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Title	A QUASI-DECOMPOSABLE ABELIAN GROUP WITHOUT PROPER ISOMORPHIC QUOTIENT GROUPS AND PROPER ISOMRPHIC SUBGROUPS II
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Citation	Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 20(4), 194-203
Issue Date	1969
Doc URL	https://hdl.handle.net/2115/56089
Type	departmental bulletin paper
File Information	JFSHIU_20_N4_194-203.pdf

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A QUASI-DECOMPOSABLE ABELIAN GROUP WITHOUT PROPER ISOMORPHIC QUOTIENT GROUPS AND PROPER ISOMRPHIC SUBGROUPS II

By

Takashi ITO and John M. IRWIN

1. Introduction

This paper is a continuation of our paper [2] with the same title. In [2] one of theorems was stated without detailed proof. The purpose of this paper is just to give a complete proof of it. So we shall omit an origin of our problem and general background for it, see the first section of [2] and references in [2].

Let $p>1$ be a fixed prime number, $C(p^n)$ be a cyclic group of order p^n, $\sum$ be the direct sum of cyclic groups $C(p^n)$, Π be the direct product of cyclic groups $C(p^n)$ and C be the torsion group of Π, that is, $\sum$ is the standard basic group and C is the torsion completion of $\sum$.

Our theorem is as following, that is Theorem 3 in [2].

Theorem. *There exists a pure subgroup G of C which contains $\sum$ and satisfies properties;*

1) *G has no proper isomorphic quotient groups,*
2) *G has no proper isomorphic subgroups,*
3) *G has a decomposition $G_1 \oplus G_2$ such that G_1 and G_2 are not bounded.*

2. Group Homomorphisms on $C[p]$.

The p-socle $C[p]$ of C is a vector space over the prime field of characteristic p and can be topologized as a totally disconnected compact topological group, because Π is clearly a totally disconnected compact topological group with respect to the product topology of compact discrete topologies and the p-socle $C[p]$ of C is the closed subgroup $\{x|x\in\Pi,\ px=0\}$ of Π. Actually $U_n=\{x|x\in C[p]$ and the hight of $x\geqq n\}=(p^nC)[p]$ $(n=1,2\cdots)$ are open compact subgroups of $C[p]$ and $\{U_n\}$ is a fundamental system of 0-neighborhoods in $C[p]$. These two structures on $C[p]$ which are a vector space and a totally disconnected compact group are used in an essential way.

Every continuous group homomorphism T on $C[p]$ defines compact subgroups $E_q(T)=\{x|x\in C[p]$ and $Tx=qx\}$ $(0\leqq q<p)$ and the compact subgroup

$E(T)=E_0(T)\oplus E_1(T)\oplus\cdots\oplus E_{p-1}(T)$. We call T is a *singular* homomorphism if $E(T)$ is an open compact subgroup of $C[p]$. For instance a continuous projection on $C[p]$ is singular. If a continuous group homomorphism T on $C[p]$ has a dense subgroup which is invariant under T and on which T is one to one, T is called a *semi-isomorphism* on $C[p]$.

In the final step in our proof of the theorem we shall use following proposition which is shown in [2] (Theorem 1 of [2]).

Proposition. *Let G be a pure subgroup of C which contains $\sum$ and $G[p]$ be the p-socle of G.*

1) *If $G[p]$ is not invariant under any non-singular onto homomorphism on $C[p]$, then G has no proper isomorphic quotient groups.*

2) *If $G[p]$ is not invariant under any non-singular semi-isomorphism on $C[p]$, then G has no proper isomorphic subgroups.*

Therefore our main work in this paper is to show existence of a group Q which is between $\sum[p]$ and $C[p]$, and which is not invariant under any non-singular homomorphism but invariant under the canonical projection P_e onto even coordinates (see Lemma 2). In our proof of this fact we have some difficulity which comes from non-commutativity between non-singular homomorphisms and P_e. So we classify degree of non-commutativity in the following section.

3. Proof of Theorem

We introduce eight classes (essentially four) of non-singular homomorphisms depending on degree of non-commutativity. Hereafter we shall use notations $A^e=P_e(A)$ $(A^o=(I-P_e)(A))$ for a subset A of $C[p]$ and $x^e=P_ex$ $(x^o=x-P_ex)$ for an element x in $C[p]$. Every continuous homomorphism T on $C[p]$ defines compact subgroups

$$F^o(T)=\{x|x\in C[p]^o \text{ and } (Tx)^e=0\},$$

$$E^o_{q,n}(T)=\{x|x\in U^o_n \text{ and } (Tx)^o=qx\} \quad \text{for} \quad q=0,1,\cdots,p-1;$$

$n=1,2,\cdots$. $F^e(T)$ and $E^e_{q,n}(T)$ are defined similarly. Let $\tilde{E}^o_n(T)=E^o_{0,n}(T)\oplus E^o_{1,n}(T)\oplus\cdots\oplus E^o_{p-1,n}(T)$ and $\tilde{E}^e_n(T)=E^e_{0,n}(T)\oplus E^e_{1,n}(T)\oplus\cdots\oplus E^e_{p-1,n}(T)$.

Following eight types about T are defined;

I° *there exists N such that $F^o(T)\supset U^o_N$ and $\tilde{E}^o_n(T)\not\supset U^o_m$ for all n, $m\geqq N$,*

II° *$F^o(T)\not\supset U^o_n$ for all $n=1,2,\cdots$ but there are N and q such that $E^o_{q,N}(T)\supset U^o_N$,*

III° *$F^o(T)\not\supset U^o_n$ for all $n=1,2,\cdots$ and $E^o_{q,n}(T)\not\supset U^o_m$ for all $n,m=$*

$1, 2, \cdots$ *and* $q=0, 1, \cdots, p-1$, *but there are* $N' \geqq N$ *such that* $\tilde{E}_N^{\circ}(T) \supset U_{N'}^{\circ}$,

IV° $F^{\circ}(T) \not\supset U_n^{\circ}$ *for all* $n=1, 2, \cdots$ *and* $\tilde{E}_n^{\circ}(T) \not\supset U_m^{\circ}$ *for all* $n, m = 1, 2, \cdots$.

I^{e}, II^{e}, III^{e} *and* IV^{e} *are defined by exchanging odd and even.*

We note that every non-singular homomorphism is at least of one of the above eight types. Because, suppose T is of none of the above eight types, then there are N, n and m such that $n, m \geqq N$ and

1) $F^{\circ}(T) \supset U_N^{\circ}$ and $F^{e}(T) \supset U_N^{e}$,

2) $\tilde{E}_n^{\circ}(T) \supset U_m^{\circ}$ and $\tilde{E}_n^{e}(T) \supset U_m^{e}$.

Then 1) implies $E_{q,n}^{\circ}(T) \subset E_q(T)$ and $E_{q,n}^{e}(T) \subset E_q(T)$ for all $q=0, 1, \cdots, p-1$. So 2) implies $E(T)=E_0(T) \oplus \cdots \oplus E_{p-1}(T) \supset U_m^{\circ} \oplus U_m^{e} = U_m$. This means that T is a singular homomorphism.

In order to construct Q explained at the end of section 2 we need the following;

Lemma 1. *For an arbitrary non-singular homomorphism* T *we can find a one-parameter family* $\Delta(T)=\{x_t | 0 \leqq t \leqq 1\}$ *of elements in* $C[p]$ *which has one of the following six properties*; 1°, 2°, 3°, 1^{e}, 2^{e} and 3^{e},

1° $x_t, Tx_t \in C[p]^{\circ}$ *for all* $0 \leqq t \leqq 1$ *and four elements* x_s, x_t, Tx_s *and* Tx_t *are linearly independent for arbitrary* $s \neq t$,

2° *there exists* q, $0 \leqq q \leqq p-1$ *such that* $x_t \in C[p]^{\circ}$ *and* $Tx_t - qx_t \in C[p]^{e}$ *for all* $0 \leqq t \leqq 1$ *and four elements* x_s, x_t, $Tx_s - qx_s$ *and* $Tx_t - qx_t$ *are linearly independent for arbitrary* $s \neq t$,

3° $x_t \in C[p]^{\circ}$ *for all* $0 \leqq t \leqq 1$ *and six elements* x_s, x_t, $(Tx_s)^{\circ}$, $(Tx_s)^{e}$, $(Tx_t)^{\circ}$ *and* $(Tx_t)^{e}$ *are linearly independent for arbitrary* $s \neq t$.

1^{e}, 2^{e} *and* 3^{e} *are dual properties of* 1°, 2° *and* 3° *by exchanging odd for even.*

Proof. Of course the choice of these six properties about $\Delta(T)$ depends on types of T defined above. Therefore the proof is divided into four cases, a)~d).

The following fact (*) which was shown already in [2] (see b) of the proof of Lemma 3 in [2]) will be used in cases a), b) and d).

(*) Suppose x_0 and Tx_0 are linearly independent and $\{z_0, z_1, \cdots, z_{p^2-1}\}$ is a group $[x_0] \oplus [Tx_0]$. Then we can take U_M such that $z_i + U_M + T(U_M)$ $(0 \leqq i \leqq p^2-1)$ are disjoint, and if $\{y_t | 0 \leqq t \leqq 1\} \subset U_M$ satisfies the condition that $y_t - y_s$ and $T(y_t - y_s)$ are linearly independent for arbitrary $s \neq t$, then four elements $x_0 + y_s$, $x_0 + y_t$, $T(x_0 + y_s)$ and $T(x_0 + y_t)$ are linearly independent for

srbitrary $s \neq t$.

a) *If T is of type I°, then $\Delta(T)$ having property 1° exists.*

If a compact group $U_n^\circ/\tilde{E}_n^\circ(T)$ is finite, then $\tilde{E}_n^\circ(T) \supset U_m^\circ$ for some m, which contradicts our assumption. Therefore, since a compact infinite group has at least cardinality $\aleph$, $\dim U_n^\circ/\tilde{E}_n^\circ(T) = \aleph$ for all $n=1,2,\cdots$, then $U_n^\circ = \tilde{E}_n^\circ(T) \oplus D_n$ for all $n=1,2,\cdots$ and $\dim D_n = \aleph$ for all $n=1,2,\cdots$. Choose $0 \neq x_0 \in D_N$, then $Tx_0 \in C[p]^\circ$ since $D_N \subset U_N^\circ \subset F^\circ(T)$ and x_0 and Tx_0 are linearly independent. By the above (*) we can determine $M \geqq N$ and choose a basis $\{y_t | 0 \leqq t \leqq 1\}$ of D_M, then $Ty_t \in C[p]^\circ$ for all $0 \leqq t \leqq 1$ and for $s \neq t$ $y_s - y_t$ and $T(y_s - y_t)$ are linearly independent, because $D_M \cap \tilde{E}_M^\circ(T) = \{0\}$. Therefore by (*) $\Delta(T) = \{x_0 + y_t | 0 \leqq t \leqq 1\}$ has the property 1°.

b) *If T is of type II°, then $\Delta(T)$ having property 2° exists.*

Our assumption $F^\circ(T) \not\supset U_n^\circ$ for all $n=1,2,\cdots$ implies that compact groups $U_n^\circ/F^\circ(T) \cap U_n^\circ$ for all $n=1,2,\cdots$ are infinite, so $U_n^\circ = (F^\circ(T) \cap U_n^\circ) \oplus H_n$ for all $n=1,2,\cdots$ and $\dim H_n = \aleph$. And we have $(Ty)^e \neq 0$ for all $0 \neq y \in H_n$. Let $\tilde{T} = T - q(I - P_e)$, then our assumption $E_{q,N}^\circ(T) \supset U_N^\circ$ implies $E_{0,N}^\circ(\tilde{T}) = U_N^\circ$, and $\tilde{T}x = (Tx)^e$ for all $x \in U_N^\circ$. Choose $0 \neq x_0 \in H_N$, then $0 \neq \tilde{T}x_0 \in C[p]^e$ and x_0 and $\tilde{T}x_0$ are linearly independent. We can apply (*) to $\tilde{T}$, so M can be chosen as in (*) and $\{y_t | 0 \leqq t \leqq 1\}$ can be a basis of H_N. Then $\tilde{T}y_t \in C[p]^e$ for all $0 \leqq t \leqq 1$ and for $s \neq t$, $y_s - y_t$ and $\tilde{T}(y_s - y_t)$ are linearly independent, because $y_s - y_t \in H_N \subset U_N^\circ$ and $\tilde{T}(y_s - y_t) \in C[p]^e$. Therefore by (*) we can see that $\Delta(T) = \{x_0 + y_t | 0 \leqq t \leqq 1\}$ is a one-parameter family having the property 2°.

The following fact (**) will be used in cases c) and d).

(**) Suppose $0 \neq x_0 \in C[p]^\circ$ such that x_0, $(Tx_0)^\circ$ *and* $(Tx_0)^e$ are linearly independent and suppose $\{z_0, z_1, \cdots, z_{p^3-1}\}$ is the group $[x_0] \oplus [(Tx_0)^\circ] \oplus [(Tx_0)^e]$. Then we can take U_M such that $z_i + U_M + (T(U_M))^\circ + (T(U_M))^e$ $(0 \leqq i \leqq p^3 - 1)$ are mutually disjoint, and if $\{y_t | 0 \leqq t \leqq 1\} \subset U_M^\circ$ satisfies the condition that three elements $y_t - y_s$, $(T(y_t - y_s))^\circ$ and $(T(y_t - y_s))^e$ are linearly independent for arbitrary $s \neq t$. Thus $\Delta(T) = \{x_0 + y_t | 0 \leqq t \leqq 1\}$ is a one-parameter family having the property 3°.

Now (**) can be shown as follows; it is clear that there exists U_N such that $z_i + U_N$ $(0 \leqq i \leqq p^3 - 1)$ are mutually disjoint. By continuity of P_eT and $(I - P_e)T$ there is $M \geqq N$ such that $(T(U_M))^e$, $(T(U_M))^\circ \subset U_N$, then U_M is as desired, since $z_i + U_M + (T(U_M))^\circ + (T(U_M))^e \subset z_i + U_N + U_N + U_N = z_i + U_N$ $(0 \leqq i \leqq p^3 - 1)$. Next, suppose $n_1(x_0 + y_t) + n_2(T(x_0 + y_t))^\circ + n_3(T(x_0 + y_t))^e = n_1'(x_0 + y_s) + n_2'(T(x_0 + y_s))^\circ + n_3'(T(x_0 + y_s))^e$ for some $s \neq t$, where n_1, n_2, n_3, n_1', n_2' and n_3' are integers, then $n_1x_0 + n_2(Tx_0)^\circ + n_3(Tx_0)^e = z_i$ for some i and $n_1'x_0 + n_2'(Tx_0)^\circ + n_3'(Tx_0)^e = z_j$ for some j. So $z_i + n_1y_t + n_2(Ty_t)^\circ + n_3(Ty_t)^e = z_j + n_1'y_s + n_2'(Ty_s)^\circ + n_3'(Ty_s)^e$ and $n_1y_t + n_2(Ty_t)^\circ + n_3(Ty_t)^e$, $n_1'y_s + n_2'(Ty_s)^\circ + n_3'(Ty_s)^e \in U_M + (T(U_M))^\circ$

$+(T(U_M))^e$. By our choice of U_M, z_i must be equal to z_j, this implies $n_1=n_1'$ mod p, $n_2=n_2'$ mod p and $n_3=n_3'$ mod p. Therefore we have $n_1(y_t-y_s)+n_2(T(y_t-y_s))^\circ+n_3(T(y_t-y_s))^e=0$, but our assumption about $\{y_t\}$ implies $n_1=n_2=n_3=0$ mod p. Hence $\Delta(T)=\{x_0+y_t|0\leqq t\leqq 1\}$ satisfies property 3°.

c) *If T is of type III°, then $\Delta(T)$ having property 3° exists.*

$F^\circ(T)\not\supset U_n^\circ$ for all $n=1,2,\cdots$ yields a decomposition of U_n° such that

(1) $U_n^\circ=(F^\circ(T)\cap U_n^\circ)\oplus H_n$ and $\dim H_n=\aleph$ for all $n=1,2,\cdots$. $\tilde{E}_N^\circ(T)\supset U_{N'}^\circ$ yields a decomposition of U_n° for $n\geqq N'$ such that

(2) $U_n^\circ=\tilde{E}_n^\circ(\tilde{T})\oplus D_n$ and $\dim D_n<\aleph_0$ for all $n\geqq N'$.

Because, $U_n^\circ/\tilde{E}_n^\circ(T)\subset U_{N'}^\circ/\tilde{E}_n^\circ(T)\subset\tilde{E}_N^\circ(T)/\tilde{E}_n^\circ(T)$

$\cong\sum_{q=0}^{p-1}(E_{q,N}^\circ(T)/E_{q,n}^\circ(T))$, but for each q, $E_{q,N}^\circ(T)/E_{q,n}^\circ(T)$

$=E_{q,N}^\circ(T)/E_{q,N}^\circ(T)\cap U_n^\circ\cong(E_{q,N}^\circ(T)+U_n^\circ)/U_n^\circ\subset U_N^\circ/U_n^\circ$ and U_N°/U_n° is finite.

And assumptions $\tilde{E}_N^\circ(T)\supset U_{N'}^\circ$ and $E_{q,n}^\circ(T)\not\supset U_m^\circ$ for all $n,m=1,2,\cdots$ and $q=0,1,\cdots,p-1$ imply that there are $q'\neq q''$ not depending on n such that

(3) $\dim E_{q',n}^\circ(T)=\dim E_{q'',n}^\circ(T)=\aleph$ for all $n\geqq N'$.

This (3) can be shown easily, see a) in the proof of Lemma 3 in [2].

(1) and (2) imply that there is q''' not depending on n such that

(4) $\dim(E_{q''',n}^\circ(T)/E_{q''',n}^\circ(T)\cap F^\circ(T))=\aleph$ for all $n\geqq N'$.

Because, suppose

$E_{q,N'}^\circ(T)/E_{q,N'}^\circ(T)\cap F^\circ(T)$ are finite for all $0\leqq q\leqq p-1$.

Then, since

$E_{q,N'}^\circ(T)/E_{q,N'}^\circ(T)\cap F^\circ(T)\cong(E_{q,N'}^\circ(T)+F^\circ(T)\cap U_{N'}^\circ)/F^\circ(T)\cap U_{N'}^\circ$,

$\left(\sum_{q=0}^{p-1}E_{q,N'}^\circ(T)+F^\circ(T)\cap U_{N'}^\circ\right)/F^\circ(T)\cap U_{N'}^\circ=(\tilde{E}_{N'}^\circ(T)+F^\circ(T)\cap U_{N'}^\circ)/F^\circ(T)\cap U_{N'}^\circ$

must be finite. However $D_{N'}$ is finite by (2), so we see that $(\tilde{E}_{N'}^\circ(T)+D_{N'}+F^\circ(T)\cap U_{N'}^\circ)/F^\circ(T)\cap U_{N'}^\circ=U_{N'}^\circ/F^\circ(T)\cap U_{N'}^\circ$ is finite. This contradicts the facts that $U_{N'}^\circ/F^\circ(T)\cap U_{N'}^\circ\cong H_{N'}$ and $H_{N'}$ is infinite by (1). Therefore we have that there exists q''' such that $E_{q''',N'}^\circ(T)/E_{q''',N'}^\circ(T)\cap F^\circ(T)$ is finite. If $n\geqq N'$, then $E_{q''',N'}^\circ(T)/E_{q''',n}^\circ(T)\cap F^\circ(T)$ is infinite, because $E_{q''',n}^\circ(T)\subset E_{q''',N'}^\circ(T)$. On the other hand $E_{q''',N'}^\circ(T)/E_{q''',n}^\circ(T)$ is finite, because $E_{q''',N'}^\circ(T)/E_{q''',n}^\circ(T)=E_{q''',N'}^\circ(T)/E_{q''',N'}^\circ(T)\cap U_n^\circ\cong(E_{q''',N'}^\circ(T)+U_n^\circ)/U_n^\circ\subset U_{N'}^\circ/U_n^\circ$ and $U_{N'}^\circ/U_n^\circ$ is finite. These two facts imply clearly that $E_{q''',n}^\circ(T)/E_{q''',n}^\circ(T)\cap F^\circ(T)$ is infinite for all $n\geqq N'$. Therefore (4) is shown (note that every compact infinite group has at least cardinality $\aleph$).

Without loss of generality we can assume $q'''=q'$ in (3) and combining

(3) and (4) we can see that there exist $q' \neq q''$ not depending on n such that for all $n \geqq N'$ we have

(5) $\dim E^{\circ}_{q',n}(T) = \dim E^{\circ}_{q'',n}(T) = \dim (E^{\circ}_{q',n}(T)/E^{\circ}_{q',n}(T) \cap F^{\circ}(T)) = \aleph$.

Since $E^{\circ}_{q',n}(T)$ and $E^{\circ}_{q'',n}(T)$ $(n=1, 2, \cdots)$ are decreasing and $\bigcap_{n=1}^{\infty} U^{\circ}_n = \{0\}$, the following three cases may occur.

(6)
(α) $E^{\circ}_{q'',n}(T) \cap F^{\circ}(T)$ is infinite for all $n \geqq N'$,
(β) $E^{\circ}_{q'',n}(T) \cap F^{\circ}(T)$ is finite but $E^{\circ}_{q',n}(T) \cap F^{\circ}(T)$ is infinite for all sufficiently large $n \geqq N'$,
(γ) $E^{\circ}_{q'',n}(T) \cap F^{\circ}(T) = E^{\circ}_{q',n}(T) \cap F^{\circ}(T) = \{0\}$ for all sufficiently large $n \geqq N'$.

We want to show that

(7) If n satisfies (6), then there exists $\{y_t | 0 \leqq t \leqq 1\} \subset U^{\circ}_n$ such that $y_s - y_t$, $(T(y_s - y_t))^{\circ}$ and $(T(y_s - y_t))^{e}$ are linearly independent for arbitrary $s \neq t$.

Suppose n satisfies (α). Then $\dim (E^{\circ}_{q'',n}(T) \cap F^{\circ}(T)) = \aleph$. By (5) $E^{\circ}_{q',n}(T)$ has a decomposition $E^{\circ}_{q',n}(T) = (E^{\circ}_{q',n}(T) \cap F^{\circ}(T)) \oplus A_n$ and $\dim (A_n) = \aleph$. Let $\{z'_t | 0 \leqq t \leqq 1\}$ be a basis of A_n and $\{z''_t | 0 \leqq t \leqq 1\}$ be a basis of $E^{\circ}_{q'',n}(T) \cap F^{\circ}(T)$, then $\{y_t = z'_t + z''_t | 0 \leqq t \leqq 1\}$ has the desired property. Because, if $n_1(y_s - y_t) + n_2(T(y_s - y_t))^{\circ} + n_3(T(y_s - y_t))^{e} = 0$, then $n_1(y_s - y_t) + n_2(T(y_s - y_t))^{\circ} = 0$ and $n_3(T(y_s - y_t))^{e} = 0$. $n_1(y_s - y_t) + n_2(T(y_s - y_t))^{\circ} = 0$ implies that $(n_1 + n_2 q')(z'_s - z'_t) + (n_1 + n_2 q'')(z''_s - z''_t) = 0$, but $z'_s - z'_t \neq 0$, $z''_s - z''_t \neq 0$ and $E^{\circ}_{q',n}(T) \cap E^{\circ}_{q'',n}(T) = \{0\}$, so $n_1 + n_2 q' = n_1 + n_2 q'' = 0 \mod p$. This implies $n_1 = n_2 = 0 \mod p$ since $q' \neq q'' \mod p$. And $n_3(T(y_s - y_t))^{e} = 0$ implies $0 = n_3(T(z'_s - z'_t))^{e} + n_3(T(z''_s - z''_t))^{e} = n_3(T(z'_s - z'_t))^{e}$, but $(T(z'_s - z'_t))^{e} \neq 0$ since $0 \neq z'_s - z'_t \in A_n$, therefore we have $n_3 = 0 \mod p$. Suppose n satisfies (β). Then automatically $\dim (E^{\circ}_{q'',n}(T)/E^{\circ}_{q'',n}(T) \cap F^{\circ}(T)) = \aleph$. Therefore the same argument as in (α) applies by replacing q'' with q'. Suppose n satisfies (γ). Let φ_n be a canonical projection onto H_n from U°_n, which is well defined by a decomposition $U^{\circ}_n = (F^{\circ}(T) \cap U^{\circ}_n) \oplus H_n$ in (1). The condition in (γ) means that φ_n is one to one on $E^{\circ}_{q',n}(T)$ and $E^{\circ}_{q'',n}(T)$. This implies $\dim (\varphi_n(E^{\circ}_{q',n}(T))) = \dim (\varphi_n(E^{\circ}_{q'',n}(T))) = \aleph$. Let $X'_n = \varphi_n(E^{\circ}_{q',n}(T))$, $X''_n = \varphi_n(E^{\circ}_{q'',n}(T))$, $X'_n = Y'_n \oplus (X'_n \cap X''_n)$ and $X''_n = Y''_n \oplus (X'_n \cap X''_n)$. If $\dim (X'_n \cap X''_n) = \aleph$, then choose $\{y_t | 0 \leqq t \leqq 1\} \subset U^{\circ}_n$ as follows. Let $\{u_t | 0 \leqq t \leqq 2\}$ be a basis of $X'_n \cap X''_n$. Then there exists uniquely z'_t in $E^{\circ}_{q',n}(T)$ such that $u_t = \varphi_n(z'_t)$ for all $0 \leqq t \leqq 1$ and uniquely z''_t in $E^{\circ}_{q'',n}(T)$ such that $u_{t+1} = \varphi_n(z''_t)$ for all $0 \leqq t \leqq 1$. Let $y_t = z'_t + z''_t$ for $0 \leqq t \leqq 1$, then it is easy to see the linear independence of $(y_s - y_t)$ and $(T(y_s - y_t))^{\circ}$ for $s \neq t$

(see case (α)). And $(T(y_s-y_t))^e=(T(z'_s-z'_t+z''_s-z''_t))^e=(T(u_s-u_t+u_{s+1}-u_{t+1}))^e \neq 0$, because $0\neq u_s-u_t+u_{s+1}-u_{t+1}\in H_n$. Therefore y_s-y_t, $(T(y_s-y_t))^o$ and $(T(y_s-y_t))^e$ are linearly independent for arbitrary $s\neq t$. If $\dim(X'_n\cap X''_n)<\aleph$, then $\dim(Y'_n)=\dim(Y''_n)=\aleph$. Let $\{u_t|0\leqq t\leqq 1\}$ be a basis of Y'_n and $\{v_t|0\leqq t\leqq 1\}$ be a basis of Y''_n, then $\{u_t\}\cup\{v_t\}$ is a linearly independent system in H_n. We can take unique z'_t in $E^o_{q',n}(T)$ and z''_t in $E^o_{q'',n}(T)$ such that $u_t=\varphi_n(z'_t)$ and $v_t=\varphi_n(z''_t)$ for all $0\leqq t\leqq 1$. Then the system $\{y_t=z'_t+z''_t|0\leqq t\leqq 1\}$ is as desired, which is proved in the same way as above.

Now we can present the proof of c). By (7) there exists $0\neq x_0\in C[p]^o$ such that x_0, $(Tx_0)^o$ and $(Tx_0)^e$ are linearly independent. By (**) we can determine a large M which satisfies the condition stated in (**). Without loss of generality we can assume this M satisfies one of (α), (β) and (γ), because M can be replaced by a larger one if it is necessary. (7) guarantees the existence of $\{y_t|0\leqq t\leqq 1\}\subset U^o_M$ for which y_s-y_t, $(T(y_s-y_t))^o$ and $(T(y_s-y_t))^e$ are linearly independent. Therefore (**) shows us that $\Delta(T)=\{x_0+y_t|0\leqq t\leqq 1\}$ is a one-parameter family having property 3°.

d) *If T is of type IV° then $\Delta(T)$ having property* 1° *or* 3° *exists.*

Our assumptions yield decompositions of U^o_n;

$$U^o_n=(F^o(T)\cap U^o_n)\oplus H_n \quad \text{and} \quad \dim H_n=\aleph \quad \text{for all} \quad n=1,2,\cdots,$$
$$U^o_n=\tilde{E}^o_n(T)\oplus D_n \quad \text{and} \quad \dim D_n=\aleph \quad \text{for all} \quad n=1,2,\cdots.$$

We can see that for all $n=1,2,\cdots$

$$\dim(F^o(T)\cap U^o_n/F^o(T)\cap\tilde{E}^o_n(T))=\aleph \quad \text{or} \quad \dim(H_n/H_n\cap\tilde{E}^o_n(T))=\aleph.$$

Because, suppose both dimensions are less than $\aleph$, then $\dim((F^o(T)\cap U^o_n+\tilde{E}^o_n(T))/\tilde{E}^o_n(T))<\aleph$ and $\dim((H_n+\tilde{E}^o_n(T))/\tilde{E}^o_n(T))<\aleph$ since $F^o(T)\cap U^o_n/F^o(T)\cap\tilde{E}^o_n(T)\cong(F^o(T)\cap U^o_n+\tilde{E}^o_n(T))/\tilde{E}^o_n(T)$ and $H_n/H_n\cap\tilde{E}^o_n(T)\cong(H_n+\tilde{E}^o_n(T))/\tilde{E}^o_n(T)$. Therefore $\aleph>\dim(((F^o(T)\cap U^o_n)+H_n+\tilde{E}^o_n(T))/\tilde{E}^o_n(T))=\dim(U^o_n/\tilde{E}^o_n(T))=\dim D_n=\aleph$. This is a contradiction. Next, if $\dim(F^o(T)\cap U^o_n/F^o(T)\cap\tilde{E}^o_n(T))<\aleph$ for some n, then $\dim(F^o(T)\cap U^o_m/F^o(T)\cap\tilde{E}^o_m(T))<\aleph$ for all $m\geqq n$. Because $F^o(T)\cap U^o_m/F^o(T)\cap\tilde{E}^o_m(T)=F^o(T)\cap U^o_m/F^o(T)\cap U^o_n\cap\tilde{E}^o_n(T)\cong(F^o(T)\cap U^o_m+\tilde{E}^o_n(T))/\tilde{E}^o_n(T)\subset(F^o(T)\cap U^o_n+\tilde{E}^o_n(T))/\tilde{E}^o_n(T)\cong F^o(T)\cap U^o_n/F^o(T)\cap\tilde{E}^o_n(T)$. Therefore the following two cases may occur;

(δ) $\dim(F^o(T)\cap U^o_n/F^o(T)\subset\tilde{E}^o_n(T))=\aleph$ for all $n=1,2,\cdots$,

(ε) There exists N such that $\dim(H_n/H_n\cap\tilde{E}^o_n(T))=\aleph$ for all $n\geqq N$.

If (δ) happens, then we can show the existence of $\Delta(T)$ having property 1°. (δ) implies that

$$F^o(T)\cap U^o_n=(F^o(T)\cap\tilde{E}^o_n(T))\oplus X_n \quad \text{and} \quad \dim(X_n)=\aleph \quad \text{for all} \quad n=1,2,\cdots.$$

We take $0\neq x_0\in X_1$, then $(Tx_0)^e=0$ and x_0 and Tx_0 are linearly independent.

According to (*) we can determine M which satisfies the condition stated in (*). For this M we take a basis $\{y_t|0\leqq t\leqq 1\}$ of X_M, then $\{y_t\}\subset U_M^\circ$ and $\{Ty_t\}\subset C[p]^\circ$. Suppose $n_1(y_s-y_t)+n_2T(y_s-y_t)=0$ for $s\neq t$, then we have $n_1=n_2=0 \bmod p$, because, if $n_2\neq 0 \bmod p$, then $T(y_s-y_t)=q(y_s-y_t)$ for some q, $0\leqq q\leqq p-1$. This leads to a contradiction $0\neq y_s-y_t\in\tilde{E}_M^\circ(T)\cap X_M^\circ=\{0\}$. By (*) $\Delta(T)=\{x_0+y_t|0\leqq t\leqq 1\}$ is a one-parameter family having property 1°. If (ε) happens, then we can show the existence of $\Delta(T)$ having property 3°. (ε) implies that

$$H_n=(H_n\cap\tilde{E}_n^\circ(T))\oplus Y_n \quad\text{and}\quad \dim Y_n=\aleph \quad\text{for all}\quad n\geqq N.$$

We can take $0\neq x_0\in X_N$, then we can see that x_0, $(Tx_0)^\circ$ and $(Tx_0)^e$ are linearly independent. According to (**) we can determine M. Take a basis $\{y_t|0\leqq t\leqq 1\}$ from Y_M, then we see also that y_s-y_t, $(T(y_s-y_t))^\circ$ and $(T(y_s-y_t))^e$ are linearly indepenpent for $s\neq t$. By (**) $\Delta(T)=\{x_0+y_t|0\leqq t\leqq 1\}$ is a one-parameter family having property 3°.

Exchanging odd and even in a)~d) we have

e) *If T is of one of types I^e~IV^e, then $\Delta(T)$ having one of properties 1^e~3^e exists.*

Q. E. D.

Now we are in a position to show existence of a group Q which plays fundamental role to our ideas.

Lemma 2. *For any family $\{T_\lambda|\lambda\in\Lambda\}$ of non-singular homomorphisms on $C[p]$ there exists a subgroup Q between $\sum[p]$ and $C[p]$ such that Q is not invariant under any $T_\lambda(\lambda\in\Lambda)$ but invariant under the canonical projection P_e onto even coordinates.*

Proof. $\{T_\lambda|\lambda\in\Lambda\}$ is a given family of non-singular homomorphisms on $C[p]$. We can assume that Λ is a well ored set of ordinal numbers which are less than the first ordinal number whose cardinality is $\aleph$. Choose $c\in C[p]$ but c°, $c^e\notin\sum[p]$. By transfinite induction we can construct the following family of subgroups $R_\lambda(\lambda\in\Lambda)$;

a) $\sum[p]=R_0\subset R_\lambda\subset R_\mu\subset C[p]$ if $0\leqq\lambda<\mu$ $(\lambda,\mu\in\Lambda)$,

b) card $R_\lambda\leqq$ (card λ)$\cdot\aleph_0<\aleph$ for all $\lambda\in\Lambda$,

c) R_λ is invariant under P_e for all $\lambda\in\Lambda$,

d) c° and $c^e\notin R_\lambda$ but there exists $x_\lambda\in R_\lambda\cap\Delta(T_\lambda)$ such that $c^\circ-T_\lambda x_\lambda$ or $c^e-T_\lambda x_\lambda$ or $c-T_\lambda x_\lambda\in R_\lambda$ for all $\lambda\in\Lambda$.

Suppose R_λ has been constructed for all $\lambda<\mu\in\Lambda$. Let $R'_\mu=\bigcup_{\lambda<\mu}R_\lambda$. Then card $R'_\lambda\leqq$ (card λ)$\cdot\aleph_0<c$ and R'_λ is invariant under P_e and c° and $c^e\notin R'_\lambda$. By Lemma 1 $\Delta(T_\mu)$ having one of properties 1°~3° and 1^e~3^e exists. Suppose $\Delta(T_\mu)$

has property 1°, then we can find $x_\mu \in \Delta(T_\mu)$ such that $(R'_\mu + [c^\circ] + [c^e]) \cap ([x_\mu] \oplus [T_\mu x_\mu]) = \{0\}$. Let $R_\mu = R'_\mu + [x_\mu] + [c^\circ - T_\mu x_\mu]$, then clearly R_μ satisfies above a), b) and c). And c° and $c^e \notin R_\mu$ also holds. Suppose $c^\circ \in R_\mu$, then $c^\circ = x + nx_\mu + m(c^\circ - T_\mu x_\mu)$ for some $x \in R'_\mu$ and some integers n and m, so $-x + (1-m)c^\circ = nx_\mu - mT_\mu x_\mu$, but by our choice of x_μ, $nx_\mu - mT_\mu x_\mu = 0$ and $x + (m-1)c^\circ = 0$. This implies $n = m = 0 \bmod p$ and $c^\circ = x \in R'_\mu$ which is a contradiction. Suppose $c^e \in R_\mu$, then $c^e = x + nx_\mu + m(c^\circ - T_\mu x_\mu)$ for some $x \in R'_\mu$ and some integers n and m, but x_μ and $T_\mu x_\mu \in C[p]^\circ$, so $c^e = x \in R'_\mu$ which is also a contradiction. Suppose $\Delta(T_\mu)$ has property 2°, then we can find $x_\mu \in \Delta(T_\mu)$ such that $(R'_\mu + [c^\circ] + [c^e]) \cap ([x_\mu] \oplus [T_\mu x_\mu - qx_\mu]) = \{0\}$. Let $R_\mu = R'_\mu + [x_\mu] + [c^e - T_\mu x_\mu + qx_\mu]$, then clearly R_μ satisfies above a), b) and c). And c° and $c^e \notin R_\mu$ also holds. Suppose $c^\circ \in R_\mu$, then $c^\circ = x + nx_\mu + m(c^e - T_\mu x_\mu + qx_\mu)$ for some $x \in R'_\mu$ and some integers n and m, but $x_\mu \in C[p]^\circ$ and $T_\mu x_\mu - qx_\mu \in C[p]^e$, hence we have $c^\circ = x^\circ + nx_\mu$, that is, $-x^\circ + c^\circ = nx_\mu$. Our choice of x_μ implies $nx_\mu = 0 = -x^\circ + c^\circ$, so we have $c^\circ = x^\circ \in R'_\mu$ which is a contradiction. Suppose $c^e \in R_\mu$, then $c^e = x + nx_\mu + m(c^e - T_\mu x_\mu + qx_\mu)$ for some $x \in R'_\mu$ and some integers n and m. Hence $-x + (1-m)c^e = nx_\mu - m(T_\mu x_\mu - qx_\mu)$, but by our choice of x_μ we see $-x + (1-m)c^e = 0 = nx_\mu - m(T_\mu x_\mu - qx_\mu)$. This implies $n = m = 0 \bmod p$, so $c^e = x \in R'_\mu$ which is also a contradiction. Suppose $\Delta(T_\mu)$ has property 3°, then we can find $x_\mu \in \Delta(T_\mu)$ such that $(R'_\mu + [c^\circ] + [c^e]) \cap ([x_\mu] \oplus [(T_\mu x_\mu)^\circ] \oplus [(T_\mu x_\mu)^e]) = \{0\}$. Let $R_\mu = R'_\mu + [x_\mu] + [c^\circ - (T_\mu x_\mu)^\circ] + [c^e - (T_\mu x_\mu)^e]$. Then R_μ clearly satisfies a), b) and c). And c° and $c^e \notin R_\mu$ can be seen as follows. Suppose $c^\circ = x + nx_\mu + m(c^\circ - (T_\mu x_\mu)^\circ) + m'(c^e - (T_\mu x_\mu)^e)$ for some $x \in R'_\mu$ and integers n, m and m', then $c^\circ = x^\circ + nx_\mu + m(c^\circ - (T_\mu x_\mu)^\circ)$, so $-x^\circ + (1-m)c^\circ = nx_\mu - m(T_\mu x_\mu)^\circ$. This implies $nx_\mu - m(T_\mu x_\mu)^\circ = 0 = -x^\circ + (1-m)c^\circ$ by our choice of x_μ. Hence $m = 0$ and $c^\circ = x^\circ \in R'_\mu$ which is a contradiction. We can see also $c^e \notin R_\mu$ for same reason. And x_μ and $c - T_\mu x_\mu \in R_\mu$ is clear. The construction of R_μ for $\Delta(T_\mu)$ having one of properties $1^e \sim 3^e$ is exactly similar by exchanging odd for even.

Let $Q = \bigcup_{\lambda \in \Lambda} R_\lambda$. Then the above properties a)$\sim$d) for all R_λ guarantee that Q is a subgroup between $\sum[p]$ and $C[p]$ not invariant under any $T_\lambda (\lambda \in \Lambda)$ but invariant under P_e.

Q. E. D.

Proof of Theorem. We have clearly $C = C^e \oplus C^\circ$ and $\sum = \sum^e \oplus \sum^\circ$. By Lemma 2 there exists a subgroup Q between $\sum[p]$ and $C[p]$ such that Q is not invariant under any non-singular homomorphisms on $C[p]$ but is invariant under P_e, therefore $\sum^e[p] = \sum[p]^e \subset Q^e \subset C[p]^e = C^e[p]$, $\sum^\circ[p] = \sum[p]^\circ \subset Q^\circ \subset C[p]^\circ = C^\circ[p]$ and $Q = Q^e \oplus Q^\circ$. We can show that there exists a pure snbgroup G_1 (G_2) of C^e (C°) which contains $\sum^e$ ($\sum^\circ$) and $G_1[p] = Q^e$

$(G_2[p]=Q^0)$. This fact is known as the purification property, see Lemma 1 in [2] and more general form in [1]. Clearly G_1 and G_2 are not bounded. Let $G=G_1\oplus G_2$, then G is a pure subgroup of C which contains $\sum$ and $G[p]=G_1[p]\oplus G_2[p]=Q^e\oplus Q^0=Q$. By Proposition G has the properties 1) and 2) in Theorem.

Q. E. D.

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(Received Dec. 24, 1968)