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ON CERTAIN CONDITIONS FOR A SUBMANIFOLD IN A RIEMANN SPACE TO BE ISOMETRIC TO A SPHERE

By

Tamao NAGAI

Introduction. In an Euclidean space of three dimensions E^3 a sphere is characterized by certain special properties of a closed surface. It has been proved by H. Liebmann [1][1] that the only ovaloid with constant mean curvature H in E^3 is a sphere. W. Süss [2] generalized this result for a closed convex hypersurface in an n-dimensional Euclidean space E^n. The analogous problem for a closed orientable hypersurface in E^n has been investigated by T. Bonnesen and W. Fenchel [3], H. Hopf [4], C. C. Hsiung [5] and A. D. Alexandrov [6]. The characterization of a sphere has been studied by many investigators and it is one of the interesting problem within the differential geometry in the large. There are also investigations about generalizing the condition $H=const.$ in the Liebmann-Süss theorem. The interesting results of this problem were given by A. D. Alexandrov [7] and S. S. Chern [9]. In the field of these investigations the integral formulas of Minkowski type has played one of the important role.

Let F be an ovaloid in E^3 and H and K the mean curvature and the Gauss curvature at a point P of F respectively. Then the integral formula of Minkowski is

$$\iint_F (Kp+H)dA=0\,,$$

where p denotes the oriented distance from a fixed point O in E^3 to the tangent space of F at P and dA is the area element of F at P. C. C. Hsiung [5] derived the generalization of this formula for a closed orientable hypersurface in E^n and gave certain characterizations of hyperspheres in E^n. Afterward Y. Katsurada [10], [11] derived the integral formulas of Minkowski type which are valid for a closed orientable hypersurface V^{n-1} in an n-dimensional Riemann space R^n and proved the following two theorems:

Theorem 0.1. (Y. Katsurada) *Let R^n be an Einstein space which ad-*

1) Numbers in brackets refer to the references at the end of the paper.

mits a vector field ξ^i generating a continuous one-parameter group of conformal transformations in R^n and V^{n-1} a closed orientable hypersurface in R^n such that

(i) $H_1 = const.$,

(ii) $n_i\xi^i$ *has fixed sign on* V^{n-1}.

Then every point of V^{n-1} is umbilic, where H_1 and n_i denote the first mean curvature of V^{n-1} and covariant component of a unit normal vector of V^{n-1} respectively.

Theorem 0. 2. (Y. Katsurada) *Let R^n be a constant Riemann curvature space which admits a vector field ξ^i generating a continuous one-parameter group of conformal transformations in R^n and V^{n-1} a closed orientable hypersurface in R^n such that*

(i) $k_1, k_2, \cdots, k_{n-1} > 0$ *on* V^{n-1} *and* $H_\nu = const.$ *for any* ν $(1 < \nu \leqq n-2)$,

(ii) $n_i\xi^i$ *has fixed sign on* V^{n-1}.

Then every point of V^{n-1} is umbilic, where k_p $(p=1, 2, \cdots, n-1)$ and H_ν $(1<\nu\leqq n-2)$ denote principal curvature of V^{n-1} and the ν-th mean curvature of V^{n-1} respectively.

The analogous problems for a closed orientable hypersurface V^{n-1} in R^n have been discussed by A. D. Alexandrov [8], K. Yano [14], T. Ôtsuki [19], M. Tani [20] and T. Koyanagi [21]. Most of these investigations are related to the characterization of an umbilical hypersurface in R^n.

Certain generalization of Theorem 0. 1 and Theorem 0. 2 for an m-dimensional closed orientable submanifold V^m in R^n have been studied by Y. Katsurada, H. Kôjyô and the present author [12], [13] and the following three theorems were proved:

Theorem 0. 3. (Y. Katsurada and H. Kôjyô) *Let R^n be a constant Riemann curvature space which admits a vector field ξ^i generating a continuous one-parameter group of conformal transformations in R^n and V^m a closed orientable submanifold in R^n such that*

(i) $H_1 = const.$,

(ii) $n_i\xi^i$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by m independent tangent vectors and* $\underset{E}{n}{}^i$ *at each point on* V^m.

Then every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n}{}^i$, where H_1 and $\underset{E}{n}{}^i$ denote the first mean curvature of V^m and covariant component of a unit normal vector which has the same direction with Euler-

Schouten vector of V^m respectively[2].

Theorem 0.4. (Y. Katsurada and H. Kôjyô) *Let R^n be a constant Riemann curvature space which admits a vector field ξ^i generating a continuous one-parameter group of conformal transformations in R^n and V^m a closed orientable submanifold in R^n such that*

(i) $\underset{E}{k_1}, \underset{E}{k_2}, \cdots, \underset{E}{k_m} > 0$ *on* V^m *and* $\underset{E}{H_\nu} = const.$ *for any* ν $(1 < \nu \leqq m-1)$,

(ii) $\underset{E}{n_i}\xi^i$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by m independent tangent vectors and* $\underset{E}{n^i}$ *at each point on* V^m.

Then every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$, where $\underset{E}{k_p}$ $(p=1, 2, \cdots, m)$ and $\underset{E}{H_\nu}$ $(1<\nu\leqq m-1)$ denote principal curvature of V^m for the normal vector $\underset{E}{n^i}$ and the ν-th mean curvature of V^m for the normal vector $\underset{E}{n^i}$ respectively.

Theorem 0.5. (Y. Katsurada and T. Nagai) *Let R^n be a Riemann space which admits a vector field ξ^i generating a continuous one-parameter group of homothetic transformations in R^n and V^m a closed orientable submanifold in R^n such that*

(i) $H_1 = const.$,

(ii) $\underset{E}{n_i}\xi^i$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by m independent tangent vectors and* $\underset{E}{n^i}$ *at each point on* V^m,

(iv) $R_{ijhk}\underset{E}{n^i}\underset{E}{n^h}g^{\alpha\beta}B^j_\alpha B^k_\beta \geqq 0$ *at each point on* V^m[3].

Then every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$.

Because of the existence of cartesian coordinate system in E^n we can prove that if every point of a closed orientable hypersurface in E^n is umbilic, then the hypersurface is isometric to a sphere. However, in a general Riemann space we can not expect the validity of the same result even if every point of a closed orientable hypersurface is umbilic. It is one of the interesting problem to find certain conditions for an umbilical hypersurface in a Riemann space to

2) With respect to these objects we shall find again in §1 of the present paper.

3) Throughout the present paper the Latin indices run from 1 to n and the Greek indices from 1 to m $(m \leqq n-1)$. R_{ijhk}, $g^{\alpha\beta}$ and B^i_α denote curvature tensor of R^n, contravariant component of induced metric tensor in V^m and $\dfrac{\partial x^i}{\partial u^\alpha}$ respectively, where x^i are local coordinates in R^n and u^α are local coordinates on V^m. (Cf. §1 of the present paper.)

be isometric to a sphere. Recently one result on this problem was given by K. Yano [15], that is,

Theorem 0.6. (K. Yano) *Let R^n be an n-dimensional Riemann space admitting a non-constant scalar field v such that*

$$v_{;i;j}=f(v)g_{ij}$$ [4]

and V^{n-1} a closed orientable hypersurface in R^n such that

(i) $H_1=const.$,

(ii) $v_{;i}n^i$ *has fixed sign on* V^{n-1},

(iii) $\left\{(n-1)f'(v)g_{ij}+R_{ij}\right\}n^in^j\geqq 0$ *at each point on* V^{n-1} [5].

Then every point of V^{n-1} is umbilic. If $f(v)=kv$ or $f(v)=k$ ($k=const.$) and $v\neq const.$ on V^{n-1}, the hypersurface V^{n-1} is isometric to a sphere.

To prove that the hypersurface under consideration is isometric to a sphere, the following theorem due to M. Obata [22] has been used:

Theorem (M. Obata) *If R^n ($n\geqq 2$) is complete and admits a non-null function φ such that $\varphi_{;i;j}=-c^2\varphi g_{ij}$ ($c=const.$), then R^n is isometric to a sphere of radius* $\frac{1}{c}$.

The purpose of the present paper is to investigate certain conditions for a closed orientable submanifold V^m in R^n to be isometric to a sphere. **§1** is devoted to give notations and fundamental formulas in the theory of submanifolds in a general Riemann space. In **§2** we give certain generalization of Theorem 0.6 due to K. Yano. The method of proof of our theorem is learned much from the paper of K. Yano [15].

Let us denote by M^n an n-dimensional Riemann space which admits a vector field ξ^i generating a continuous one-parameter group G of conformal transformations in M^n such that there exist a family of hypersurfaces of M^n orthogonal to the trajectories of the group G. Analytic representations of characteristic properties of M^n are shown in **§3**. In **§4** we derive the integral formulas which are valid for a closed orientable submanifold V^m in M^n. In **§5** we apply the integral formulas obtained in **§4** to a closed orientable submanifold V^m whose first mean curvature H_1 is constant and give some conditions for every point of V^m to be umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$. Making use of results in **§2** and **§5**, we give in the last

4) ";" denotes the covariant differentiation with respect to the Christoffel symbol formed with metric tensor g_{ij} of R^n.

5) $f'(v)$ denotes $\frac{df}{dv}$ and R_{ij} is the Ricci tensor.

section §6 certain conditions for a closed orientable submanifold V^m in M^n to be isometric to a sphere. Especially when M^n is a constant Riemann curvature space or admits a homothetic Killing vector field, making use of Theorem 0. 3, Theorem 0. 4 and Theorem 0. 5 we can arrive at a conclusion such that V^m is isometric to a sphere.

The present author wishes to express his very sincere thanks to Professor Y. Katsurada for her many valuable advices and constant guidances.

§ 1. Notations and fundamental formulas in the theory of submanifolds. Let R^n $(n\geqq 3)$ be an n-dimensional Riemann space of class C^r $(r\geqq 3)$ and x^i, g_{ij} and R_{ijhk} be local coordinates, positive definite metric tensor and curvature tensor of R^n respectively. We now consider an m-dimensional closed orientable submanifold V^m $(m\leqq n-1)$ in R^n whose local expression is

$$x^i = x^i(u^\alpha),$$

where u^α denotes local coordinate on V^m. If we put

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha},$$

then B^i_α $(\alpha=1, 2, \cdots, m)$ are m independent vectors tangent to V^m and an induced metric tensor $g_{\alpha\beta}$ is given by

$$g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta .$$

We indicate by $\underset{P}{n^i}$ $(P=m+1, m+2, \cdots, n)$[6] the contravariant components of $n-m$ unit vectors which are normal to V^m and mutually orthogonal. Hence they satisfy the following relations:

$$g_{ij}B^i_\alpha \underset{P}{n^j} = 0, \qquad g_{ij}\underset{P}{n^i}\underset{Q}{n^j} = \delta_{PQ},$$

where δ_{PQ} means the Kronecker delta. In this case a set of n independent vectors

$$(1.1) \qquad (B^i_1, B^i_2, \cdots, B^i_m, \underset{m+1}{n^i}, \underset{m+2}{n^i}, \cdots, \underset{n}{n^i})$$

determines an ennuple at each point on V^m. We put

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \qquad \underset{P}{n_i} = g_{ij}\underset{P}{n^j},$$

where $g^{\alpha\beta}$ are defined by the equations $g^{\alpha\beta}g_{\beta\gamma}=\delta^\alpha_\gamma$. Then we have

$$(1.2) \qquad \begin{aligned} g^{ij} &= g^{\alpha\beta}B^i_\alpha B^j_\beta + \sum_{P=m+1}^{n} \underset{P}{n^i}\underset{P}{n^j}, \\ g_{ij} &= g_{\alpha\beta}B^\alpha_i B^\beta_j + \sum_{P=m+1}^{n} \underset{P}{n_i}\underset{P}{n_j}, \end{aligned}$$

6) Throughout the present paper the capital Latin indices run from $m+1$ to n.

$$B^i_j + C^i_j = \delta^i_j , \tag{1.3}$$

where we put

$$B^i_j = B^i_\alpha B^\alpha_j , \qquad C^i_j = \sum_{P=m+1}^{n} n^i_P n_{jP} .$$

Denoting by " ; " the operation of D-symbol due to van der Waerden-Bortolotti [23], from the definition we have

$$B^i_{\alpha;\beta} = B^i_{j;k} B^j_\alpha B^k_\beta , \tag{1.4}$$

$$n^i_{P;\alpha} = C^i_{j;k} n^j_P B^k_\alpha . \tag{1.5}$$

Putting $H^i_{\alpha\beta} = B^i_{\alpha;\beta}$, we call $H^i_{\alpha\beta}$ the Euler-Schouten curvature tensor. By means of (1.3) and (1.4) it follows that

$$\begin{aligned} H^i_{\alpha\beta} &= (\delta^i_j - C^i_j)_{;k} B^j_\alpha B^k_\beta \\ &= -\sum_{P=m+1}^{n} (n_{Pj;k} B^j_\alpha B^k_\beta) n^i_P . \end{aligned}$$

Therefore if we put $b_{P\alpha\beta} = H^i_{\alpha\beta} n_{Pi}$, we have

$$H^i_{\alpha\beta} = \sum_{P=m+1}^{n} b_{P\alpha\beta} n^i_P . \tag{1.6}$$

Multiplying (1.6) by $g^{\alpha\beta}$ and summing for α and β, we get

$$g^{\alpha\beta} H^i_{\alpha\beta} = \sum_{P=m+1}^{n} m H_{1P} n^i_P , \tag{1.7}$$

where we put $H_{1P} = \frac{1}{m} g^{\alpha\beta} b_{P\alpha\beta}$. H_{1P} is called the first mean curvature of V^m for the normal vector n^i_P.

Let n^i_E be a unit vector which has the same direction to the vector $g^{\alpha\beta} H^i_{\alpha\beta}$ and we call it the Euler-Schouten unit vector. Then the components of the vector n^i_E are independent of a change of parameters u^α on V^m, that is, the vector n^i_E is determined uniquely at each point on V^m. By means of (1.6) n^i_E is a unit normal vector of V^m. In the set of n independent vectors (1.1), $n-m$ unit normal vectors n^i_P may be chosen in a multiply infinite number of ways. Hence we may consider n^i_E as one of n^i_P in (1.1). Consequently, putting $n^i_{m+1} = n^i_E$ we take a set of n independent vectors

$$(B^i_1, B^i_2, \cdots, B^i_m, n^i_E, n^i_{m+2}, \cdots, n^i_n) \tag{1.8}$$

as an ennuple at each point on V^m.

The first mean curvature of V^m for normal vector $\underset{E}{n^i}$ is the socalled first mean curvature of V^m. Hence we denote it by H_1 without subscript E. In this case, with respect to the ennuple (1. 8) we get from (1. 7)

$$g^{\alpha\beta}H_{\alpha\beta}^{i} = mH_1\underset{E}{n^i}\,. \tag{1. 9}$$

If we denote by $\underset{E}{k_1}, \underset{E}{k_2}, \cdots, \underset{E}{k_m}$ the principal curvatures of V^m for the normal vector $\underset{E}{n^i}$, that is, the roots of the characteristic equation

$$\det.\ (\underset{E}{b_{\alpha\beta}} - k g_{\alpha\beta}) = 0\,,$$

the ν-th mean curvature $\underset{E}{H_\nu}$ of V^m for the normal vector $\underset{E}{n^i}$ is defined to be the ν-th elementary symmetric function of $\underset{E}{k_1}, \underset{E}{k_2}, \cdots, \underset{E}{k_m}$ divided by the number of terms, i. e.,

$$\binom{m}{\nu}\underset{E}{H_\nu} = \sum_{\alpha_1<\alpha_2<\cdots<\alpha_\nu} \underset{E}{k_{\alpha_1}}\,\underset{E}{k_{\alpha_2}}\cdots\underset{E}{k_{\alpha_\nu}} \qquad (1\leqq\nu\leqq m)\,.$$

From the above definition, especially H_1 and $\underset{E}{H_2}$ satisfy the following relations:

$$mH_1 = \sum_\alpha \underset{E}{k_\alpha} = \underset{E}{b_\alpha^\alpha}\,, \tag{1. 10}$$

$$\binom{m}{2}\underset{E}{H_2} = \sum_{\alpha<\beta} \underset{E}{k_\alpha}\underset{E}{k_\beta} = \frac{1}{2}(\underset{E}{b_\alpha^\alpha}\underset{E}{b_\beta^\beta} - \underset{E}{b_\beta^\alpha}\underset{E}{b_\alpha^\beta})\,, \tag{1. 11}$$

where $\underset{E}{b_\alpha^\beta} = \underset{E}{g^{\beta\gamma}}b_{\alpha\gamma}$. By virtue of (1. 10) and (1. 11) it follows that

$$\underset{E}{b_\beta^\alpha}\underset{E}{b_\alpha^\beta} = m\left\{mH_1^2 - (m-1)\underset{E}{H_2}\right\}. \tag{1. 12}$$

On the other hand, by means of (1. 3) and (1. 5) it follows that

$$\begin{aligned}\underset{P}{n^i}{}_{;\alpha} &= (\delta_j^i - B_j^i)_{;k}\underset{P}{n^j}B_\alpha^k \\ &= -(B_{j;k}^\beta \underset{P}{n^j}B_\alpha^k)B_\beta^i\,.\end{aligned}$$

Therefore we may put as follows:

$$\underset{P}{n^i}{}_{;\alpha} = \underset{P}{\gamma_\alpha^\beta}B_\beta^i\,. \tag{1. 13}$$

Multiplying (1. 13) by $g_{ij}B_\gamma^j$ and contracting, we have

$$g_{ij}B_\gamma^j\underset{P}{n^i}{}_{;\alpha} = \underset{P}{\gamma_\alpha^\beta}g_{\beta\gamma}\,. \tag{1. 14}$$

Since we have

$$\underset{P}{b_{\gamma\alpha}} = g_{ij}B_{\gamma;\alpha}^j\underset{P}{n^i} = -\,g_{ij}B_\gamma^j\underset{P}{n^i}{}_{;\alpha}\,,$$

by means of (1. 14) we get

$$\underset{P}{\gamma}{}^{\beta}_{\alpha\gamma}g_{\beta\gamma} = -\underset{P}{b}_{\alpha\gamma}.$$

Consequently we obtain

$$\underset{P}{n}{}^{i}_{;\alpha} = -\underset{P}{b}{}^{\gamma}_{\alpha}B^{i}_{\gamma}. \tag{1. 15}$$

By virtue of (1. 4) and (1. 5), after some calculations we get

$$\begin{aligned} B^{i}_{\alpha;\beta} &= \frac{\partial B^{i}_{\alpha}}{\partial u^{\beta}} + \Gamma^{i}_{hj}B^{h}_{\alpha}B^{j}_{\beta} - \Gamma'{}^{\gamma}_{\alpha\beta}B^{i}_{\gamma}, \\ \underset{P}{n}{}^{i}_{;\alpha} &= \frac{\partial \underset{P}{n}{}^{i}}{\partial x^{j}}B^{j}_{\alpha} + \Gamma^{i}_{hj}\underset{P}{n}{}^{h}B^{j}_{\alpha} - \Gamma''{}^{Q}_{P\alpha}\underset{Q}{n}{}^{i}, \end{aligned} \tag{1. 16}$$

where Γ^{i}_{hj} are the Christoffel symbols formed with g_{ij} and

$$\Gamma'{}^{\gamma}_{\alpha\beta} = \Gamma^{i}_{hj}B^{h}_{\alpha}B^{j}_{\beta}B^{\gamma}_{i} + \frac{\partial B^{i}_{\alpha}}{\partial u^{\beta}}B^{\gamma}_{i},$$

$$\Gamma''{}^{Q}_{P\alpha} = \frac{\partial \underset{P}{n}{}^{i}}{\partial x^{j}}B^{j}_{\alpha}\underset{Q}{n}{}_{i} + \Gamma^{i}_{hj}\underset{P}{n}{}^{h}B^{j}_{\alpha}\underset{Q}{n}{}_{i}.$$

Since $\underset{P}{n}{}^{i}\underset{Q}{n}{}_{i} = \delta_{PQ}$, from the last relation we can easily find

$$\Gamma''{}^{Q}_{P\alpha} + \Gamma''{}^{P}_{Q\alpha} = 0. \tag{1. 17}$$

By virtue of (1. 6) and (1. 15) we have

$$\underset{P}{b}_{\alpha\delta;\beta} - \underset{P}{b}_{\alpha\beta;\delta} = (H_{\alpha\delta}{}^{i}{}_{;\beta} - H_{\alpha\beta}{}^{i}{}_{;\delta})\underset{P}{n}{}_{i}.$$

Consequently we obtain the equation of Codazzi in the form

$$\underset{P}{b}_{\alpha\delta;\beta} - \underset{P}{b}_{\alpha\beta;\delta} = -R_{ikjl}\underset{P}{n}{}^{i}B^{k}_{\alpha}B^{j}_{\delta}B^{l}_{\beta} \quad ([23], \text{ p. } 266). \tag{1. 18}$$

§ 2. Some properties of V^m in R^n admitting a scalar field v such that $v_{;i;j} = f(v)g_{ij}$. Let us consider that R^n admit a scalar field v such that

$$v_{;i;j} = f(v)g_{ij} \tag{2. 1}$$

and V^m be a closed orientable submanifold in R^n. We have

$$v_{;\alpha} = v_{;i}B^{i}_{\alpha}.$$

Covariantly differentiating[7] the vector $v_{;\alpha}$ we have

$$v_{;\alpha;\beta} = v_{;i;j}B^{i}_{\alpha}B^{j}_{\beta} + v_{;i}H_{\alpha\beta}{}^{i}. \tag{2. 2}$$

7) In the present paper, covariant differentiation means always the operation of D-symbol.

Multiplying (2. 2) by $g^{\alpha\beta}$ and contracting, by means of (1. 9) and (2. 1) we get

$$v^{\alpha}{}_{;\alpha} = mf(v) + mH_1 v_{;i} \underset{E}{n^i} . \qquad (v^{\alpha} = g^{\alpha\gamma} v_{;\gamma})$$

Hence by virtue of Green's theorem ([16], p. 31) we obtain the following integral formula:

$$\int_{V^m} f(v) dA + \int_{V^m} H_1 v_{;i} \underset{E}{n^i} dA = 0 , \qquad (I)$$

where dA is the area element of V^m.

Now we put

$$w_{\alpha} = \left(\frac{\partial \underset{E}{n^i}}{\partial x^j} B^j_{\alpha} + \Gamma^i_{hj} \underset{E}{n^h} B^j_{\alpha} \right) v_{;i} .$$

By means of (1. 16) we have

$$w_{\alpha} = (\underset{E}{n^i}{}_{;\alpha} + \Gamma''_{E\,\alpha}{}^{P}_{P} \underset{E}{n^i}) v_{;i} .$$

We assume that at each point of V^m the vector $v^i (= g^{ij} v_{;j})$ is contained in the vector space spanned by $m+1$ vectors B^i_{α} $(\alpha = 1, 2, \cdots, m)$ and $\underset{E}{n^i}$. Then we put as follows:

$$v^i = v^{\alpha} B^i_{\alpha} + \phi \underset{E}{n^i} . \qquad (2. 3)$$

Then by means of (1. 15), (1. 17) and (2. 3) we have

$$w_{\alpha} = -\underset{E}{b^{\gamma}_{\alpha}} B^i_{\gamma} v_{;i} .$$

From the definition of w_{α}, it is remarkable that the subscript E in the last relation does not mean the vector component. Accordingly, covariantly differentiating the covariant vector w_{α}, we have

$$w_{\alpha;\beta} = -(\underset{E}{b^{\gamma}_{\alpha;\beta}} B^i_{\gamma} v_{;i} + \Gamma''_{E\,\beta}{}^{P}_{P} \underset{E}{b^{\gamma}_{\alpha}} B^i_{\gamma} v_{;i} + \underset{E}{b^{\gamma}_{\alpha}} H^i_{\gamma\beta} v_{;i} + \underset{E}{b^{\gamma}_{\alpha}} B^i_{\gamma} B^j_{\beta} v_{;i;j}) . \qquad (2. 4)$$

Making use of (1. 15), (1. 17), (2. 1) and (2. 3) we can easily find

$$\Gamma''_{E\,\beta}{}^{P}_{P} \underset{E}{b^{\gamma}_{\alpha}} B^i_{\gamma} v_{;i} = 0$$

and

$$\underset{E}{b^{\gamma}_{\alpha}} H^i_{\gamma\beta} v_{;i} = \underset{E}{b^{\gamma}_{\alpha}} \underset{E}{b_{\gamma\beta}} \underset{E}{n^i} v_{;i} , \qquad \underset{E}{b^{\gamma}_{\alpha}} B^i_{\gamma} B^j_{\beta} v_{;i;j} = f(v) \underset{E}{b_{\alpha\beta}} .$$

Then, multiplying (2. 4) by $g^{\alpha\beta}$ and contracting we have

$$w^{\alpha}{}_{;\alpha} = -\underset{E}{g^{\alpha\beta} b^{\gamma}_{\alpha;\beta}} B^i_{\gamma} v_{;i} - \underset{E}{b^{\gamma}_{\alpha}} \underset{E}{b^{\alpha}_{\gamma}} \underset{E}{n^i} v_{;i} - f(v) \underset{E}{b^{\alpha}_{\alpha}} . \qquad (2. 5)$$

By means of (1. 10) and the equations of Codazzi, the first term of the right

hand side of (2. 5) becomes

$$g^{\alpha\beta}\underset{E}{b^{\gamma}_{\alpha;\beta}}B^{i}_{\gamma}v_{;i}=(mH_{1;\delta}g^{\gamma\delta}-g^{\alpha\beta}g^{\gamma\delta}\underset{E}{R_{ikjl}n^{i}}B^{k}_{\alpha}B^{j}_{\delta}B^{l}_{\beta})B^{h}_{\gamma}v_{;h}\,.$$

By virtue of (1. 2) the last term of the right hand side in the preceding equation is rewritten as follows:

$$g^{\alpha\beta}g^{\gamma\delta}\underset{E}{R_{ikjl}n^{i}}B^{k}_{\alpha}B^{j}_{\delta}B^{l}_{\beta}B^{h}_{\gamma}v_{;h}=\underset{E}{R_{ikjl}n^{i}}v^{j}g^{*kl}-\underset{E\ E}{R_{ikjl}n^{i}n^{j}}g^{*kl}\underset{E}{n^{h}}v_{;h}\,,$$

where $g^{*kl}=g^{\alpha\beta}B^{k}_{\alpha}B^{l}_{\beta}$. Making use of the Ricci identity [24] and (2. 1) we can easily find

$$\underset{E}{R_{ikjl}n^{i}}v^{j}g^{*kl}=-mf'(v)\underset{E}{n^{i}}v_{;i}\,.$$

Consequently by means of (1. 10), (1. 12) and (2. 5) we have

$$\begin{aligned}w^{\alpha}{}_{;\alpha}=-\Big[&mH_{1;\delta}v^{\delta}+\underset{E\ E}{R_{ikjl}n^{i}n^{j}}g^{*kl}\phi+mf'(v)\phi\\&+m\Big\{mH_{1}^{2}-(m-1)\underset{E}{H_{2}}\Big\}\phi+mH_{1}f(v)\Big]\end{aligned}$$

Then, by virtue of Green's theorem we obtain

$$\begin{aligned}\int_{V^{m}}\Big[&mH_{1;\delta}v^{\delta}+\underset{E\ E}{R_{ikjl}n^{i}n^{j}}g^{*kl}\phi+mf'(v)\phi\\&+m\Big\{mH_{1}^{2}-(m-1)\underset{E}{H_{2}}\Big\}\phi+mH_{1}f(v)\Big]dA=0\,.\end{aligned}\qquad\text{(II)}$$

If the first mean curvature H_1 of V^m is constant, we have from (I) and (II)

$$\int_{V^{m}}mH_{1}f(v)dA+\int_{V^{m}}mH_{1}^{2}\phi dA=0\,.$$

and

$$\int_{V^{m}}\Big[\underset{E\ E}{R_{ikjl}n^{i}n^{j}}g^{*kl}\phi+mf'(v)\phi+m\Big\{mH_{1}^{2}-(m-1)\underset{E}{H_{2}}\Big\}\phi+mH_{1}f(v)\Big]dA=0\,.$$

Eliminating $\int_{V^{m}}mH_{1}f(v)dA$ from above two equations, we obtain

$$\int_{V^{m}}\Big\{\underset{E\ E}{R_{ikjl}n^{i}n^{j}}g^{*kl}+m(m-1)(H_{1}^{2}-\underset{E}{H_{2}})+mf'(v)\Big\}\phi dA=0\,.\qquad(2.6)$$

Hence we have

Theorem 2. 1. *Let R^n be an n-dimensional Riemann space which admits a non-constant scalar field v such that*

$$v_{;i;j}=f(v)g_{ij}$$

and V^m *a closed orientable submanifold such that*

(i) $H_1 = const.$,

(ii) $v_{;i}\underset{E}{n^i}$ *has fixed sign on* V^m,

(iii) v^i *is contained in the vector space spanned by* $m+1$ *vectors* B^i_α $(\alpha=1, 2, \cdots, m)$ *and* $\underset{E}{n^i}$ *at each point on* V^m,

(iv) $mf'(v)+R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl}\geqq 0$ *at each point on* V^m.

Then every point of V^m *is umbilic with respect to Euler-Schouten unit vector* $\underset{E}{n^i}$.

Proof. By means of (1. 10) and (1. 11) we have

$$H_1^2-\underset{E}{H_2}=\frac{1}{m^2(m-1)}\sum_{\alpha<\beta}(\underset{E}{k_\alpha}-\underset{E}{k_\beta})^2\geqq 0\,. \tag{2. 7}$$

Hence from (2. 6) and our hypothesis it should be satisfied that $H_1^2-\underset{E}{H_2}=0$. Therefore by means of (2. 7) we obtain $\underset{E}{k_1}=\underset{E}{k_2}=\cdots=\underset{E}{k_m}$.

Theorem 2. 2. *Let* R^n *be an n-dimensional Riemann space which admits a non-constant scalar field* v *such that*

$$v_{;i;j}=kvg_{ij} \qquad (k=const.) \tag{2. 8}$$

and V^m *a closed orientable submanifold such that*

(i) $H_1 = const.$,

(ii) $v_{;i}\underset{E}{n^i}$ *has fixed sign on* V^m,

(iii) v^i *is contained in the vector space spanned by* $m+1$ *vectors* B^i_α $(\alpha=1, 2, \cdots, m)$ *and* $\underset{E}{n^i}$ *at each point on* V^m,

(iv) $mk+R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl}\geqq 0$ *at each point on* V^m,

where $v\neq const.$ *on* V^m. *Then* V^m *is isometric to a sphere.*

Proof. By virtue of Theorem 2. 1, every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$. Since $H_1=const.$, we have

$$\underset{E}{b_{\alpha\beta}}=\lambda g_{\alpha\beta} \qquad (\lambda=const.) \tag{2. 9}$$

Then from (2. 2) and (2. 9) we get

$$v_{;\alpha;\beta}=(kv+\lambda v_{;i}\underset{E}{n^i})g_{\alpha\beta}\,. \tag{2. 10}$$

Covariantly differentiating the scalar $v_{;i}\underset{E}{n^i}$ we have

$$(v_{;i}\underset{E}{n^i})_{;\alpha} = v_{;i;j}\underset{E}{n^i}B^j_\alpha + v_{;i}\underset{E}{n^i}_{;\alpha} + v_{;i}\Gamma''^{\,P}_{E\alpha}\underset{P}{n^i}\,.$$

By means of our hypothesis (iii), (1. 15), (1. 17), (2. 8) and (2. 9) we have

$$(v_{;i}\underset{E}{n^i})_{;\alpha} = -\lambda v_{;\alpha}\,.$$

Hence we obtain

$$v_{;i}\underset{E}{n^i} = -\lambda v + c\,. \qquad (c = const.) \tag{2. 11}$$

Substituting (2. 11) into (2. 10) we have

$$v_{;\alpha;\beta} = \left\{(k-\lambda^2)v + c\lambda\right\} g_{\alpha\beta}\,. \tag{2. 12}$$

If $k-\lambda^2=0$, then from (2. 12) we have $\Delta v = mc\lambda$, where Δ means the Laplacian operator. This is impossible unless $v=const.$ [16]. Then $k-\lambda^2\neq 0$, and from (2. 12) we have

$$\left(v+\frac{c\lambda}{k-\lambda^2}\right)_{;\alpha;\beta} = (k-\lambda^2)\left(v+\frac{c\lambda}{k-\lambda^2}\right)g_{\alpha\beta}\,. \tag{2. 13}$$

Therefore we obtain

$$\Delta\left(v+\frac{c\lambda}{k-\lambda^2}\right) = m(k-\lambda^2)\left(v+\frac{c\lambda}{k-\lambda^2}\right).$$

Consequently it follows that $k-\lambda^2<0$ [17]. Hence, by virtue of Obata's theorem V^m is isometric to a sphere.

Theorem 2. 3. *Let R^n be an n-dimensional Riemann space which admits a non-constant scalar field v such that*

$$v_{;i;j} = kg_{ij} \qquad (k = const.) \tag{2. 14}$$

and V^m a closed orientable submanifod such that

(i) $H_1 = const.$,

(ii) $v_{;i}\underset{E}{n^i}$ *has fixed sign on* V^m,

(iii) v^i *is contained in the vector space spanned by* $m+1$ *vectors* B^i_α $(\alpha=1, 2, \cdots, m)$ *and* $\underset{E}{n^i}$ *at each point on* V^m,

(iv) $R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl} \geqq 0$ *at each point on* V^m,

where $v\neq const.$ on V^m. Then V^m is isometric to a sphere.

Proof. By virtue of Theorem (2. 1), every point of V^m is umbilic with respect to Euler-Schuten unit vector $\underset{E}{n^i}$. Since $H_1=const.$, we have

$$\underset{E}{b_{\alpha\beta}} = \lambda g_{\alpha\beta}. \qquad (\lambda = const.)$$

Hence from (2. 2) and (2. 14) we get

$$v_{;\alpha;\beta} = (k + \lambda v_{;i} \underset{E}{n^i}) g_{\alpha\beta} . \tag{2. 15}$$

As we have (2. 11), from (2. 15) it follows that

$$v_{;\alpha;\beta} = (-\lambda^2 v + c\lambda + k) g_{\alpha\beta} . \tag{2. 16}$$

If $\lambda = 0$, from (2. 16) we get $\Delta v = m(c\lambda + k)$ and this is impossible unless $v =$ *const.* [16]. Therefore $\lambda \neq 0$ and (2.16) is rewritten as follows:

$$\left(v - \frac{c\lambda + k}{\lambda^2} \right)_{;\alpha;\beta} = -\lambda^2 \left(v - \frac{c\lambda + k}{\lambda^2} \right) g_{\alpha\beta} .$$

Consequently by virtue of Obata's theorem, V^m is isometric to a sphere.

Remark. When $m = n-1$, that is, V^m is a closed orientable hypersurface in R^n, Euler-Schouten unit vector $\underset{E}{n^i}$ is the unit normal vector n^i of V^{n-1} and $g^{*kl} = g^{kl} - n^k n^l$. In this case our hypothesis (iii) in the above three theorems are satisfied identically and (iv) becomes

$$mf'(v) + R_{ikjl} \underset{E}{n^i} \underset{E}{n^j} g^{*kl} = (mf'(v) g_{ij} + R_{ij}) n^i n^j \geqq 0 .$$

Therefore if $m = n-1$, our results coincide with those of Theorem 0. 6 (K. Yano).

§ 3. Properties of the vector field of M^n. Let ξ^i be a vector field in R^n such that

$$\xi_{i;j} + \xi_{j;i} = 2\Phi g_{ij}$$

where Φ is a scalar field in R^n. Then ξ^i is called a conformal Killing vector field and a continuous one-parameter group G generated by an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i \delta\tau$$

is called a conformal transformation group. If $\Phi = c$ ($c =$ *const.*), ξ^i is called a homothetic Killing vector field and the group G is called a homothetic transformation group [18].

With respect to a scalar field σ in R^n, if a system of differential equations

$$\frac{\partial f}{\partial x^i} = e^{2\sigma} \xi_i$$

is integrable, then R^n admits a family of hypersurfaces defined by

$$f(x^1, x^2, \cdots, x^n) = \text{const.} ,$$

where $f(x^i)$ is a solution of the above system of differential equations. In

this case every hypersurface belonging to the family is orthogonal to the trajectories of the group G.

Theorem 3.1. *Let ξ^i and σ be a vector field and a scalar field in R^n respectively. In order that ξ^i is a conformal Killing vector field and a system of differential eqnations*

$$\frac{\partial f}{\partial x^i} = e^{2\sigma}\xi_i \tag{3.1}$$

is integrable, it is necessary and sufficient that there exist a scalar field Φ in R^n snch that

$$\xi_{i;j} = \Phi g_{ij} + \sigma_{;i}\xi_j - \sigma_{;j}\xi_i \,. \tag{3.2}$$

Proof. If ξ^i is a conformal Killing vector field, there exists a scalar field Φ and

$$\xi_{i;j} + \xi_{j;i} = 2\Phi g_{ij} \,. \tag{3.3}$$

Covariantly differentiating (3.1) with respect to x^j, we obtain

$$f_{;i;j} = e^{2\sigma}(2\sigma_{;j}\xi_i + \xi_{i;j}) \,. \tag{3.4}$$

If the system of differential equations (3.1) is integrable, by means of its integrability conditions and (3.4) we have

$$2\sigma_{;j}\xi_i + \xi_{i;j} - 2\sigma_{;i}\xi_j - \xi_{j;i} = 0 \,. \tag{3.5}$$

Consequently, by means of (3.3) and (3.5) we obtain (3.2).

Conversely, if there exists a scalar field Φ and if (3.2) is satisfied we can easily obtain (3.3). Hence ξ^i is a conformal Killing vector field. On the other hand we have

$$(e^{2\sigma}\xi_i)_{;j} - (e^{2\sigma}\xi_j)_{;i} = e^{2\sigma}(2\sigma_{;j}\xi_i + \xi_{i;j} - 2\sigma_{;i}\xi_j - \xi_{j;i}) \,.$$

By means of (3.2) we find that the right hand side of the above relation vanishes identically. Therefore a system of differential equations (3.1) is integrable.

Let M^n be an n-dimensional Riemann space admitting a conformal Killing vector field ξ^i and a scalar field σ such that a system of differential equations (3.1) is integrable. Then by virtue of Theorem 3.1, M^n is characterized by admitting a vector field ξ^i and scalar field Φ and σ such that (3.2) is satisfied.

As a special case of Theorem 3.1 we have the following Corollary:

Corollary 3.1. *Let ξ^i and σ be a vector field and a scalar field such that $\sigma_{;i} = \mu\xi_i$. In order that ξ^i is a conformal Killing vector field and a system of differential equations* (3.1) *is integrable, it is necessary and suf-*

ficient that there exist a scalar field Φ *such that* $\xi_{i;j}=\Phi g_{ij}$.

Remark. Let a point O of an n-dimensional Euclidean space E^n be origin of cartesian coordinate system in E^n and y^i the coordinate of E^n with respect to the coordinate system. If we take the position vector y^i as the vector ξ^i, then we have

$$y_{i;j}=g_{ij},$$

where g_{ij} denotes covariant compoment of the metaic tensor in E^n with respect to the above cartesian coordinate system. Hence the vector field y^i is a homothetic Killing vector field in E^n, where we have $\Phi=1$. The trajectories of the homothetic transformation group G are straight lines which pass through the origin O. If we put $\sigma=\frac{1}{4}g_{jh}y^jy^h$, we have

$$\sigma_{;i}=\frac{1}{2}y_i.$$

This is the case when $\mu=\frac{1}{2}$ in Corollary 3. 1. Consequently a system of differential equations

$$\frac{\partial f}{\partial y^i}=e^{2\sigma}y_i$$

is integrable. If we put

$$f(y^1, y^2, \cdots, y^n)=e^{\frac{1}{2}(g_{jh}y^jy^h)},$$

we can easily verify that $f(y^i)$ satisfies the above system of differential equations. In this case a family of hypersurfaces defined by $f(y^1, y^2, \cdots, y^n)=const.$ is a family of hyperspheres for which the origin O is common centre. It is evident that each hypersphere belonging to this family be orthogonal to the trajectories of the homothetic transformation group G. In consequence of the above observations, E^n may be considered as a special case of M^n.

§ 4. Integral formulas for a closed orientable submanifold in M^n. Let M^n be an n-dimensional Riemann space admitting a vector field ξ^i and scalar fields Φ and σ such that

$$\xi_{i;j}=\Phi g_{ij}+\sigma_{;i}\xi_j-\sigma_{;j}\xi_i \tag{4. 1}$$

and V^m an m-dimensional closed orientable submanifold in M^n whose local expression is $x^i=x^i(u^\alpha)$. In this section we derive some integral formulas which are valid for V^m in M^n.

To the vector ξ^i, there is a covariant vector $\bar{\xi}_\alpha$ of V^m with the components

$$\bar{\xi}_\alpha = \xi_i B^i_\alpha .$$

Covariantly differentiating the vector $\bar{\xi}_\alpha$, we have

$$\bar{\xi}_{\alpha;\beta} = \xi_{i;j} B^i_\alpha B^j_\beta + \xi_i H_{\alpha\beta}{}^i . \tag{4.2}$$

Multiplying (4.2) by $g^{\alpha\beta}$ and contracting, by means of (1.9) and (4.1) we obtain

$$\bar{\xi}^\alpha{}_{;\alpha} = (\Phi g_{ij} + T_{ij}) B^i_\alpha B^j_\beta g^{\alpha\beta} + m H_1 \xi_i \underset{E}{n^i} ,$$

where we put

$$T_{ij} = \sigma_{;i}\xi_j - \sigma_{;j}\xi_i .$$

Since T_{ij} is skew-symmetric with respect to its indices, we obtain

$$\bar{\xi}^\alpha{}_{;\alpha} = m\Phi + m H_1 \xi_i \underset{E}{n^i} .$$

As V^m is closed and orientable, by virtue of Green's theorem we get the following integral formula:

$$\int_{V^m} \Phi dA + \int_{V^m} H_1 \xi_i \underset{E}{n^i} dA = 0 , \tag{I'}$$

where dA is the area element of V^m.

Now we put

$$\eta_\alpha = \left(\frac{\partial \underset{E}{n^i}}{\partial x^j} B^j_\alpha + \Gamma^i_{hj} \underset{E}{n^h} B^j_\alpha \right) \xi_i .$$

By means of (1.16) we have

$$\eta_\alpha = (\underset{E}{n^i}{}_{;\alpha} + \Gamma''_{E\alpha}{}^{P} \underset{P}{n^i}) \xi_i . \tag{4.3}$$

We assume that at each point of V^m the vector ξ^i is contained in the vector space spanned by $m+1$ independent vectors B^i_α $(\alpha = 1, 2, \cdots, m)$ and $\underset{E}{n^i}$. Then we put

$$\xi^i = \bar{\xi}^\alpha B^i_\alpha + \rho \underset{E}{n^i} . \tag{4.4}$$

By means of (1.15), (1.17) and (4.3) we have

$$\eta_\alpha = \underset{E}{n^i}{}_{;\alpha} \xi_i = - \underset{E}{b^r_\alpha} B^i_r \xi_i . \tag{4.5}$$

From the definition of η_α the subscript E in (4.5) does not mean vector component. Accordingly, covariantly differentiating the covariant vector η^α we get

$$\eta_{\alpha;\beta} = -(\underset{E}{b^r_{\alpha;\beta}} B^i_r \xi_i + \Gamma''_{E\beta}{}^{P} \underset{P}{b^r_\alpha} B^i_r \xi_i + \underset{E}{b^r_\alpha} H_{r\beta}{}^i \xi_i + \underset{E}{b^r_\alpha} B^i_r B^j_\beta \xi_{i;j}) .$$

Putting

$$F_{\alpha\beta} = \Gamma''_{E\beta}{}^{P}_{P} n^i_{;\alpha}\xi_i = -\Gamma''_{E\beta}{}^{P}_{P} b^\gamma_\alpha B^i_\gamma \xi_i \,, \tag{4.6}$$

by means of (4. 1) we have

$$\begin{aligned}\eta^\alpha{}_{;\alpha} = & -g^{\alpha\beta}_{E} b^\gamma_{\alpha;\beta} B^i_\gamma \xi_i + F^\alpha_\alpha \\ & - g^{\alpha\beta}_{E} b^\gamma_\alpha H_{\gamma\beta}{}^i \xi_i - g^{\alpha\beta}_{E} b^\gamma_\alpha B^i_\gamma B^j_\beta (\Phi g_{ij} + T_{ij}) \,.\end{aligned} \tag{4.7}$$

By means of (1. 2), (1. 10) and the equations of Codazzi, the first term of the right hand side of (4. 7) becomes

$$g^{\alpha\beta}_{E} b^\gamma_{\alpha;\beta} B^i_\gamma \xi_i = mH_{1;\delta}\bar{\xi}^\delta + R_{ikjl}\underset{E}{n^i} \underset{E}{n^j} g^{*kl}\rho - R_{ikjl}\underset{E}{n^i}\xi^j g^{*kl} \,.$$

Making use of the Ricci identity we have

$$R_{jlik}\xi^j \underset{E}{n^i} g^{*kl} = (\xi_{l;i;k} - \xi_{l;k;i})\underset{E}{n^i} g^{*kl} \,.$$

Hence by means of (4. 1) we get

$$\begin{aligned} R_{ikjl}\underset{E}{n^i}\xi^j g^{*kl} &= (\Phi_{;k} g_{li} + T_{li;k} - \Phi_{;i} g_{lk} - T_{lk;i})\underset{E}{n^i} g^{*kl} \\ &= T_{li;k}\underset{E}{n^i} g^{*kl} - m\Phi_{;i}\underset{E}{n^i} \,. \end{aligned}$$

Consequently we have

$$g^{\alpha\beta}_{E} b^\gamma_{\alpha;\beta} B^i_\gamma \xi_i = mH_{1;\delta}\bar{\xi}^\delta + R_{ikjl}\underset{E}{n^i}\underset{E}{n^j} g^{*kl}\rho - T_{li;k}\underset{E}{n^i} g^{*kl} + m\Phi_{;i}\underset{E}{n^i} \,. \tag{4.8}$$

By means of (1. 7), (1. 12) and (4. 4) the third term of the right hand side of (4. 7) becomes

$$g^{\alpha\beta}_{E} b^\gamma_\alpha H_{\gamma\beta}{}^i \xi_i = m\left\{mH_1^2 - (m-1)\underset{E}{H_2}\right\}\rho \,. \tag{4.9}$$

From (1. 10) and skew-symmetric property of T_{ij} the last term of the right hand side of (4. 7) is reduced as follows:

$$g^{\alpha\beta}_{E} b^\gamma_\alpha B^i_\gamma B^j_\beta (\Phi g_{ij} + T_{ij}) = m\Phi H_1 \,. \tag{4.10}$$

By virtue of (4. 7), (4. 8), (4. 9) and (4. 10) we have

$$\begin{aligned} -\eta^\alpha{}_{;\alpha} = & mH_{1;\delta}\bar{\xi}^\delta + R_{ikjl}\underset{E}{n^i}\underset{E}{n^j} g^{*kl}\rho - T_{li;k}\underset{E}{n^i} g^{*kl} \\ & + m\Phi_{;i}\underset{E}{n^i} - F^\alpha_\alpha + m\left\{mH_1^2 - (m-1)\underset{E}{H_2}\right\}\rho + m\Phi H_1 \,. \end{aligned}$$

Hence, by means of Green's theorem we get the following integral formula:

$$\begin{aligned} \int_{V^m} \Big[& mH_{1;\delta}\bar{\xi}^\delta + R_{ikjl}\underset{E}{n^i}\underset{E}{n^j} g^{*kl}\rho - T_{li;k}\underset{E}{n^i} g^{*kl} + m\Phi_{;i}\underset{E}{n^i} \\ & - F^\alpha_\alpha + m\left\{mH_1^2 - (m-1)\underset{E}{H_2}\right\}\rho + m\Phi H_1 \Big] dA = 0 \,. \end{aligned} \tag{4.11}$$

From (4. 4) we have

$$\rho = \underset{E}{\xi_i n^i} .$$

Covariantly differentiating the scalar ρ, by means of (4. 1) and (4. 5) we get

$$\rho_{;\alpha} = \underset{E}{T_{ij} n^i B^j_\alpha} + \eta_\alpha .$$

Then we have

$$\rho_{;\alpha;\beta} = \underset{E}{T_{ij;k} n^i B^j_\alpha B^k_\beta} + \underset{E}{T_{ij} n^i{}_{;\beta} B^j_\alpha} + \underset{P}{T_{ij} \Gamma''^{P}_{E\beta} n^i B^j_\alpha} + \underset{E}{T_{ij} n^i H_{\alpha\beta}{}^j} + \eta_{\alpha;\beta} .$$

Multiplying the above relations by $g^{\alpha\beta}$ and contracting, by virtue of (1. 9) and (1. 15) we get

$$\begin{aligned} \rho^\alpha{}_{;\alpha} = & \underset{E}{T_{ij;k} n^i g^{*jk}} - \underset{E}{T_{ij} b^\gamma_\beta B^i_\gamma B^j_\alpha g^{\alpha\beta}} \\ & + \underset{P}{T_{ij} \Gamma''^{P}_{E\beta} n^i B^j_\alpha g^{\alpha\beta}} + m H_1 \underset{E\ E}{T_{ij} n^i n^j} + \eta^\alpha{}_{;\alpha} . \end{aligned} \tag{4. 12}$$

By means of the skew-symmetric property of T_{ij}, the second and the fourth term of the right hand side of (4. 12) are vanish and the first term is rewritten as follows:

$$\underset{E}{T_{ij;k} n^i g^{*jk}} = -\underset{E}{T_{ji;k} n^i g^{*kj}} . \tag{4. 13}$$

Now, we calculate the third term of the right hand side of (4. 12). By virtue of (4. 1) it follows that

$$\underset{P}{T_{ij} \Gamma''^{P}_{E\beta} n^i B^j_\alpha g^{\alpha\beta}} = \underset{P}{\xi_{i;j} \Gamma''^{P}_{E\beta} n^i B^j_\alpha g^{\alpha\beta}} .$$

By means of (1. 17) and (4. 4) we get

$$\underset{P}{\xi_{i;j} \Gamma''^{P}_{E\beta} n^i B^j_\alpha} = -\underset{P}{\Gamma''^{P}_{E\beta} n^i{}_{;\alpha} \xi_i} .$$

Consequently from (4. 6) we have

$$\underset{P}{T_{ij} \Gamma''^{P}_{E\beta} n^i B^j_\alpha g^{\alpha\beta}} = -F^\alpha_\alpha . \tag{4. 14}$$

Hence by means of (4. 12), (4. 13) and (4. 14) we obtain the following integral formula:

$$\int_{V^m} (\underset{E}{T_{li;k} n^i g^{*kl}} + F^\alpha_\alpha) dA = 0 . \tag{4. 15}$$

By means of (4. 11) and (4. 15) we obtain the following integral formula for V^m in M^n:

$$\int_{V^m}\Big[mH_{1;\delta}\bar{\xi}^{\delta}+m\Phi_{;i}\underset{E}{n^i}+R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl}\rho \\ +m\Big\{mH_1^2-(m-1)\underset{E}{H_2}\Big\}\rho+m\Phi H_1\Big]dA=0\,. \tag{II'}$$

§ 5. Some properties of closed orientable submanifolds in M^n. We suppose that V^m be an m-dimensional closed orientable submanifold in M^n and the first mean curvature H_1 of V^m is constant. Then from the integral formula (I') and (II') we gst

$$\int_{V^m} m\Phi H_1 dA+\int_{V^m} mH_1^2\rho dA=0$$

and

$$\int_{V^m}\Big[m\Phi_{;i}\underset{E}{n^i}+R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl}+m\Big\{mH_1^2-(m-1)\underset{E}{H_2}\Big\}\rho+m\Phi H_1\Big]dA=0\,.$$

Eliminating $\int_{V^m} m\Phi H_1 dA$ from above two equations, we obtain

$$\int_{V^m}\Big\{m(m-1)(H_1^2-\underset{E}{H_2})\rho+m\Phi_{;i}\underset{E}{n^i}+R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl}\rho\Big\}dA=0\,. \tag{5.1}$$

Theorem 5.1. *Let M^n be an n-dimensional Riemann space admitting a vector field ξ^i and scalar fields Φ and σ such that*

$$\xi_{i;j}=\Phi g_{ij}+\sigma_{;i}\xi_j-\sigma_{;j}\xi_i\,,$$

and V^m an m-dimensional closed orientable submanifold in M^n such that

(i) $H_1=const.$,

(ii) *ξ^i is contained in the vector space spanned by $m+1$ vectors B^i_α $(\alpha=1,2,\cdots,m)$ and $\underset{E}{n^i}$ at each point on V^m*[8],

(iii) $m\Phi_{;i}\underset{E}{n^i}+R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl}\rho\geqq 0$ *(or* $\leqq 0$*) and* $\rho>0$ *(or* <0*) at each point on* V^m.

Then every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$.

Proof. By virtue of (2.7), (5.1) and our hypothesis it should be satisfied that

$$H_1^2-\underset{E}{H_2}=0\,.$$

8) If $m=n-1$ we have Theorem 5.1 without hypothesis (ii) because it is satisfied identically for a hypersurface. This remark is available for every theorems in **§5** and **§6** of the present paper.

Consequently we get $\underset{E}{k_1}=\underset{E}{k_2}=\cdots=\underset{E}{k_m}$.

Theorem 5.2. *Let M^n be an n-dimensional Riemann space admitting a vector field ξ^i and scalar fields Φ (Φ is not const.) and σ such that*

$$\xi_{i;j}=\Phi g_{ij}+\sigma_{;i}\xi_j-\sigma_{;j}\xi_i$$
$$\Phi_{;i}=\kappa\xi_i$$

and V^m an m-dimensional closed orientable submanifold in M^n such that

(i) $H_1=const.$,

(ii) *ξ^i is contained in the vector space spanned by $m+1$ vectors B^i_α ($\alpha=1,2,\cdots,m$) and $\underset{E}{n^i}$ at each point on V^m,*

(iii) *$\xi_i\underset{E}{n^i}$ has fixed sign on V^m,*

(iv) $m\kappa+R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl}\geqq 0$ *at each point on V^m.*

Then every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$.

Proof. By means of (5.1) and our hypothsis we should have $H_1^2-\underset{E}{H_2}=0$. Hence we obtain $\underset{E}{k_1}=\underset{E}{k_2}=\cdots=\underset{E}{k_m}$.

Let S^2 be a sphere in E^3 and P be a point on S^2. We denote by ξ^i a field of tangent vectors to great circles passing through the point P. This is an example of a vector fields which satisfy the hypothesis in Theorem 5.2.

If we suppose that $\kappa=0$ in Theorem 5.2, we have $\Phi=c$ ($c=$const.) and a continuous one-parameter group of transformations generated by an infinitesimal transformation $\bar{x}^i=x^i+\xi^i\delta\tau$ is homothetic transformation group. In this case we have the following theorem analogous to Theorem 0.5:

Theorem 5.3. *Let M^n be an n-dimensional Riemann space admitting a vector field ξ^i and a scalar field σ such that*

$$\xi_{i;j}=cg_{ij}+\sigma_{;i}\xi_j-\sigma_{;j}\xi_i\,,$$

where $c=const.$, and V^m an m-dimensional closed orientable submanifold such that

(i) $H_1=const.$,

(ii) *ξ^i is contained in the vector space spanned by $m+1$ vectors B^i_α ($\alpha=1,2,\cdots,m$) and $\underset{E}{n^i}$ at each point on V^m,*

(iii) *$\xi_i\underset{E}{n^i}$ has fixed sign on V^m,*

(iv) $R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl} \geqq 0$ *at each point on* V^m.

Then every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$.

§ 6. Certain characterizations of an m-dimensional sphere in M^n. In this section we give certain conditions for an m-dimensional closed orientable submanifold V^m in M^n to be isometric to a sphere.

Theorem 6. 1. *Let M^n be an n-dimensional Riemann space admitting a vector field ξ^i and scalar fields Φ (Φ is not constant) and σ such that*

$$\xi_{i;j} = \Phi g_{ij} + \sigma_{;i}\xi_j - \sigma_{;j}\xi_i , \tag{6. 1}$$

$$\Phi_{;i} = c\xi_i \qquad (c = \text{const.} \neq 0) \tag{6. 2}$$

and V^m an m-dimensional closed orientable submanifold such that

(i) $H_1 = const.$,

(ii) $\xi_i\underset{E}{n^i}$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by* $m+1$ *vectors* B^i_α $(\alpha = 1, 2, \cdots, m)$ *and* $\underset{E}{n^i}$ *at each point on* V^m,

(iv) $mc + R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl} \geqq 0$ *at each point on* V^m,

where $\Phi \neq const.$ on V^m. Then V^m is isometric to a sphere.

Proof. By virtue of Theorem 5. 2, we can see that every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$. By means of (6.2), we have

$$\Phi_{;i;j} = c\xi_{i;j} . \tag{6. 3}$$

This shows that $\xi_{i;j}$ is symmetric with respect to its indices. Hence, from (6. 1) it should be satisfied that

$$\sigma_{;i}\xi_j - \sigma_{;j}\xi_i = 0 .$$

Consequently we have

$$\xi_{i;j} = \Phi g_{ij} . \tag{6. 4}$$

By means of (6. 3) and (6. 4) we obtain

$$\Phi_{;i;j} = c\Phi g_{ij} .$$

Because of (6. 2) and hypothesis (iii), the vector Φ^i $(= g^{ij}\Phi_{;j})$ is contained in the vector space spanned by $m+1$ vectors B^i_α $(\alpha = 1, 2, \cdots, m)$ and $\underset{E}{n^i}$. Therefore, by virtue of Theorem 2. 2, V^m is isometric to a sphere.

Theorem 6.2. *Let M^n be an n-dimensional Riemann space admitting a homothetic Killing vector field ξ^i and a scalar field σ such that $\sigma_{;i}=\mu\xi_i$. We suppose that a system of differential equations $\frac{\partial f}{\partial x^i}=e^{2\sigma}\xi_i$ is integrable and V^m is an m-dimensional closed orientable submanifold such that*

(i) $H_1=const.$,

(ii) $\xi_i \underset{E}{n^i}$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by* $m+1$ *vectors* B^i_α $(\alpha=1,2,\cdots,m)$ *and* $\underset{E}{n^i}$ *at each point on* V^m,

(iv) $R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl}\geqq 0$ *at each point on* V^m,

where $\xi_\alpha\not\equiv 0$ on V^m. Then V^m is isometric to a sphere.

Proof. From our hypothesis and Theorem 5.3, we can see that every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$. By virtue of Corollary 3.1, there exists a constant field c in M^n and we have

$$\xi_{i;j}=cg_{ij}$$

Then $\xi_{i;j}$ is symmetric with respect to its indices and we can see that there exists a scalar field φ satisfying $\varphi_{;i}=\xi_i$. Consequently we obtain

$$\varphi_{;i;j}=cg_{ij}\,.$$

Furthermore, from our hypothesis (iii) the vector φ^i $(=g^{ij}\varphi_{;j})$ is contained in the vector space spanned by $m+1$ vectors B^i_α $(\alpha=1,2,\cdots,m)$ and $\underset{E}{n^i}$. Then by virtue of Theorem 2.3, V^m is isometric to a sphere.

Theorem 6.3. *Let M^n be an n-dimensional constant Riemann curvature space admitting a conformal Killing vector field ξ^i and a scalar field σ such that $\sigma_{;i}=\mu\xi_i$. We suppose that a system of differential equations $\frac{\partial f}{\partial x^i}=e^{2\sigma}\xi_i$ is integrable and V^m is an m-dimensional closed orientable submanifold such that*

(i) $H_1=const.$,

(ii) $\xi_i \underset{E}{n^i}$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by* $m+1$ *vectors* B^i_α $(\alpha=1,2,\cdots,m)$ *and* $\underset{E}{n^i}$ *at each point on* V^m,

where $\xi_\alpha\not\equiv 0$ on V^m. Then V^m is isometric to a sphere.

Proof. From our hypothesis and Theorem 0.3, we can see that every

point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$. By virtue of Corollary 3. 1, there exists a scalar field Φ in M^n and we have

$$\xi_{i;j} = \Phi g_{ij} . \tag{6. 5}$$

Hence, by means of the Ricci identity it follows that

$$R_{lijk}\xi^l = \Phi_{;k}g_{ij} - \Phi_{;j}g_{ik} .$$

Since M^n is constant Riemann curvature space we have

$$R_{lijk} = c(g_{lj}g_{ik} - g_{lk}g_{ij}). \qquad (c=\text{const.})$$

Then from above two relations we get

$$\Phi_{;i} = c\xi_i . \tag{6. 6}$$

By means of (6. 5) and (6. 6) we obtain

$$\Phi_{;i;j} = c\Phi g_{ij} .$$

Because of (6. 6) and hypothesis (iii), the vector Φ^i $(=g^{ij}\Phi_{;j})$ is contained in the vector space spanned by $m+1$ vectors B^i_α $(\alpha=1, 2, \cdots, m)$ and $\underset{E}{n^i}$. Consequently, by virtue of Theorem 2. 2, V^m is isometric to a sphere.

Theorem 6. 4. *Let M^n be an n-dimensional constant Riemann curvature space admitting a conformal Killing vector field ξ^i and a scalar field σ such that $\sigma_{;i}=\mu\xi_i$. We suppose that a system of differential equations $\dfrac{\partial f}{\partial x^i} = e^{2\sigma}\xi_i$ is integrable and V^m is an m-dimensional closed submanifold such that*

(i) $\underset{E}{k_1}, \underset{E}{k_2}, \cdots, \underset{E}{k_m} > 0$ *on* V^m *and* $\underset{E}{H_\nu} = const.$ *for any* ν $(1<\nu\leqq m-1)$,

(ii) $\xi_i \underset{E}{n^i}$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by* $m+1$ *vectors* B^i_α $(\alpha=1, 2, \cdots, m)$ *and* $\underset{E}{n^i}$ *at each point on* V^m,

where $\xi_\alpha \not\equiv 0$ on V^m. Then V^m is isometric to a sphere.

Proof. From our hypothesis and Theorem 0. 4, we can see that every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$. In this case, because of the relation

$$\underset{E}{H_1}\underset{E}{H_\nu} - \underset{E}{H_{\nu+1}} = \frac{\nu!(m-\nu-1)!}{m!} \sum \underset{E}{k_{\alpha_1}}\underset{E}{k_{\alpha_2}}\cdots\underset{E}{k_{\alpha_{\nu-1}}}(\underset{E}{k_{\alpha_\nu}} - \underset{E}{k_{\alpha_{\nu+1}}})^2 \quad \text{(cf. [10])}$$

it follows that

$$\underset{E}{H_1}\underset{E}{H_\nu}-\underset{E}{H_{\nu+1}}=0 \qquad (1\leqq\nu\leqq m-1).$$

Consequently we get $\underset{E}{H_\nu}=(\underset{E}{H_1})^\nu$ for any ν. Then $\underset{E}{H_\nu}=$const. means $H_1=$const. Therefore, by virtue of Theorem 6. 3, V^m is isometric to a sphere.

Remark. By virtue of the Remark stated at the end of §3, E^n admits a homothetic Killing vector field ξ^i and a scalar field σ such that $\sigma_{;i}=\mu\xi_i$, where $\mu=\frac{1}{2}$. If ξ^i is a proper homothetic Killing vector, then constant Riemann curvature space M^n becomes E^n. Let V^{n-1} be a closed convex hypersurface in E^n. We take the origin O of cartesian coordinate system of E^n in the interior of V^{n-1}. Then hypothesis (ii) and (iv) of Theorem 6.2 is satisfied. It is evident that the hypothesis (iii) of Theorem 6. 2 is satisfied identically for a hypersurface. Because of the above observation, when M^n is E^n, Theorem 6. 2 give us the fact that a closed convex hypersurface with constant mean curvature H_1 in E^n is isometric to a sphere. This means that when M^n is E^n, Theorem 6. 2 coincides with Liebmann-Süss theorem.

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