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Semimartingales from the Fokker-Planck equation

Dedicated to Professor Wendell H. Fleming
on the occasion of his seventy seventh birthday

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Abstract

We show the existence of a semimartingale of which one-dimensional marginal distributions are given by the solution of the Fokker-Planck equation with the p -th integrable drift vector ($p > 1$).

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A shortened version of the title: Semimartingales from the FP equation

1 Introduction.

Let $\mathcal{M}_1(\mathbf{R}^d)$ denote the complete separable metric space, with a weak topology, of Borel probability measures on \mathbf{R}^d ($d \geq 1$).

Let $b : [0, 1] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ be measurable and $\{P_t(dx)\}_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbf{R}^d)$, satisfy the following Fokker-Planck equation: for $f \in C_b^{1,2}([0, 1] \times \mathbf{R}^d)$ and $t \in [0, 1]$,

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$$\begin{aligned} & \int_{\mathbf{R}^d} f(t, x) P_t(dx) - \int_{\mathbf{R}^d} f(0, x) P_0(dx) \\ &= \int_0^t ds \int_{\mathbf{R}^d} \left(\frac{\partial f(s, x)}{\partial s} + \frac{1}{2} \Delta f(s, x) + \langle b(s, x), D_x f(s, x) \rangle \right) P_s(dx), \end{aligned} \quad (1.1)$$

where $\Delta := \sum_{i=1}^d \partial^2 / \partial x_i^2$, $D_x := (\partial / \partial x_i)_{i=1}^d$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbf{R}^d .

Inspired by Born's probabilistic interpretation of a solution to Schrödinger's equation, Nelson proposed the problem of the construction of a diffusion process $\{X(t)\}_{0 \leq t \leq 1}$ for which the following holds (see [20]):

$$X(t) = X(0) + \int_0^t b(s, X(s)) ds + W(t) \quad (t \in [0, 1]), \quad (1.2)$$

$$P(X(t) \in dx) = P_t(dx) \quad (t \in [0, 1]), \quad (1.3)$$

where $\{W(t)\}_{0 \leq t \leq 1}$ is a $\sigma[X(s) : 0 \leq s \leq t]$ -Wiener process.

The first result was given by Carlen [2] (see also [23]). It was generalized, by Mikami [12], to the case where the second order differential operator has a variable coefficient. The further generalization and almost complete resolution was made by Cattiaux and Léonard [3-6] (see also [1, 13-15] for the related topics). But in these papers, they assumed that

$$\int_0^1 dt \int_{\mathbf{R}^d} |b(t, x)|^2 P_t(dx) < \infty \quad (1.4)$$

for some b for which (1.1) holds. This is called the **finite energy condition** for $\{P_t(dx)\}_{0 \leq t \leq 1}$.

Remark 1.1 *It is known that b is not unique for $\{P_t(dx)\}_{0 \leq t \leq 1}$ in (1.1) (see [12] or [3-6]).*

In this paper we consider Nelson's problem under a weaker assumption than (1.4): there exists $p > 1$ such that

$$\int_0^1 dt \int_{\mathbf{R}^d} |b(t, x)|^p P_t(dx) < \infty \quad (1.5)$$

for some b for which (1.1) holds. We call (1.5) the **generalized finite energy condition** for $\{P_t(dx)\}_{0 \leq t \leq 1}$.

Let $L(t, x; u) : [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d \mapsto [0, \infty)$ be continuous and be convex in u . Let \mathcal{A} denote the set of all \mathbf{R}^d -valued, continuous semimartingales $\{X(t)\}_{0 \leq t \leq 1}$ on a complete filtered probability space such that there exists a Borel measurable $\beta_X : [0, 1] \times C([0, 1]) \mapsto \mathbf{R}^d$ for which

- (i) $\omega \mapsto \beta_X(t, \omega)$ is $\mathcal{B}(C([0, t]))_+$ -measurable for all $t \in [0, 1]$, where $\mathcal{B}(C([0, t]))$ denotes the Borel σ -field of $C([0, t])$ and $\mathcal{B}(C([0, t]))_+$ denotes the left hand side limit of $t \mapsto \mathcal{B}(C([0, t]))$,
- (ii) $\{W_X(t) := X(t) - X(0) - \int_0^t \beta_X(s, X) ds\}_{0 \leq t \leq 1}$ is a $\sigma[X(s) : 0 \leq s \leq t]$ -Wiener process.

For P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$, put

$$V(P_0, P_1) := \inf \left\{ E \left[\int_0^1 L(t, X(t); \beta_X(t, X)) dt \right] \middle| \begin{array}{l} PX(t)^{-1} = P_t(t = 0, 1), \\ X \in \mathcal{A} \end{array} \right\}, \quad (1.6)$$

$$\begin{aligned} & v(P_0, P_1) & (1.7) \\ := & \inf \left\{ \int_0^1 \int_{\mathbf{R}^d} L(t, x; b(t, x)) P(t, dx) dt \middle| \begin{array}{l} P(t, dx) = P_t(dx)(t = 0, 1), \\ \{P(t, dx)\}_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbf{R}^d), \\ (b(t, x), P(t, dx)) \text{ satisfies (1.1)} \end{array} \right\}. \end{aligned}$$

In [12] where $u \mapsto L$ is quadratic, we proved and used the following:

$$V(P_0, P_1) = v(P_0, P_1). \quad (1.8)$$

Remark 1.2 *As a typical case, when $L = |u|^2$, the minimizer of $V(P_0, P_1)$ is known to be the h -path process for the space-time Brownian motion (see [7, 18] and the references therein). It is known that its zero-noise limit exists and is the unique minimizer of Monge's problem (see [16, 19]).*

In this paper we prove (1.8) for a more general function L by the duality theorem for V . To make the point clearer, we describe [18] briefly. For P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$, put

$$\mathcal{V}(P_0, P_1) := \sup \left\{ \int_{\mathbf{R}^d} \varphi(1, y) P_1(dy) - \int_{\mathbf{R}^d} \varphi(0, x) P_0(dx) \right\}, \quad (1.9)$$

where the supremum is taken over all classical solutions φ to the following Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} \frac{\partial \varphi(t, x)}{\partial t} + \frac{1}{2} \Delta \varphi(t, x) + H(t, x; D_x \varphi(t, x)) &= 0 \quad ((t, x) \in (0, 1) \times \mathbf{R}^d) \\ \varphi(1, \cdot) &\in C_b^\infty(\mathbf{R}^d) \end{aligned} \quad (1.10)$$

(see Lemma 3.1). Here for $(t, x, z) \in [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d$,

$$H(t, x; z) := \sup_{u \in \mathbf{R}^d} \{ \langle z, u \rangle - L(t, x; u) \}. \quad (1.11)$$

The following was proved in [18] and is called the duality theorem for the stochastic optimal control problem (1.6).

Theorem 1.1 (Duality Theorem) *Suppose that (A.1)-(A.4) in section 2 hold. Then for any P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$,*

$$V(P_0, P_1) = \mathcal{V}(P_0, P_1) \in [0, \infty]. \quad (1.12)$$

Suppose in addition that $V(P_0, P_1)$ is finite. Then $V(P_0, P_1)$ has a minimizer and for any minimizer $\{X(t)\}_{0 \leq t \leq 1}$ of $V(P_0, P_1)$,

$$\beta_X(t, X) = b_X(t, X(t)) := E[\beta_X(t, X) | (t, X(t))]. \quad (1.13)$$

Remark 1.3 *(1.12) can be considered as a counterpart in the stochastic optimal control theory of the duality theorem in the Monge-Kantorovich problem (see [10, 17, 21, 22] and the references therein).*

Using a similar result to (1.8) on small time intervals $\subset [0, 1]$, we prove that for $\mathbf{P} := \{P_t(dx)\}_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbf{R}^d)$,

$$\mathbf{V}(\mathbf{P}) = \mathbf{v}(\mathbf{P}), \quad (1.14)$$

where

$$\mathbf{V}(\mathbf{P}) := \inf \left\{ E \left[\int_0^1 L(t, X(t); \beta_X(t, X)) dt \right] \middle| PX(t)^{-1} = P_t(0 \leq t \leq 1), X \in \mathcal{A} \right\}, \quad (1.15)$$

$$\mathbf{v}(\mathbf{P}) := \inf \left\{ \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b(t, x)) P_t(dx) \mid b \text{ satisfies (1.1)} \right\}. \quad (1.16)$$

In particular, the existence of a minimizer of $\mathbf{V}(\mathbf{P})$ implies that of a semimartingale for which (1.2)-(1.3) hold. When $p = 2$ in (1.5), this semimartingale is Markovian. But we do not know if it is also true even when $1 < p < 2$. This is our future problem.

In section 2 we state our result which will be proved in section 4. Technical lemmas are given in section 3.

I would like to dedicate this paper to Professor Wendell H. Fleming on the occasion of his seventy seventh birthday. I would like to thank him for his constant encouragement since I was a student of his.

2 Main result.

In this section we state our result.

We state assumptions on L .

(A.1). There exists $p > 1$ such that

$$\liminf_{|u| \rightarrow \infty} \frac{\inf\{L(t, x; u) : (t, x) \in [0, 1] \times \mathbf{R}^d\}}{|u|^p} > 0.$$

(A.2).

$$\Delta L(\varepsilon_1, \varepsilon_2) := \sup \frac{L(t, x; u) - L(s, y; u)}{1 + L(s, y; u)} \rightarrow 0 \quad \text{as } \varepsilon_1, \varepsilon_2 \rightarrow 0,$$

where the supremum is taken over all (t, x) and $(s, y) \in [0, 1] \times \mathbf{R}^d$, for which $|t - s| \leq \varepsilon_1$, $|x - y| < \varepsilon_2$ and all $u \in \mathbf{R}^d$.

(A.3). (i) $L(t, x; u) \in C^3([0, 1] \times \mathbf{R}^d \times \mathbf{R}^d : [0, \infty))$,

(ii) $D_u^2 L(t, x; u)$ is positive definite for all $(t, x, u) \in [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d$,

(iii) $\sup\{L(t, x; 0) : (t, x) \in [0, 1] \times \mathbf{R}^d\}$ is finite,

(iv) $|D_x L(t, x; u)| / (1 + L(t, x; u))$ is bounded,

(v) $\sup\{|D_u L(t, x; u)| : (t, x) \in [0, 1] \times \mathbf{R}^d, |u| \leq R\}$ is finite for all $R > 0$.

(A.4). (i) $\Delta L(0, \infty)$ is finite, or (ii) $p = 2$ in (A.1).

Remark 2.1 (i). (A.3, ii) implies that $L(t, x; u)$ is strictly convex in u . (ii). $(1 + |u|^2)^{p/2}$ ($p > 1$) satisfies (A.1)-(A.3) and (A.4, i).

We state that (1.8) holds.

Theorem 2.1 *Suppose that (A.1)-(A.4) hold. Then for any P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$,*

$$V(P_0, P_1) = v(P_0, P_1) \in [0, \infty]. \quad (2.1)$$

The following is our main result (see (1.15)-(1.16) for notations).

Theorem 2.2 *Suppose that (A.1)-(A.4) hold. Then (i) for any $\mathbf{P} := \{P_t(dx)\}_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbf{R}^d)$,*

$$\mathbf{V}(\mathbf{P}) = \mathbf{v}(\mathbf{P}) \in [0, \infty]. \quad (2.2)$$

(ii) For any $\mathbf{P} := \{P_t(dx)\}_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbf{R}^d)$, for which $\mathbf{v}(\mathbf{P})$ is finite, there exist a unique minimizer $b_o(t, x)$ of $\mathbf{v}(\mathbf{P})$ and a minimizer $X \in \mathcal{A}$, of $\mathbf{V}(\mathbf{P})$. In particular, for any minimizer $X \in \mathcal{A}$, of $\mathbf{V}(\mathbf{P})$,

$$\beta_X(t, X) = b_o(t, X(t)) \quad (2.3)$$

and (1.2)-(1.3) with $b = b_o$ hold.

Remark 2.2 *If $\mathbf{v}(\mathbf{P})$ is finite, then the generalized finite energy condition (1.5) holds from (A.1).*

3 Lemmas.

In this section we give technical lemmas.

In the same way as to \mathcal{A} , we define the set of semimartingales \mathcal{A}_t in $C([t, 1])$. We recall the following result.

Lemma 3.1 ([8, p. 210, Remark 11.2]) *Suppose that (A.1) and (A.3) hold. Then for any $f \in C_b^\infty(\mathbf{R}^d)$, the HJB equation (1.10) with $\varphi(1, \cdot) = f$ has a unique solution $\varphi \in C^{1,2}([0, 1] \times \mathbf{R}^d) \cap C_b^{0,1}([0, 1] \times \mathbf{R}^d)$, which can be written as follows:*

$$\begin{aligned} \varphi(t, x) = \sup_{X \in \mathcal{A}_t} & \left\{ E[f(X(1)) | X(t) = x] \right. \\ & \left. - E \left[\int_t^1 L(s, X(s); \beta_X(s, X)) ds \middle| X(t) = x \right] \right\}, \end{aligned} \quad (3.1)$$

where for the maximizer $X \in \mathcal{A}_t$, the following holds:

$$\beta_X(s, X) = D_z H(s, X(s); D_x \varphi(s, X(s))).$$

Fix $P_0 \in \mathcal{M}_1(\mathbf{R}^d)$. For $f \in C_b(\mathbf{R}^d)$, put

$$V^*(f) := \sup_{P \in \mathcal{M}_1(\mathbf{R}^d)} \left\{ \int_{\mathbf{R}^d} f(x) P(dx) - V(P_0, P) \right\}, \quad (3.2)$$

$$v^*(f) := \sup_{P \in \mathcal{M}_1(\mathbf{R}^d)} \left\{ \int_{\mathbf{R}^d} f(x) P(dx) - v(P_0, P) \right\}. \quad (3.3)$$

The following lemma plays a crucial role in the proof of Theorem 2.1.

Lemma 3.2 (i) *Suppose that (A.3, i, ii) hold. Then for any Q_0 and $Q_1 \in \mathcal{M}_1(\mathbf{R}^d)$,*

$$V(Q_0, Q_1) \geq v(Q_0, Q_1). \quad (3.4)$$

(ii) *Suppose in addition that (A.1) and (A.3) hold. Then for any $f \in C_b^\infty(\mathbf{R}^d)$,*

$$V^*(f) \geq v^*(f). \quad (3.5)$$

(Proof) We first prove (i). For $X \in \mathcal{A}$ for which $E[\int_0^1 L(t, X(t); \beta_X(t, X)) dt]$ is finite and for which $PX(t)^{-1} = Q_t$ ($t = 0, 1$), $(b_X(t, x), P(X(t) \in dx))$ satisfies (1.1) with $(b(t, x), P_t(dx)) = (b_X(t, x), P(X(t) \in dx))$ (see (1.13) for notation). Indeed, for any $f \in C_b^{1,2}([0, 1] \times \mathbf{R}^d)$ and $t \in [0, 1]$, by Itô's formula,

$$\begin{aligned} & \int_{\mathbf{R}^d} f(t, x) P(X(t) \in dx) - \int_{\mathbf{R}^d} f(0, x) P(X(0) \in dx) \\ &= E[f(t, X(t)) - f(0, X(0))] \\ &= \int_0^t ds E \left[\frac{\partial f(s, X(s))}{\partial s} + \frac{1}{2} \Delta f(s, X(s)) + \langle \beta_X(s, X), D_x f(s, X(s)) \rangle \right] \\ &= \int_0^t ds E \left[\frac{\partial f(s, X(s))}{\partial s} + \frac{1}{2} \Delta f(s, X(s)) + \langle b_X(s, X(s)), D_x f(s, X(s)) \rangle \right] \\ &= \int_0^t ds \int_{\mathbf{R}^d} \left(\frac{\partial f(s, x)}{\partial s} + \frac{1}{2} \Delta f(s, x) + \langle b_X(s, x), D_x f(s, x) \rangle \right) P(X(s) \in dx). \end{aligned} \quad (3.6)$$

Hence, from Remark 2.1, (i), by Jensen's inequality,

$$\begin{aligned}
& E \left[\int_0^1 L(t, X(t); \beta_X(t, X)) dt \right] \\
& \geq E \left[\int_0^1 L(t, X(t); b_X(t, X(t))) dt \right] \\
& = \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b_X(t, x)) P(X(t) \in dx) \geq v(Q_0, Q_1).
\end{aligned} \tag{3.7}$$

Next we prove (ii). For φ in (3.1) and $\{(b(t, x), P(t, dx))\}_{0 \leq t \leq 1}$ for which $\{P(t, dx)\}_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ and (1.1) with $P(0, dx) = P_0$ holds,

$$\int_{\mathbf{R}^d} f(x) P(1, dx) - \int_{\mathbf{R}^d} \varphi(0, x) P_0(dx) \leq \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b(t, x)) P(t, dx). \tag{3.8}$$

Indeed, take $\psi \in C_o^\infty(\mathbf{R}^d : [0, \infty))$ for which $\psi(x) = 1$ ($|x| \leq 1$) and $\psi(x) = 0$ ($|x| \geq 2$), and put $\psi_R(x) := \psi(x/R)$ for $R > 0$. Then from (1.1),

$$\begin{aligned}
& \int_{\mathbf{R}^d} \psi_R(x) f(x) P(1, dx) - \int_{\mathbf{R}^d} \psi_R(x) \varphi(0, x) P(0, dx) \\
& = \int_0^1 dt \int_{\mathbf{R}^d} \psi_R(x) \left[\frac{\partial \varphi(t, x)}{\partial t} + \frac{1}{2} \Delta \varphi(t, x) + \langle b(t, x), D_x \varphi(t, x) \rangle \right] P(t, dx) \\
& \quad + \int_0^1 dt \int_{\mathbf{R}^d} \left[\langle D_x \psi_R(x), D_x \varphi(t, x) \rangle + \frac{1}{2} \Delta \psi_R(x) \varphi(t, x) \right. \\
& \quad \left. + \langle b(t, x), D_x \psi_R(x) \rangle \varphi(t, x) \right] P(t, dx).
\end{aligned} \tag{3.9}$$

Let $R \rightarrow \infty$. Then we obtain (3.8) from (1.10), (A.1) and Lemma 3.1.

Lemma 3.1 and (3.8) implies (ii). Indeed,

$$\begin{aligned}
v^*(f) & = \sup \left\{ \int_{\mathbf{R}^d} f(x) P(1, dx) - \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b(t, x)) P(t, dx) \right. \\
& \quad \left. P(0, dx) = P_0(dx), \{P(t, dx)\}_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbf{R}^d), \right. \\
& \quad \left. (b(t, x), P(t, dx)) \text{ satisfies (1.1)}. \right\} \\
& \leq \int_{\mathbf{R}^d} \varphi(0, x) P_0(dx) \quad (\text{from (3.8)})
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&= \sup \left\{ E \left[f(X(1)) - \int_0^1 L(t, X(t); \beta_X(t, X)) dt \right] \right\} \\
&\quad \left. PX(0)^{-1} = P_0, X \in \mathcal{A} \right\} \quad (\text{from Lemma 3.1}) \\
&= V^*(f). \square
\end{aligned}$$

Let $(\Omega, \mathbf{B}, \{\mathbf{B}_t\}_{t \geq 0}, P)$ be a complete filtered probability space, X_o be a (\mathbf{B}_0) -adapted random variable, and $\{W(t)\}_{t \geq 0}$ denote a d -dimensional (\mathbf{B}_t) -Wiener process for which $W(0) = o$ (see e.g., [11]). For a \mathbf{R}^d -valued, (\mathbf{B}_t) -progressively measurable stochastic process $\{u(t)\}_{0 \leq t \leq 1}$, put

$$X^u(t) = X_o + \int_0^t u(s) ds + W(t) \quad (t \in [0, 1]). \quad (3.11)$$

Then the following is known.

Lemma 3.3 *Suppose that $E[\int_0^1 |u(t)| dt]$ is finite. Then $\{X^u(t)\}_{0 \leq t \leq 1} \in \mathcal{A}$ and*

$$\beta_{X^u}(t, X^u) = E[u(t) | X^u(s), 0 \leq s \leq t] \quad (3.12)$$

(see [11, p. 270]). Besides, by Jensen's inequality,

$$E \left[\int_0^1 L(t, X^u(t); u(t)) dt \right] \geq E \left[\int_0^1 L(t, X^u(t); \beta_{X^u}(t, X^u)) dt \right]. \quad (3.13)$$

For $\mathbf{P} := \{P_t(dx)\}_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ and $n \geq 1$, put

$$\begin{aligned}
V_n(\mathbf{P}) &:= \inf \left\{ E \left[\int_0^1 L(t, X(t); \beta_X(t, X)) dt \right] \right\} \\
&\quad \left. PX(t)^{-1} = P_t \left(t = \frac{i}{2^n}, i = 0, \dots, 2^n \right), X \in \mathcal{A} \right\},
\end{aligned} \quad (3.14)$$

$$\begin{aligned}
v_n(\mathbf{P}) &:= \inf \left\{ \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b(t, x)) P(t, dx) \right\} \\
&\quad \left. \begin{aligned} P(t, dx) &= P_t(dx) \left(t = \frac{i}{2^n}, i = 0, \dots, 2^n \right), \\ \{P(t, dx)\}_{0 \leq t \leq 1} &\subset \mathcal{M}(\mathbf{R}^d), (b(t, x), P(t, dx)) \text{ satisfies (1.1)} \end{aligned} \right\}.
\end{aligned} \quad (3.15)$$

Then we have

Lemma 3.4 *Suppose that (A.1)-(A.4) hold. Then for any $\mathbf{P} := \{P_t(dx)\}_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ and $n \geq 1$,*

$$v_n(\mathbf{P}) = V_n(\mathbf{P}). \quad (3.16)$$

(Proof) For $i = 0, \dots, 2^n - 1$, put

$$V_{n,i}(\mathbf{P}) := \inf \left\{ E \left[\int_0^{\frac{1}{2^n}} L(t, X(t); \beta_X(t, X)) dt \right] \middle| \right. \\ \left. PX(t)^{-1} = P_{t+\frac{i}{2^n}} \left(t = 0, \frac{1}{2^n} \right), X \in \mathcal{A} \right\}, \quad (3.17)$$

$$v_{n,i}(\mathbf{P}) \quad (3.18) \\ := \inf \left\{ \int_0^{\frac{1}{2^n}} dt \int_{\mathbf{R}^d} L(t, x; b(t, x)) P(t, dx) \middle| \right. \\ \left. P(t, dx) = P_{t+\frac{i}{2^n}}(dx) \left(t = 0, \frac{1}{2^n} \right), \{P(t, dx)\}_{0 \leq t \leq \frac{1}{2^n}} \subset \mathcal{M}(\mathbf{R}^d), \right. \\ \left. (b(t, x), P(t, dx)) \text{ satisfies (1.1) on } [0, 1/2^n] \right\}.$$

Then, from Theorem 2.1,

$$v_n(\mathbf{P}) = \sum_{i=0}^{2^n-1} v_{n,i}(\mathbf{P}) = \sum_{i=0}^{2^n-1} V_{n,i}(\mathbf{P}). \quad (3.19)$$

Since $V_n(\mathbf{P}) \geq v_n(\mathbf{P})$ from (3.6)-(3.7), we only have to prove the following:

$$\sum_{i=0}^{2^n-1} V_{n,i}(\mathbf{P}) \geq V_n(\mathbf{P}). \quad (3.20)$$

Suppose that the left hand side of (3.20) is finite. For $i = 0, \dots, 2^n - 1$, take a minimizer $X_{n,i}$ of $V_{n,i}(\mathbf{P})$ (see Theorem 1.1), and put

$$P_{n,i} := PX_{n,i} \left(\cdot - \frac{i}{2^n} \right)^{-1} \quad \text{on } (C([\frac{i}{2^n}, \frac{i+1}{2^n}] : \mathbf{R}^d), \mathcal{B}(C([\frac{i}{2^n}, \frac{i+1}{2^n}] : \mathbf{R}^d))), \quad (3.21)$$

$$\begin{aligned}
P_n\left(dX|_{C([0,1]:\mathbf{R}^d)}\right) &:= P_{n,0}\left(dX|_{C([0,\frac{1}{2^n}]:\mathbf{R}^d)}\right) \\
&\quad \times \prod_{i=1}^{2^n-1} P_{n,i}\left(dX|_{C([\frac{i}{2^n},\frac{i+1}{2^n}]:\mathbf{R}^d)}\middle|X_{n,i}\left(\frac{i}{2^n}\right) = X\left(\frac{i}{2^n}\right)\right)
\end{aligned} \tag{3.22}$$

on $(C([0,1]:\mathbf{R}^d), \mathcal{B}(C([0,1]:\mathbf{R}^d)))$. Under the completion of this measure, the coordinate process $\{X_n(t)\}_{0 \leq t \leq 1}$ satisfies the following:

$$X_n(t) = X_n(0) + \sum_{i=0}^{2^n-1} \int_{\min(\frac{i}{2^n}, t)}^{\min(\frac{i+1}{2^n}, t)} b_{n,i}\left(s - \frac{i}{2^n}, X_n(s)\right) ds + W_{X_n}(t) \quad (0 \leq t \leq 1), \tag{3.23}$$

where $b_{n,i}$ denotes the drift vector of $X_{n,i}$ (see Theorem 1.1). In particular, $PX_n(t)^{-1} = P_t$ ($t = i/2^n, i = 0, \dots, 2^n$), which implies (3.20). \square

4 Proofs.

In this section we prove our results given in section 2.

When $L = |u|^2$, the following proof extremely simplifies that of [12, Lemma 2.5].

(Proof of Theorem 2.1). Lemma 3.2, (i) and the following complete the proof:

$$\begin{aligned}
&v(P_0, P_1) \tag{4.1} \\
&\geq \sup_{f \in C_b^\infty(\mathbf{R}^d)} \left\{ \int_{\mathbf{R}^d} f(x) P_1(dx) - v^*(f) \right\} \quad (\text{from (3.3)}) \\
&\geq \sup_{f \in C_b^\infty(\mathbf{R}^d)} \left\{ \int_{\mathbf{R}^d} f(x) P_1(dx) - V^*(f) \right\} \quad (\text{from Lemma 3.2, (ii)}) \\
&= V(P_0, P_1) \quad (\text{from Theorem 1.1 (see (3.10))}). \square
\end{aligned}$$

(Proof of Theorem 2.2). We first prove (i). From (3.6)-(3.7), $\mathbf{V}(\mathbf{P}) \geq \mathbf{v}(\mathbf{P})$. Therefore we only have to show that

$$\mathbf{v}(\mathbf{P}) \geq \mathbf{V}(\mathbf{P}). \tag{4.2}$$

Suppose that $\mathbf{v}(\mathbf{P})$ is finite. Then, from Lemma 3.4,

$$\mathbf{v}(\mathbf{P}) \geq v_n(\mathbf{P}) = V_n(\mathbf{P}) \quad (4.3)$$

and X_n constructed in (3.23) is a minimizer of $V_n(\mathbf{P})$.

Let b_n denote the drift vector of $\{X_n(t)\}_{0 \leq t \leq 1}$. It is easy to see that $\{(X_n(t), \int_0^t b_n(s, X_n(s)) ds) : t \in [0, 1]\}_{n \geq 1}$ is tight in $C([0, 1] : \mathbf{R}^{2d})$ from (A.1) (see [23, Theorem 3] or [9]). Take a weakly convergent subsequence $\{(X_{n_k}(t), \int_0^t b_{n_k}(s, X_{n_k}(s)) ds) : t \in [0, 1]\}_{k \geq 1}$ such that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} E \left[\int_0^1 L(t, X_n(t); b_n(s, X_n(s))) dt \right] \\ &= \lim_{k \rightarrow \infty} E \left[\int_0^1 L(t, X_{n_k}(t); b_{n_k}(s, X_{n_k}(s))) dt \right]. \end{aligned} \quad (4.4)$$

Let $\{(X(t), A(t))\}_{t \in [0, 1]}$ denote the limit of $\{(X_{n_k}(t), \int_0^t b_{n_k}(s, X_{n_k}(s)) ds) : t \in [0, 1]\}_{k \geq 1}$ as $k \rightarrow \infty$. Then $\{X(t) - X(0) - A(t)\}_{t \in [0, 1]}$ is a $\sigma[X(s) : 0 \leq s \leq t]$ -Wiener process and $\{A(t)\}_{t \in [0, 1]}$ is absolutely continuous (see [23, Theorem 5] or [9]). We can also prove, in the same way as in the proof of [15, (3.17)], the following: from (4.3)-(4.4), (A.2) and (A.3, ii) (see Remark 2.1, (i)),

$$\begin{aligned} \mathbf{v}(\mathbf{P}) &\geq \liminf_{n \rightarrow \infty} E \left[\int_0^1 L(t, X_n(t); b_n(t, X_n(t))) dt \right] \\ &\geq E \left[\int_0^1 L\left(t, X(t); \frac{dA(t)}{dt}\right) dt \right] \\ &\geq \tilde{E} \left[\int_0^1 L\left(t, X(t); \beta_X(t, X)\right) dt \right] \quad (\text{from Lemma 3.3}) \\ &\geq \mathbf{V}(\mathbf{P}). \end{aligned} \quad (4.5)$$

Here \tilde{E} denotes the mean value by the completion of $PX(\cdot)^{-1}$ and we used the fact that $P(X(t) \in dx) = P_t(dx)$ for all $t \in [0, 1]$. Indeed,

$$P(X(t) \in dx) = \lim_{n \rightarrow \infty} P\left(X\left(\frac{[2^n t]}{2^n}\right) \in dx\right) \quad \text{weakly,}$$

$$P\left(X\left(\frac{[2^n t]}{2^n}\right) \in dx\right) = P_{\frac{[2^n t]}{2^n}}(dx) \rightarrow P_t(dx) \quad \text{as } n \rightarrow \infty \text{ weakly.}$$

Next we prove (ii). Suppose that $\mathbf{v}(\mathbf{P})$ is finite. Then (2.2) and (4.5) show the existence of a minimizer X of $\mathbf{V}(\mathbf{P})$. In the same way as in (3.7), Theorem 2.2, (i) and the strict convexity of $u \mapsto L(t, x; u)$ (see Remark 2.1, (i)) imply that $\beta_X(t, X) = b_X(t, X(t))$ and $b_X(t, x)$ is a minimizer of $\mathbf{v}(\mathbf{P})$.

Let b_1 and b_2 be minimizers of $\mathbf{v}(\mathbf{P})$. Then for any $\lambda \in (0, 1)$, $\lambda b_1(t, x) + (1 - \lambda)b_2(t, x)$ satisfies (1.1), and

$$\begin{aligned} & \mathbf{v}(\mathbf{P}) && (4.6) \\ & \leq \int_0^1 \int_{\mathbf{R}^d} L(t, x; \lambda b_1(t, x) + (1 - \lambda)b_2(t, x)) P_t(dx) \\ & \leq \lambda \int_0^1 \int_{\mathbf{R}^d} L(t, x; b_1(t, x)) P_t(dx) + (1 - \lambda) \int_0^1 \int_{\mathbf{R}^d} L(t, x; b_2(t, x)) P_t(dx) \\ & = \mathbf{v}(\mathbf{P}). \end{aligned}$$

The strict convexity of $u \mapsto L(t, x; u)$ implies the uniqueness of a minimizer of $\mathbf{v}(\mathbf{P})$. \square

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