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A QUASI-DECOMPOSABLE ABELIAN GROUP WITHOUT PROPER ISOMORPHIC QUOTIENT GROUPS AND PROPER ISOMORPHIC SUBGROUPS II

By

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1. Introduction

This paper is a continuation of our paper [2] with the same title. In [2] one of theorems was stated without detailed proof. The purpose of this paper is just to give a complete proof of it. So we shall omit an origin of our problem and general background for it, see the first section of [2] and references in [2].

Let $p > 1$ be a fixed prime number, $C(p^n)$ be a cyclic group of order p^n , Σ be the direct sum of cyclic groups $C(p^n)$, Π be the direct product of cyclic groups $C(p^n)$ and C be the torsion group of Π , that is, Σ is the standard basic group and C is the torsion completion of Σ .

Our theorem is as following, that is Theorem 3 in [2].

Theorem. *There exists a pure subgroup G of C which contains Σ and satisfies properties;*

- 1) G has no proper isomorphic quotient groups,
- 2) G has no proper isomorphic subgroups,
- 3) G has a decomposition $G_1 \oplus G_2$ such that G_1 and G_2 are not bounded.

2. Group Homomorphisms on $C[p]$.

The p -socle $C[p]$ of C is a vector space over the prime field of characteristic p and can be topologized as a totally disconnected compact topological group, because Π is clearly a totally disconnected compact topological group with respect to the product topology of compact discrete topologies and the p -socle $C[p]$ of C is the closed subgroup $\{x | x \in \Pi, px = 0\}$ of Π . Actually $U_n = \{x | x \in C[p] \text{ and the height of } x \geq n\} = (p^n C)[p]$ ($n = 1, 2, \dots$) are open compact subgroups of $C[p]$ and $\{U_n\}$ is a fundamental system of 0-neighbors in $C[p]$. These two structures on $C[p]$ which are a vector space and a totally disconnected compact group are used in an essential way.

Every continuous group homomorphism T on $C[p]$ defines compact subgroups $E_q(T) = \{x | x \in C[p] \text{ and } Tx = qx\}$ ($0 \leq q < p$) and the compact subgroup

$E(T) = E_0(T) \oplus E_1(T) \oplus \cdots \oplus E_{p-1}(T)$. We call T is a *singular* homomorphism if $E(T)$ is an open compact subgroup of $C[p]$. For instance a continuous projection on $C[p]$ is singular. If a continuous group homomorphism T on $C[p]$ has a dense subgroup which is invariant under T and on which T is one to one, T is called a *semi-isomorphism* on $C[p]$.

In the final step in our proof of the theorem we shall use following proposition which is shown in [2] (Theorem 1 of [2]).

Proposition. *Let G be a pure subgroup of C which contains Σ and $G[p]$ be the p -socle of G .*

1) *If $G[p]$ is not invariant under any non-singular onto homomorphism on $C[p]$, then G has no proper isomorphic quotient groups.*

2) *If $G[p]$ is not invariant under any non-singular semi-isomorphism on $C[p]$, then G has no proper isomorphic subgroups.*

Therefore our main work in this paper is to show existence of a group Q which is between $\Sigma[p]$ and $C[p]$, and which is not invariant under any non-singular homomorphism but invariant under the canonical projection P_e onto even coordinates (see Lemma 2). In our proof of this fact we have some difficulty which comes from non-commutativity between non-singular homomorphisms and P_e . So we classify degree of non-commutativity in the following section.

3. Proof of Theorem

We introduce eight classes (essentially four) of non-singular homomorphisms depending on degree of non-commutativity. Hereafter we shall use notations $A^\circ = P_e(A)$ ($A^\circ = (I - P_e)(A)$) for a subset A of $C[p]$ and $x^\circ = P_e x$ ($x^\circ = x - P_e x$) for an element x in $C[p]$. Every continuous homomorphism T on $C[p]$ defines compact subgroups

$$F^\circ(T) = \{x | x \in C[p]^\circ \text{ and } (Tx)^\circ = 0\},$$

$$E_{q,n}^\circ(T) = \{x | x \in U_n^\circ \text{ and } (Tx)^\circ = qx\} \quad \text{for } q=0, 1, \dots, p-1;$$

$n=1, 2, \dots$. $F^\circ(T)$ and $E_{q,n}^\circ(T)$ are defined similarly. Let $\tilde{E}_n^\circ(T) = E_{0,n}^\circ(T) \oplus E_{1,n}^\circ(T) \oplus \cdots \oplus E_{p-1,n}^\circ(T)$ and $\tilde{E}_n^\circ(T) = E_{0,n}^\circ(T) \oplus E_{1,n}^\circ(T) \oplus \cdots \oplus E_{p-1,n}^\circ(T)$.

Following eight types about T are defined;

- I° *there exists N such that $F^\circ(T) \supset U_N^\circ$ and $\tilde{E}_n^\circ(T) \not\supset U_m^\circ$ for all $n, m \geq N$,*
- II° *$F^\circ(T) \not\supset U_n^\circ$ for all $n=1, 2, \dots$ but there are N and q such that $E_{q,N}^\circ(T) \supset U_N^\circ$,*
- III° *$F^\circ(T) \not\supset U_n^\circ$ for all $n=1, 2, \dots$ and $E_{q,n}^\circ(T) \not\supset U_m^\circ$ for all $n, m =$*

$1, 2, \dots$ and $q=0, 1, \dots, p-1$, but there are $N' \geq N$ such that $\tilde{E}_N^\circ(T) \supset U_{N'}^\circ$,

IV $^\circ$ $F^\circ(T) \not\supset U_n^\circ$ for all $n=1, 2, \dots$ and $\tilde{E}_n^\circ(T) \not\supset U_m^\circ$ for all $n, m=1, 2, \dots$.

I $^\circ$, II $^\circ$, III $^\circ$ and IV $^\circ$ are defined by exchanging odd and even.

We note that every non-singular homomorphism is at least of one of the above eight types. Because, suppose T is of none of the above eight types, then there are N, n and m such that $n, m \geq N$ and

- 1) $F^\circ(T) \supset U_N^\circ$ and $F^e(T) \supset U_N^\circ$,
- 2) $\tilde{E}_n^\circ(T) \supset U_m^\circ$ and $\tilde{E}_n^e(T) \supset U_m^\circ$.

Then 1) implies $E_{q,n}^\circ(T) \subset E_q(T)$ and $E_{q,n}^e(T) \subset E_q(T)$ for all $q=0, 1, \dots, p-1$. So 2) implies $E(T) = E_0(T) \oplus \dots \oplus E_{p-1}(T) \supset U_m^\circ \oplus U_m^e = U_m$. This means that T is a singular homomorphism.

In order to construct Q explained at the end of section 2 we need the following ;

Lemma 1. For an arbitrary non-singular homomorphism T we can find a one-parameter family $\Delta(T) = \{x_t | 0 \leq t \leq 1\}$ of elements in $C[p]$ which has one of the following six properties; 1 $^\circ$, 2 $^\circ$, 3 $^\circ$, 1 $^\circ$, 2 $^\circ$ and 3 $^\circ$,

- 1 $^\circ$ $x_t, Tx_t \in C[p]^\circ$ for all $0 \leq t \leq 1$ and four elements x_s, x_t, Tx_s and Tx_t are linearly independent for arbitrary $s \neq t$,
- 2 $^\circ$ there exists $q, 0 \leq q \leq p-1$ such that $x_t \in C[p]^\circ$ and $Tx_t - qx_t \in C[p]^\circ$ for all $0 \leq t \leq 1$ and four elements $x_s, x_t, Tx_s - qx_s$ and $Tx_t - qx_t$ are linearly independent for arbitrary $s \neq t$,
- 3 $^\circ$ $x_t \in C[p]^\circ$ for all $0 \leq t \leq 1$ and six elements $x_s, x_t, (Tx_s)^\circ, (Tx_s)^e, (Tx_t)^\circ$ and $(Tx_t)^e$ are linearly independent for arbitrary $s \neq t$.

1 $^\circ$, 2 $^\circ$ and 3 $^\circ$ are dual properties of 1 $^\circ$, 2 $^\circ$ and 3 $^\circ$ by exchanging odd for even.

Proof. Of course the choice of these six properties about $\Delta(T)$ depends on types of T defined above. Therefore the proof is divided into four cases, a)~d).

The following fact (*) which was shown already in [2] (see b) of the proof of Lemma 3 in [2]) will be used in cases a), b) and d).

(*) Suppose x_0 and Tx_0 are linearly independent and $\{z_0, z_1, \dots, z_{p^2-1}\}$ is a group $[x_0] \oplus [Tx_0]$. Then we can take U_M such that $z_i + U_M + T(U_M)$ ($0 \leq i \leq p^2-1$) are disjoint, and if $\{y_t | 0 \leq t \leq 1\} \subset U_M$ satisfies the condition that $y_t - y_s$ and $T(y_t - y_s)$ are linearly independent for arbitrary $s \neq t$, then four elements $x_0 + y_s, x_0 + y_t, T(x_0 + y_s)$ and $T(x_0 + y_t)$ are linearly independent for

arbitrary $s \neq t$.

a) If T is of type I° , then $\Delta(T)$ having property 1° exists.

If a compact group $U_n^\circ/\tilde{E}_n^\circ(T)$ is finite, then $\tilde{E}_n^\circ(T) \supset U_n^\circ$ for some m , which contradicts our assumption. Therefore, since a compact infinite group has at least cardinality $\aleph$, $\dim U_n^\circ/\tilde{E}_n^\circ(T) = \aleph$ for all $n=1, 2, \dots$, then $U_n^\circ = \tilde{E}_n^\circ(T) \oplus D_n$ for all $n=1, 2, \dots$ and $\dim D_n = \aleph$ for all $n=1, 2, \dots$. Choose $0 \neq x_0 \in D_N$, then $Tx_0 \in C[p]^\circ$ since $D_N \subset U_N^\circ \subset F^\circ(T)$ and x_0 and Tx_0 are linearly independent. By the above (*) we can determine $M \geq N$ and choose a basis $\{y_t | 0 \leq t \leq 1\}$ of D_M , then $Ty_t \in C[p]^\circ$ for all $0 \leq t \leq 1$ and for $s \neq t$ $y_s - y_t$ and $T(y_s - y_t)$ are linearly independent, because $D_M \cap \tilde{E}_M^\circ(T) = \{0\}$. Therefore by (*) $\Delta(T) = \{x_0 + y_t | 0 \leq t \leq 1\}$ has the property 1° .

b) If T is of type II° , then $\Delta(T)$ having property 2° exists.

Our assumption $F^\circ(T) \not\supset U_n^\circ$ for all $n=1, 2, \dots$ implies that compact groups $U_n^\circ/F^\circ(T) \cap U_n^\circ$ for all $n=1, 2, \dots$ are infinite, so $U_n^\circ = (F^\circ(T) \cap U_n^\circ) \oplus H_n$ for all $n=1, 2, \dots$ and $\dim H_n = \aleph$. And we have $(Ty)^\circ \neq 0$ for all $0 \neq y \in H_n$. Let $\tilde{T} = T - q(I - P_e)$, then our assumption $E_{q,N}^\circ(T) \supset U_N^\circ$ implies $E_{0,N}^\circ(\tilde{T}) = U_N^\circ$, and $\tilde{T}x = (Tx)^\circ$ for all $x \in U_N^\circ$. Choose $0 \neq x_0 \in H_N$, then $0 \neq \tilde{T}x_0 \in C[p]^\circ$ and x_0 and $\tilde{T}x_0$ are linearly independent. We can apply (*) to $\tilde{T}$, so M can be chosen as in (*) and $\{y_t | 0 \leq t \leq 1\}$ can be a basis of H_N . Then $\tilde{T}y_t \in C[p]^\circ$ for all $0 \leq t \leq 1$ and for $s \neq t$, $y_s - y_t$ and $\tilde{T}(y_s - y_t)$ are linearly independent, because $y_s - y_t \in H_N \subset U_N^\circ$ and $\tilde{T}(y_s - y_t) \in C[p]^\circ$. Therefore by (*) we can see that $\Delta(T) = \{x_0 + y_t | 0 \leq t \leq 1\}$ is a one-parameter family having the property 2° .

The following fact (**) will be used in cases c) and d).

(**) Suppose $0 \neq x_0 \in C[p]^\circ$ such that $x_0, (Tx_0)^\circ$ and $(Tx_0)^\circ$ are linearly independent and suppose $\{z_0, z_1, \dots, z_{p^3-1}\}$ is the group $[x_0] \oplus [(Tx_0)^\circ] \oplus [(Tx_0)^\circ]$. Then we can take U_M such that $z_i + U_M + (T(U_M))^\circ + (T(U_M))^\circ$ ($0 \leq i \leq p^3 - 1$) are mutually disjoint, and if $\{y_t | 0 \leq t \leq 1\} \subset U_M^\circ$ satisfies the condition that three elements $y_t - y_s, (T(y_t - y_s))^\circ$ and $(T(y_t - y_s))^\circ$ are linearly independent for arbitrary $s \neq t$. Thus $\Delta(T) = \{x_0 + y_t | 0 \leq t \leq 1\}$ is a one-parameter family having the property 3° .

Now (**) can be shown as follows; it is clear that there exists U_N such that $z_i + U_N$ ($0 \leq i \leq p^3 - 1$) are mutually disjoint. By continuity of $P_e T$ and $(I - P_e)T$ there is $M \geq N$ such that $(T(U_M))^\circ, (T(U_M))^\circ \subset U_N$, then U_M is as desired, since $z_i + U_M + (T(U_M))^\circ + (T(U_M))^\circ \subset z_i + U_N + U_N + U_N = z_i + U_N$ ($0 \leq i \leq p^3 - 1$). Next, suppose $n_1(x_0 + y_t) + n_2(T(x_0 + y_t))^\circ + n_3(T(x_0 + y_t))^\circ = n'_1(x_0 + y_s) + n'_2(T(x_0 + y_s))^\circ + n'_3(T(x_0 + y_s))^\circ$ for some $s \neq t$, where $n_1, n_2, n_3, n'_1, n'_2$ and n'_3 are integers, then $n_1x_0 + n_2(Tx_0)^\circ + n_3(Tx_0)^\circ = z_i$ for some i and $n'_1x_0 + n'_2(Tx_0)^\circ + n'_3(Tx_0)^\circ = z_j$ for some j . So $z_i + n_1y_t + n_2(Ty_t)^\circ + n_3(Ty_t)^\circ = z_j + n'_1y_s + n'_2(Ty_s)^\circ + n'_3(Ty_s)^\circ$ and $n_1y_t + n_2(Ty_t)^\circ + n_3(Ty_t)^\circ, n'_1y_s + n'_2(Ty_s)^\circ + n'_3(Ty_s)^\circ \in U_M + (T(U_M))^\circ$

$+(T(U_M))^e$. By our choice of U_M , z_i must be equal to z_j , this implies $n_1=n'_1 \bmod p$, $n_2=n'_2 \bmod p$ and $n_3=n'_3 \bmod p$. Therefore we have $n_1(y_t-y_s)+n_2(T(y_t-y_s))^e+n_3(T(y_t-y_s))^e=0$, but our assumption about $\{y_t\}$ implies $n_1=n_2=n_3=0 \bmod p$. Hence $\Delta(T)=\{x_0+y_t|0 \leq t \leq 1\}$ satisfies property 3° .

c) If T is of type III° , then $\Delta(T)$ having property 3° exists.

$F^\circ(T) \not\subset U_n^\circ$ for all $n=1, 2, \dots$ yields a decomposition of U_n° such that

(1) $U_n^\circ = (F^\circ(T) \cap U_n^\circ) \oplus H_n$ and $\dim H_n = \aleph$ for all $n=1, 2, \dots$. $\tilde{E}_N^\circ(T) \supset U_{N'}^\circ$ yields a decomposition of U_n° for $n \geq N'$ such that

(2) $U_n^\circ = \tilde{E}_n^\circ(\tilde{T}) \oplus D_n$ and $\dim D_n < \aleph_0$ for all $n \geq N'$.

Because, $U_n^\circ / \tilde{E}_n^\circ(T) \subset U_{N'}^\circ / \tilde{E}_n^\circ(T) \subset \tilde{E}_N^\circ(T) / \tilde{E}_n^\circ(T)$

$$\cong \sum_{q=0}^{p-1} (E_{q,N}^\circ(T) / E_{q,n}^\circ(T)), \text{ but for each } q, E_{q,N}^\circ(T) / E_{q,n}^\circ(T)$$

$$= E_{q,N}^\circ(T) / E_{q,N}^\circ(T) \cap U_n^\circ \cong (E_{q,N}^\circ(T) + U_n^\circ) / U_n^\circ \subset U_{N'}^\circ / U_n^\circ \text{ and } U_{N'}^\circ / U_n^\circ$$

is finite.

And assumptions $\tilde{E}_N^\circ(T) \supset U_{N'}^\circ$ and $E_{q,n}^\circ(T) \not\subset U_m^\circ$ for all $n, m=1, 2, \dots$ and $q=0, 1, \dots, p-1$ imply that there are $q' \neq q''$ not depending on n such that

(3) $\dim E_{q',n}^\circ(T) = \dim E_{q'',n}^\circ(T) = \aleph$ for all $n \geq N'$.

This (3) can be shown easily, see a) in the proof of Lemma 3 in [2].

(1) and (2) imply that there is q''' not depending on n such that

(4) $\dim (E_{q''',n}^\circ(T) / E_{q''',n}^\circ(T) \cap F^\circ(T)) = \aleph$ for all $n \geq N'$.

Because, suppose

$$E_{q,N'}^\circ(T) / E_{q,N'}^\circ(T) \cap F^\circ(T) \text{ are finite for all } 0 \leq q \leq p-1.$$

Then, since

$$E_{q,N'}^\circ(T) / E_{q,N'}^\circ(T) \cap F^\circ(T) \cong (E_{q,N'}^\circ(T) + F^\circ(T) \cap U_{N'}^\circ) / F^\circ(T) \cap U_{N'}^\circ, \\ \left(\sum_{q=0}^{p-1} E_{q,N'}^\circ(T) + F^\circ(T) \cap U_{N'}^\circ \right) / F^\circ(T) \cap U_{N'}^\circ = (\tilde{E}_N^\circ(T) + F^\circ(T) \cap U_{N'}^\circ) / F^\circ(T) \cap U_{N'}^\circ,$$

must be finite. However $D_{N'}$ is finite by (2), so we see that

$(\tilde{E}_N^\circ(T) + D_{N'} + F^\circ(T) \cap U_{N'}^\circ) / F^\circ(T) \cap U_{N'}^\circ = U_{N'}^\circ / F^\circ(T) \cap U_{N'}^\circ$ is finite. This contradicts the facts that $U_{N'}^\circ / F^\circ(T) \cap U_{N'}^\circ \cong H_{N'}$ and $H_{N'}$ is infinite by (1). Therefore we have that there exists q''' such that $E_{q''',N'}^\circ(T) / E_{q''',N'}^\circ(T) \cap F^\circ(T)$ is finite. If $n \geq N'$, then $E_{q''',N'}^\circ(T) / E_{q''',n}^\circ(T) \cap F^\circ(T)$ is infinite, because $E_{q''',n}^\circ(T) \subset E_{q''',N'}^\circ(T)$. On the other hand $E_{q''',N'}^\circ(T) / E_{q''',n}^\circ(T)$ is finite, because $E_{q''',N'}^\circ(T) / E_{q''',n}^\circ(T) = E_{q''',N'}^\circ(T) / E_{q''',N'}^\circ(T) \cap U_n^\circ \cong (E_{q''',N'}^\circ(T) + U_n^\circ) / U_n^\circ \subset U_{N'}^\circ / U_n^\circ$ and $U_{N'}^\circ / U_n^\circ$ is finite. These two facts imply clearly that $E_{q''',n}^\circ(T) / E_{q''',n}^\circ(T) \cap F^\circ(T)$ is infinite for all $n \geq N'$. Therefore (4) is shown (note that every compact infinite group has at least cardinality $\aleph$).

Without loss of generality we can assume $q''' = q'$ in (3) and combining

(3) and (4) we can see that there exist $q' \neq q''$ not depending on n such that for all $n \geq N'$ we have

$$(5) \quad \dim E_{q',n}^\circ(T) = \dim E_{q'',n}^\circ(T) = \dim (E_{q',n}^\circ(T)/E_{q',n}^\circ(T) \cap F^\circ(T)) = \aleph.$$

Since $E_{q',n}^\circ(T)$ and $E_{q'',n}^\circ(T)$ ($n=1, 2, \dots$) are decreasing and $\bigcap_{n=1}^\infty U_n^\circ = \{0\}$, the following three cases may occur.

- (6) (α) $E_{q'',n}^\circ(T) \cap F^\circ(T)$ is infinite for all $n \geq N'$,
 (β) $E_{q'',n}^\circ(T) \cap F^\circ(T)$ is finite but $E_{q',n}^\circ(T) \cap F^\circ(T)$ is infinite for all sufficiently large $n \geq N'$,
 (γ) $E_{q'',n}^\circ(T) \cap F^\circ(T) = E_{q',n}^\circ(T) \cap F^\circ(T) = \{0\}$ for all sufficiently large $n \geq N'$.

We want to show that

- (7) If n satisfies (6), then there exists $\{y_t | 0 \leq t \leq 1\} \subset U_n^\circ$ such that $y_s - y_t$, $(T(y_s - y_t))^\circ$ and $(T(y_s - y_t))^\circ$ are linearly independent for arbitrary $s \neq t$.

Suppose n satisfies (α) . Then $\dim (E_{q'',n}^\circ(T) \cap F^\circ(T)) = \aleph$. By (5) $E_{q',n}^\circ(T)$ has a decomposition $E_{q',n}^\circ(T) = (E_{q',n}^\circ(T) \cap F^\circ(T)) \oplus A_n$ and $\dim(A_n) = \aleph$. Let $\{z'_t | 0 \leq t \leq 1\}$ be a basis of A_n and $\{z''_t | 0 \leq t \leq 1\}$ be a basis of $E_{q'',n}^\circ(T) \cap F^\circ(T)$, then $\{y_t = z'_t + z''_t | 0 \leq t \leq 1\}$ has the desired property. Because, if $n_1(y_s - y_t) + n_2(T(y_s - y_t))^\circ + n_3(T(y_s - y_t))^\circ = 0$, then $n_1(y_s - y_t) + n_2(T(y_s - y_t))^\circ = 0$ and $n_3(T(y_s - y_t))^\circ = 0$. $n_1(y_s - y_t) + n_2(T(y_s - y_t))^\circ = 0$ implies that $(n_1 + n_2q')(z'_s - z'_t) + (n_1 + n_2q'')(z''_s - z''_t) = 0$, but $z'_s - z'_t \neq 0$, $z''_s - z''_t \neq 0$ and $E_{q',n}^\circ(T) \cap E_{q'',n}^\circ(T) = \{0\}$, so $n_1 + n_2q' = n_1 + n_2q'' = 0 \pmod p$. This implies $n_1 = n_2 = 0 \pmod p$ since $q' \neq q'' \pmod p$. And $n_3(T(y_s - y_t))^\circ = 0$ implies $0 = n_3(T(z'_s - z'_t))^\circ + n_3(T(z''_s - z''_t))^\circ = n_3(T(z'_s - z'_t))^\circ$, but $(T(z'_s - z'_t))^\circ \neq 0$ since $0 \neq z'_s - z'_t \in A_n$, therefore we have $n_3 = 0 \pmod p$. Suppose n satisfies (β) . Then automatically $\dim (E_{q'',n}^\circ(T)/E_{q'',n}^\circ(T) \cap F^\circ(T)) = \aleph$. Therefore the same argument as in (α) applies by replacing q'' with q' . Suppose n satisfies (γ) . Let φ_n be a canonical projection onto H_n from U_n° , which is well defined by a decomposition $U_n^\circ = (F^\circ(T) \cap U_n^\circ) \oplus H_n$ in (1). The condition in (γ) means that φ_n is one to one on $E_{q',n}^\circ(T)$ and $E_{q'',n}^\circ(T)$. This implies $\dim(\varphi_n(E_{q',n}^\circ(T))) = \dim(\varphi_n(E_{q'',n}^\circ(T))) = \aleph$. Let $X'_n = \varphi_n(E_{q',n}^\circ(T))$, $X''_n = \varphi_n(E_{q'',n}^\circ(T))$, $X'_n = Y'_n \oplus (X'_n \cap X''_n)$ and $X''_n = Y''_n \oplus (X'_n \cap X''_n)$. If $\dim(X'_n \cap X''_n) = \aleph$, then choose $\{y_t | 0 \leq t \leq 1\} \subset U_n^\circ$ as follows. Let $\{u_t | 0 \leq t \leq 2\}$ be a basis of $X'_n \cap X''_n$. Then there exists uniquely z'_t in $E_{q',n}^\circ(T)$ such that $u_t = \varphi_n(z'_t)$ for all $0 \leq t \leq 1$ and uniquely z''_t in $E_{q'',n}^\circ(T)$ such that $u_{t+1} = \varphi_n(z''_t)$ for all $0 \leq t \leq 1$. Let $y_t = z'_t + z''_t$ for $0 \leq t \leq 1$, then it is easy to see the linear independence of $(y_s - y_t)$ and $(T(y_s - y_t))^\circ$ for $s \neq t$.

(see case (α)). And $(T(y_s - y_t))^{\circ} = (T(z'_s - z'_t + z''_s - z''_t))^{\circ} = (T(u_s - u_t + u_{s+1} - u_{t+1}))^{\circ} \neq 0$, because $0 \neq u_s - u_t + u_{s+1} - u_{t+1} \in H_n$. Therefore $y_s - y_t$, $(T(y_s - y_t))^{\circ}$ and $(T(y_s - y_t))^{\circ}$ are linearly independent for arbitrary $s \neq t$. If $\dim(X'_n \cap X''_n) < \aleph$, then $\dim(Y'_n) = \dim(Y''_n) = \aleph$. Let $\{u_t | 0 \leq t \leq 1\}$ be a basis of Y'_n and $\{v_t | 0 \leq t \leq 1\}$ be a basis of Y''_n , then $\{u_t\} \cup \{v_t\}$ is a linearly independent system in H_n . We can take unique z'_t in $E_{q',n}^{\circ}(T)$ and z''_t in $E_{q'',n}^{\circ}(T)$ such that $u_t = \varphi_n(z'_t)$ and $v_t = \varphi_n(z''_t)$ for all $0 \leq t \leq 1$. Then the system $\{y_t = z'_t + z''_t | 0 \leq t \leq 1\}$ is as desired, which is proved in the same way as above.

Now we can present the proof of c). By (7) there exists $0 \neq x_0 \in C[p]^{\circ}$ such that x_0 , $(Tx_0)^{\circ}$ and $(Tx_0)^{\circ}$ are linearly independent. By (**) we can determine a large M which satisfies the condition stated in (**). Without loss of generality we can assume this M satisfies one of (α) , (β) and (γ) , because M can be replaced by a larger one if it is necessary. (7) guarantees the existence of $\{y_t | 0 \leq t \leq 1\} \subset U_M^{\circ}$ for which $y_s - y_t$, $(T(y_s - y_t))^{\circ}$ and $(T(y_s - y_t))^{\circ}$ are linearly independent. Therefore (**) shows us that $\Delta(T) = \{x_0 + y_t | 0 \leq t \leq 1\}$ is a one-parameter family having property 3 $^{\circ}$.

d) If T is of type IV $^{\circ}$ then $\Delta(T)$ having property 1 $^{\circ}$ or 3 $^{\circ}$ exists.

Our assumptions yield decompositions of U_n° ;

$$\begin{aligned} U_n^{\circ} &= (F^{\circ}(T) \cap U_n^{\circ}) \oplus H_n \quad \text{and} \quad \dim H_n = \aleph \quad \text{for all } n = 1, 2, \dots, \\ U_n^{\circ} &= \tilde{E}_n^{\circ}(T) \oplus D_n \quad \text{and} \quad \dim D_n = \aleph \quad \text{for all } n = 1, 2, \dots. \end{aligned}$$

We can see that for all $n = 1, 2, \dots$

$$\dim(F^{\circ}(T) \cap U_n^{\circ} / F^{\circ}(T) \cap \tilde{E}_n^{\circ}(T)) = \aleph \quad \text{or} \quad \dim(H_n / H_n \cap \tilde{E}_n^{\circ}(T)) = \aleph.$$

Because, suppose both dimensions are less than $\aleph$, then

$\dim((F^{\circ}(T) \cap U_n^{\circ} + \tilde{E}_n^{\circ}(T)) / \tilde{E}_n^{\circ}(T)) < \aleph$ and $\dim((H_n + \tilde{E}_n^{\circ}(T)) / \tilde{E}_n^{\circ}(T)) < \aleph$ since $F^{\circ}(T) \cap U_n^{\circ} / F^{\circ}(T) \cap \tilde{E}_n^{\circ}(T) \cong (F^{\circ}(T) \cap U_n^{\circ} + \tilde{E}_n^{\circ}(T)) / \tilde{E}_n^{\circ}(T)$ and $H_n / H_n \cap \tilde{E}_n^{\circ}(T) \cong (H_n + \tilde{E}_n^{\circ}(T)) / \tilde{E}_n^{\circ}(T)$. Therefore $\aleph > \dim(((F^{\circ}(T) \cap U_n^{\circ}) + H_n + \tilde{E}_n^{\circ}(T)) / \tilde{E}_n^{\circ}(T)) = \dim(U_n^{\circ} / \tilde{E}_n^{\circ}(T)) = \dim D_n = \aleph$. This is a contradiction. Next, if $\dim(F^{\circ}(T) \cap U_n^{\circ} / F^{\circ}(T) \cap \tilde{E}_n^{\circ}(T)) < \aleph$ for some n , then $\dim(F^{\circ}(T) \cap U_m^{\circ} / F^{\circ}(T) \cap \tilde{E}_m^{\circ}(T)) < \aleph$ for all $m \geq n$. Because $F^{\circ}(T) \cap U_m^{\circ} / F^{\circ}(T) \cap \tilde{E}_m^{\circ}(T) = F^{\circ}(T) \cap U_m^{\circ} / F^{\circ}(T) \cap U_m^{\circ} \cap \tilde{E}_n^{\circ}(T) \cong (F^{\circ}(T) \cap U_m^{\circ} + \tilde{E}_n^{\circ}(T)) / \tilde{E}_n^{\circ}(T) \subset (F^{\circ}(T) \cap U_n^{\circ} + \tilde{E}_n^{\circ}(T)) / \tilde{E}_n^{\circ}(T) \cong F^{\circ}(T) \cap U_n^{\circ} / F^{\circ}(T) \cap \tilde{E}_n^{\circ}(T)$. Therefore the following two cases may occur;

(δ) $\dim(F^{\circ}(T) \cap U_n^{\circ} / F^{\circ}(T) \cap \tilde{E}_n^{\circ}(T)) = \aleph$ for all $n = 1, 2, \dots$,

(ε) There exists N such that $\dim(H_n / H_n \cap \tilde{E}_n^{\circ}(T)) = \aleph$ for all $n \geq N$.

If (δ) happens, then we can show the existence of $\Delta(T)$ having property 1 $^{\circ}$. (δ) implies that

$$F^{\circ}(T) \cap U_n^{\circ} = (F^{\circ}(T) \cap \tilde{E}_n^{\circ}(T)) \oplus X_n \quad \text{and} \quad \dim(X_n) = \aleph \quad \text{for all } n = 1, 2, \dots.$$

We take $0 \neq x_0 \in X_1$, then $(Tx_0)^{\circ} = 0$ and x_0 and Tx_0 are linearly independent.

According to (*) we can determine M which satisfies the condition stated in (*). For this M we take a basis $\{y_t | 0 \leq t \leq 1\}$ of X_M , then $\{y_t\} \subset U_M^\circ$ and $\{Ty_t\} \subset C[p]^\circ$. Suppose $n_1(y_s - y_t) + n_2T(y_s - y_t) = 0$ for $s \neq t$, then we have $n_1 = n_2 = 0 \pmod p$, because, if $n_2 \neq 0 \pmod p$, then $T(y_s - y_t) = q(y_s - y_t)$ for some q , $0 \leq q \leq p-1$. This leads to a contradiction $0 \neq y_s - y_t \in \tilde{E}_M^\circ(T) \cap X_M^\circ = \{0\}$. By (*) $\Delta(T) = \{x_0 + y_t | 0 \leq t \leq 1\}$ is a one-parameter family having property 1° . If (ε) happens, then we can show the existence of $\Delta(T)$ having property 3° . (ε) implies that

$$H_n = (H_n \cap \tilde{E}_n^\circ(T)) \oplus Y_n \text{ and } \dim Y_n = \aleph \text{ for all } n \geq N.$$

We can take $0 \neq x_0 \in X_N$, then we can see that x_0 , $(Tx_0)^\circ$ and $(Tx_0)^\circ$ are linearly independent. According to (**) we can determine M . Take a basis $\{y_t | 0 \leq t \leq 1\}$ from Y_M , then we see also that $y_s - y_t$, $(T(y_s - y_t))^\circ$ and $(T(y_s - y_t))^\circ$ are linearly independent for $s \neq t$. By (**) $\Delta(T) = \{x_0 + y_t | 0 \leq t \leq 1\}$ is a one-parameter family having property 3° .

Exchanging odd and even in a)~d) we have

e) If T is of one of types $I^\circ \sim IV^\circ$, then $\Delta(T)$ having one of properties $1^\circ \sim 3^\circ$ exists.

Q. E. D.

Now we are in a position to show existence of a group Q which plays fundamental role to our ideas.

Lemma 2. For any family $\{T_\lambda | \lambda \in \Lambda\}$ of non-singular homomorphisms on $C[p]$ there exists a subgroup Q between $\Sigma[p]$ and $C[p]$ such that Q is not invariant under any $T_\lambda (\lambda \in \Lambda)$ but invariant under the canonical projection P_e onto even coordinates.

Proof. $\{T_\lambda | \lambda \in \Lambda\}$ is a given family of non-singular homomorphisms on $C[p]$. We can assume that Λ is a well ordered set of ordinal numbers which are less than the first ordinal number whose cardinality is $\aleph$. Choose $c \in C[p]$ but $c^\circ, c^\circ \notin \Sigma[p]$. By transfinite induction we can construct the following family of subgroups $R_\lambda (\lambda \in \Lambda)$;

- a) $\Sigma[p] = R_0 \subset R_\lambda \subset R_\mu \subset C[p]$ if $0 \leq \lambda < \mu$ ($\lambda, \mu \in \Lambda$),
- b) $\text{card } R_\lambda \leq (\text{card } \lambda) \cdot \aleph_0 < \aleph$ for all $\lambda \in \Lambda$,
- c) R_λ is invariant under P_e for all $\lambda \in \Lambda$,
- d) c° and $c^\circ \notin R_\lambda$ but there exists $x_\lambda \in R_\lambda \cap \Delta(T_\lambda)$ such that $c^\circ - T_\lambda x_\lambda$ or $c^\circ - T_\lambda x_\lambda$ or $c - T_\lambda x_\lambda \in R_\lambda$ for all $\lambda \in \Lambda$.

Suppose R_λ has been constructed for all $\lambda < \mu \in \Lambda$. Let $R'_\mu = \bigcup_{\lambda < \mu} R_\lambda$. Then $\text{card } R'_\mu \leq (\text{card } \mu) \cdot \aleph_0 < c$ and R'_μ is invariant under P_e and c° and $c^\circ \notin R'_\mu$. By Lemma 1 $\Delta(T_\mu)$ having one of properties $1^\circ \sim 3^\circ$ and $1^\circ \sim 3^\circ$ exists. Suppose $\Delta(T_\mu)$

has property 1°, then we can find $x_\mu \in \mathcal{A}(T_\mu)$ such that $(R'_\mu + [c^\circ] + [c^\circ]) \cap ([x_\mu] \oplus [T_\mu x_\mu]) = \{0\}$. Let $R_\mu = R'_\mu + [x_\mu] + [c^\circ - T_\mu x_\mu]$, then clearly R_μ satisfies above a), b) and c). And c° and $c^\circ \notin R_\mu$ also holds. Suppose $c^\circ \in R_\mu$, then $c^\circ = x + nx_\mu + m(c^\circ - T_\mu x_\mu)$ for some $x \in R'_\mu$ and some integers n and m , so $-x + (1-m)c^\circ = nx_\mu - mT_\mu x_\mu$, but by our choice of x_μ , $nx_\mu - mT_\mu x_\mu = 0$ and $x + (m-1)c^\circ = 0$. This implies $n = m = 0 \pmod p$ and $c^\circ = x \in R'_\mu$ which is a contradiction. Suppose $c^\circ \in R_\mu$, then $c^\circ = x + nx_\mu + m(c^\circ - T_\mu x_\mu)$ for some $x \in R'_\mu$ and some integers n and m , but x_μ and $T_\mu x_\mu \in C[p]^\circ$, so $c^\circ = x \in R'_\mu$ which is also a contradiction. Suppose $\mathcal{A}(T_\mu)$ has property 2°, then we can find $x_\mu \in \mathcal{A}(T_\mu)$ such that $(R'_\mu + [c^\circ] + [c^\circ]) \cap ([x_\mu] \oplus [T_\mu x_\mu - qx_\mu]) = \{0\}$. Let $R_\mu = R'_\mu + [x_\mu] + [c^\circ - T_\mu x_\mu + qx_\mu]$, then clearly R_μ satisfies above a), b) and c). And c° and $c^\circ \notin R_\mu$ also holds. Suppose $c^\circ \in R_\mu$, then $c^\circ = x + nx_\mu + m(c^\circ - T_\mu x_\mu + qx_\mu)$ for some $x \in R'_\mu$ and some integers n and m , but $x_\mu \in C[p]^\circ$ and $T_\mu x_\mu - qx_\mu \in C[p]^\circ$, hence we have $c^\circ = x^\circ + nx_\mu$, that is, $-x^\circ + c^\circ = nx_\mu$. Our choice of x_μ implies $nx_\mu = 0 = -x^\circ + c^\circ$, so we have $c^\circ = x^\circ \in R'_\mu$ which is a contradiction. Suppose $c^\circ \in R_\mu$, then $c^\circ = x + nx_\mu + m(c^\circ - T_\mu x_\mu + qx_\mu)$ for some $x \in R'_\mu$ and some integers n and m . Hence $-x + (1-m)c^\circ = nx_\mu - m(T_\mu x_\mu - qx_\mu)$, but by our choice of x_μ we see $-x + (1-m)c^\circ = 0 = nx_\mu - m(T_\mu x_\mu - qx_\mu)$. This implies $n = m = 0 \pmod p$, so $c^\circ = x \in R'_\mu$ which is also a contradiction. Suppose $\mathcal{A}(T_\mu)$ has property 3°, then we can find $x_\mu \in \mathcal{A}(T_\mu)$ such that $(R'_\mu + [c^\circ] + [c^\circ]) \cap ([x_\mu] \oplus [(T_\mu x_\mu)^\circ] \oplus [(T_\mu x_\mu)^\circ]) = \{0\}$. Let $R_\mu = R'_\mu + [x_\mu] + [c^\circ - (T_\mu x_\mu)^\circ] + [c^\circ - (T_\mu x_\mu)^\circ]$. Then R_μ clearly satisfies a), b) and c). And c° and $c^\circ \notin R_\mu$ can be seen as follows. Suppose $c^\circ = x + nx_\mu + m(c^\circ - (T_\mu x_\mu)^\circ) + m'(c^\circ - (T_\mu x_\mu)^\circ)$ for some $x \in R'_\mu$ and integers n , m and m' , then $c^\circ = x^\circ + nx_\mu + m(c^\circ - (T_\mu x_\mu)^\circ)$, so $-x^\circ + (1-m)c^\circ = nx_\mu - m(T_\mu x_\mu)^\circ$. This implies $nx_\mu - m(T_\mu x_\mu)^\circ = 0 = -x^\circ + (1-m)c^\circ$ by our choice of x_μ . Hence $m = 0$ and $c^\circ = x^\circ \in R'_\mu$ which is a contradiction. We can see also $c^\circ \notin R_\mu$ for same reason. And x_μ and $c - T_\mu x_\mu \in R_\mu$ is clear. The construction of R_μ for $\mathcal{A}(T_\mu)$ having one of properties 1°~3° is exactly similar by exchanging odd for even.

Let $Q = \bigcup_{\lambda \in \Lambda} R_\lambda$. Then the above properties a)~d) for all R_λ guarantee that Q is a subgroup between $\Sigma[p]$ and $C[p]$ not invariant under any $T_\lambda (\lambda \in \Lambda)$ but invariant under P_e .

Q. E. D.

Proof of Theorem. We have clearly $C = C^\circ \oplus C^\circ$ and $\Sigma = \Sigma^\circ \oplus \Sigma^\circ$. By Lemma 2 there exists a subgroup Q between $\Sigma[p]$ and $C[p]$ such that Q is not invariant under any non-singular homomorphisms on $C[p]$ but is invariant under P_e , therefore $\Sigma^\circ[p] = \Sigma[p]^\circ \subset Q^\circ \subset C[p]^\circ = C^\circ[p]$, $\Sigma^\circ[p] = \Sigma[p]^\circ \subset Q^\circ \subset C[p]^\circ = C^\circ[p]$ and $Q = Q^\circ \oplus Q^\circ$. We can show that there exists a pure snbgroup G_1 (G_2) of C° (C°) which contains Σ° (Σ°) and $G_1[p] = Q^\circ$

$(G_2[p] = Q^\circ)$. This fact is known as the purification property, see Lemma 1 in [2] and more general form in [1]. Clearly G_1 and G_2 are not bounded. Let $G = G_1 \oplus G_2$, then G is a pure subgroup of C which contains Σ and $G[p] = G_1[p] \oplus G_2[p] = Q^\circ \oplus Q^\circ = Q$. By Proposition G has the properties 1) and 2) in Theorem.

Q. E. D.

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