $A=\cup_\lambda A^{N_\lambda}$, we have $\overline{Y}=\cup_\lambda(\overline{Y}\cap A^{N_\lambda})=\cup_\lambda\overline{Y}^{N_\lambda}$, and $\overline{Y}^{N_\lambda}/(\overline{Y}\cap B)$ is finite G/N_λ-Galois ([22; Th. 5.1]. Hence $\overline{Y}/(\overline{Y}\cap B)$ is locally finite G-Galois. Thus we have obtained the following.

Theorem 2.1. *Let A_0 be a G^*-invariant subring of A such that $\sigma\to\sigma|A_0$ $(\sigma\in G^*)$is 1–1 and such that A_0/B_0 is locally finite G-Galois where $B_0=A_0^G$, and let G^* be compact. Then there hold the following*:

(1) *A/B is locally finite G-Galois.*

(2) *$A^H=B\otimes_{B_0}A_0^H=A_0^H\otimes_{B_0}B$ for any subgroup H of G. In particular, $A=B\otimes_{B_0}A_0=A_0\otimes_{B_0}B$.*

(3) *Let $\{\overline{X}\}$ and $\{X\}$ be the set of all A_0G-left submodules of A and the set of all B_0-left submodules of B, respectively. Then, $\overline{X}\to\overline{X}\cap B$ and $X\to A_0X=A_0\otimes_{B_0}X$ are mutually converse order isomorphisms between $\{\overline{X}\}$ and X.*

(4) *Let $\{\overline{Y}\}$ and $\{Y\}$ be the set of all G-invariant intermediate rings of A/A_0 and the set of all intermediate rings of B/B_0 such that $A_0Y=YA_0$, respectively. Then $\overline{Y}/(\overline{Y}\cap B)$ is locally finite G-Galois, and $\overline{Y}\to\overline{Y}\cap B$ and $Y\to A_0Y=YA_0$ are mutually converse order isomorphisms between $\{\overline{Y}\}$ and $\{Y\}$.*

Let A, A' be R-algebras such that $A\otimes_R A'\neq 0$. Assume that A/B is a locally finite G-Galois extension such that $R\cdot 1\subseteq B$, and assume that A' is a locally finite G'-Galois extension such that $R\cdot 1\subseteq B'$. Then each $\sigma\times\tau$ in $G\times G'$ induces an automorphism of $A\otimes_R A'$. Let $A=\cup_\alpha A^{N_\alpha}$ and $A'=\cup_\beta A'^{N'_\beta}$ be representations of A/B and A'/B' respectively. Then, by [22; Th. 5.2], $(A^{N_\alpha}\otimes_R A'^{N'_\beta})/(B\otimes B')$ is a finite $(G/N_\alpha)\times(G/N'_\beta)$-Galois extension. Let $\varphi_{\alpha\beta}$ be the canonical R-algebra homomorphism from $A^{N_\alpha}\otimes_R A'^{N'_\beta}$ to $A^{N_\alpha}\otimes A'^{N'_\beta}$ $(\subseteq A\otimes_R A')$. We put $(A\otimes_R A'\supseteq)$ $A^{N_\alpha}\otimes A'^{N'_\beta}=A_{\alpha\beta}$ and $(A\otimes_R A'\supseteq)$ $B\otimes B'=B^*$. To be easily seen, Ker $\varphi_{\alpha\beta}$ is a $(G/N_\alpha)\times(G'/N'_\beta)$-invariant ideal of $A^{N_\alpha}\otimes_R A'^{N'_\beta}$. Hence $A_{\alpha\beta}/B^*$ is $(G/N_\alpha)\times(G'/N'_\beta)$-Galois ([22; Th. 5.6]). There are elements c and c' in A^{N_α} and $A'^{N'_\beta}$ respectively such that $t_{G/N_\alpha}(c)=1$ and $t_{G'/N'_\beta}(c')=1$. Then $c\otimes c'\in A_{\alpha\beta}$ and $t_{(G/N_\alpha)\times(G'/N'_\beta)}(c\otimes c')=1\otimes 1$. Hence $A_{\alpha\beta}/B^*$ is a finite $(G/N_\alpha)\times(G'/N'_\beta)$-Galois extension, and $\{\sigma\times\tau\in G\times G';\ \sigma\times\tau\,|\,A_{\alpha\beta}=1_{A_{\alpha\beta}}\}=N_\alpha\times N'_\beta$. Since $\cap_{\alpha,\beta}(N_\alpha\times N'_\beta)=(\cap_\alpha N_\alpha)\times(\cap_\beta N'_\beta)=1$, $G\times G'$ may be considered as a group of automorphisms of $A\otimes_R A'$. Let H and H' be subgroups of G and G', respectively. Then, $(A\otimes_R A')^{H\times H'}=\cup_{\alpha,\beta}A_{\alpha,\beta}^{H\times H'}=\cup_{\alpha,\beta}(A^{N_\alpha H}\otimes A'^{N'_\beta H'})=(\cup_\alpha A^{N_\alpha H})\otimes(\cup_\beta A'^{N'_\beta H'})=A^H\otimes A'^{H'}$ by [22; Th. 5.2]. In particular $(A\otimes_R A')^{N_\alpha\times N'_\beta}=A^{N_\alpha}\otimes A'^{N'_\beta}=A_{\alpha\beta}$, and evidently $(G\times G': N_\alpha\times N'_\beta)<\infty$. Since $A\otimes_R A'=\cup_{\alpha,\beta}A^{N_\alpha}\otimes A'^{N'_\beta}$ is a directed union, $A\otimes_R A'/B\otimes B'$ is a locally finite $G\times G'$-Galois extension. Let $a\in A$ and $a'\in A'$. Then it is evident that $\{\sigma\times\tau\in G\times G'; \sigma(a)\otimes\tau(a')=a\otimes a'\}\supseteq\{\sigma\in G; \sigma(a)=a\}\times\{\tau\in G'; \tau(a')=a'\}$. Put $\{\sigma\in G; \sigma(a)=a\}=K$ and $\{\tau\in G'; \tau(a')=a'\}=K'$. Then $A^K\subseteq A^{N_\alpha}$ and $A'^{K'}\subseteq A'^{N'_\beta}$ for some α, β (Prop. 1.3), so that $N_\alpha\subseteq K$ and $N'_\beta\subseteq K'$. By [22; Th. 5.2], $(G/N_\alpha\times G'/N'_\beta)^{A^K\otimes A'^{K'}}=K/N_\alpha\times K'/N'_\beta$, and hence $(G\times G')^{A^K\otimes A'^{K'}}=K\times K'$. Since $(A^K)_B$ and $(A'^{K'})_{B'}$ are finitely generated, $(A^K\otimes A'^{K'})_{B\otimes B'}$ is finitely generated. Hence the finite topology of $G\times G'$ with respect to $A\otimes_R A'$ is the product topology of the finite topology of G with respect to A and the finite topology of G' with respect to A'. Thus we have proved the following

Theorem 2.2. *Let A andA' be R-algebras such that $A\otimes_R A'\neq 0$. If A/B is a locally finite G-Galois extension such that $R\cdot 1\subseteq B$, and A'/B' is a locally finite G'-Galois extension such that $R\cdot 1\subseteq B'$, then $(A\otimes_R A')/(B\otimes B')$ is a locally finite $G\times G'$-Galois extension, and $(A\otimes_R A')^{H\times H'}=A^H\otimes A'^{H'}$ for any subgroup H of G and any subgroup H' of G'. The finite topology of $G\times G'$ with respect to $A\otimes_R A'$ is the product topology of the finite topology of G with respect to A and the finite topology of G' with respect to A'.*

Corollary. *Let A/B be a locally finite G-Galois extension such that $B\subseteq C$, and A' a B-algebra such that $A\otimes_B A'\neq 0$. Then $(A\otimes_B A')/(1\otimes A')$ is a locally finite G-Galois extension, and $(A_B\otimes A')^H=A^H\otimes A'$ for any subgroup H of G.*

Proposition 2.3. *Let A/B be locally finite G-Galois, and $G=G^*$. If H and K are closed subgroups of G, then $A^{H\cap K}=A^H\cdot A^K=A^K\cdot A^H$. In particular, if $H\cap K=1$ then $A=A^H\cdot A^K=A^K\cdot A^H$.*

Proof. Let $A=\cup_{\lambda\in\Lambda}A^{N_\lambda}$ be a representation of the locally finite G-Galois extension A/B. First we assume that $(G:K)<\infty$. Then, by Prop. 1.3, $A^K\subseteq A^{N_\mu}$ for some $\mu\in\Lambda$. Since $(A^{N_\mu})_B$ is finitely generated and $(A^K)_{A^K}$ is a direct summand of $(A^{N_\mu})_{A^K}$ ([22; §2. p. 118]), $(A^K)_B$ is finitely generated. Therefore we may assume that $A^K\subseteq A^{N_\lambda}$ for all $\lambda\in\Lambda$. Then $N_\lambda\subseteq K$ for $\lambda\in\Lambda$, and $A^H\cdot A^K=(\cup_{\lambda\in\Lambda}A^{N_\lambda H})(\cup_{\mu\in\Lambda}A^{N_\mu K})=\cup_{\lambda\in\Lambda}(A^{N_\lambda H}\cdot A^{N_\lambda K})=\cup_\lambda A^{N_\lambda H\cap N_\lambda K}=\cup_\lambda A^{N_\lambda H\cap K}$ by [22; Prop. 5.3]. Since $N_\lambda H\cap K=N_\lambda(H\cap K)$ for all λ, we have $A^H\cdot A^K=\cup_\lambda A^{N_\lambda(H\cap K)}=A^{H\cap K}$. Next we return to general case· For any finite subset F of A^K, we put $K_F=\{\sigma\in G; \sigma|F=1_F\}$. Then $(G:K_F)<\infty$, $A^{K_F}\subseteq A^K$, and $(A^{K_F})_B$ is finitely generated. Therefore $A^K=\cup_F A^{K_F}$ is a directed union, and hence $A^H\cdot A^K=A^H(\cup_F A^{K_F})=\cup_F(A^H\cdot A^{K_F})$ is also a directed union. Since each $A^H\cdot A^{K_F}$ $(=A^{H\cap K_F})$ is a fixed subring of A, $A^H\cdot A^K$ is a fixed subring of A (Prop. 1.1). Hence, as is easily seen, $A^H\cdot A^K=A^{H\cap K}$. Symmetrically we have $A^{H\cap K}=A^K\cdot A^H$.

Corollary. *Let A/B be locally finite G-Galois, $G=G^*$, and H_γ $(\gamma\in\Gamma)$ be closed subgroups of G. Then, $[\cup_\gamma A^{H_\gamma}]=A^{\cap H_\gamma}$, where $[\cup_\gamma A^{H_\gamma}]$ means the subring of A generated by $\cup_\gamma A^{H_\gamma}$.*

Proof. Evidently $[\cup_\gamma A^{H_\gamma}]=\cup[A^{H_{\gamma_1}}\cup\cdots\cup A^{H_{\gamma_n}}]$, where $\{\gamma_1,\cdots,\gamma_n\}$ ranges over all finite subsets of Γ. By Prop. 2.3, $A^{H_{\gamma_1}\cap\cdots\cap H_{\gamma_n}}=A^{H_{\gamma_1}}\cdots A^{H_{\gamma_n}}=[A^{H_{\gamma_1}}\cup\cdots\cup A^{H_{\gamma_n}}]$, and therefore $[\cup_\gamma A^{H_\gamma}]$ is a directed union of fixed subrings of A. Hence, by Prop. 1.1, $[\cup_\gamma A^{H_\gamma}]$ is a fixed subring. Since $\{\sigma\in G;\ \sigma|[\cup_\gamma A^{H_\gamma}]=1\}=\cap_\gamma H_\gamma$, we obtain $[\cup_\gamma A^{H_\gamma}]=A^{\cap_\gamma H_\gamma}$, as desired.

Proposition 2.4. *Let A/B be locally finite G-Galois, $\mathfrak{A}$ a G-invariant proper ideal of A, K a closed subgroup of G, and N a closed normal subgroup of G such that $(G:N)<\infty$. Then there hold the following*:

(1) *$A^{K\cap N}/A^K$ is finite $K/(K\cap N)$-Galois. In particular, A^N/B is finite G/N-Galois.*

(2) *$((A^N+\mathfrak{A})/\mathfrak{A})/((B+\mathfrak{A})/\mathfrak{A})$ is finite G/N-Galois, and $((A^N+\mathfrak{A})/\mathfrak{A})^H=(A^{NH}+\mathfrak{A})/\mathfrak{A}$ for any subgroup H of G.*

Proof. Let $A=\cup_{\lambda\in\Lambda}A^{N_\lambda}$ be a representation of the locally finite G-Galois extension A/B. (1) By Prop. 1.3, $A^N\subseteq A^{N_\mu}$ for some $\mu\in\Lambda$, and then $N_\mu\subseteq N$, $A^N=(A^{N_\mu})^{N/N_\mu}$. Therefore, by [22; Prop. 5.7], A^N/B is finite $(G/N_\mu)/(N/N_\mu)$-Galois, or equivalently, finite G/N-Galois. Accordingly, A^N/A^{NK} is finite NK/N-Galois, or equivalently, finite $K/(K\cap N)$-Galois. $K/(K\cap N)$ may be considered as a finite group of automorphisms of $A^{K\cap N}$, because $K\cap N\lhd K$. Then $A^{K\cap N}/A^K$ is finite $K/(K\cap N)$-Galois. (2) By (1), A^N/B is finite G/N-Galois. If $t_{G/N}(c)=1$ for c in A^N, then $t_{G/N}(c+\mathfrak{A})=1+\mathfrak{A}$. Then, by [22; Th. 5.6], $((A^N+\mathfrak{A})/\mathfrak{A})/((B+\mathfrak{A})/\mathfrak{A})$ is finite G/N-Galois, and $((A^N+\mathfrak{A})/\mathfrak{A})^H=(A^{NH}+\mathfrak{A})/\mathfrak{A}$ for any subgroup H of G.

Let A/B be locally finite G-Galois, K a closed subgroup of G,N a closed normal subgroup of G, and $\mathfrak{A}$ a G-invariant proper ideal of A. Let $A=\cup_{\lambda\in\Lambda}A^{N_\lambda}$ be a representation of the locally finite G-Galois extension A/B. Then $A^N=\cup_\lambda(A^N\cap A^{N_\lambda})=\cup_\lambda A^{NN_\lambda}$ is a directed union, and each NN_λ is a closed normal subgroup of G, because $(G:N_\lambda)<\infty$. Then, by Prop. 2.4 (1), A^{NN_λ}/B is finite G/NN_λ-Galois. Therefore there are elements $a_1,\cdots,a_m;\ b_1,\cdots,b_m$ in A^{NN_λ} such that $\sum_i a_i\cdot\sigma(b_i)=\delta_{NN_\lambda,\sigma}$ for σ in G. Hence A^{NN_λ}/B is finite $(G/N)/((NN_\lambda)/N)$-Galois. Hence A^N/B is locally finite G/N-Galois. Next we consider K. $A=\cup_\lambda A^{N_\lambda\cap K}$ is a directed union, and each $N_\lambda\cap K$ is a fixed normal subgroup of K such that $(K:N_\lambda\cap K)<\infty$. By Prop. 2.4 (1), each $A^{N_\lambda\cap K}/A^K$ is finite $K/(N_\lambda\cap K)$-Galois. Hence A/A^K is locally finite K-Galois. Finally we consider $\mathfrak{A}$. Evidently, $A/\mathfrak{A}=\cup_\lambda((A^{N_\lambda}+\mathfrak{A})/\mathfrak{A})$. By Prop. 2.4 (2), $((A^{N_\lambda}+\mathfrak{A})/\mathfrak{A})/((B+\mathfrak{A})/\mathfrak{A})$ is finite G/N_λ-Galois, and $((A^{N_\lambda}+\mathfrak{A})/\mathfrak{A})^H=(A^{N_\lambda H}+\mathfrak{A})/\mathfrak{A}$ for any subgroup H of G. Therefore $(A/\mathfrak{A})^H=\cup_\lambda((A^{N_\lambda}+\mathfrak{A})/\mathfrak{A})^H=\cup_\lambda(A^{N_\lambda H}+\mathfrak{A})/\mathfrak{A}=(A^H+\mathfrak{A})/\mathfrak{A}$ for any subgroup H of G. Hence $((A+\mathfrak{A})/\mathfrak{A})/((B+\mathfrak{A})/\mathfrak{A})$ is locally finite G-Galois. Thus we have proved the following

Theorem 2.5. *Let A/B be locally finite G-Galois, N a closed normal subgroup of G, K a closed subgroup of G, and $\mathfrak{A}$ a G-invariant proper ideal of A. Then there hold the following:*

(1) *A^N/B is locally finite G/N-Galois.*

(2) *A/A^K is locally finite K-Galois.*

(3) *$((A+\mathfrak{A})/\mathfrak{A})/((B+\mathfrak{A})/\mathfrak{A})$ is locally finite G-Galois, and $((A+\mathfrak{A})/\mathfrak{A})^H=(A^H+\mathfrak{A})/\mathfrak{A}$ for any subgroup H of G.*

Corollary. *Let A/B be locally finite G-Galois, and e a non-zero idempotent in $B\cap C$. Then Ae/Be is locally finite G-Galois, and $(Ae)^H=A^H\cdot e$ for any subgroup H of G.*

Let A/B be locally finite G-Galois, n a positive integer, and J the ring of rational integers. Then, $(J)_n$ is a J-algebra, and $(J)_n\otimes_J A\simeq(A)_n\neq 0$. If we define $\sigma((a_{ik}))=(\sigma(a_{ik}))$ for any σ in G and any (a_{ik}) in $(A)_n$, then $(A)_n/(B)_n$ is locally finite G-Galois and $((A)_n)^H=(A^H)_n$ for any subgroup H of G (Th. 2.2). Now, let $\{e_{ik};\ i,k=1,\cdots,m\}$ a system of matrix units contained in B, and $A=\cup_{\lambda\in\Lambda}A^{N_\lambda}$ a representation of A/B. Put $A_0=V_A(\{e_{ik}\})$ and $B_0=B\cap A_0$. Then, as is well known, $A=\sum_{i,k}\oplus A_0e_{ik}$, $A_0\simeq A_0e_{ik}$ by the right multiplication of e_{ik}. To be easily seen, $A^{N_\lambda}=\sum_{i,k}A_0^{N_\lambda}e_{ik}$, and $A_0^{N_\lambda}=V_{A^{N_\lambda}}(\{e_{ik}\})$. There is an element c in A^{N_λ} such that $t_{G:N_\lambda}(c)=1$. Let $c=\sum_{i,k}x_{ik}e_{ik}$ $(x_{ik}\in A_0^{N_\lambda})$. Then $1=t_{G:N_\lambda}(c)=\sum_{i,k}t_{G:N_\lambda}(x_{ik})e_{ik}$, and so $t_{G:N_\lambda}(x_{11})=1$. Thus, by [22; Th. 5.8], $A_0^{N_\lambda}/B_0$ is finite G/N_λ-Galois. Since $A_0=\cup_\lambda A_0^{N_\lambda}$ is a directed union, A_0/B_0 is locally finite G-Galois. Therefore, by Th. 2.1, $A=A_0\otimes_{B_0}B$. Thus we have obtained the following

Theorem 2.6. *Let A/B be locally finite G-Galois.*

(1) *For any positive integer n, $(A)_n/(B)_n$ is locally finite G-Galois, and $((A)_n)^H=(A^H)_n$ for any subgroup H of G.*

(2) *If $\{e_{ik};\ i,k=1,\cdots,m\}$ is a system of matrix units contained in B, $A_0=V_A(\{e_{ik}\})$, and $B_0=B\cap A_0$, then A_0/B_0 is locally finite G-Galois, and $A=A_0\otimes_{B_0}B$.*

Let A/B be finite G-Galois, and M a Δ-left module. For any subgroup H of G, we put $M^H=\{m\in M;\ u_\tau m=m$ for all $\tau\in H\}$, which is an A^H-

submodule of M. Evidently $M^H \supseteq A^H \cdot M^G$, and the mapping $\varphi: A^H \otimes_B M^G \to M^H$ defined by $a \otimes m \to am$ ($a \in A$, $m \in M^G$) is an A^H-left homomorphism. By assumption there are elements $a_1, \cdots, a_n$; $a_1^*, \cdots, a_n^*$, d in A such that $\sum_i a_i \cdot \sigma(a_i^*) = \delta_{1,\sigma}$ ($\sigma \in G$), $t_H(d)=1$. Put $t_i = t_H(a_i)$. Then, $t_i \in A^H$ and $\sum_i t_i \cdot \sigma(a_i^*) = \delta_{H,\sigma}$ for σ in G. If m is in M^H, then $A^H \cdot M^G \ni t_i \sum_{\sigma \in G} u_\sigma(a_i^* dm) = \sum_i t_i \sum_{\sigma \in G} \sigma(a_i^* d) u_\sigma m = t_H(d)m = m$. Hence φ is an epimorphism. If $a \in A^H$ and $m_0 \in M^G$, then $\sum_i t_i \otimes \sum_{\sigma \in G} u_\sigma(a_i^* dam_0) = \sum_i t_i \otimes \sum_{\sigma \in G} \sigma(a_i^* da) m_0 = \sum_i t_i \sum_{\sigma \in G} \sigma(a_i^* da) \otimes m_0 = t_H(da) \otimes m_0 = a \otimes m_0$. From this fact, as is easily seen, φ is 1–1. Thus we have $M^H = A^H \otimes_B M^G$. Next we proceed to more general case.

Let A/B be locally finite G-Galois, $A = \cup_{\lambda \in \Lambda} A^{N_\lambda}$ its representation, and M a Δ-left module. Let $G = \sigma_1 N_\lambda \cup \cdots \cup \sigma_r N_\lambda$ be the coset decomposition of G, and let Δ_λ be the trivial crossed product of A^{N_λ} with G/N_λ: $\Delta_\lambda = \sum_i \oplus A^{N_\lambda} v_{\bar{\sigma}_i}$, $v_{\bar{\sigma}_i} v_{\bar{\sigma}_k} = v_{\overline{\sigma_i \sigma_k}}$, $v_{\bar{\sigma}_i} a = \sigma_i(a) v_{\bar{\sigma}_i}$ ($\bar{\sigma}_i = \sigma_i N$, $\bar{\sigma}_k = \sigma_k N \in G/N_\lambda$, $a \in A^{N_\lambda}$). For any m in M^{N_λ} and any $\sum_i x_i v_{\bar{\sigma}_i}$ in Δ_λ, we define $\sum_i x_i v_{\bar{\sigma}_i}(m) = \sum_i x_i u_{\sigma_i} m$. Then, as is easily seen, M^{N_λ} is a Δ_λ-left module. Since A^{N_λ}/B is finite G/N_λ-Galois, we obtain that $M^{N_\lambda} = A^{N_\lambda} \otimes_B M^G$ and $M^{N_\lambda H} = A^{N_\lambda H} \otimes_B M^G$ for any subgroup H of G. Since $A = \cup_\lambda A^{N_\lambda}$ is a directed union, so is $\cup_\lambda M^{N_\lambda}$. For any subgroup H of G, $(\cup_\lambda M^{N_\lambda})^H = \cup_\lambda M^{N_\lambda H} = \cup_\lambda A^{N_\lambda H} \cdot M^G = A^H \cdot M^G$, and $A^{N_\lambda H} \otimes_B M^G \simeq A^{N_\lambda H} \cdot M^G$ canonically. The last isomorphism may be considered as $A^{N_\lambda H} \otimes_B M^G \to A^H \otimes_B M^G \to A^H \cdot M^G$, and hence we see that $(\cup_\lambda M^{N_\lambda})^H = A^H \otimes_B M^G$. For any m in M we put $H_m = \{\sigma \in G; u_\sigma m = m\}$, which is a subgroup of G. Assume that $(G : H_m) < \infty$ and that H_m is closed in G. Then, by Prop. 1.3, $H_m \supseteq N_\nu$ for some $\nu \in \Lambda$, so that $m \in M^{N_\nu}$. Conversely, if m is in $\cup_\lambda M^{N_\lambda}$, then $m \in M^{N_\mu}$ for some N_μ, so that $H_m \supseteq N_\mu$. Then, since $(G : N_\mu) < \infty$ and N_μ is closed in G, $(G : H_m) < \infty$ and H_m is closed in G. Thus we have proved the following

Theorem 2.7. *Let A/B be locally finite G-Galois, and M a Δ-left module. Then there hold the following*:

(1) *$A \cdot M^G$ is a Δ-submodule of M, and $(A \cdot M^G)^H = A^H \otimes_B M^G$ for any subgroup H of G.*

(2) *$A \cdot M^G = \{m \in M; (G : H_m) < \infty$ and H_m is closed in $G\}$, where $H_m = \{\sigma \in G; u_\sigma m = m\}$.*

Corollary. *Let A/B be finite G-Galois, and M a Δ-left module. Then, $M^H = A^H \otimes_B M^G$ for any subgroup H of G, in particular, $M = A \otimes_B M^G$* (cf. [4; Th. 1.3] and [22; Th. 5.1 (2)]).

Proposition 2.8. *Let A/B be finite G-Galois. Then the following are equivalent.*

(i) *There are elements $a_1, \cdots, a_n$; $a_1^*, \cdots, a_n^*$ in $\mathrm{V}_A(B)$ such that $\sum_i a_i \cdot \sigma(a_i^*) = \delta_{1,\sigma}$ ($\sigma \in G$)* (cf. [22; Cor. to Th. 5.1]).

(ii) ${}_BA_B|{}_BB_B$.

Proof. Since $(\Delta\supseteq)$ $(\sum_\sigma u_\sigma)A\simeq \mathrm{Hom}(A_B, B_B)$ by j, it follows that $(\sum_\sigma u_\sigma)$ $\mathrm{V}_A(B)\simeq\mathrm{Hom}({}_BA_B, {}_BB_B)$, and it is evident that $\mathrm{V}_A(B)\simeq\mathrm{Hom}({}_BB_B, {}_BA_B)$ canonically. To be easily seen, ${}_BA_B|{}_BB_B$ if and only if there are elements $f_1,\cdots,f_n$ in $\mathrm{Hom}({}_BA_B, {}_BB_B)$ and $g_1,\cdots,g_n$ in $\mathrm{Hom}({}_BB_B, {}_BA_B)$ such that $\sum_i g_i f_i(x)=x$ for all x in A. Consequently (ii) is equivalent to that $u_1=\sum_i a_i(\sum_\sigma u_\sigma)a_i^*$ $(=\sum_\sigma\sum_i a_i\cdot\sigma(a_i^*)u_\sigma)$ for some $a_1,\cdots,a_n$; $a_1^*,\cdots,a_n^*$ in $\mathrm{V}_A(B)$. Hence (i) and (ii) are equivalent.

Corollary. *Let G be finite. Then the following are equivalent.*

(i) *A/B is outer G-Galois, and ${}_BA_B|{}_BB_B$.*

(ii) *There are elements $a_1,\cdots,a_n$; $a_1^*,\cdots,a_n^*$ in C such that $\sum_i a_i\cdot\sigma(a_i^*)$ $=\delta_{1,\sigma}$ $(\sigma\in G)$.*

Proof. This follows from [22; Prop. 6.4 and Prop. 6.5] and Prop. 2.8.

A/B is called a *completely outer G-Galois extension* if G is finite and completely outer (cf. [22]).

Theorem 2.9. *Let B' be a ring with identity, Z its center, and G' a finite group.*

(1) *If A'/B' is completely outer G'-Galois and ${}_{B'}A'_{B'}|{}_{B'}B'_{B'}$, then $A'=B'\otimes_Z C'$, where C' is the center of A', and C'/Z is G'-Galois.*

(2) *If C'/Z is G'-Galois and C' is commutative, then $A'=B'\otimes_Z C'$ is a completely outer G'-Galois extension over B', ${}_{B'}A'_{B'}|{}_{B'}B'_{B'}$ and $1\otimes C'$ is the center of A'.*

Proof. (1) By [22; Prop. 6.4], A'/B' is outer G'-Galois and $\mathrm{V}_{A'}(B')=C'$, where C' is the center of A'. Then, by Cor. to Prop. 2.8 and [22; Th. 5.1], C'/Z is G'-Galois and $A'=B'\otimes_Z C'$. (2) By [22; Th. 5.2 and Prop. 6.5], $A'/(B'\otimes 1)$ is completely outer G'-Galois. Since ${}_ZZ$ is a direct summand of ${}_ZC'$, $B'\simeq B'\otimes 1$ canonically, and ${}_{B'}A'_{B'}|{}_{B'}B'_{B'}$, because ${}_ZC'|{}_ZZ$. Then, by Cor. to Prop. 2.8, C^*/Z is C'-Galois, where C^* is the center of A'. Since $C^*\supseteq 1\otimes C'\supseteq Z$ and $(1\otimes C')/Z$ is G'-Galois ([22; Th. 5.1 or Th. 5.6]), we have $C^*=Z\cdot(1\otimes C')=1\otimes C'$ ([22; Th. 5.1]).

Lemma 2.10. *Let T be a ring, and U a subring of T.*

(1) *Let T/U be a separable extension. If a T-left module M is U-projective, then M is T-projective.*

(2) *If ${}_TT\otimes_U T_T|{}_TT_T$ and ${}_UU|{}_UM$ for a T-left module M, then ${}_TT|{}_TM$.*

(3) *Let T_0 be an intermediate ring of T/U. If T is (U, T_0)-projective and T_0 is a T_0-T_0-direct summand of T, then T_0/U is a separable extension.*

Proof. (1) Since the mapping $x\otimes y\to xy$ form $T\otimes_U T$ to T splits as a T-T-homomorphism, the mapping $x\otimes m\to xm$ from $T\otimes_U M$ to M splits as a T-left homomorphism. Since ${}_UM$ is projective, so is ${}_TT\otimes_U M$. Therefore M is T-projective. (2) Since ${}_UU|_UM$, ${}_TT|_TT\otimes_U M$. Since ${}_TT\otimes_U T_T|_TT_T$, we have ${}_TT\otimes_U M|_TM$. Hence we have ${}_TT|_TM$. (3) Let φ be the canonical homomorphism from $T_0\otimes_U T$ to T defined by $\varphi(t_0\otimes t)=t_0t$, and let ψ be a T_0-T_0-homomorphism from T to $T_0\otimes_U T$ such that $\varphi\psi(x)=x$ for all x in T. If $\psi(1)=\sum_i a_i\otimes b_i$ ($a_i\in T_0$, $b_i\in T$), then $\sum_i a_ib_i=1$ and $\sum_i ya_i\otimes b_i=\sum_i a_i\otimes b_iy$ ($\in T_0\otimes_U T$) for all y in T_0. Let π be a T_0-T_0-homomorphism from T to T_0 such that $\pi|T_0=1_{T_0}$. Then, since $\sum_i ya_i\otimes b_i=\sum_i a_i\otimes b_iy$ ($\in T_0\otimes_U T$) for all y in T_0, we have $\sum_i a_i\cdot\pi(b_i)=1$ and $\sum_i ya_i\otimes\pi(b_i)=\sum_i a_i\otimes\pi(b_i)y$ ($\in T_0\otimes_U T_0$) for y in T_0. Then the mapping $y\to\sum_i a_i\otimes\pi(b_i)y$ from T_0 to $T_0\otimes_U T_0$ is a T_0-T_0-homomorphism, and $\sum_i a_i\cdot\pi(b_i)y=y$. Hence T_0/U is a separable extension.

Proposition 2.11. *Let A/B be finite G-Galois, and Z the center of B. If B is a separable Z-algebra and $Z\subseteq C$, then $\mathrm{V}_A(B)/Z$ is finite G-Galois.*

Proof. By [2; Prop. 1.5], $B\otimes_Z B^0$ is a central separable Z-algebra, where B^0 is the opposite ring of B. Since ${}_BA$ and ${}_ZB$ are finitely generated and projective, so is ${}_ZA$. Then, by Lemma 2.10 (1), ${}_{B\otimes_Z B^0}A$ is finitely generated and projective. By [2; Th. 2.1], ${}_{B\otimes_Z B^0}B\otimes_Z B^0|_{B\otimes_Z B^0}B$, and hence ${}_BA_B|_BB_B$. Then, by Prop. 2.8, $\mathrm{V}_A(B)/Z$ is finite G-Galois (cf. S. 3).

Theorem 2.12. *Let G be finite, π the group homomorphism defined by $\pi(\sigma)=\sigma|C$ ($\sigma\in G$), Z the center of B, and $Z_0=C^G$, and assume that A is indecomposable. Then the following statements are equivalent.*

(i) *A/Z_0 is separable, and π is 1–1.*

(ii) *$\mathrm{V}_A(B)=C$, A/Z is separable, and ${}_BA_B|_BB_B$.*

(iii) *$\mathrm{V}_A(B)=C$, and both B/Z and C/Z are separable.*

(iv) *Both B/Z and C/Z_0 are separable, and π is 1–1.*

(v) *$\mathrm{V}_A(B)=C$, A/B is separable, A is (Z, B)-projective, and ${}_BB_B|_BA_B$.*

(vi) *$A=B\cdot C$, and A/Z is separable.*

(vii) *${}_{A\otimes_{Z_0}A^0}A\otimes_{Z_0}A^0|_{A\otimes_{Z_0}A^0}A$, and $\mathrm{Hom}({}_AAu_{\sigma_A}, {}_AAu_{1_A})=0$ for any σ in G such that $\sigma\neq 1$.*

Proof. (i)$\Longrightarrow$(ii) By [2; Th. 2.3], A/C and C/Z_0 are separable. Therefore, by [4; Th. 1.3], C/Z_0 is G-Galois. Then, by [22; Th. 5.1], $A=B\otimes_{Z_0}C$. Hence $\mathrm{V}_A(B)=C$, and $Z=Z_0$. Since ${}_ZC$ is finitely generated and projective, ${}_BA_B|_BB_B$. (ii)$\Longrightarrow$(iii) $\mathrm{V}_A(B)=C$ implies $Z=Z_0$ ($\subseteq C$). By [22; Lemma 2.7], A/C and A/B are separable, so that A/B is outer G-Galois ([22; Th. 1.5]). Then, by Prop. 2.8, C/Z is G-Galois, so that C/Z is separable. Since A/C is separable, B/Z is separable ([22; Cor. to Th. 5.1]). (iii)$\Longrightarrow$(iv) In this case, $Z=Z_0$. By [2; Th. 3.1], $A=B\cdot C$, whence π is 1–1. (iv)$\Longrightarrow$(v) By

[4; Th. 1.3], C/Z_0 is G-Galois. Hence, by [22; Th. 5.1], A/B is G-Galois, and $A=B\cdot C$. Then A/B is separable, $V_A(B)=C$, and $Z=Z_0$. Since Z is commutative, ${}_ZZ$ is a direct summand of ${}_ZC$ (S. 3), so that $t_G(c)=1$ for some c in C. Then B is a B-B-direct summand of A (cf. [22; § 2. p. 118]). Since B/Z is separable, A is (Z, B)-projective ([22; Lemma 2.7]). (v)$\Longrightarrow$(vi) By Lemma 2.10 (3), B/Z is separable. Then, by [2; Th. 3.1], $A=B\otimes_Z C$. Since both A/B and B/Z are separable, A/Z is separable ([22; Lemma 2.7]). (vi)$\Longrightarrow$(i) As $A=B\cdot C$, $V_A(B)=C$, $Z=Z_0$, and π is 1–1. Thus we know that (i)$\sim$(vi) are equivalent. (i)$\Longrightarrow$(vii) In this case, $V_A(B)=C$, $Z=Z_0$, and B/Z is separable. Then, by [2; Th. 2.1], ${}_{B\otimes_Z B^0}B\otimes_Z B^0|_{B\otimes_Z B^0}B$. Therefore ${}_BB\otimes_Z B_B|_BB_B$, and then ${}_AA\otimes_Z A_A|_AA\otimes_B A_A$. By [22; Prop. 1.3], ${}_A\Delta_A\simeq {}_AA\otimes_B A_A$. Hence ${}_{A\otimes_Z A^0}A\otimes_Z A^0|_{A\otimes_Z A^0}\Delta$. The second assertion follows from [22; Prop. 6.3]. (vii)$\Longrightarrow$(i) By assumption, $\mathrm{End}({}_{A\otimes_{Z_0}A^0}\Delta)\simeq\bigoplus_{\sigma\in G}\mathrm{End}({}_{A\otimes_{Z_0}A^0}Au_\sigma)$ (external direct sum as rings). To be easily seen, $\mathrm{End}({}_{A\otimes_{Z_0}A^0}Au_\sigma)\simeq C$, which is commutative. Hence $\mathrm{End}({}_{A\otimes_{Z_0}A^0}\Delta)$ is a commutative ring. Then, by S. 1 and S. 3, ${}_{A\otimes_{Z_0}A^0}\Delta$ is finitely generated and projective. Hence ${}_{A\otimes_{Z_0}A^0}A$ is finitely generated and projective, that is, A/Z_0 is separable. Let f be the projection from Δ to Au_1 with respect to the decomposition $\Delta=\sum_\sigma\oplus Au_\sigma$. Then, since $\mathrm{End}({}_{A\otimes_{Z_0}A^0}\Delta)$ is commutative, f is in the center of $A\otimes_{Z_0}A^0$ (cf. S. 1). By [2; Prop. 1.5], the center of $A\otimes_{Z_0}A^0$ is $C\otimes C$, so that f is written as $f=\sum_i a_i\otimes a_i^*$ (a_i, $a_i^*\in C$). Then, $u_1=\sum_i a_i(\sum_\sigma u_\sigma)a_i^*$ $(=\sum_\sigma(\sum_i a_i\cdot\sigma(a_i^*))u_\sigma)$, and hence $\sum_i a_i\cdot\sigma(a_i^*)=\delta_{1,\sigma}$. This completes the proof of the theorem.

Pemark. The following are also equivalent to (i) ($\longleftrightarrow$(iii)).

(viii) A/C is separable, and C/Z_0 is G-Galois (cf. Kanzaki [8]).

(ix) A/B is outer G-Galois, and B/Z is separable.

Proposition 2.13. *Let A/B be locally finite G-Galois, and b an element of B which is not a right zero divisor of B. Tnen b is not a right zero divisor of A.*

Proof. Let a be an element of A such that $ab=0$. Then $Aab=0$, and so $\sigma(Aa)b=0$ for all σ in G. Hence, $((\sum_\sigma\sigma(Aa))\cap B)b=0$. Then, by assumption, $(\sum_\sigma\sigma(Aa))\cap B=0$. Then, by Th. 2.1 (3), $\sum_\sigma\sigma(Aa)=A((\sum_\sigma\sigma(Aa))\cap B)=0$. Hence $a=0$.

Let A/B be locally finite G-Galois, and $S\ni 1$ a G-invariant multiplicative system of regular elements in A such that a left quotient ring $\bar{A}$ of A with respect to S exists. Then G may be regarded as a group of automorphisms of $\bar{A}$. To be easily seen, $\{\sigma(x);\ \sigma\in G\}$ is finite for any x in $\bar{A}$. Then, by Th. 2.1, $\bar{A}/\bar{B}$ is locally finite G-Galois and $\bar{A}=\bar{B}\otimes_B A=A\otimes_B\bar{B}$, where $\bar{B}=\bar{A}^G$. To be easily seen, any element in $B\cap S$ is a unit of $\bar{B}$. For b in $\bar{B}$, we put

$\mathfrak{L}=\{x \in A ; xb \in A\}$, which is a Δ-left submodule of A. Then $(\mathfrak{L} \cap B)b \subseteq B$. If $\mathfrak{L} \cap B \cap S \neq \emptyset$, then $sb \in B$ for some s in $B \cap S$. Therefore, if we assume that $\Delta(s) \cap B \cap S \neq \emptyset$ for all $s \in S$, then $\bar{B}$ is a left quotient ring of B with respect to $B \cap S$. Thus we obtain the following

Theorem 2.14. *Let A/B be locally finite G-Galois, and $S \ni 1$ a G-invariant multiplicative system of regular elements of A such that a left quotient ring $\bar{A}$ of A with respect to S exists. Further, assume that $\Delta(s) \cap B \cap S \neq \emptyset$ for all s in S. Then there hold the following*:

(1) *$\bar{A}/\bar{B}$ is locally finite G-Galois and $\bar{A}=\bar{B} \otimes_B A = A \otimes_B \bar{B}$*, where $\bar{B}=\bar{A}^G$.

(2) *$\bar{A}$ is a left quotient ring of A with respect to $B \cap S$. $\bar{B}$ is a left quotient ring of B with respect to $B \cap S$.*

Remark. Let A/B be locally finite G-Galois, and S a G-invariant multiplicative system of regular elements in A such that $S \subseteq C$ and $S \ni 1$. Then S satisfies the conditions in Th. 2.14. To see this, we put $H=\{\sigma \in G ; \sigma(s)=s\}$ for s in S. If $G=\sigma_1 H \cup \cdots \cup \sigma_r H$ is the left coset decomposition of G, then $\prod_i \sigma_i(s) \in \Delta(s) \cap B \cap S$.

A non-zero ring T with 1 is called a *left Goldie ring* if T satisfies the following conditions: (1) T is a semi-prime ring. (2) Any independent set of non-zero left ideals is finite (i.e., ${}_T T$ is *finite dimensional*). (3) T satisfies the ascending chain condition for annihilator left ideals.

A left Goldie ring has a complete left quotient ring which is a semi-simple ring with minimum condition for left ideals, and conversely (Goldie [17]). (Cf. [7])

Theorem 2.15. *Let A/B be locally finite G-Galois, A a left Goldie ring, $\bar{A}$ a complete left quotient ring of A, and B a semi-prime ring. Then there hold the following*:

(1) *$\bar{A}/\bar{B}$ is locally finite G-Galois, where $\bar{B}=\bar{A}^G$.*

(2) *B is a left Goldie ring, and $\bar{B}$ is a complete left quotient ring of B.*

Proof. Let S be the set of all regular elements of A. First we shall prove that B is a left Goldie ring. Since ${}_A A$ is finite dimensional, ${}_\Delta A$ is finite dimensional. Then, by Th. 2.1 (3), ${}_B B$ is finite dimensional. Let $\mathfrak{l} \subseteq \mathfrak{l}'$ be left ideals of B. Then $l_A(r_B(\mathfrak{l})) \subseteq l_A(r_B(\mathfrak{l}'))$, where $r_B(\mathfrak{l})=\{y \in B ; \mathfrak{l}y=0\}$ and $l_A(r_B(\mathfrak{l}))=\{x \in A ; x \cdot r_B(\mathfrak{l})=0\}$. From this fact, B satisfies the ascending chain condition for annihilator left ideals of B. Hence B is a left Goldie ring. By Prop. 2.13, $S \cap B$ is the set of all regular elements of B. For any s in S, ${}_A As$ is essential in ${}_A A$, so that ${}_\Delta \Delta(s)$ is essential in ${}_\Delta A$. Then, by Th. 2.1 (3), ${}_B(\Delta(s) \cap B)$ is essential in ${}_B B$, so that $\Delta(s) \cap B$ contains a regular element of B ([17; Th. (3.9)]). Hence $\varDelta(s)\cap B\cap S\neq\emptyset$ for any s in S. Thus the present theorem follows from Th. 2.14.

Remark. In the following cases, the condition that B is semi-prime is superfluous.

(1) G is finite and completely outer (cf. [22; p. 132]).

(2) B is contained in the center of A.

Let T be a ring. If T-left modules M and N have no non-zero isomorphic subquotients, we say that ${}_TM$ and ${}_TN$ are *unrelated* (cf. [22]).

Lemma 2.16. *Let T be a ring, and let M and N be T-left modules, and W a T-submodule of M. If ${}_T(M/W)$ and ${}_TN$ are unrelated, and ${}_TW$ and ${}_TN$ are unrelated, then ${}_TM$ and ${}_TN$ are unrelated.*

Proof. Assume that there are isomorphic subquotients X/X_0 and Y/Y_0 of ${}_TM$ and ${}_TN$, respectively. Then, as is easily seen, $X+W\supsetneqq X_0+W$ or $X\cap W\supsetneqq X_0\cap W$. If $X+W\supsetneqq X_0+W$, then $Y/Y_0\simeq X/X_0\twoheadrightarrow(X+W)/(X_0+W)\neq 0$, a contradiction. If $X\cap W\supsetneqq X_0\cap W$, then $(X\cap W)/(X_0\cap W)\simeq(X_0+(X\cap W))/X_0\subseteq X/X_0\simeq Y/Y_0$, which is also a contradiction.

Proposition 2.17. *Let σ,τ be in G, and assume that ${}_AAu_{\sigma A}$ and ${}_AAu_{\tau A}$ are unrelated. Then, for any finite subset $\{x_1,\cdots,x_r;\ y_1,\cdots,y_s\}$ of A, there are elements a_k, b_k $(k=1,\cdots,t)$ in A such that $\sum_k a_kx_i\cdot\sigma(b_k)=x_i$ and $\sum_k a_ky_h\cdot\tau(b_k)=0$ for all x_i, y_h.*

Poof. By Lemma 2.16, ${}_A(Au_\sigma)^r_A$ and ${}_A(Au_\tau)^s_A$ are unrelated. Then, since $A(x_1u_\sigma,\cdots,x_ru_\sigma,\ y_1u_\tau,\cdots,y_su_\tau)A$ is an A-A-submodule of ${}_A(Au_\sigma)^r\oplus(Au_\sigma)^s_A$, $(x_1u_\sigma,\cdots,x_ru_\sigma,\ 0,\cdots,0)\in A(x_1u_\sigma,\cdots,x_ru_\sigma,\ y_1u_\tau,\cdots,y_su_\tau)A$ (cf. [22; Prop. 6.1]). Therefore there are elements a_k, b_k $(k=1,\cdots,t)$ in A such that $\sum_k a_k(x_1u_\sigma,\cdots,x_ru_\sigma,\ y_1u_\tau,\cdots,y_su_\tau)b_k=(x_1u_\sigma,\cdots,x_ru_\sigma,\ 0,\cdots,0)$. Then, $\sum_k a_kx_i\cdot\sigma(b_k)=x_i$ and $\sum_k a_ky_h\cdot\tau(b_k)=0$ for all x_i, y_h.

Combining Prop. 2.17 with [22; Prop. 6.11] we can easily see the following

Proposition 2.18. *Let A and A' be R-algebras with $A\otimes_RA'\neq 0$, and let G and G' be completely outer finite groups of R-automorphisms of A and A', respectively. Then, $G\times G'$ is completely outer as an automorphism group of $A\otimes_RA'$.*

§ 3.

Proposition 3.1. *Let A/B be locally finite G-Galois, and X a $\varDelta$-left submodule of A. Then $X=A(X\cap B)$.*

Proof. This follows from Th. 2.1 (3).

Proposition 3.2. *Let A/B be locally finite G-Galois, $\{\mathfrak{P}\}$ the set of* *all maximal ideals of* A, *and* $\{\mathfrak{p}\}$ *the set of all maximal ideals of* B. *Then the following are equivalent*:

(i) $\mathfrak{P}\to\mathfrak{P}\cap B$ *is a mapping from* $\{\mathfrak{P}\}$ *onto* $\{\mathfrak{p}\}$.

(ii) $A\mathfrak{p}A\neq A$ *for all* $\mathfrak{p}\in\{\mathfrak{p}\}$, *and* $\cap_{\sigma\in G}\sigma(\mathfrak{P})$ *is* Δ-A-*maximal for all* $\mathfrak{P}\in\{\mathfrak{P}\}$.

If (i) *holds, then the following are true*:

(1) $\mathfrak{p}A=A\mathfrak{p}\neq A$ *for any* $\mathfrak{p}\in\{\mathfrak{p}\}$.

(2) $\{\cap_\sigma\sigma(\mathfrak{P});\ \mathfrak{P}\in\{\mathfrak{P}\}\}$ *is the set of all maximal* Δ-A-*snbmodules of* A.

(3) $\mathfrak{R}({}_\Delta A_A)=\mathfrak{R}({}_A A_A)=\mathfrak{R}({}_B B_B)A=A\cdot\mathfrak{R}({}_B B_B)$, *and* $\mathfrak{R}({}_A A_A)\cap B=\mathfrak{R}({}_B B_B)$.

(4) B *is* B-B-*completely reducible if and only if* $\cap_i\cap_\sigma\sigma(\mathfrak{P}_i)=0$ *for some* $\mathfrak{P}_i$ $(i=1,\cdots,n)$ *in* $\{\mathfrak{P}\}$.

Proof. (i)$\Longrightarrow$(ii) If $\mathfrak{P}$ is in $\{\mathfrak{P}\}$, then $\mathfrak{P}\cap B=\sigma(\mathfrak{P})\cap B$ for any σ in G, and so $\mathfrak{P}\cap B=(\cap_\sigma\sigma(\mathfrak{P}))\cap B$. By Prop. 3.1, $A((\cap_\sigma\sigma(\mathfrak{P}))\cap B)=\cap_\sigma\sigma(\mathfrak{P})=((\cap_\sigma\sigma(\mathfrak{P}))\cap B)A$. Hence $A\mathfrak{p}=\mathfrak{p}A\neq A$ for all $\mathfrak{p}$ in $\{\mathfrak{p}\}$. Let X be a Δ-A-submodule of A with $A\supsetneqq X\supseteq\cap_\sigma\sigma(\mathfrak{p})$. Then $B\supsetneqq X\cap B\supseteq(\cap_\sigma\sigma(\mathfrak{P}))\cap B=\mathfrak{P}\cap B$, and so $X\cap B=(\cap_\sigma\sigma(\mathfrak{P}))\cap B$. Then, by Prop. 3.1, $X=\cap_\sigma\sigma(\mathfrak{P})$. Thus $\cap_\sigma\sigma(\mathfrak{P})$ is Δ-A-maximal. Let Y be a maximal Δ-A-submodule of A. Take a maximal ideal $\mathfrak{P}_1$ of A with $\mathfrak{P}_1\supseteq Y$. Then $\cap_\sigma\sigma(\mathfrak{P}_1)\supseteq Y$, and so $\cap_\sigma\sigma(\mathfrak{P}_1)=Y$. Thus we obtain (2). Therefore $\mathfrak{R}({}_\Delta A_A)=\mathfrak{R}({}_A A_A)$. Since $\mathfrak{R}({}_A A_A)\cap B=\mathfrak{R}({}_B B_B)$, we have $\mathfrak{R}({}_A A_A)=A\cdot\mathfrak{R}({}_B B_B)=\mathfrak{R}({}_B B_B)A$ (Prop. 3.1). B is B-B-completely reducible if and only if $\cap_i\mathfrak{p}_i=0$ for some $\mathfrak{p}_1,\cdots,\mathfrak{p}_n$ in $\{\mathfrak{p}\}$. Thus we obtain (4) (cf. Prop. 3.1). (ii)$\Longrightarrow$(i) Let $\mathfrak{p}\in\{\mathfrak{p}\}$. Then, as $A\mathfrak{p}A\neq A$, $\mathfrak{p}\subseteq\mathfrak{P}$ for some $\mathfrak{P}\in\{\mathfrak{P}\}$, and so $\mathfrak{p}=\mathfrak{P}\cap B$ by the maximality of $\mathfrak{p}$. Let $\mathfrak{Q}$ be in $\{\mathfrak{P}\}$. Then $\mathfrak{q}\supseteq\mathfrak{Q}\cap B$ for some $\mathfrak{q}\in\{\mathfrak{p}\}$. There is a $\mathfrak{Q}'\in\{\mathfrak{P}\}$ with $\mathfrak{Q}'\cap B=\mathfrak{q}$. Then $(\cap_\sigma\sigma(\mathfrak{Q}'))\cap B=\mathfrak{Q}'\cap B\supseteq\mathfrak{Q}\cap B=(\cap_\sigma\sigma(\mathfrak{Q}))\cap B$, and therefore $\cap_\sigma\sigma(\mathfrak{Q}')\supseteq\cap_\sigma\sigma(\mathfrak{Q})$ by Prop. 3.1. By assumption, $\cap_\sigma\sigma(\mathfrak{Q}')=\cap_\sigma\sigma(\mathfrak{Q})$. Hence $\mathfrak{q}=\mathfrak{Q}'\cap B=\mathfrak{Q}\cap B$. This completes the proof.

Concerning Prop. 3.2, we state the following

Lemma 3.3. *Let* $\mathfrak{P}$ *be a maximal ideal of* A *such that* $\cap_{\sigma\in G}\sigma(\mathfrak{P})=\cap_i\sigma_i(\mathfrak{P})$ *for some* $\sigma_1,\cdots,\sigma_n$ *in* G. *Then* $\cap_\sigma\sigma(\mathfrak{P})$ *is* Δ-A-*maximal, and* $\{\sigma_i(\mathfrak{P});\ i=1,\cdots,n\}$ *is the set of all maximal ideals containing* $\cap_\sigma\sigma(\mathfrak{P})$.

Proof. Let $\mathfrak{Q}$ be a maximal ideal of A with $\mathfrak{Q}\supseteq\cap_\sigma\sigma(\mathfrak{P})$. If $\mathfrak{Q}\neq\sigma_i(\mathfrak{P})$ for all i, then $\mathfrak{Q}+\sigma_i(\mathfrak{P})=A$ for all i. Then we have a contradiction $A=\mathfrak{Q}+\cap_i\sigma_i(\mathfrak{P})=\mathfrak{Q}+\cap_\sigma\sigma(\mathfrak{P})$.

Remark. In the following cases, the assumption in Lemma 3.3 holds. (1) G is finite. (2) The ring $A/\mathfrak{R}({}_A A_A)$ satisfies the descending chain condition for ideals. (3) G^* is compact, and every maximal ideal of A is A-A-finitely generated. (Cf. Prop. 1.1).

Proposition 3.4.

(1) *Let A/B be locally finite outer G-Galois, and B B-B-completely reducible. Assume that, for any maximal ideal $\mathfrak{P}$ of A, there are elements $\sigma_1, \cdots, \sigma_n$ in G such that $\cap_i \sigma_i(\mathfrak{P}) = \cap_\sigma \sigma(\mathfrak{P})$. Then A is A-A-completely reducible.*

(2) *Let G be finite and completely outer, and $B_B | A_B$. Then A is A-A-completely reducible if and only if B is B-B-completely reducible. If there is a maximal ideal $\mathfrak{P}$ of A such that $\cap_\sigma \sigma(\mathfrak{P}) = 0$, then B is B-B-minimal, and conversely.*

Proof. (1) Any maximal ideal $\mathfrak{p}$ of B is written as $\mathfrak{p} = Be$ with a central idempotent e of B. Then, by assumption, $(1 \neq)$ $e \in V_A(B) = C$. Therefore, $A\mathfrak{p} = Ae = eA = \mathfrak{p}A \neq A$. Thus, by Prop. 3.2 and Lemma 3.3, A is A-A-completely reducible. (2) In this case, $\mathfrak{a}A = A\mathfrak{a} \neq A$ for any proper ideal $\mathfrak{a}$ of B (cf. [22; p. 132]). Then, by Prop. 3.2 and Lemma 3.3, the first assertion is evident (cf. [22; Prop. 6.4]). For any $\mathfrak{P}$ in $\{\mathfrak{P}\}$, $((\cap_\sigma \sigma(\mathfrak{P})) \cap B =)$ $\mathfrak{P} \cap B = 0$ if and only if $\cap_\sigma \sigma(\mathfrak{P}) = 0$ (Prop. 3.1). Thus we know the second assertion.

Theorem 3.5. *Let A/B be finite G-Galois, B a semi-primary ring, and $A\mathfrak{p}A \neq A$ for any maximal ideal $\mathfrak{p}$ of B. Then ${}_{BG}A \simeq {}_{BG}BG$, that is, A has a normal basis.* (Cf. [13; Th. 1]).

Proof. By [22; Th. 1.7], it suffices to prove that ${}_BA$ is free. Let $g = (G:1)$. (1) First we assume that $\mathfrak{R}(B) = 0$. Then B is a direct sum of simple rings: $B = \mathfrak{a}_1 + \cdots + \mathfrak{a}_n$. Let $1 = \sum_i e_i$, $e_i \in \mathfrak{a}_i$. Then $\mathfrak{a}_i = Be_i = e_iB$ and $e_i^2 = e_i$. By assumption we have $(1 - e_i)A = A(1 - e_i)$ (Prop. 3.2 and Lemma 3.3), so that e_i is a central idempotent of A contained in B. Then each Ae_i/Be_i is G-Galois ([22; Cor. to Th. 5.6]). Since Be_i is a simple ring, ${}_{Be_i}Ae_i$ is free (cf. [7]). Hence Ae_i has a normal basis, so that ${}_{Be_i}Ae_i \simeq {}_{Be_i}(Be_i)^g$ for all i ([22; Th. 1.7]). Hence ${}_BA \simeq {}_BB^g$. (2) Next we proceed to general case. Since A and B are semi-primary ([22; Prop. 7.3]), $\mathfrak{R}({}_AA_A) = \mathfrak{R}(A)$ and $\mathfrak{R}({}_BB_B) = \mathfrak{R}(B)$. Then, by Prop. 3.2 and Lemma 3.3, $\mathfrak{R}(A) = \mathfrak{R}(B)A = A \cdot \mathfrak{R}(B)$ and $\mathfrak{R}(A) \cap B = \mathfrak{R}(B)$. By [22; Th. 5.6], $(A/\mathfrak{R}(A))/((B + \mathfrak{R}(A))/\mathfrak{R}(A))$ is G-Galois, and satisfies the same conditions as A/B, because $(B + \mathfrak{R}(A))/\mathfrak{R}(A) \simeq B/(\mathfrak{R}(A) \cap B) = B/\mathfrak{R}(B)$ canonically. By (1), we have ${}_BA/\mathfrak{R}(A) \simeq {}_B(B/\mathfrak{R}(B))^g \simeq {}_BB^g/\mathfrak{R}(B)^g$. Since $\mathfrak{R}(A) = \mathfrak{R}(B)A$ and ${}_BA$ is finitely generated and projective, we have ${}_BA \simeq {}_BB^g$ by the uniqueness of projective cover. This completes the proof.

Corollary. *Let A/B be finite G-Galois, B a semi-primary ring, and Z the center of B. Assume that $Z \subseteq C$ and that B is a central separable Z-algebra. Then A has a normal basis.*

Proof. In this case, any proper ideal of B is written as $\mathfrak{a}B$ with an ideal

$\mathfrak{a}$ of Z (cf. [2]). Then, as $Z\subseteq C$, $(\mathfrak{a}B)A=\mathfrak{a}A=A\mathfrak{a}=A(B\mathfrak{a})\neq A$ ([22; Lemma 7.1]).

Let A/B be finite G-Galois, $B\subseteq C$, and $g=(G:1)$. For any prime ideal $\mathfrak{p}$ of B, we denote by B_p the quotient extension of B with respect to $\mathfrak{p}$. Then B_p is a B-algebra, canonically. By [22; Cor. to Th. 5.2], $(B_p\otimes_B A)/B_p$ is G-Galois. Since B_p is a local ring, ${}_{B_p}B_p\otimes_B A\simeq{}_{B_p}(B_p)^g$ (Cor. to Th. 3.5). We denote by K_p the quotient field of $B/\mathfrak{p}$. Then we have ${}_{K_p}K_p\otimes_B A\simeq{}_{K_q}(K_p)^g$ similarly. Thus we obtain the following

Proposition 3.6. *Let A/B be finite G-Galois, $B\subseteq C$, and $g=(G:1)$. Then, ${}_{B_p}B_p\otimes_B A\simeq{}_{B_p}(B_p)^g$ and ${}_{K_p}K_p\otimes_B A\simeq{}_{K_p}(K_p)^g$ for any prime ideal $\mathfrak{p}$ of B, where B_p is the quotient extension of B with respect to $\mathfrak{p}$ and K_p is the quotient field of $B/\mathfrak{p}$.*

The following lemma is of some interest.

Lemma 3.7. *Let $R\supseteq S$ be rings, R_S is finitely generated and projective, and ${}_SS$ is a direct summand of ${}_SR$. If ${}_RR$ is injective, then ${}_SS$ is injective.*

Proof. Let $\mathfrak{l}$ be any left ideal of S, and f any S-left homomorphism from $\mathfrak{l}$ to ${}_SR$. Since R_S is finitely generated and projective, we have $R\mathfrak{l}=R\otimes_S\mathfrak{l}$. Therefore f can be extended to an R-left homomorphism from $R\mathfrak{l}$ to R, canonically. Then, by assumption, there is an element a in R such that $r\cdot(s)f=rsa$ for r in R and s in $\mathfrak{l}$, so that $(s)f=sa$ for all s in $\mathfrak{l}$. Therefore, as is well known, ${}_SR$ is injective. Since ${}_SS$ is a direct summand of ${}_SR$, ${}_SS$ is injective.

Lemma 3.8. $\mathfrak{R}(A)\cap B\subseteq\mathfrak{R}(B)$.

Proof. Let b be in $\mathfrak{R}(R)\cap B$. Then $1-b$ has an inverse in A. Since $B=A^G$, $1-b$ has an inverse in B. Hence $\mathfrak{R}(A)\cap B$ is a quasi-regular ideal of B, that is, $\mathfrak{R}(A)\cap B\subseteq\mathfrak{R}(B)$.

Proposition 3.9. *Let G be finite. If there is an element c in A such that $1-t_G(c)\in\mathfrak{R}(A)$, then there is an element d in A such that $t_G(d)=1$.*

Proof. By Lemma 3.8, we have $1-t_G(c)\in\mathfrak{R}(A)\cap B\subseteq\mathfrak{R}(B)$, so that $t_G(A)+\mathfrak{R}(B)=B$. Since $t_G(A)$ is an ideal of B, we have $t_G(A)=B$. Hence $t_G(d)=1$ for some d in A.

Theorem 3.10. *Let A/B be G-Galois, A a commutative ring, H a subgroup of G, and A' a B-algebra. Then, $A'\otimes_B A^H$ is a direct sum of minimal ideals if and only if A' is a direct sum of minimal ideals* (cf. [7; p. 178. Th. 2]).

Proof. In this case, $(A'\otimes_B A)/A'$ is finite G-Galois, G is completely outer as an automorphism group of $A'\otimes_B A$, and $(A'\otimes_B A)^H=A'\otimes_B A^H$ (cf. [22; Th.

5.2 and Prop. 6.5]). Thus the present theorem is an easy consequence from Prop. 3.4 (2).

Concerning [22; Th. 6.9], we note the following

Lemma 3.11. *Let A/C be separable, and e an idempotent of A such that $eA \subseteq Ae$. Then e is a central idempotent of A.*

Proof. Since $A/\mathfrak{R}(A)$ is a semi-prime ring, we have $(eA+\mathfrak{R}(A))/\mathfrak{R}(A) = (Ae+\mathfrak{R}(A))/\mathfrak{R}(A)$, that is, $eA+\mathfrak{R}(A)=Ae+\mathfrak{R}(A)$, and so $Ae=eA+(Ae\cap \mathfrak{R}(A))=eA+\mathfrak{R}(A)e$. Since A is a central separable C-algebra, $\mathfrak{R}({}_AA_A)=\mathfrak{R}(C)A$ by [2; Cor. 3.2]. Since $\mathfrak{R}({}_AA_A)\supseteq\mathfrak{R}(A)\supseteq\mathfrak{R}(C)A$, we have $\mathfrak{R}(A)=\mathfrak{R}(C)A$, and $Ae=eA+\mathfrak{R}(C)Ae$. Hence $Ae=eA$, because ${}_CAe$ is finitely generated. Consequently, e is a central idempotent of A.

Proposition 3.12. *Let A/B be locally finite G-Galois, and assume that there is a representation $A=\cup_{\lambda\in\Lambda}A^{N_\lambda}$ of A/B such that each $\mathfrak{R}(B)A^{N_\lambda}$ is an ideal of A^{N_λ}. Then $\mathfrak{R}(A)=\mathfrak{R}(B)A=A\cdot\mathfrak{R}(B)$, and $\mathfrak{R}(A)\cap B=\mathfrak{R}(B)$.*

Proof. Let $\mathfrak{J}$ be a right ideal of A such that $\mathfrak{R}(B)A+\mathfrak{J}=A$. Then $\mathfrak{R}(B)A^{N_\lambda}+(\mathfrak{J}\cap A^{N_\lambda})\ni 1$ for some λ in Λ, so that $\mathfrak{R}(B)A^{N_\lambda}+(\mathfrak{J}\cap A^{N_\lambda})=A^{N_\lambda}$ Since $\mathfrak{R}(B)A^{N_\lambda}\subseteq\mathfrak{R}(A^{N_\lambda})$, we have $\mathfrak{J}\cap A^{N_\lambda}=A^{N_\lambda}$, and hence $\mathfrak{J}=A$. Thus we know that $\mathfrak{R}(B)A\subseteq\mathfrak{R}(A)$. Combining this with Lemma 3.8, we have $\mathfrak{R}(A)\cap B=\mathfrak{R}(B)$. Hence $\mathfrak{R}(A)=\mathfrak{R}(B)A=A\cdot\mathfrak{R}(B)$ (Prop. 3.1).

Theorem 3.13. *Let A/B be lacally finite G-Galois, $B\subseteq C$, and A' a B-algebra such that $A'\simeq A'\otimes 1$ $(\subseteq A'\otimes_B A)$ canonically.*

(1) $\mathfrak{R}(A'\otimes_B A)=\mathfrak{R}(A'\otimes A)$, *and* $\mathfrak{R}(A'\otimes A)\cap(A'\otimes 1)=\mathfrak{R}(A')\otimes 1$.

(2) *If A is commutative, then $\mathfrak{R}(A'\otimes A^H)=\mathfrak{R}(A')\otimes A^H$ for any subgroup H of G.*

Proof. Let $A=\cup_{\lambda\in\Lambda}A^{N_\lambda}$ be a representation of the locally finite G-Galois extension A/B. Then $(A'\otimes_B A)/(A'\otimes 1)$ is a locally finite G-Galois extension with representation $A'\otimes_B A=\cup_\lambda A'\otimes A^{N_\lambda}$, where $A'\otimes A^{N_\lambda}=(A'\otimes_B A)^{N_\lambda}$ is a finite G/N_λ-Galois extension over $A'\otimes 1$. (1) This will be easily seen by Prop. 3.12. (2) We may assume that H is closed in G. Then each $A^{H\cap N_\lambda}/A^H$ is finite $H/(H\cap N_\lambda)$-Galois, and $H/(H\cap N_\lambda)$ is completely outer as an automorphism group of $A^{H\cap N_\lambda}$ ([22; Th. 6.6]). Then $H/(H\cap N_\lambda)$ is completely outer as an automorphism group $A'\otimes_B A^{H\cap N_\lambda}$ (Prop. 2.18), and so $H/(H\cap N_\lambda)$ is completely outer as an automorphism group of $A'\otimes A^{H\cap N_\lambda}$ $(\subseteq A'\otimes_B A)$ ([22; Prop. 6.11]). Now, $(A'\otimes_B A)/(A'\otimes A^H)$ is a locally finite H-Galois extension with representation $A'\otimes_B A=\cup_\lambda A'\otimes A^{H\cap N_\lambda}$, where $A'\otimes A^{H\cap N_\lambda}=(A'\otimes_B A)^{H\cap N_\lambda}$ is a finite $H/(H\cap N_\lambda)$-Galois extension over $A'\otimes A^H$. Then, by [22; Th. 7.10] and Prop. 3.12, $\mathfrak{R}(A'\otimes_B A)=\mathfrak{R}(A'\otimes A^H)(A'\otimes_B A)$. On the other hand,

$\mathfrak{R}(A'\otimes_B A)=\mathfrak{R}(A')\otimes A=(\mathfrak{R}(A')\otimes A^H)(A'\otimes_B A)$. Hence $\mathfrak{R}(A'\otimes A^H)=\mathfrak{R}(A')\otimes A^H$, as desired (cf. [22; Lemma 7.1]).

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