Construction of canonical systems of basic invariants for finite reflection groups

（有限鏡映群の標準不変式系の構成）

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Preface

For a polytope P whose symmetry group is a reflection group W, a canonical system of basic invariants for W plays an important role to determine if the space of all P-harmonic functions coincides with the space of all W-harmonic functions. The main purpose of this thesis is to describe explicit formulas for a canonical system of basic invariants for a finite irreducible reflection group.

This thesis is mainly based on [17] and [18]. The paper [18] is a joint-work with Norihiro Nakashima and describes formulas for canonical systems of finite Euclidean reflection groups. The paper [17] is a joint-work with Norihiro Nakashima and Hiroaki Terao and proves the existence of canonical systems for unitary reflection groups. These materials are included in the thesis with a kind permission of those coworkers.

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Contents

Chapter 1

Introduction

In this chapter we give the definition of a canonical system for a finite reflection group. Throughout this paper, $\mathbb{K}$ denotes the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$ and V an n-dimensional vector space over $\mathbb{K}$. Lehrer and Taylor [15] describe the theory of unitary reflection groups in detail.

1.1 Finite reflection groups

In this section, we recall the definition of a reflection and the classification of finite reflection groups.

Definition 1.1. A linear transformation $s \in \mathrm{GL}(V)$ is a *reflection* (or a *pseudo reflection*) if $\dim_{\mathbb{K}} \mathrm{Im}(1-s) = 1$ and the order of s is finite. For a reflection s, the hyperplane $\mathrm{Ker}(1-s)$ is called the *reflecting hyperplane* of s. A finite subgroup $W \subseteq \mathrm{GL}(V)$ is a *finite reflection group* if W is generated by reflections. When $\mathbb{K} = \mathbb{R}$ a reflection group is sometimes called a *Euclidean reflection group* and when $\mathbb{K} = \mathbb{C}$ a reflection group is called a *unitary reflection group* or a *complex reflection group*.

Let $W \subseteq \mathrm{GL}(V)$ be a finite reflection group. There exists a W-invariant inner product $\langle \cdot, \cdot \rangle$, that is, there exists a bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ satisfying the following:

(1) $\overline{\langle u, v \rangle} = \langle v, u \rangle$,

(2) $\langle u_1 + u_2, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle$,

(3) $\langle au, v \rangle = a \langle u, v \rangle$,

(4) $\langle v, v\rangle \in \mathbb{R}$ and $\langle v, v\rangle \geq 0$,

(5) $\langle v, v\rangle = 0$ if and only if $v = 0$,

(6) $\langle wu, wv\rangle = \langle u, v\rangle$

for all $u, u_1, u_2, v \in V, a \in \mathbb{K}$ and $w \in W$, where $\overline{a} = a$ when $\mathbb{K} = \mathbb{R}$ or $\overline{a}$ denotes the complex conjugate of a when $\mathbb{K} = \mathbb{C}$.

Let $s \in W$ be a reflection and H the reflecting hyperplane of s. The orthogonal complement $H^{\perp}$ has dimension 1.

Definition 1.2. A generator α of $H^{\perp}$ is called a *root* of the reflection s.

Let α be a root of s. For any $u \in H$,

$$0 = \langle u, \alpha\rangle = \langle su, s\alpha\rangle = \langle u, s\alpha\rangle.$$

Therefore $s\alpha \in H^{\perp} = \mathbb{K}\alpha$. Hence α is an eigenvector of s. Assume that $s\alpha = \zeta\alpha$ for some $\zeta \in \mathbb{K}, \zeta \neq 1$. An arbitrary vector $v \in V$ has a unique decomposition

$$v = u + a\alpha,$$

where $u \in H$ and $a \in \mathbb{K}$. Since $\langle u, \alpha\rangle = 0$, we have $a = \frac{\langle v,\alpha\rangle}{\langle \alpha,\alpha\rangle}$. Therefore

$$sv = su + as\alpha = u + \zeta a\alpha = v - (1-\zeta)\frac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha. \tag{1.1}$$

Proposition 1.3. *Let s_1, s_2 be reflections with roots α_1, α_2 respectively. Then s_1 and s_2 commute if and only if $\mathbb{K}\alpha_1 = \mathbb{K}\alpha_2$ or $\langle \alpha_1, \alpha_2\rangle = 0$.*

Proof. From (1.1), we have

$$\begin{aligned} s_2 s_1 v = v &- (1-\zeta_1)\frac{\langle v, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle}\alpha_1 - (1-\zeta_2)\frac{\langle v, \alpha_2\rangle}{\langle \alpha_2, \alpha_2\rangle}\alpha_2 \\ &+ (1-\zeta_1)(1-\zeta_2)\frac{\langle v, \alpha_1\rangle\langle \alpha_1, \alpha_2\rangle}{\langle \alpha_1, \alpha_1\rangle\langle \alpha_2, \alpha_2\rangle}\alpha_2, \end{aligned}$$

where $s_i\alpha_i = \zeta_i\alpha_i \, (i = 1, 2)$. By symmetry a similar expression holds for $s_1 s_2 v$. Therefore

$$\begin{aligned} & s_1 \text{ and } s_2 \text{ commute} \\ \Leftrightarrow & \langle v, \alpha_1\rangle\langle \alpha_1, \alpha_2\rangle\alpha_2 = \langle v, \alpha_2\rangle\langle \alpha_2, \alpha_1\rangle\alpha_1 \text{ for any } v \in V \\ \Leftrightarrow & \mathbb{K}\alpha_1 = \mathbb{K}\alpha_2 \text{ or } \langle \alpha_1, \alpha_2\rangle = 0. \end{aligned}$$

$\square$

Proposition 1.4. *Let s be a reflection with root α and $U \subseteq V$ a subspace. Then U is s-stable, that is, $sU = U$ if and only if $\alpha \in U$ or $\alpha \in U^{\perp}$.*

Proof. Assume that $sU = U$. Then $sU^{\perp} = U^{\perp}$. Write

$$\alpha = u + v \quad (u \in U, v \in U^{\perp}),$$

then

$$su + sv = s\alpha = \zeta\alpha = \zeta u + \zeta v.$$

Since $sU = U$ and $sU^{\perp} = U^{\perp}$, we have $su = \zeta u$ and $sv = \zeta v$. By definition $\dim_{\mathbb{K}} \operatorname{Im}(1-s) = 1$, one of u and v must be 0 since $\zeta \neq 1$. Hence $\alpha \in U$ or $\alpha \in U^{\perp}$.

Conversely, assume that $\alpha \in U^{\perp}$. Then the reflecting hyperplane of s contains U. Therefore U is s-stable. If $\alpha \in U$ then $U^{\perp}$ is s-stable so is U. □

Proposition 1.5. *Let $U \subseteq V$ be a W-stable subspace and $W' \subseteq W$ the subgroup generated by reflections with roots in U. Then W' is a normal subgroup of W.*

Proof. Let $s \in W'$ with root $\alpha \in U$. For any $w \in W$ we have

$$s(w^{-1}v) = w^{-1}v - (1-\zeta)\frac{\langle w^{-1}v, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle}\alpha_1.$$

Therefore

$$(wsw^{-1})v = v - (1-\zeta)\frac{\langle v, w\alpha_1\rangle}{\langle w\alpha_1, w\alpha_1\rangle}w\alpha_1.$$

Thus wsw^{-1} is a reflection with root $w\alpha$. Since U is W-stable, we have $w\alpha \in U$. By definition of W' we can conclude that $wsw^{-1} \in W'$. Hence W' is a normal subgroup of W. □

Definition 1.6. Let $W \subseteq \mathrm{GL}(V)$ be a finite reflection group. We say that W is *essential* if $V^W := \{v \in V \mid wv = v \text{ for all } w \in W\} = 0$ and W is *reducible* if $W = W_1 \times W_2$ for some non-trivial reflection subgroups $W_1, W_2 \subseteq W$. When W is not reducible, we say that W is *irreducible*.

Proposition 1.7. *The $\mathbb{K}[W]$-module V is irreducible if and only if W is essential and irreducible.*

Proof. By Maschke's theorem V has the orthogonal decomposition

$$V = V^W \perp V_1 \perp \cdots \perp V_k,$$

where $V_i\,(1 \leq i \leq k)$ are non-trivial irreducible submodules. Assume that the $\mathbb{K}[W]$-module V is reducible. We may assume that W is essential and $k \geq 2$. Let $W_i \subseteq W$ be the subgroup generated by reflections with roots in V_i. Then the following three conditions hold:

- When $i \neq j$, elements $w \in W_i$ and $w' \in W_j$ commute by Proposition 1.3.

- $W = W_1 \cdots W_k$ by Proposition 1.4.

- W_i is a normal subgroup of W by Proposition 1.5.

Hence $W = W_1 \times \cdots \times W_k$. Thus W is a reducible reflection group. Conversely, we may assume that W is essential and reducible. Then W has a decomposition $W = W_1 \times W_2$, where W_1 and W_2 are non-trivial reflection subgroups. Let $V_1 \subseteq V$ be the subgroup spanned by roots of reflections in W_1. The subspace V_1 is W_1-stable by Proposition 1.4. Since an element in W_1 and one in W_2 commute, the subspaces V_1 and V_2 are orthogonal by Proposition 1.3. Hence V_1 is contained in the reflecting hyperplanes of reflections in W_2. Therefore V_1 is also W_2-stable. Hence V_1 is a non-trivial W-stable subspace. Thus the $\mathbb{K}[W]$-module V is reducible. □

Irreducible finite Euclidean reflection groups were classified by Coxeter [3] [4] in terms of Coxeter-Dynkin diagrams. They consist of four infinite families $A_n, B_n, D_n.I_2(m)$, and six exceptional groups $E_6, E_7, E_8, F_4, H_3, H_4$.

Shepherd and Todd [22] classified irreducible finite unitary reflection groups into 37 cases, three infinite families (symmetric, imprimitive, cyclic) and 34 exceptional groups (denoted by $G_4, \ldots, G_{37}$). The three infinite families are described by an infinite family $G(m, p, n)$ depending on three positive integer parameters with p dividing m.

Proposition 1.8 (Bourbaki [1]). *Let V be an Euclidean space and $W \subseteq \mathrm{O}(V)$ a finite Euclidean reflection group. If V is an irreducible $\mathbb{R}[W]$-module then $\mathrm{End}_W(V) = \mathbb{R}$. Hence V is absolutely irreducible, that is, $V \otimes_{\mathbb{R}} \mathbb{C}$ is also irreducible as a $\mathbb{C}[W]$-module.*

Proposition 1.8 provides that each irreducible finite Euclidean reflection group can be regarded as an irreducible finite unitary group. The correspondence is listed below:

Infinite family		Exceptional group	
A_n	$G(1,1,n+1)$	E_6	G_{35}
B_n	$G(2,1,n)$	E_7	G_{36}
D_n	$G(2,2,n)$	E_8	G_{37}
$I_2(m)$	$G(m,m,2)$	F_4	G_{28}
		H_3	G_{23}
		H_4	G_{30}

1.2 Chevalley's Theorem

Let S be the symmetric algebra $S(V^*)$ of the dual space V^*. For any basis $x = (x_1, \ldots, x_n)$ for V^*, we may regard S as the ring of polynomial functions $\mathbb{K}[x]$ of V^*. A finite reflection group W acts on S in a usual manner:

$$wf := f \circ w^{-1}$$

for all $w \in W$ and $f \in S$. Let S^W be the ring of W invariants and S^W_+ its irrelevant ideal, that is, the ideal generated by homogeneous invariants of positive degrees. The following classical theorem is due to Chevalley [2].

Theorem 1.9 (Chevalley [2])**.** *The following assertions hold:*

(1) Let $\{f_1, \ldots, f_m\}$ be a minimal system of homogeneous generators of S^W_+. Then $m = n$ and $f_1, \ldots, f_n$ are algebraically independent.

(2) There exist algebraically independent homogeneous invariants $f_1, \ldots, f_n$ of positive degrees such that

$$S^W = \mathbb{K}[f_1, \ldots, f_n].$$

(3) The multiset of the degrees $\{\deg f_1, \ldots, \deg f_n\}$ is independent of a choice of generators for S^W_+.

Definition 1.10. The set of generators $\{f_1, \dots, f_n\}$ in Theorem 1.9 is called *a system of basic invariants* (or a *basic invariants* shortly) for W. The multiset $\{\deg f_1, \dots, \deg f_n\}$ is called the *degrees* of W.

Assume that V is reducible as a $\mathbb{K}[W]$-module. Write $V = V_1 \oplus V_2$, where V_i is a non-trivial $\mathbb{K}[W]$-module. From the proof of Proposition 1.7, we have $W = W_1 \times W_2$, where each W_i is a reflection group on V_i (allowed to be the unit group when $V_i \subseteq V^W$). Then

$$S^W = S(V_1^*)^{W_1} \otimes_{\mathbb{K}} S(V_2^*)^{W_2}.$$

Hence without loss of generality we may assume that V is an irreducible $\mathbb{K}[W]$-module to determine a system of basic invariants for W. Many researchers determined systems of basic invariants explicitly (Coxeter [2], Lee [14], Lehner and Taylor [15], Mehta [16], Saito, Yano and Sekiguchi [19], Sekiguchi and Yano [20, 21]). Determining systems of basic invariants is not difficult for infinite families $A_n, B_n, D_n, I_2(m)$ and $G(m,p,n)$ but determining ones for exceptional groups needs further considerations.

1.3 Definition of canonical systems

For $v \in V$ we define a $\mathbb{K}$-linear map $D_v \colon S \to S$ to be the directional derivative in direction v,

$$D_v f \coloneqq \lim_{t \to 0} \frac{1}{t}(f \circ T_{tv} - f),$$

where $T_v(u) \coloneqq u + v$. The operator D_v is a derivation of S, that is, D_v satisfies

$$D_v(fg) = (D_v f)g + f(D_v g) \tag{1.2}$$

for all $f, g \in S$. Note that

$$D_v L = L(v) \tag{1.3}$$

for all $L \in V^* \subseteq S$.

Definition 1.11. We denote by $\mathcal{D}_S$ the commutative algebra which is generated by directional derivatives $D_v\,(v \in V)$ regarding the product as map composition. An element of $\mathcal{D}_S$ is called a *differential operator*.

The linear map $D\colon V \to \mathcal{D}_S$ can be extended to a $\mathbb{K}$-algebra isomorphism defined by the linear extension of

$$\begin{aligned} D\colon S(V) &\to \mathcal{D}_S \\ v_1 \otimes \cdots \otimes v_k &\mapsto D_{v_1} \cdots D_{v_k}. \end{aligned} \tag{1.4}$$

A finite reflection group W acts on $\mathcal{D}_S$ as

$$wD_p \coloneqq D_{wp},$$

where $w \in W$ and $p \in S(V)$ so that (1.4) becomes a $\mathbb{K}[W]$-algebra isomorphism.

Proposition 1.12. *Let $p \in S(V), f \in S = S(V^*), w \in W$. Then*

$$w(D_p f) = D_{wp}(wf).$$

Proof. Without loss of generality we may assume that p is homogeneous. By induction on the degree of p it suffices to show that $w(D_v f) = D_{wv}(wf)$ for all $p = v \in V$. We have

$$\begin{aligned} w(D_v f) &= \lim_{t\to 0} \frac{1}{t} \left(f \circ T_{tv} \circ w^{-1} - f \circ w^{-1} \right) \\ &= \lim_{t\to 0} \frac{1}{t} \left(f \circ w^{-1} \circ T_{twv} - f \circ w^{-1} \right) \\ &= D_{wv}(wf). \end{aligned}$$

Hence $w(D_v f) = D_{wf}(wf)$. □

Define a map $\iota : V \to V^*$ by

$$\iota(v) := \langle \cdot, v \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the W-invariant inner product on V.

Proposition 1.13. *The map ι is a W-invariant (anti)isomorphism, that is, ι satisfying the following:*

(1) $\iota(v_1 + v_2) = \iota(v_1) + \iota(v_2)$,

(2) $\iota(av) = \overline{a}\iota(v)$,

(3) *ι is bijective,*

(4) $\iota(wv) = w\iota(v)$

for all $v, v_1, v_2 \in V, a \in \mathbb{K}$ *and* $w \in W$.

Proof. (1) For any $u, v_1, v_2 \in V$,

$$\begin{aligned}\iota(v_1+v_2)(u) &= \langle u, v_1+v_2\rangle = \overline{\langle v_1+v_2, u\rangle}_V = \overline{\langle v_1, u\rangle}_V + \overline{\langle v_2, u\rangle}_V \\ &= \langle v_1, u\rangle + \langle v_2, u\rangle = \iota(v_1)(u) + \iota(v_2)(u).\end{aligned}$$

Therefore $\iota(v_1+v_2) = \iota(v_1) + \iota(v_2)$.

(2) For any $u, v \in V$ and $a \in \mathbb{K}$,

$$\begin{aligned}\iota(av)(u) &= \langle u, av\rangle = \overline{\langle av, u\rangle}_V = \overline{a\langle v, u\rangle}_V \\ &= \overline{a}\langle u, v\rangle = \overline{a}\iota(v)(u).\end{aligned}$$

Hence $\iota(av) = \overline{a}\iota(v)$.

(3) By (1) and (2) the subset $\operatorname{Im}\iota \subseteq V^*$ is a subspace of V^*. Hence it suffices to verify that ι is injective. Assume that $\iota(v) = 0$. Then $\langle v, v\rangle = \iota(v)(v) = 0$, so that $v = 0$. Thus ι is injective.

(4) For any $u, v \in V$ and $w \in W$ we have

$$\iota(wv)(u) = \langle u, wv\rangle = \langle w^{-1}u, v\rangle = \iota(v)(w^{-1}u) = w\iota(v).$$

Hence $\iota(wv) = w\iota(v)$. □

The map ι can be extended to a W-invariant (anti)isomorphism defined by the linear extension of

$$\begin{array}{rcl} S(V^*) & \to & S(V) \\ L_1 \otimes \cdots \otimes L_k & \mapsto & \iota^{-1}(L_1) \otimes \cdots \otimes \iota^{-1}(L_k) \end{array}$$

Definition 1.14. We define a W-invariant (anti)isomorphism $*\colon S \to \mathcal{D}_S$ by

$$f^* := D_{\iota^{-1}(f)},$$

where $f \in S = S(V^*)$.

Let $e_1, \ldots, e_n$ be an orthonormal basis for V and $x_1, \ldots, x_n$ its dual basis. Then we have

$$x_i^* = D_{e_i} = \frac{\partial}{\partial x_i}.$$

For convenience we put $\partial_i \coloneqq \frac{\partial}{\partial x_i}$. For a multi-index $\boldsymbol{i} = (i_1, \ldots, i_n) \in \mathbb{N}_0 \coloneqq \mathbb{Z}^n_{\geq 0}$ we write

$$x^{\boldsymbol{i}} \coloneqq x_1^{i_1} \cdots x_n^{i_n}.$$

For a polynomial $f = \sum_{\boldsymbol{i}} a_{\boldsymbol{i}} x^{\boldsymbol{i}} \in S$ we have

$$f^* = \sum_{\boldsymbol{i}} \overline{a}_{\boldsymbol{i}} \partial^{\boldsymbol{i}},$$

where $\partial \coloneqq (\partial_1, \ldots, \partial_n)$.

Proposition 1.15. *For $f \in S$*

$$f^* x_i - x_i f^* = (\partial_i f)^*.$$

as operators on S.

Proof. We may assume that f is a monomial. Since the operators x_i and ∂_j commute when $i \neq j$ it suffices to show that

$$\partial_i^m x_i - x_i \partial_i^m = m \partial_i^{m-1}$$

by induction on m.

Let $m = 1$ then for any $f \in S$

$$\partial_i x_i f = f + x_i \partial_i f.$$

Hence $\partial_i x_i - x_i \partial_i = 1$ as operators on S.

For $m \geq 2$

$$\begin{aligned}
\partial_i^m x_i &= \partial_i \left(\partial_i^{m-1} x_i \right) \\
&= \partial_i \left(x_i \partial_i^{m-1} + (m-1) \partial_i^{m-2} \right) \\
&= (x_i \partial_i + 1) \partial_i^{m-1} + (m-1) \partial_i^{m-1} \\
&= x_i \partial_i^m + m \partial_i^{m-1}
\end{aligned}$$

as operators on S. Therefore $\partial_i^m x_i - x_i \partial_i^m = m \partial_i^{m-1}$.

$\square$

Definition 1.16. Define a map $(\cdot,\cdot)\colon S \times S \to S$ by

$$(f,g) \coloneqq f^* g \quad (f, g \in S)$$

and define a map $(\cdot,\cdot)_0\colon S \times S \to \mathbb{K}$ by

$$(f,g)_0 \coloneqq (f,g)(0) = (f^* g)(0) \quad (f, g \in S).$$

Proposition 1.17. *The map $(\cdot,\cdot)$ is a W-invariant sesquilinear map, that is, the following hold:*

(1) $(f_1 + f_2, g) = (f_1, g) + (f_2, g)$ *and* $(f, g_1 + g_2) = (f, g_1) + (f, g_2)$*,*

(2) $(af, g) = \overline{a}(f,g)$ *and* $(f, ag) = a(f,g)$*,*

(3) $w(f,g) = (wf, wg)$

for all $f, f_1, f_2, g, g_1, g_2 \in S, a \in \mathbb{K}, w \in W$*.*

The operator f^* is the adjoint operator of the multiplication-by-f map with respect to $(\cdot,\cdot)$:

(4) $(fg, h) = (g, f^* h)$

for all $f, g, h \in S$*.*

Moreover, if $f, g \in S$ are homogeneous and have the same degree then

(5) $(f,g) = (f,g)_0$*.*

Proof. (1) and (2) follow immediately from the linearity of differential operators and the (anti)linearity of the map $*$. To verify (3) we take $f, g \in S$ and $w \in W$. By Proposition 1.12 we have

$$w(f,g) = w(D_{\iota^{-1}(f)} g) = D_{w\iota^{-1}(f)}(wg) = D_{\iota^{-1}(wf)}(wg) = (wf, wg).$$

(4) We have $(fg, h) = (fg)^* h = g^*(f^* h) = (g, fh)$.

(5) If $f, g \in S$ are homogeneous and have the same degree then (f,g) is a constant. Hence $(f,g) = (f,g)_0$. $\square$

Proposition 1.18. *The map $(\cdot,\cdot)_0$ is a W-invariant inner product on S, that is, $(\cdot,\cdot)_0$ satisfies the following:*

(1) $\overline{(f,g)}_0 = (g,f)_0$*,*

(2) $(f_1 + f_2, g)_0 = (f_1, g)_0 + (f_2, g)_0$,

(3) $(af, g)_0 = \overline{a}(f, g)_0$,

(4) $(f, f)_0 \in \mathbb{R}$ *and* $(f, f)_0 \geq 0$,

(5) $(f, f)_0 = 0$ *if and only if* $f = 0$,

(6) $(wf, wg)_0 = (f, g)_0$

for all $f, f_1, f_2, g \in S, a \in \mathbb{K}, w \in W$.

The operator f^* *is the adjoint operator of the multiplication-by-*f *map with respect to* $(\cdot, \cdot)_0$*:*

(7) $(fg, h)_0 = (g, f^*h)_0$ *and* $(g, fh)_0 = (f^*g, h)_0$.

for all $f, g, h \in S$.

Proof. (1) We may assume that $f, g \in S$ are homogeneous. If $\deg f \neq \deg g$ then $\overline{(f, g)_0} = 0 = (g, f)_0$. Assume that $k \coloneqq \deg f = \deg g$ and write

$$f = \sum_{|\boldsymbol{i}|=k} a_{\boldsymbol{i}} x^{\boldsymbol{i}}, g = \sum_{|\boldsymbol{i}|=k} b_{\boldsymbol{i}} x^{\boldsymbol{i}},$$

where $|\boldsymbol{i}| \coloneqq i_1 + \cdots + i_n$. Then we have

$$(f, g)_0 = \left(\sum_{|\boldsymbol{i}|=k} \overline{a}_{\boldsymbol{i}} \partial^{\boldsymbol{i}} \right) \left(\sum_{|\boldsymbol{i}|=k} b_{\boldsymbol{i}} x^{\boldsymbol{i}} \right) = \sum_{|\boldsymbol{i}|=k} \boldsymbol{i}! \overline{a}_{\boldsymbol{i}} b_{\boldsymbol{i}}, \tag{1.5}$$

where $\boldsymbol{i}! \coloneqq i_1 \cdots i_n$. By symmetry a similar expression holds for $(g, f)_0$. Hence $\overline{(f, g)_0} = (g, f)_0$.

(2)(3) These conditions follow immediately from Proposition 1.17.

(4)(5) By expression (1.5) we have

$$(f, f)_0 = \sum_{|\boldsymbol{i}|=\deg f} \boldsymbol{i}! |a_{\boldsymbol{i}}|^2.$$

Hence (4) and (5) hold.

(6) We may assume that $f, g \in S$ are homogeneous. If $\deg f \neq \deg g$ then $(wf, wg)_0 = 0 = (f, g)_0$. Hence we may assume that $\deg f = \deg g$. Then

$(f,g)_0 = (f,g)$ is a constant. Therefore $(wf,wg)_0 = (wf,wg) = w(f,g) = (f,g)_0$.

(7) The first equality $(fg,h)_0 = (g,f^*h)_0$ follows immediately from Proposition 1.17(4). We shall prove the second equality. We may assume that f,g,h are all homogeneous and $\deg g = \deg f + \deg h$ since $(g,fh)_0 = 0 = (f^*g,h)$ if $\deg g \neq \deg f + \deg h$. By Proposition 1.17(4)(5) we have

$$(g,fh)_0 = \overline{(fh,g)_0} = \overline{(h,f^*g)_0} = (f^*g,h)_0.$$

□

Definition 1.19. A system of basic invariants $\{f_1,\dots,f_n\}$ is said to be a *canonical system of basic invariants* (or *canonical invariants* shortly) if

$$(f_i,f_j) = 0 \quad (i \neq j). \tag{1.6}$$

A canonical system of basic invariants $\{f_1,\dots,f_n\}$ is normalized if

$$(f_i,f_i) = 1$$

for all $i \in \{1,\dots,n\}$.

This notion was introduced by Flatto [5, 6] and by Flatto and Wiener[7] for the first time, but in a quite different manner. Later Iwasaki [8] formulated it as in Definition 1.6, introducing the terminology ”canonical system”. When $\mathbb{K} = \mathbb{R}$, according to [5, 6, 7] it was also proven that the existence and the uniqueness (up to equivalence) of canonical systems, where two systems $\{f_1,\dots,f_n\}$ and $\{g_1,\dots,g_n\}$ are equivalent if there exists $A \in \mathrm{GL}_n(\mathbb{K})$ such that

$$[f_1\ \dots\ f_n] = [g_1\ \dots\ g_n]A.$$

Canonical systems are determined explicitly by Iwasaki [8] for types A_n, B_n, D_n and $I_2(m)$ and by Iwasaki, Kenma and Matsumoto [13] for types H_3, H_4 and F_4.

1.4 Steinberg's Theorem

To state Steinberg's theorem we introduce the notion of a W-harmonic polynomial.

Definition 1.20. We call $D_p \in \mathcal{D}_S$ a *W-invariant differential operator* if $wD_p = D_p$ for all $w \in W$.

Definition 1.21. A polynomial $f \in S$ is *W-harmonic* if $D_p f = 0$ for all W-invariant differential operator D_p without a constant term. The space of W-harmonic polynomials is denoted by $\mathcal{H}_W$.

Let $I \coloneqq SS_+^W$, where S_+^W is the irrelevant ideal of S^W. Making use of the sesquilinear map $(\cdot, \cdot)$ we may express

$$\mathcal{H}_W = \{f \in S \mid (g, f) = 0 \text{ for all } g \in I\}. \tag{1.7}$$

Let $\mathcal{A}_W$ be the set of the reflecting hyperplanes of all reflections in W. For any $H \in \mathcal{A}_W$ we define a subgroup $W_H \subseteq W$ by the pointwise stabilizer of H, that is,

$$W_H \coloneqq \{w \in W \mid wv = v \text{ for all } v \in H\}.$$

For a root α of H (a generator of 1-dimensional subspace $H^\perp$) we have

$$w\alpha = \zeta_w \alpha \qquad (w \in W_H)$$

for some $\zeta_w \in \mathbb{K}^\times$ and this is independent of a choice of a root α. The finiteness of W implies that ζ_w is a root of unity. Hence we have a group inclusion

$$\begin{array}{rcl} W_H & \to & \boldsymbol{\mu}(\mathbb{K}), \\ w & \mapsto & \zeta_w \end{array}$$

where $\boldsymbol{\mu}(\mathbb{K})$ denotes the group of roots of unity in $\mathbb{K}$. Therefore W_H is a cyclic group. Define e_H to be the order of W_H.

We define a polynomial $\Delta \in S$ by

$$\Delta \coloneqq \prod_{H \in \mathcal{A}_W} L_H^{e_H - 1},$$

where $L_H \in V^*$ satisfying $\operatorname{Ker} L_H = H$.

Definition 1.22. A polynomial $f \in S$ is skew if $wf = (\det w)f$ for all $w \in W$.

Proposition 1.23 (Steinberg [23])**.** *The polynomial Δ is skew, and every skew polynomial is divisible by Δ.*

The skew polynomial Δ is quite important to study W-harmonic polynomials. Steinberg also proved the significant theorem below.

Theorem 1.24 (Steinberg [23]). *The following hold:*

(1) $f \in I$ *if and only if* $(f, \Delta) = 0$,

(2) $\mathcal{H}_W = \mathcal{D}_S \Delta$.

By Theorem 1.24 we have an orthogonal decomposition

$$S = I \oplus \mathcal{H}_W \tag{1.8}$$

Theorem 1.25 (Chevalley [2]). *The space* S/I *affords the regular representation of* W.

From (1.8) and Theorem 1.25 we can deduce

Proposition 1.26. *The space* $\mathcal{H}_W$ *of* W*-harmonic polynomials affords the regular representation of* W.

Chapter 2

Background

In this chapter we introduce the mean value property for polytopes and its relation with the canonical system of basic invariants for a finite reflection group.

2.1 The mean value property for a polytope

Let V be an n-dimensional Euclidean space. An n-dimensional *convex polytope* is the convex hull of a finite set of points in V whose affine hull is the entire space V. An n-dimensional *polytope* P is a finite union of n-dimensional convex polytopes. For each $k \in \{0, 1, \ldots, n\}$, the *$k$-skeleton* of P is the union of all k-dimensional faces of P, which is denoted by $P(k)$.

For a polytope P the *symmetry group* $\Gamma(P)$ of P is defined by

$$\Gamma(P) \coloneqq \{\sigma \in \mathrm{GL}(V) \mid \sigma P = P\}\,.$$

Definition 2.1. Let $P \subseteq V$ be an n-dimensional polytope and $k = 0, \ldots, n$. A continuous function $f\colon V \to \mathbb{R}$ is said to be *$P(k)$-harmonic* if it satisfies the mean value property:

$$f(x) = \frac{1}{|P(k)|} \int_{P(k)} f(x + ry) d\mu_k(y)$$

for any $x \in V$ and $r > 0$, where μ_k is the k-dimensional Euclidean measure and $|P(k)| \coloneqq \mu_k(P(k))$ is the k-dimensional volume of $P(k)$. The space of all $P(k)$-harmonic functions is denoted by $\mathcal{H}_{P(k)}$.

Theorem 2.2 (Iwasaki [9]). *Let $P \subseteq V$ be an n-dimensional polytope and $k = 0, \ldots, n$. Then the following assertions hold:*

(1) $\mathcal{H}_{P(k)}$ is a finite-dimensional linear space of polynomials,

(2) A basis of $\mathcal{H}_{P(k)}$ can be taken from homogeneous polynomials,

(3) $\mathcal{H}_{P(k)}$ admits a structure of $\mathcal{D}_S$-module, where $S = S(V^*)$.

Let $G \subseteq \mathrm{GL}(V)$ be a finite group. A polynomial $f \in S$ is said to be *G-harmonic* if

$$(g, f) = g(\partial)f = 0$$

holds for any $g \in S_+^G$. Let $\mathcal{H}_G$ denote the space of all G-harmonic polynomials. Iwasaki [9] showed that

$$\mathcal{H}_{\Gamma(P)} \subseteq \mathcal{H}_{P(k)}. \tag{2.1}$$

The inclusion (2.1) yields a problem to find sufficient conditions for the equality $\mathcal{H}_{\Gamma(P)} = \mathcal{H}_{P(k)}$.

2.2 Relation with a canonical system

Iwasaki [11, Theorem 1.1] stated a method to decide if the inclusion (2.1) is actually an equality when P admits a finite reflection group W as its symmetry group.

Theorem 2.3 (Iwasaki [11]). *Let $P \subseteq V$ be an n-dimensional polytope and assume that $W := \Gamma(P)$ is a finite reflection group. Let $\{f_1, \ldots, f_n\}$ be a canonical system of basic invariants for W. Assume that the degrees of W are strictly increasing:*

$$d_1 < d_2 < \cdots < d_n. \tag{2.2}$$

Then for every integer $k = 0, \ldots, n$, the equality

$$\mathcal{H}_W = \mathcal{H}_{P(k)} \tag{2.3}$$

is equivalent to the condition

$$\int_{P(k)} f_i(x) d\mu_k(x) \neq 0 \quad (i = 1, \ldots, n). \tag{2.4}$$

The symmetry group of a regular convex polytope is a finite irreducible reflection group satisfying the condition (2.2). The equality (2.3) is proven for regular convex polytopes except for measure polytopes (n-dimensional cubes) by verifying the condition (2.4)[10, 13]. Iwasaki [12] verified (2.3) for measure polytopes by using another method, hence for all regular convex polytopes. The symmetry group of a regular convex polytope is an irreducible reflection group of type A_n, B_n, F_4, H_3, H_4 or $I_2(m)$. For polytopes admitting reflection groups of the other types as their symmetry group, for example Gosset 4_{21} polytope (admitting a reflection group of type E_8), the problem to verify (2.3) is still open.

Chapter 3

Existence of a canonical system

This chapter is based on a joint-work with Nakashima and Terao [17]. Let V be an n-dimensional vector space over $\mathbb{K}\,(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$ and W an essential irreducible finite reflection group of V. Existence of a canonical system was proven by Flatto [5, 6] and by Flatto and Wiener[7] for $\mathbb{K} = \mathbb{R}$. In this chapter we shall prove the existence including the case $\mathbb{K} = \mathbb{C}$.

3.1 Preliminaries

Let $e_1, \ldots, e_n$ be an orthonormal basis for V with respect to the W-invariant inner product $\langle \cdot, \cdot \rangle$ on V, $x_1, \ldots, x_n$ the dual basis and $\partial_1, \ldots, \partial_n$ the corresponding derivations. For $S = S(V^*)$ we define the module of regular 1-forms Ω_S^1 as

$$\Omega_S^1 \coloneqq S \otimes_{\mathbb{K}} V^*$$

equipped with the differential map $d\colon S \to \Omega_S^1$ defined by

$$df \coloneqq \sum_{i=1}^{n} \partial_i f \otimes x_i = \sum_{i=1}^{n} (\partial_i f) dx_i.$$

Define

$$f\omega = f \sum_{i=1}^{n} g_i dx_i \coloneqq \sum_{i=1}^{n} (f g_i) dx_i$$

for all $f \in S$ and $\omega = \sum_{i=1}^{n} g_i dx_i \in \Omega_S^1$. Thus Ω_S^1 has a structure of a graded S-module. The differential map d satisfies

$$d(fg) = f(dg) + g(df) \tag{3.1}$$

for all $f, g \in S$ since

$$d(fg) = \sum_{i=1}^{n} \partial_i(fg)dx_i = \sum_{i=1}^{n}(f\partial_i g + g\partial_i f)dx_i = f(dg) + g(df).$$

The reflection group W acts on Ω_S^1 as

$$w\omega \coloneqq \sum_{i=1}^{n} wg_i \otimes wx_i,$$

where $w \in W$ and $\omega = \sum_{i=1}^{n} g_i \otimes x_i$. Because of (3.1) and

$$d(wx_i) = 1 \otimes wx_i = w(1 \otimes x_i) = w(dx_i),$$

the differential map d is a $\mathbb{K}[W]$-homomorphism.

Define a $\mathbb{K}[W]$-homomorphism $\varepsilon \colon \Omega_S^1 \to S$ by

$$\varepsilon\left(\sum_{i=1}^{n} f_i dx_i\right) \coloneqq \sum_{i=1}^{n} x_i f_i.$$

Then the composition $\varepsilon \circ d$ is the Euler operator:

$$\varepsilon(df) = \sum_{i=1}^{n} x_i \partial_i f.$$

One can verify that for a homogeneous polynomial $f \in S$

$$\varepsilon(df) = (\deg f)f.$$

We define a subspace Ω_W as below to show the existence of a canonical system.

Definition 3.1. Define

$$\Omega_W \coloneqq \left(\mathcal{H}_W \otimes_{\mathbb{K}} V^*\right)^W,$$

$$\mathcal{F}_W \coloneqq \bigoplus_{k=1}^{\infty} \left\{ f \in S_k^W \mid (g, f) = 0 \text{ for all } g \in S_\ell^W \text{ with } 0 < \ell < k \right\},$$

where S_k^W denotes the space generated by homogeneous invariants of degree k and $(\cdot, \cdot)$ is the sesquilinear map defined by Definition 1.16.

3.2 Main theorems

Our purpose in this chapter is to prove the theorems:

Theorem 3.2. *We have* $\dim_{\mathbb{K}} \Omega_W = \dim_{\mathbb{K}} \mathcal{F}_W = n$. *The restrictions* $d|_{\mathcal{F}_W} : \mathcal{F}_W \to \Omega_W$ *and* $\varepsilon|_{\Omega_W} : \Omega_W \to \mathcal{F}_W$ *are isomorphisms.*

Theorem 3.3. *Any orthogonal homogeneous basis for the vector space* $\mathcal{F}_W$ *is a canonical system of basic invariants for the reflection group* W*, and vice versa. In particular there exists a unique canonical system up to equivalence.*

First we shall prove the following lemma:

Lemma 3.4. *The following hold:*

(1) $d(\mathcal{F}_W) \subseteq \Omega_W$ *and the restriction* $d|_{\mathcal{F}_W}$ *is an injection,*

(2) $\varepsilon(\Omega_W) \subseteq \mathcal{F}_W$.

Proof. (1) Let $f \in \mathcal{F}_W$ and $g \in I$. To prove $df \in \Omega_W$ it suffices to show that $(g, \partial_i f) = 0$ for all $i \in \{1, \ldots, n\}$. We may assume that f and g are homogeneous. If $\deg g \geq \deg f$ then $(g, \partial_i f) = 0$. Suppose $\deg g < \deg f$. Then $(g, \partial_i f) = \partial_i g^* f = 0$ since $f \in \mathcal{F}_W$ and $d(\Omega) \subseteq \mathcal{F}_W$.

Suppose that $df = 0$ for a polynomial in $\mathcal{F}_W$. Then f is a constant. Since every non-zero homogeneous polynomial in $\mathcal{F}_W$ has positive degree, we can conclude that $f = 0$ and $d|_{\mathcal{F}_W}$ is injective.

(2) Let $f \in \varepsilon(\Omega_W)$ and $g \in S_+^W$ with $\deg g < \deg f$. We may assume that both of f and g are homogeneous. The polynomial $(g, f) = g^* f$ is a non-constant invariant since f, g are invariants with $\deg g < \deg f$. Hence $g^* f \in I$. We shall prove that $g^* f \in \mathcal{H}_W$. Since $f \in \varepsilon(\Omega_W)$ we may write $f = \sum_{i=1}^n x_i f_i$ for some $f_i \in \mathcal{H}_W$. By Proposition 1.15 and Theorem 1.24 we have

$$g^* f = \sum_{i=1}^n g^* x_i f_i = \sum_{i=1}^n (x_i (g^* f_i) + (\partial_i g)^* f_i) = \sum_{i=1}^n (\partial_i g)^* f_i.$$

Hence $g^* f \in \mathcal{H}_W$. Therefore $g^* f \in I \cap \mathcal{H}_W = 0$. Thus $(g, f) = g^* f = 0$ and $f \in \mathcal{F}_W$. Hence $\varepsilon(\Omega_W) \subseteq \mathcal{F}_W$.

$\square$

Recall the orthogonal decomposition (1.8) $S = I \oplus \mathcal{H}_W$. Let π the projection map from S to $\mathcal{H}_W$ with respect to the decomposition $S = I \oplus \mathcal{H}_W$. Define a $\mathbb{K}[W]$-endomorphism $\tilde{\pi}$ on $\Omega_S^1 = S \otimes_{\mathbb{K}} V^*$ by

$$\tilde{\pi}(\sum_{i=1}^{n} f_i dx_i) \coloneqq \sum_{i=1}^{n} \pi(f_i) dx_i.$$

Then we have

$$\operatorname{Ker} \tilde{\pi} = I \otimes_{\mathbb{K}} V^* \text{ and } \operatorname{Im} \tilde{\pi} = \mathcal{H}_W \otimes V^*.$$

For a system of basic invariants $\{h_1, \dots, h_n\}$ we put

$$dh_i = \eta_i + \omega_i,$$

where $\eta_i \in (I \otimes_{\mathbb{K}} V^*)^W$ and $\omega_i \in \Omega_W = (\mathcal{H}_W \otimes_{\mathbb{K}} V^*)^W$

For each $i \in \{1, \dots, n\}$ we define

$$g_i \coloneqq \varepsilon(\eta_i) \in S^W, \quad f_i \coloneqq \varepsilon(\omega_i) = (\varepsilon \circ \tilde{\pi})(dh_i) \in \mathcal{F}_W,$$

where $f_i \in \mathcal{F}_W$ follows from Lemma 3.4(2). Then we have

$$(\deg h_i) h_i = \varepsilon(h_i) = g_i + f_i.$$

Lemma 3.5. *The subset $\{f_1, \dots, f_n\} \subseteq \mathcal{F}_W$ is a system of basic invariants.*

Proof. We may write $\eta_i = \sum_{j=1}^{n} g_{ij} dx_j$ For every $i, j \in \{1, \dots, n\}$ the polynomial $g_{ij} \in I$. We may assume that $\{h_1, \dots, h_\ell\}$ is a subset of the system $\{h_1, \dots, h_n\}$ consisting of invariants h_j with $\deg h_j < \deg h_i$. Then there exist polynomials $g_{ijk} \in S$ such that

$$g_{ij} = \sum_{k=1}^{\ell} g_{ijk} h_k.$$

Then we have

$$g_i = \varepsilon(\eta_i) = \sum_{j=1}^{n} x_j g_{ij} = \sum_{j=1}^{n} \sum_{k=1}^{\ell} x_j g_{ijk} h_k.$$

Since g_i is invariant we have

$$g_i = \mathrm{Av}(g_i) = \sum_{j=1}^{n} \sum_{k=1}^{\ell} \mathrm{Av}(x_j g_{ijk}) h_k,$$

where Av is the averaging operator defined by

$$\mathrm{Av} := \frac{1}{|W|} \sum_{w \in W} w,$$

where $|W|$ denotes the order of W. The invariants $\mathrm{Av}(x_j g_{ijk})$ are polynomials in $\{h_1, \ldots, h_\ell\}$ so is g_i. By induction on $\deg h_i$ all of $\{h_1, \ldots, h_n\}$ are polynomials in $\{f_1, \ldots, f_n\}$. Hence $\{f_1, \ldots, f_n\}$ is a generating system of S_+^W. By Theorem 1.9 we can conclude that $\{f_1, \ldots, f_n\}$ is a system of basic invariants. □

We now prove Theorems 3.2 and 3.3.

Proof of Theorem 3.2. By Lemma 3.5 the space $\mathcal{F}_W$ contains a system of basic invariants. Hence $\dim_{\mathbb{K}} \mathcal{F}_W \geq n$. Lemma 3.4(1) implies $\dim_{\mathbb{K}} \mathcal{F}_W \leq \dim_{\mathbb{K}} \Omega_W$. We shall show that $\dim_{\mathbb{K}} \Omega_W = n$. There exists a natural isomorphism

$$\Omega_W = (\mathcal{H}_W \otimes_{\mathbb{K}} V^*)^W \simeq \mathrm{Hom}_{\mathbb{K}[W]}(V, \mathcal{H}_W). \tag{3.2}$$

The dimension of $\mathrm{Hom}_{\mathbb{K}[W]}(V, \mathcal{H}_W)$ coincides with the multiplicity of V in $\mathcal{H}_W$ since V is (absolutely) irreducible by Proposition 1.8. From Proposition 1.26 the space of W-harmonic polynomials $\mathcal{H}_W$ affords the regular representation of W. Therefore the multiplicity of V in $\mathcal{H}_W$ is $\dim_{\mathbb{K}} V = n$, so $\dim_{\mathbb{K}} \Omega_W = n$. Thus the estimate $n \leq \dim_{\mathbb{K}} \mathcal{F}_W \leq \dim_{\mathbb{K}} \Omega_W = n$ forces $\dim_{\mathbb{K}} \mathcal{F}_W = \dim_{\mathbb{K}} \Omega_W = n$. Hence the restriction $d|_{\mathcal{F}_W} \colon \mathcal{F}_W \to \Omega_W$ becomes an isomorphism, since it is injective by Lemma 3.4(1). From Lemma 3.4(2), we have $\varepsilon(\Omega_W) \subseteq \mathcal{F}_W$. To show that $\varepsilon|_{\Omega_W} \colon \Omega_W \to \mathcal{F}_W$ is an isomorphism, note that there is a commutative diagram

$$\begin{tikzcd}
\mathcal{F}_W \arrow[d, "d|_{\mathcal{F}_W}"', "\wr"] \arrow[dr, "\text{Euler op.}", "\sim"'] & \\
\Omega_W \arrow[r, "\varepsilon|_{\Omega_W}"'] & \mathcal{F}_W
\end{tikzcd}$$

Since $d|_{\mathcal{F}_W}$ and the Euler operator in the diagram are isomorphic, so is $\varepsilon|_{\Omega_W}$. □

Proof of Theorem 3.3. By Lemma 3.5, the vector space $\mathcal{F}_W$ contains a system $\{f_1, \ldots, f_n\}$ of basic invariants for the reflection group W. This system becomes a homogeneous linear basis for $\mathcal{F}_W$, since $\dim_{\mathbb{K}} \mathcal{F}_W = n$ by Theorem 3.2 and algebraic independence implies linear independence. Therefore any homogeneous linear basis for $\mathcal{F}_W$ gives a system of basic invariants for W.

Take an orthogonal homogeneous basis $\mathcal{B}$ for $\mathcal{F}_W$. Then $\mathcal{B}$ is a system of basic invariants satisfying the condition (1.6). Therefore $\mathcal{B}$ is a canonical system of basic invariants. In particular, a canonical system exists.

Conversely, let $\mathcal{B}$ be a canonical system. The condition (1.6) implies $\mathcal{B} \subseteq \mathcal{F}_W$. Hence $\mathcal{B}$ is an orthogonal homogeneous basis for the vector space $\mathcal{F}_W$. In particular, there exists a unique canonical system up to equivalence. □

Corollary 3.6. *Let π be the projection onto $\mathcal{H}_W$ with respect to the decomposition $S = I \oplus \mathcal{H}_W$ and $\tilde{\pi} := \pi \otimes 1_{V^*}$. For an arbitrary system $\{h_1, \ldots, h_n\}$ of basic invariants, the orthogonalization of the system*

$$\{(\varepsilon \circ \tilde{\pi})(dh_i) \mid 1 \leq i \leq n\}$$

yields a canonical system of basic invariants.

Chapter 4

Explicit formulas for canonical systems

This chapter is based on a joint-work with Nakashima [18]. Let V be an n-dimensional vector space over $\mathbb{K}\,(\mathbb{K}=\mathbb{R}$ or $\mathbb{C})$ and W an essential irreducible finite reflection group of V. By Corollary 3.6 a canonical system may be expressed by making use of an arbitrary system of basic invariants, but this formula is not explicit because of abstractness of the projection π. In this chapter we shall create a computable map which refines Corollary 3.6.

4.1 Preliminaries

Recall the W-invariant sesquilinear map $(\cdot,\cdot)\colon S\times S\to S$ defined by Definition 1.16. The restriction of $(\cdot,\cdot)$ to $\mathcal{F}_W\times\mathcal{F}_W$ is a inner product on $\mathcal{F}_W$ since for homogeneous polynomials $f,g\in\mathcal{F}_W$

$$(f,g)=\begin{cases}(f,g)_0 & \text{if }\deg f=\deg g\\ 0 & \text{if }\deg f\neq\deg g\end{cases}.$$

We introduce a sesquilinear map $(\cdot,\cdot)\colon \Omega_W\times\Omega_W\to S$ defined by

$$\left(\sum_{i=1}^{n} f_i dx_i, \sum_{j=1}^{n} g_j dx_j\right) := \sum_{i=1}^{n}\sum_{j=1}^{n}(f_i,g_j)(x_i,x_j)=\sum_{i=1}^{n}(f_i,g_i), \tag{4.1}$$

where $(\cdot,\cdot)$ in the middle and the right hand is the sesquilinear map defined by Definition 1.16. By Theorem 3.2, any element in Ω_W is of the form

df ($f \in \mathcal{F}_W$). For any homogeneous polynomials $f, g \in \mathcal{F}_W$,

$$\begin{aligned}(df, dg) &= \left(\sum_{i=1}^{n} \partial_i f dx_i, \sum_{i=1}^{n} \partial_i g dx_j \right) = \sum_{i=1}^{n} (\partial_i f, \partial_i g) \\ &= \sum_{i=1}^{n} (x_i \partial_i f, g) = (\deg f)(f, g).\end{aligned}$$

Hence the sesquilinear map (4.1) actually determines an inner product on Ω_W. For homogeneous elements $\omega_1, \omega_2 \in \Omega_W$ satisfying $\deg \omega_1 \neq \deg \omega_2$, we have $(\omega_1, \omega_2) = 0$. Therefore

$$f \perp g \text{ if and only if } df \perp dg$$

for homogeneous polynomials $f, g \in \mathcal{F}_W$ and

$$\omega_1 \perp \omega_2 \text{ if and only if } \varepsilon(\omega_1) \perp \varepsilon(\omega_2) \tag{4.2}$$

for homogeneous elements $\omega_1, \omega_2 \in \Omega_W$.

Proposition 4.1. *Let $\{\omega_1, \ldots, \omega_n\}$ be an orthogonal homogeneous basis for Ω_W. Then $\{\varepsilon(\omega_1), \ldots, \varepsilon(\omega_n)\}$ is a canonical system of basic invariants.*

Proof. It follows from Theorem 3.2, Theorem 3.3 and (4.2). □

4.2 Construction of canonical systems

We will construct a homogeneous basis for $\mathcal{F}_W$ from arbitrary basic invariants. The following map plays an important roll for the construction.

Definition 4.2. Define a linear map $\phi \colon S \to S$ by

$$\phi(f) = ((f, \Delta)\, \Delta)\,.$$

Proposition 4.3. *The map ϕ has the following properties:*

(1) ϕ is a $\mathbb{K}[W]$-homomorphism,

(2) ϕ is homogeneous and degree-preserving,

(3) $\operatorname{Ker} \phi = I$ *and* $\operatorname{Im} \phi = \mathcal{H}_W$. *In particular the restriction $\phi|_{\mathcal{H}_W} : \mathcal{H}_W \to \mathcal{H}_W$ is a $\mathbb{K}[W]$-isomorphism.*

(4) ϕ is a symmetric (Hermitian) transformation with respect to the inner product $(\cdot,\cdot)_0$, that is,

$$(\phi(f), g)_0 = (f, \phi(g))_0, \qquad (f, g \in S).$$

In particular, $\phi \colon S \to S$ can be diagonalized by homogeneous polynomials.

Proof. (1) For any $f \in S$ and $w \in W$,

$$\begin{aligned} w\phi(f) &= w((f, \Delta), \Delta) \\ &= ((wf, (\det w)\Delta), (\det w)\Delta) \\ &= ((\det w)(wf, \Delta), (\det w)\Delta) \\ &= \overline{(\det w)}(\det w)((wf, \Delta), \Delta) \\ &= ((wf, \Delta), \Delta) \\ &= \phi(wf). \end{aligned}$$

(2) For any homogeneous polynomial $f \in S$,

$$\deg \phi(f) = \deg \Delta - \deg(f, \Delta) = \deg \Delta - (\deg \Delta - \deg f) = \deg f.$$

(3) By Theorem 1.24,

$$f \in \operatorname{Ker} \phi \Leftrightarrow ((f, \Delta), \Delta) = 0 \Leftrightarrow (f, \Delta) \in I \cap \mathbb{K}[\partial]\Delta = 0 \Leftrightarrow f \in I.$$

By Theorem 1.24 again, we have $\operatorname{Im} \phi \subseteq \mathcal{H}_W$. Since $\operatorname{Ker} \phi = I$ we have $\dim_{\mathbb{K}} \operatorname{Im} \phi = \dim_{\mathbb{K}} \mathcal{H}_W$. Then $\operatorname{Im} \phi = \mathcal{H}_W$ and the restriction $\phi|_{\mathcal{H}_W} \colon \mathcal{H}_W \to \mathcal{H}_W$ is an isomorphism.

(4) We may assume that f, g are homogeneous polynomials having the same degree. Put $F \coloneqq (f, \Delta)$ and $G \coloneqq (g, \Delta)$. Then

$$(f, \phi(g))_0 = (f, (G, \Delta))_0 = (fG, \Delta)_0 = (G, (f, \Delta))_0 = (G, F)_0.$$

By symmetry

$$(\phi(f), g)_0 = \overline{(g, \phi(f))_0} = \overline{(F, G)_0} = (G, F)_0 = (f, \phi(g))_0.$$

Thus ϕ is a symmetric (Hermitian) transformation with respect to the inner product $(\cdot,\cdot)_0$. $\square$

Remark. In the proof of (2) and (4), we did not use any property of Δ other than homogeneity. Hence (2) and (4) are true for $\phi_h(f) \coloneqq ((f,h),h)$ with any homogeneous polynomial $h \in S$.

The map ϕ induces a linear map $\tilde{\phi}\colon (\Omega_S^1)^W \to \Omega_W$ defined by

$$\tilde{\phi}\left(\sum_{i=1}^{n} f_i dx_i\right) \coloneqq \sum_{i=1}^{n} \phi(f_i) dx_i.$$

By Proposition 4.3, we have

Proposition 4.4. *The following conditions hold:*

(1) $\tilde{\phi}$ *is homogeneous and degree-preserving,*

(2) $\operatorname{Ker}\tilde{\phi} = (I \otimes_{\mathbb{K}} V^*)^W$ *and* $\operatorname{Im}\tilde{\phi} = \Omega_W$. *In particular the restriction* $\tilde{\phi}|_{\Omega_W}\colon \Omega_W \to \Omega_W$ *is an isomorphism.*

(3) $\tilde{\phi}$ *is a symmetric (Hermitian) transformation with respect to the inner product* $(\cdot,\cdot)\colon \Omega_W \times \Omega_W \to \mathbb{K}$, *that is,*

$$(\tilde{\phi}(\omega_1), \omega_2) = (\omega_1, \tilde{\phi}(\omega_2)), \qquad (\omega_1, \omega_2 \in \Omega_W).$$

In particular, $\tilde{\phi}|_{\Omega_W}\colon \Omega_W \to \Omega_W$ *can be diagonalized by homogeneous elements.*

Proof. (1) and (2) follow immediately by Proposition 4.3.

(3) Let $\omega_1, \omega_2 \in \Omega_W$ be homogeneous elements. If $\deg\omega_1 \neq \deg\omega_2$ then

$$(\tilde{\phi}(\omega_1), \omega_2) = 0 = (\omega_1, \tilde{\phi}(\omega_2)).$$

When $\deg\omega_1 = \deg\omega_2$,

$$\begin{aligned}\left(\tilde{\phi}(\omega_1), \omega_2\right) &= \left(\sum_{i=1}^{n} \phi(f_i) dx_i, \sum_{i=1}^{n} g_i dx_i\right) = \sum_{i=1}^{n} (\phi(f_i), g_i)_0 \\ &= \sum_{i=1}^{n} (f_i, \phi(g_i))_0 = \left(\sum_{i=1}^{n} f_i dx_i, \sum_{i=1}^{n} \phi(g_i) dx_i\right) \\ &= \left(\omega_1, \tilde{\phi}(\omega_2)\right).\end{aligned}$$

Thus $\tilde{\phi}$ is symmetric (Hermitian). $\square$

Proposition 4.5. *There exists a canonical system* $\{f_1, \ldots, f_n\}$ *such that* $df_1, \ldots, df_n$ *are eigenvectors of* $\tilde{\phi}$.

Proof. Let $U_{d,\lambda}$ be the subspace of Ω_W consisting of homogeneous eigenvectors of $\tilde{\phi}$ with degree d and eigenvalue λ. By Proposition 4.4,

$$\Omega_W = \bigoplus_{d,\lambda} U_{d,\lambda},$$

where every λ is real since $\tilde{\phi}$ is symmetric (Hermitian). We assume that $U_{d_1,\lambda_1} \neq U_{d_2,\lambda_2}$. If $d_1 \neq d_2$, then U_{d_1,λ_1} is orthogonal to U_{d_2,λ_2}. When $\lambda_1 \neq \lambda_2$, let $\omega_1 \in U_{d_1,\lambda_1}$ and $\omega_2 \in U_{d_2,\lambda_2}$. By Proposition 4.4 (3), we have

$$\lambda_1 (\omega_1, \omega_2) = \left(\tilde{\phi}(\omega_1), \omega_2\right) = \left(\omega_1, \tilde{\phi}(\omega_2)\right) = \lambda_2 (\omega_1, \omega_2).$$

Then $(\omega_1, \omega_2) = 0$. This means that U_{d_1,λ_1} is orthogonal to U_{d_2,λ_2}. Hence the subspaces $U_{d,\lambda}$ are orthogonal to each other.

The union of orthogonal bases for $U_{d,\lambda}$ is an orthogonal basis for Ω_W consisting of eigenvectors of $\tilde{\phi}$. Let $\{\omega_1 \ldots, \omega_n\}$ be such an orthogonal basis for Ω_W. Then $\varepsilon(\omega_1) \ldots, \varepsilon(\omega_n)$ is a required canonical system. □

Theorem 4.6. *For an arbitrary system of basic invariants* $\{h_1, \ldots, h_n\}$, *the system*

$$\{\tilde{\phi}(dh_1), \ldots, \tilde{\phi}(dh_n)\}$$

is a basis for Ω_W. *Therefore the orthogonalization of*

$$\left\{ (\varepsilon \circ \tilde{\phi})(dh_i) = \sum_{j=1}^{n} x_j \phi(\partial_j h_i) \,\middle|\, 1 \leq i \leq n \right\} \tag{4.3}$$

is a canonical system of basic invariants.

Proof. We can take a canonical system $\{f_1 \ldots, f_n\}$ satisfying that $df_1, \ldots, df_n$ are eigenvectors of $\tilde{\phi}$ by Proposition 4.5. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\tilde{\phi}$ corresponding to $df_1, \ldots, df_n$. All of these are nonzero since $\tilde{\phi}|_{\Omega_W}$ is an isomorphism. Let $d_1 \leq \cdots \leq d_n$ are the degrees of W. We may assume that $\deg f_i = \deg h_i = d_i$. Let r be the multiplicity of d_i. Then we may assume that $d_i = \cdots = d_{i+r-1}$. We only need to verify that the linear independence of $\{\tilde{\phi}(dh_i), \ldots, \tilde{\phi}(dh_{i+r-1})\}$ since homogeneous elements having different degrees are orthogonal to each other.

For $j = 0, \ldots, r-1$, we may write

$$h_{i+j} = \sum_{k=0}^{r-1} a_{jk} f_{i+k} + P_j(f_1, \ldots, f_{i-1}), \tag{4.4}$$

where $P_j = P_j(f_1, \ldots, f_{i-1})$ is a polynomial in $f_1, \ldots, f_{i-1}$ and $a_{jk} \in \mathbb{K}$. Since $\{h_1, \ldots, h_n\}$ is algebraically independent, we have $\det[a_{jk}] \neq 0$. The 1-form dP_j is in $\ker \tilde{\phi} = (I \otimes_{\mathbb{K}} V^*)^W$ since each term of P_j is a product of two or more invariants in $\{f_1, \ldots, f_r\}$. Applying $\tilde{\phi} \circ d$ on the equality (4.4), we have

$$\tilde{\phi}(dh_{i+j}) = \sum_{k=0}^{r-1} a_{jk} \lambda_{i+k} df_{i+k}. \tag{4.5}$$

Therefore, $\tilde{\phi}(dh_i), \ldots, \tilde{\phi}(dh_{i+r-1})$ are linearly independent since $\det[a_{jk}] \neq 0$. $\square$

Corollary 4.7. *Suppose that the degrees of W are all different. Then the system* (4.3) *is a canonical system of basic invariants.*

Chapter 5

Canonical systems for $G(m, p, n)$

For the infinite family of irreducible finite Euclidean reflection groups (of types A_n, B_n, D_n, I_2), Iwasaki [8] constructs explicit formulas for the canonical systems. In this chapter, we construct explicit formulas for canonical systems for unitary reflection groups $G(m, p, n)$ by making use of a similar method to Iwasaki's one.

5.1 Preliminaries

For positive integers m, n, let $\boldsymbol{\mu}_m := \boldsymbol{\mu}_m(\mathbb{C})$ be the cyclic group of m-th roots of unity and $\mathfrak{S}_n$ the symmetric group of degree n. The group $\mathfrak{S}_n$ acts on $\boldsymbol{\mu}_m^n$ by changing coordinates. For a divisor p of m, define

$$A(m, p, n) := \left\{ \zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n) \in \boldsymbol{\mu}_m^n \;\mid (\zeta_1 \zeta_2 \cdots \zeta_n)^{m/p} = 1 \right\}.$$

It is easy to show that $A(m, p, n)$ is a subgroup of $\boldsymbol{\mu}_m^n$ and is stable under the action of $\mathfrak{S}_n$.

Definition 5.1. For positive integers m, n and a divisor p of m, we define a group $G(m, p, n)$ by the semidirect product of $A(m, p, n)$ and $\mathfrak{S}_n$:

$$G(m, p, n) := A(m, p, n) \rtimes \mathfrak{S}_n.$$

The group structure is given by

$$(\zeta, \sigma)(\zeta', \sigma') = (\zeta\,(\sigma\zeta'), \sigma\sigma').$$

For each permutation $\sigma \in \mathfrak{S}_n$, E_σ denotes the permutation matrix with respect to σ. Note that both of $\mathrm{diag}(\zeta)$ and E_σ are unitary matrices for $\zeta \in \boldsymbol{\mu}_m^n$ and $\sigma \in \mathfrak{S}_n$. We define a map $\rho\colon G(m,p.n) \to U_n(\mathbb{C})$ by

$$\rho(\zeta,\sigma) \coloneqq \mathrm{diag}(\zeta)E_\sigma.$$

Proposition 5.2. *The map ρ is an injective group homomorphism.*

Proof. For any $(\zeta,\sigma),(\zeta',\sigma') \in G(m,p,n)$,

$$\begin{aligned}
\rho((\zeta,\sigma)(\zeta',\sigma')) &= \rho(\zeta(\sigma\zeta'),\sigma\sigma') \\
&= \mathrm{diag}(\zeta(\sigma\zeta'))E_{\sigma\sigma'} \\
&= \mathrm{diag}(\zeta)\,\mathrm{diag}(\sigma\zeta')E_\sigma E_{\sigma'} \\
&= \mathrm{diag}(\zeta)E_\sigma\,\mathrm{diag}(\zeta')E_{\sigma'} \\
&= \rho(\zeta,\sigma)\rho(\zeta',\sigma').
\end{aligned}$$

Hence ρ is a homomorphism. Assume that $\rho(\zeta,\sigma) = \mathrm{diag}(\zeta)E_\sigma = I$. Then $\mathrm{diag}(\zeta) = E_{\sigma^{-1}}$. Hence both of $\mathrm{diag}(\zeta)$ and $E_{\sigma^{-1}}$ are equal to the identity matrix. Therefore (ζ,σ) is equal to the identity of $G(m,p,n)$. Thus the homomorphism ρ is injective. $\square$

Let $V \coloneqq \mathbb{C}^n$ and $e_1,\dots,e_n$ the standard basis for V. Let $x = (x_1,\dots,x_n)$ be the dual basis of $e_1,\dots,e_n$ and S the ring of polynomial function $\mathbb{C}[x]$. From the previous proposition, we can regard the group $G(m,p,n)$ as a subgroup of $U(V) \cong U_n(\mathbb{C})$. The group $G(m,p,n)$ contains two kinds of reflections:

$$s_{i,\zeta}\,e_k = \begin{cases} \zeta e_i & (k=i) \\ e_k & (k \neq i) \end{cases} \quad \text{for } \zeta \in \boldsymbol{\mu}_{m/p},\ 1 \le i \le n,$$

$$s_{i,j,\zeta}\,e_k = \begin{cases} \zeta e_j & (k=i) \\ \zeta^{-1} e_i & (k=j) \\ e_k & (k \neq i,j) \end{cases} \quad \text{for } \zeta \in \boldsymbol{\mu}_m,\ 1 \le i < j \le n.$$

All reflections of $G(m,p,n)$ are of types above and $G(m,p,n)$ are generated by those reflections (see for example [15]). Therefore the skew polynomial Δ for $G(m,p,n)$ is given by

$$\Delta = \prod_{i=1}^n x_i^{m/p-1} \prod_{\substack{1\le i<j\le n \\ \zeta\in\boldsymbol{\mu}_m}} (x_i - \zeta x_j) = \prod_{i=1}^n x_i^{m/p-1} \prod_{1\le i<j\le n} (x_i^m - x_j^m). \tag{5.1}$$

According to [15], a system of basic invariants for $G(m,p,n)$ is given by

$$\sum_{i=1}^{n} x_i^m, \quad \sum_{i=1}^{n} x_i^{2m}, \quad \ldots, \quad \sum_{i=1}^{n} x_i^{(n-1)m}, \quad (x_1 \cdots x_n)^{m/p}. \tag{5.2}$$

Hence the degrees of $G(m,p,n)$ are

$$m, 2m, \ldots, (n-1)m, n\frac{m}{p}. \tag{5.3}$$

When $p = 1$, replacing the invariant $(x_1 \cdots x_n)^m$ by $\sum_{i=1}^{n} x_i^{nm}$, we have another system of basic invariants for $G(m,1,n)$:

$$\sum_{k=1}^{n} x_k^{im} \qquad (1 \le i \le n). \tag{5.4}$$

5.2 Canonical systems for $G(m,1,n)$

In this section we set $G := G(m,1,n)$. Let $\{f_1, \ldots, f_n\}$ be a canonical system for G. From (5.3), we may assume that $\deg f_i = im$. Since every f_i is homogeneous, we have

$$(\deg f_i) f_i = \sum_{j=1}^{n} x_j \partial_j f_i.$$

Hence it suffices to determine $\partial_1 f_i, \ldots, \partial_n f_i$ to determine f_i. From the proof of Lemma 3.4(1), we have $\partial_j f_i \in \mathcal{H}_G$. By Steinberg's Theorem 1.24 (2) there exists a homgeneous polynomial $g_{ij} \in S$ such that

$$\partial_j f_i = g_{ij}^* \Delta \qquad (1 \le i, j \le n). \tag{5.5}$$

From (5.1) the skew polynomial Δ for $G(m,1,n)$ is given by

$$\Delta = \prod_{i=1}^{n} x_i^{m-1} \prod_{1 \le i < j \le n} (x_i^m - x_j^m).$$

Let

$$G_j := \mathrm{Stab}_G(x_j) = \{w \in G \mid w x_j = x_j\}.$$

The subgroup G_j is a reflection group isomorphic to $G(m,1,n-1)$. So the order of G_j is independent of j. The skew polynomial Δ_j for G_j is given by

$$\Delta_j = \prod_{\substack{1\le k\le n\\ k\ne j}} x_k^{m-1} \prod_{\substack{1\le k<\ell\le n\\ k,\ell\ne j}} (x_k^m - x_\ell^m).$$

Define the averaging operator Av_j and the skew operator Sk_j for G_j by

$$\begin{aligned}\mathrm{Av}_j &:= \frac{1}{|G_j|}\sum_{w\in G_j} w,\\ \mathrm{Sk}_j &:= \frac{1}{|G_j|}\sum_{w\in G_j} (\det w)^{-1} w,\end{aligned}$$

where $|G_j|$ is the order of G_j. Applying Av_j to (5.5), we have

$$\begin{aligned}\partial_j f_i &= \mathrm{Av}_j(g_{ij}^*\Delta)\\ &= \frac{1}{|G_j|}\sum_{w\in G_j}(wg_{ij})^* w\Delta\\ &= \frac{1}{|G_j|}\sum_{w\in G_j}(wg_{ij})^*(\det w)\Delta\\ &= \frac{1}{|G_j|}\sum_{w\in G_j}\left((\det w)^{-1}wg_{ij}\right)^*\Delta\\ &= (\mathrm{Sk}_j\, g_{ij})^*\,\Delta \qquad (1\le i,j\le n),\end{aligned} \tag{5.6}$$

where the fourth equality follows from the fact that $\det w$ is a root of unity. By Proposition 1.23, every skew polynomial $\mathrm{Sk}_j\, g_{ij}$ is divisible by the skew polynomial Δ_j. Hence there exists a G_j-invariant homogeneous polynomial h_{ij} such that $\mathrm{Sk}_j\, g_{ij} = h_{ij}\Delta_j$. From (5.6), we have

$$-1 + \deg f_i = -(\deg h_{ij} + \deg \Delta_j) + \deg \Delta.$$

Hence

$$\begin{aligned}\deg h_{ij} &= 1 - \deg f_i - \deg \Delta_j + \deg \Delta\\ &= 1 - im - (n-1)(m-1) - m\binom{n-1}{2} + n(m-1) + m\binom{n}{2}\\ &= (n-i)m.\end{aligned}$$

From (5.4), each h_{ij} is a polynomial of the G_j-invariants

$$\sum_{\substack{1\le k\le n\\ k\ne j}} x_k^{\ell m} \qquad (1\le \ell\le n-1).$$

For any positive integer ℓ, we have $\sum_{k=1}^n x_k^{\ell m}\in I$. Hence

$$h_{ij}\equiv c_{ij}x_j^{(n-i)m} \pmod I$$

for some constant $c_{ij}\in\mathbb{C}^\times$. By Steinberg's Theorem 1.24(1), the equality (5.6) yields

$$\partial_j f_i = (\mathrm{Sk}_j\, g_{ij})^*\,\Delta = (h_{ij}\Delta_j)^*\,\Delta = \left(c_{ij}x_j^{(n-i)m}\Delta_j\right)^*\Delta.$$

Finally we need to determine the coefficients c_{ij}. Without loss of generality we may assume that $c_{i1}=1$. For $j\ge 2$, let $s_j:=s_{1,j,1}$. Then

$$s_j\Delta_j=(-1)^{\#\{k\mid 1<k<j\}}\Delta_1=(-1)^{j-2}\Delta_1.$$

Therefore

$$\begin{aligned}\partial_1 f_i &= s_j(\partial_j f_i) = \left(c_{ij}x_1^{(n-i)m}(-1)^{j-2}\Delta_1\right)^*(-\Delta)\\ &= (-1)^{j-1}\overline{c_{ij}}\left(x_1^{(n-i)m}\Delta_1\right)^*\Delta = (-1)^{j-1}\overline{c_{ij}}\partial_1 f_i.\end{aligned}$$

Hence $(-1)^{j-1}\overline{c_{ij}}=1$, namely, $c_{ij}=(-1)^{j-1}$. We have concluded the following theorem.

Theorem 5.3. *Polynomials*

$$f_i=\sum_{j=1}^n(-1)^{j-1}x_j\left(x_j^{(n-i)m}\Delta_j\right)^*\Delta \qquad (1\le i\le n)$$

form a canonical system of basic invariants for $G(m,1,n)$, where the skew polynomials Δ,Δ_j are defined by

$$\begin{aligned}\Delta&=\prod_{k=1}^n x_k^{m-1}\prod_{1\le k<\ell\le n}(x_k^m-x_\ell^m),\\ \Delta_j&=\prod_{\substack{1\le k\le n\\ k\ne j}}x_k^{m-1}\prod_{\substack{1\le k<\ell\le n\\ k,\ell\ne j}}(x_k^m-x_\ell^m) \qquad (1\le j\le n).\end{aligned}$$

Corollary 5.4. *Let $W = G(1,1,n)$. Then W is a (non-essential) finite Euclidean reflection group of type A_{n-1}. A system of canonical invariants for W is given by*

$$f_i = \sum_{j=1}^{n} (-1)^{j-1} x_j \left(x_j^{n-i} \Delta_j\right)^* \Delta \qquad (1 \le i \le n),$$

where the skew polynomials Δ, Δ_j are defined by

$$\Delta = \prod_{1 \le k < \ell \le n} (x_k - x_\ell),$$
$$\Delta_j = \prod_{\substack{1 \le k < \ell \le n \\ k,\ell \ne j}} (x_k - x_\ell) \qquad (1 \le j \le n).$$

Corollary 5.5. *Let $W = G(2,1,n)$. Then W is an irreducible finite Euclidean reflection group of type B_n. A system of canonical invariants for W is given by*

$$f_i = \sum_{j=1}^{n} (-1)^{j-1} x_j \left(x_j^{2(n-i)} \Delta_j\right)^* \Delta \qquad (1 \le i \le n),$$

where the skew polynomials Δ, Δ_j are defined by

$$\Delta = \prod_{k=1}^{n} x_k \prod_{1 \le k < \ell \le n} (x_k^2 - x_\ell^2),$$
$$\Delta_j = \prod_{\substack{1 \le k \le n \\ k \ne j}} x_k \prod_{\substack{1 \le k < \ell \le n \\ k,\ell \ne j}} (x_k^2 - x_\ell^2) \qquad (1 \le j \le n).$$

5.3 Canonical systems for $G(m,p,n)$ with $p > 1$

Let m, n be positive integers and $p \ge 2$ a divisor of m. Throughout this section we set $G \coloneqq G(m,p,n)$. Let $f_1, \dots, f_n$ be a canonical system for $G(m,1,n)$ given by Theorem 5.3. We shall prove the following theorem.

Theorem 5.6. *The polynomials*

$$f_1, \dots, f_{n-1}, (x_1 \cdots x_n)^{m/p}$$

form a canonical system of basic invariants for $G = G(m,p,n)$.

Proof. We only need to verify that $f_1,\dots,f_{n-1},(x_1\cdots x_n)^{m/p}$ form an orthogonal basis for $\mathcal{F}_G$ by Theorem 3.3. Set

$$g_i := \sum_{j=1}^{n} x_j^{im} \qquad (1\le i\le n).$$

From (5.2) the polynomials $g_1,\dots,g_{n-1},(x_1\cdots x_n)^{m/p}$ form a system of basic invariants for G and from (5.4) the polynomials $g_1,\dots,g_n$ form a system of basic invariants for $G(m,1,n)$. It is immediate that $g_i^*(x_1\cdots x_n)^{m/p}=0$ for $i\in\{1,\dots,n-1\}$ since $p>1$. Hence $(x_1\cdots x_n)^{m/p}\in\mathcal{F}_G$.

To show $f_i\in\mathcal{F}_G \quad (1\le i\le n-1)$ we need to verify that

$$g_j^* f_i = 0 \quad (j<i) \qquad \text{and} \qquad (\partial_1\cdots\partial_n)^{m/p} f_i = 0 \quad (1\le i\le n-1).$$

The former follows immediately since the system $f_1,\dots,f_n$ is canonical for $G(m,1,n)$ and the polynomials $g_1,\dots,g_n$ form a system of basic invariants for $G(m,1,n)$. To prove that the latter condition holds we have only to verify

$$\partial_1\cdots\partial_n f_i = 0 \qquad (1\le i\le n-1). \tag{5.7}$$

Let ζ be a primitive m-th root of unity. Since f_i is a $G(m,1,n)$-invariant, we have

$$s_{j,\zeta}(\partial_1\cdots\partial_n f_i) = \zeta\partial_1\cdots\partial_n f_i \qquad (1\le j\le n).$$

Hence the polynomial $\partial_1\cdots\partial_n f_i$ is a skew polynomial for the cyclic reflection group $\langle s_{j,\zeta}\rangle$ of order m. From Proposition 1.23 the polynomial $\partial_1\cdots\partial_n f_i$ is divisible by the skew polynomial x_j^{m-1} for every $j\in\{1,\dots,n\}$. Therefore $\partial_1\cdots\partial_n f_i$ is divisible by $(x_1\cdots x_n)^{m-1}$. For $1\le i\le n-1$, we have $\deg\partial_1\cdots\partial_n f_i = im-n < n(m-1) = \deg(x_1\cdots x_n)^{m-1}$. Thus (5.7) holds. Hence $f_i\in\mathcal{F}_G$ for $i\in\{1,\dots,n-1\}$.

Finally we show the orthogonality of $f_1,\dots,f_{n-1},(x_1\cdots x_n)^{m/p}$. The conditions $f_i^* f_j = 0 \quad (1\le i,j\le n-1, i\ne j)$ follow from the orthogonality of the system $f_1,\dots,f_n$. The conditions $(\partial_1\cdots\partial_n)^{m/p} f_i = 0 \quad (1\le i\le n-1)$ are proved above. For $i\in\{1,\dots,n-1\}$, if $\deg f_i < \deg(x_1\cdots x_n)^{m/p}$ then $f_i^*(x_1\cdots x_n)^{m/p}=0$ since f_i is a polynomial in $g_1,\dots,g_i$. If $\deg f_i = \deg(x_1\cdots x_n)^{m/p}$ then $f_i^*(x_1\cdots x_n)^{m/p} = (\partial_1\cdots\partial_n)^{m/p} f_i = 0$. If $\deg f_i > \deg(x_1\cdots x_n)^{m/p}$ then the condition $f_i^*(x_1\cdots x_n)^{m/p}=0$ is trivial. Thus the orthogonality of the system $f_1,\dots,f_{n-1},(x_1\cdots x_n)^{m/p}$ has been proved. □

Corollary 5.7. *Let $W = G(2,2,n)$. Then W is an irreducible finite Euclidean reflection group of type D_n. A system of canonical invariants for W is given by*

$$\begin{aligned} f_i &= \sum_{j=1}^{n} (-1)^{j-1} x_j \left(x_j^{2(n-i)} \Delta_j \right)^* \Delta \qquad (1 \le i \le n-1), \\ f_n &= (x_1 \cdots x_n)^{m/p} \end{aligned}$$

where the skew polynomials Δ, Δ_j are defined by

$$\begin{aligned} \Delta &= \prod_{k=1}^{n} x_k \prod_{1 \le k < \ell \le n} (x_k^2 - x_\ell^2), \\ \Delta_j &= \prod_{\substack{1 \le k \le n \\ k \ne j}} x_k \prod_{\substack{1 \le k < \ell \le n \\ k,\ell \ne j}} (x_k^2 - x_\ell^2) \qquad (1 \le j \le n). \end{aligned}$$

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