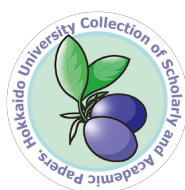




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Mean-field behavior for the survival probability and the percolation point-to-surface connectivity

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Abstract

We consider the critical survival probability (up to time t) for oriented percolation and the contact process, and the point-to-surface (of the ball of radius t) connectivity for critical percolation. Let θ_t denote both quantities. We prove in a unified fashion that, if θ_t exhibits a power law and both the two-point function and its restricted version exhibit the same mean-field behavior, then $\theta_t \asymp t^{-1}$ for the time-oriented models with $d > 4$ and $\theta_t \asymp t^{-2}$ for percolation with $d > 7$.

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1 Introduction

Percolation, oriented percolation and the contact process are known to exhibit a phase transition. Various interesting properties around the model-dependent critical point p_c have been studied and revealed, but still there are many open problems. One of the most important problems is to investigate *critical exponents* that characterize singular behavior of observables. Some of them were identified in certain situations.

In this paper, we consider the critical survival probability up to time t for oriented percolation and the contact process, and the probability of the origin $o \in \mathbb{Z}^d$ being connected to the surface of the ball of radius t , centered at the origin, for critical percolation. Since the survival probability is a time-oriented version of the point-to-surface connectivity, we denote both quantities by θ_t . It is believed that θ_t exhibits a power law: $\theta_t \approx t^{-1/\delta_r}$ as $t \rightarrow \infty$ (in some

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appropriate sense). In the percolation school, δ_r is sometimes called the *one-arm exponent*. Lawler, Schramm and Werner proved $\delta_r = 48/5$ for the two-dimensional site percolation on the triangular lattice, using the estimates for the stochastic Loewner evolution with parameter 6 (see [21] for a precise statement). Except for this result, there has been no proof of existence of δ_r , or identification of its values for finite-range models in mathematically rigorous manner, even in high dimensions.

In contrast, the behavior of the two-point function is well-understood in high dimensions. For percolation, the two-point function at p_c , denoted $\tau(x)$, is the probability of $o, x \in \mathbb{Z}^d$ being connected to each other, defined at p_c . It has been proved that $\tau(x) \asymp |x|^{-(d-2+\eta)}$ as $|x| \rightarrow \infty$ with $\eta = 0$ when $d > 6$ and the number N of neighbors is sufficiently large [9, 10], where “ $\asymp$ ” means that the left-hand side divided by the right-hand side is bounded away from zero and infinity. For the time-oriented models, the two-point function at p_c , denoted $\tau_t(x)$, is, in terms of the contact process, the probability of $x \in \mathbb{Z}^d$ being infected at time t by the infected individual at $o \in \mathbb{Z}^d$ at time 0, defined at p_c . It has been proved that $\sup_x \tau_t(x) \asymp t^{-d/\alpha}$, $\hat{\tau}_t \equiv \sum_x \tau_t(x) \asymp t^\eta$ and $\sum_x |x|^2 \tau_t(x) / \hat{\tau}_t \asymp t^{2\nu}$ as $t \rightarrow \infty$, with $\alpha = 2$, $\eta = 0$ and $\nu = 1/2$, when the spatial dimension d is above 4 and N is sufficiently large [17, 19, 20, 23]. These dimension-independent values of the critical exponents are equal to the values for branching random walk (the *mean-field model*). Let ρ ($\equiv 1/\delta_r$) be defined by $\theta_t \asymp t^{-\rho}$ as $t \rightarrow \infty$. It is not so hard to see that $\eta = 0$ implies $\rho \leq 2$ for percolation with $d > 4$ and $\rho \leq 1$ for the time-oriented models with $d \geq 1$ (see Section 3.1), where the upper bounds are the mean-field values of ρ .

On the other hand, the critical exponents are known to satisfy the so-called *hyperscaling inequalities*, e.g., $d - 2 + \eta \geq 2\rho$ for percolation [27] and $d\nu \geq \eta + 2\rho$ for the time-oriented models [25, (5.2) and (5.4)], where the critical exponents were defined in a wider sense. Other hyperscaling inequalities were also derived in [7, 25, 27]. By those inequalities, the mean-field values are known to be incompatible with $d < 6$ for percolation and with $d < 4$ for the time-oriented models. These threshold dimensions are called the *upper critical dimensions* for the corresponding models.

In this paper, we prove in a unified way that ρ takes on the mean-field values for the time-oriented models with $d > 4$ and for percolation with $d > 7$, if ρ exists and both the two-point function and its restricted version exhibit the same mean-field behavior (see Assumption 2.1). The assumption on the restricted two-point function is expected to hold above the upper critical dimension for each model, but is still insufficient to extend $\rho = 2$ for percolation down to $d > 6$. For sufficiently spread-out oriented percolation with $d > 4$, the asymptotic behavior of θ_t with $\rho = 1$ will be reported in [15, 16], without any assumption on the restricted two-point function. In this respect, our results are not so strong as the results in [15, 16] for oriented percolation. However, the approach reported in this paper is short and intuitive, and more importantly, gives a unified approach for both the time-oriented models and percolation. We expect that, with the help of the *random-current representation* [1], our unified approach could be applied to the single-spin expectation $\langle \sigma_o \rangle_t$ for Ising ferromagnets in the ball of radius t (with plus-boundary condition), and result in the mean-field behavior, i.e., $\langle \sigma_o \rangle_t \asymp t^{-1}$ as $t \rightarrow \infty$, at the critical temperature in high dimensions. This will be discussed in [26].

We organize the rest of this paper as follows. In Section 2, we define the models and state the main result. A brief explanation of the proof is given at the end of Section 2, and the detailed proof is given in Section 3.

2 Models and the results

2.1 Models

We consider the d -dimensional integer lattice $\mathbb{Z}^d$ as space. For $L \geq 1$, let

$$\Omega = \{x \in \mathbb{Z}^d : 0 < |x| \leq L\}, \quad D(x) = N^{-1} \mathbb{1}_{\{x \in \Omega\}}, \quad (2.1)$$

where $|x|$ is the Euclidean norm of x , N is the cardinality of Ω , and $\mathbb{1}_{\{\dots\}}$ is the indicator function. The model with $L = 1$ is the *nearest-neighbor model*, where $N = 2d$. We call the model with $L > 1$ the *spread-out model*, where $N = O(L^d)$ (see, e.g., [17] for a more general definition). Our models are defined in terms of D as follows.

Percolation. A bond $\{x, y\}$ is an unordered pair of distinct sites in $\mathbb{Z}^d$, and is *occupied* with probability $p D(y - x)$ and *vacant* with probability $1 - p D(y - x)$, independently of the other bonds, where $p \in [0, N]$ is the expected number of occupied bonds growing out of a single site. We denote by $\mathbb{P}_p$ the probability distribution for the bond variables. We say that x is *connected to* y , and write $x \leftrightarrow y$, if either $x = y$ or there is a path of occupied bonds between x and y . We define $\mathcal{C}(x) = \{y \in \mathbb{Z}^d : x \leftrightarrow y\}$. For $\mathcal{Z} \subset \mathbb{Z}^d$, we write $\{x \leftrightarrow \mathcal{Z}\} = \{\mathcal{C}(x) \cap \mathcal{Z} \neq \emptyset\}$.

It is known that there is a critical value $p_c = p_c(d, L) \geq 1$ such that $\sum_x \mathbb{P}_p(o \leftrightarrow x)$ is finite if and only if $p < p_c$ and diverges as $p \uparrow p_c$. Let

$$\mathcal{B}_t = \{x \in \mathbb{Z}^d : |x| \leq t\}, \quad \partial\mathcal{B}_t = \{x \in \mathbb{Z}^d : t \leq |x| \leq t + L\}. \quad (2.2)$$

and define the two-point function and the point-to-surface connectivity at p_c as

$$\tau(x) = \mathbb{P}_{p_c}(o \leftrightarrow x), \quad \theta_t = \mathbb{P}_{p_c}(o \leftrightarrow \partial\mathcal{B}_t). \quad (2.3)$$

We are interested in the critical exponents η and ρ , defined by

$$\tau(x) \asymp \|x\|^{-(d-2+\eta)}, \quad \theta_t \asymp \|t\|^{-\rho}, \quad (2.4)$$

where $f \asymp g$ means that f/g is bounded away from zero and infinity, and where $\|\cdot\| = |\cdot| \vee 1$. Note that $\|\cdot\|$ is not a norm on $\mathbb{R}^d$, but it satisfies the following properties: for $x, y \in \mathbb{R}^d$ and $r > 0$,

$$\|x + y\| \leq \|x\| + \|y\|, \quad \|rx\| \begin{cases} \leq r\|x\|, & \text{if } r \geq 1, \\ \geq r\|x\|, & \text{if } r < 1. \end{cases} \quad (2.5)$$

We also note that the above definition of ρ is based on the assumption that θ_t decays as $t \rightarrow \infty$. This has been confirmed only when $d = 2$ or $d \geq 19$ with $L = 1$, and $d > 6$ with $L \gg 1$ (see, e.g., [8, 12]).

It has been proved that $\eta = 0$ for the nearest-neighbor model with $d \gg 6$ [9] and for the spread-out model with $d > 6$ and $L \gg 1$ [10]. The critical exponent η is believed to be independent of the range L , as long as it is finite (*universality*), and thus is expected to be zero for all $d > 6$ and $L \geq 1$. This dimension-independent value of η equals the corresponding value for the mean-field model. Various other critical exponents are also known to take on their respective mean-field values, if (see [3] and references therein)

$$\nabla_\ell \equiv \sup_{x \notin \mathcal{B}_\ell} (\tau * D * \tau * \tau)(x) \rightarrow 0, \quad \text{as } \ell \rightarrow \infty, \quad (2.6)$$

where “ $*$ ” represents a convolution in $\mathbb{Z}^d$. With the help of [10, Proposition 1.7(i)], $\eta = 0$ implies $\nabla_\ell = O(\|\ell\|^{-(d-6)})$ if $d > 6$, and thus implies the mean-field values for all the other critical exponents, except for ρ until now.

Oriented percolation and the contact process. We begin with oriented percolation. A bond $((x, t), (y, t+1))$ is an ordered pair of sites in $\mathbb{Z}^d \times \mathbb{Z}_+$, and is *occupied* with probability $p D(y-x)$ and *vacant* with probability $1-p D(y-x)$, independently of the other bonds, where $p \in [0, 1]$. We say that (x, s) is *connected to* (y, t) , and write $(x, s) \rightarrow (y, t)$, if either $(x, s) = (y, t)$ or there is an oriented path of occupied bonds from (x, s) to (y, t) . Let $\mathcal{C}(x, s) = \{(y, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ : (x, s) \rightarrow (y, t)\}$. For $\mathcal{Z} \subset \mathbb{Z}^d \times \mathbb{Z}_+$, we define $\{(x, s) \rightarrow \mathcal{Z}\} = \{\mathcal{C}(x, s) \cap \mathcal{Z} \neq \emptyset\}$.

The contact process is a model for the spread of an infection in $\mathbb{Z}^d$, and is regarded as continuous-time oriented percolation in $\mathbb{Z}^d \times \mathbb{R}_+$, via the following *graphical representation*. Along each time line $\{x\} \times \mathbb{R}_+$, we place points in the manner of a Poisson process with intensity 1, independently of the other time lines. For each ordered pair of distinct time lines from $\{x\} \times \mathbb{R}_+$ to $\{y\} \times \mathbb{R}_+$, we place oriented bonds $((x, t), (y, t))$, $t \geq 0$, in the manner of a Poisson process with intensity $p D(y-x)$, independently of the other Poisson processes, where $p \geq 0$ is the infection rate. We say that (x, s) is *connected to* (y, t) , and write $(x, s) \rightarrow (y, t)$, if either $(x, s) = (y, t)$ or there is an oriented path in $\mathbb{Z}^d \times \mathbb{R}_+$ from (x, s) to (y, t) using the Poisson bonds and time-line segments traversed in the increasing-time direction without traversing the Poisson points. We define $\mathcal{C}(x, s)$ and $\{(x, s) \rightarrow \mathcal{Z}\}$ for $\mathcal{Z} \subset \mathbb{Z}^d \times \mathbb{R}_+$ similarly to oriented percolation.

We denote by $\mathbb{P}_p$ the probability distributions for these time-oriented models. It is known that there is a critical value $p_c = p_c(d, L) \geq 1$, depending on the models, such that the sum over $t \in \mathbb{Z}_+$ of $\sum_x \mathbb{P}_p((o, 0) \rightarrow (x, t))$ for oriented percolation, or the integral of $\sum_x \mathbb{P}_p((o, 0) \rightarrow (x, t))$ with respect to $t \in \mathbb{R}_+$ for the contact process, is finite if and only if $p < p_c$ and diverges as $p \uparrow p_c$. Let

$$\mathcal{B}_t = \mathbb{Z}^d \times [0, t], \quad \partial \mathcal{B}_t = \mathbb{Z}^d \times \{t\}, \quad (2.7)$$

and define the two-point function and the survival probability at p_c as

$$\tau_t(x) = \mathbb{P}_{p_c}((o, 0) \rightarrow (x, t)), \quad \theta_t = \mathbb{P}_{p_c}((o, 0) \rightarrow \partial \mathcal{B}_t). \quad (2.8)$$

We are interested in the critical exponents α , η , ν and ρ , defined by

$$\bar{\tau}_t \equiv \sup_{x \in \mathbb{Z}^d} \tau_t(x) \asymp \|t\|^{-d/\alpha}, \quad \hat{\tau}_t \equiv \sum_{x \in \mathbb{Z}^d} \tau_t(x) \asymp \|t\|^\eta, \quad (2.9)$$

$$\sum_{x \in \mathbb{Z}^d} |x|^2 \frac{\tau_t(x)}{\hat{\tau}_t} \asymp \|t\|^{2\nu}, \quad \theta_t \asymp \|t\|^{-\rho}, \quad (2.10)$$

where, by analogy, we used the same letters η and ρ for the critical exponents of the spatial sum of the two-point function and the survival probability, respectively.

It has been proved that $(\alpha, \eta, \nu) = (2, 0, \frac{1}{2})$ for the time-oriented models with $d > 4$ and $L \gg 1$ [17, 20]. The same result except for $\alpha = 2$ was proved in [23] for nearest-neighbor oriented percolation with $d \gg 4$, but there have been no results on this set of exponents for the nearest-neighbor contact process. Other critical exponents for both the nearest-neighbor and spread-out time-oriented models are known to take on their respective mean-field values, if (see [4] and references therein)

$$\nabla_\ell \equiv \sup_{\substack{x: |x| \geq \ell \\ t \geq 0}} \nabla(x, t) \rightarrow 0, \quad \text{as } \ell \rightarrow \infty, \quad (2.11)$$

where, for oriented percolation,

$$\nabla(x, t) = \sum_{\substack{s, s' \in \mathbb{Z}_+ \\ t \leq s' \leq s}} \sum_{y \in \mathbb{Z}^d} \tau_{s+1}(y) (\tau_{s-s'} * D * \tau_{s'-t})(y-x), \quad (2.12)$$

and for the contact process,

$$\nabla(x, t) = \int_t^\infty ds \int_t^s ds' \sum_{y \in \mathbb{Z}^d} \tau_s(y) (\tau_{s-s'} * D * \tau_{s'-t})(y - x). \quad (2.13)$$

Since the range of the set of infected sites almost surely grows at most linearly [5], $(\alpha, \eta) = (2, 0)$ implies $\nabla_\ell = O(\|\ell\|^{-(d-4)/2})$ if $d > 4$, and thus implies the mean-field values for all the other critical exponents than ρ .

2.2 Results

In this paper, we prove in a unified fashion for all three models that the mean-field behavior for the two-point function implies the mean-field values of ρ , assuming existence of ρ and the following assumption.

Assumption 2.1. *There are positive constants $C_1 = C_1(d, L)$ and $C_2 = C_2(d, L)$ that are independent of t such that, for the time-oriented models,*

$$\sum_{(x,s) \in \mathcal{B}_{t/2}} \mathbb{P}_{p_c}((o, 0) \rightarrow (x, s), (o, 0) \not\rightarrow \partial \mathcal{B}_t) \geq C_1 \|t\|, \quad (2.14)$$

and for percolation,

$$\sum_{x \in \mathcal{B}_{t/2+L}} \mathbb{P}_{p_c}(o \leftrightarrow x, o \not\rightarrow \partial \mathcal{B}_t) \geq C_2 \|t\|^2, \quad (2.15)$$

where $\mathcal{B}_{t/2+L} = \mathcal{B}_{t/2} \cup \partial \mathcal{B}_{t/2}$.

If $\eta = 0$, then we obtain $\sum_{s \leq t/2} \hat{\tau}_s \asymp \|t\|$ for the time-oriented models with $d \geq 1$ and $\sum_{x \in \mathcal{B}_{t/2+L}} \tau(x) \asymp \|t\|^2$ for percolation with $d \geq 2$. Therefore, Assumption 2.1 states, in a weak sense, that the above restricted two-point functions exhibit the same mean-field behavior as the unrestricted two-point functions.

Theorem 2.2. *Suppose that $\eta = 0$ and $\alpha = 2$ (the latter is only for the time-oriented models). If ρ exists and Assumption 2.1 holds, then $\rho = 1$ for the time-oriented models with $d > 4$ and $\rho = 2$ for percolation with $d > 7$.*

We briefly explain the main idea of the proof. It is easy to show that $\eta = 0$ implies $\rho \leq 1$ for the time-oriented models with $d \geq 1$ and $\rho \leq 2$ for percolation with $d > 4$ (see Section 3.1). It thus suffices to prove the opposite inequalities for ρ . Let us consider typical configurations for θ_t . When $t \gg 1$, there may be a *pivotal bond* for the connection from the origin to the boundary $\partial \mathcal{B}_t$. We take notice of the *last* pivotal bond b , where we have a connection from the origin to the first endpoint of b and two disjoint connections from the second endpoint of b to $\partial \mathcal{B}_t$ (see Figure 1). If we could bound the probability of these configurations from below by θ_t^2 times the sum of the *unrestricted* two-point function (over $b = (\underline{b}, \bar{b})$ with $\bar{b} \in \mathcal{B}_{t/2}$, as in Figure 1), then $\eta = 0$ implies

$$t^{-\rho} \geq \begin{cases} ct^{1-2\rho}, & \text{for the time-oriented models,} \\ ct^{2-2\rho}, & \text{for percolation,} \end{cases} \quad (2.16)$$

for some positive constant c , and thus $\rho \geq 1$ for the time-oriented models and $\rho \geq 2$ for percolation.

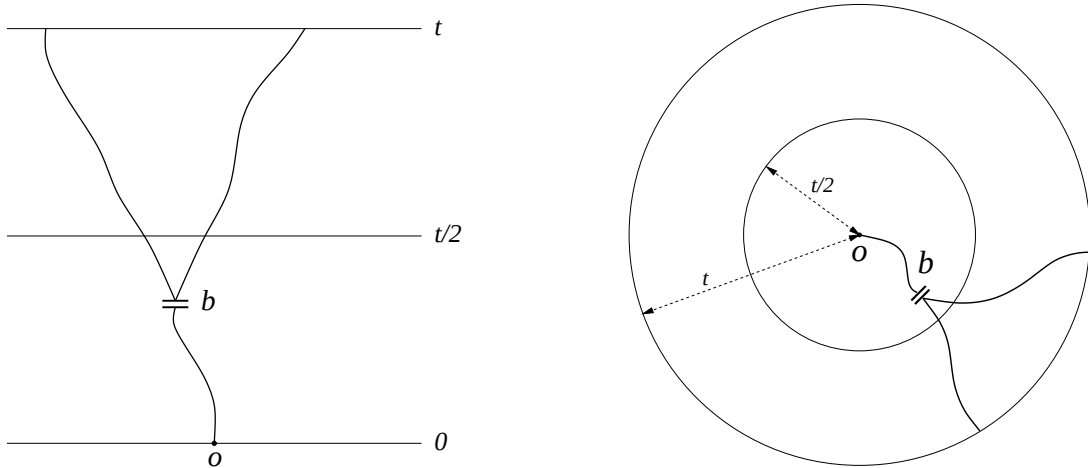


Figure 1: Typical configurations for θ_t .

To realize the above idea, we have to control the correction. As we will show in Section 3.2, most error terms can be made small by letting $\nabla_\ell \ll 1$ and $t \gg 1$ in high dimensions. However, the correction due to the above approximation using the unrestricted two-point function cannot be controlled by a finite number of applications of the *BK inequality* (see, e.g., [6, 8]), and here we will use Assumption 2.1. The desired asymptotic behavior of θ_t for spread-out oriented percolation with $d > 4$ and $L \gg 1$ will be reported in [15, 16], with no assumption on the restricted two-point function. The proof in [15, 16] is based on the *lace expansion* for θ_t , and the difference between the restricted and unrestricted two-point functions is efficiently taken into account along the expansion. Our proof of Theorem 2.2 does not depend on the full expansion as in [15, 16], and Assumption 2.1 is inevitable.

We remark that Assumption 2.1 is still insufficient to fully control the boundary effect and thus to obtain $\rho = 2$ for percolation with $d > 6$. To improve the result down to $d > 6$, we may also need some information on the restricted two-point function close to the boundary (see Remark at the end of Section 3.2).

3 Proofs

We prove Theorem 2.2 in two steps. First, in Section 3.1, we prove that $\eta = 0$ implies $\rho \leq 1$ for the time-oriented models with $d \geq 1$ and $\rho \leq 2$ for percolation with $d > 4$. Then, in Section 3.2, we prove that $\eta = 0$ and $\alpha = 2$ (the latter is only for the time-oriented models) imply the opposite inequalities for ρ , if $d > 4$ for the time-oriented models and $d > 7$ for percolation, assuming existence of ρ and Assumption 2.1.

In the rest of this paper, we omit the subscript p_c and write $\mathbb{E}$ for the expectation with respect to $\mathbb{P} = \mathbb{P}_{p_c}$. We will use c to denote a finite positive constant which may depend on d and L , but whose exact value is unimportant and may change from line to line.

3.1 Proof of the upper bound

Proof for the time-oriented models. Let

$$I_t = \mathbb{1}_{\{(o,0) \rightarrow \partial \mathcal{B}_t\}}, \quad X_t = \sum_{x \in \mathbb{Z}^d} \mathbb{1}_{\{(o,0) \rightarrow (x,t)\}}, \quad (3.1)$$

so that $\mathbb{E}(I_t) = \theta_t$ and $\mathbb{E}(X_t) = \hat{\tau}_t$. By the Schwarz inequality, we obtain

$$\hat{\tau}_t^2 = \mathbb{E}(I_t X_t)^2 \leq \mathbb{E}(I_t^2) \mathbb{E}(X_t^2) = \theta_t \sum_{x,y} \mathbb{P}((o,0) \rightarrow (x,t), (o,0) \rightarrow (y,t)). \quad (3.2)$$

If $(o,0) \rightarrow (x,t)$ and $(o,0) \rightarrow (y,t)$ occur simultaneously, then there exists a $(z,s) \in \mathcal{B}_t$ such that $(o,0) \rightarrow (z,s)$ occurs and that $(z,s) \rightarrow (x,t)$ and $(z,s) \rightarrow (y,t)$ occur *disjointly*, i.e., on disjoint sets of bonds. Using the Markov property, the BK inequality and $\eta = 0$, we can bound the sum in (3.2) by

$$\int_0^t ds \sum_{x,y,z \in \mathbb{Z}^d} \tau_s(z) \tau_{t-s}(x-z) \tau_{t-s}(y-z) = \int_0^t ds \hat{\tau}_s \hat{\tau}_{t-s}^2 \leq c \|t\|. \quad (3.3)$$

(The integral is replaced by $\sum_{s=0}^t$ for oriented percolation.) Together with (3.2), we thus obtain $\rho \leq 1$, if ρ exists. $\square$

Remark. For spread-out oriented percolation with $d > 4$ and $L \gg 1$, Theorem 4.1 and Lemma 4.2 in [14] imply that the left-hand side of (3.2) is asymptotically A^2 , while the sum in the right-hand side of (3.2) is asymptotically $A^3 V t$, where A and V are constants depending only on d and L . This leads to a lower bound on θ_t like $(AVt)^{-1}$, which is consistent with [14, Theorem 1.5], where the limit $\lim_{t \rightarrow \infty} t \theta_t$, if it exists, equals $2(AV)^{-1}$.

Proof for percolation. We follow the same strategy as above. Let

$$I_t = \mathbb{1}_{\{o \leftrightarrow \partial \mathcal{B}_t\}}, \quad X_t = \sum_{x \in \partial \mathcal{B}_t} \mathbb{1}_{\{o \leftrightarrow x\}}. \quad (3.4)$$

Using the Schwarz inequality as in (3.2), we obtain

$$\left[\sum_{x \in \partial \mathcal{B}_t} \tau(x) \right]^2 = \mathbb{E}(I_t X_t)^2 \leq \mathbb{E}(I_t^2) \mathbb{E}(X_t^2) = \theta_t \sum_{x,y \in \partial \mathcal{B}_t} \mathbb{P}(o \leftrightarrow x, o \leftrightarrow y). \quad (3.5)$$

Since $\eta = 0$, the leftmost quantity is bounded from below by $c \|t\|^2$. If $o \leftrightarrow x$ and $o \leftrightarrow y$ occur simultaneously, then there is a $z \in \mathbb{Z}^d$ such that $o \leftrightarrow z$, $z \leftrightarrow x$ and $z \leftrightarrow y$ occur disjointly. By the BK inequality and $\eta = 0$, the sum in the right-hand side of (3.5) is bounded by

$$\begin{aligned} \sum_{\substack{x,y \in \partial \mathcal{B}_t \\ z \in \mathbb{Z}^d}} \tau(z) \tau(x-z) \tau(y-z) &= \sum_{\substack{x,y \in \partial \mathcal{B}_t \\ z \in \mathcal{B}_{t/2}}} \tau(z) \tau(x-z) \tau(y-z) + \sum_{\substack{x,y \in \partial \mathcal{B}_t \\ z \notin \mathcal{B}_{t/2}}} \tau(z) \tau(x-z) \tau(y-z) \\ &\leq c \|t\|^{2(2-d)+2(d-1)} \sum_{z \in \mathcal{B}_{t/2}} \|z\|^{2-d} + c \|t\|^{2-d} \sum_{\substack{x,y \in \partial \mathcal{B}_t \\ z \in \mathbb{Z}^d}} \|x-z\|^{2-d} \|y-z\|^{2-d}, \end{aligned} \quad (3.6)$$

where we used $|x-z| \geq t/2$ and $|y-z| \geq t/2$ in the first sum, and $|z| \geq t/2$ in the second sum. By [10, Proposition 1.7(i)], the convolution of $\|x-z\|^{2-d}$ and $\|y-z\|^{2-d}$ is bounded by $c \|x-y\|^{4-d}$ if $d > 4$, whose sum over $x, y \in \partial \mathcal{B}_t$ is bounded by $c \|t\|^{2(d-1)+4-d} = c \|t\|^{d+2}$. Therefore, (3.1) is bounded by $c \|t\|^4$, and we obtain $\rho \leq 2$ using (3.5). $\square$

3.2 Proof of the lower bound

In this section, we will use $\epsilon = \epsilon(\rho)$ defined by

$$\epsilon(\rho) \begin{cases} > 0 \text{ (but arbitrarily small)}, & \text{if } \rho = 1, \\ = 0, & \text{if } \rho \neq 1, \end{cases} \quad (3.7)$$

for both the time-oriented models and percolation.

Proof for the time-oriented models. We only consider oriented percolation, since the same idea given below also applies to the time-discretized contact process in [17, 24] that weakly converges to the original contact process as the discretized-time unit tends to zero. We prove below

$$\theta_t \geq c[1 - O(\bar{\nabla}) - O(\|t\|^{-(d-4)/2+\epsilon})] \|t\|^{1-2\rho}, \quad (3.8)$$

and thus prove Theorem 2.2 for the time-oriented models, assuming $\bar{\nabla} \equiv \sup_x \nabla(x, 0) \ll 1$. In the proof of (3.8), we will require $p_c \leq 3/2$, which is a consequence of $\bar{\nabla} \ll 1$, if $d > 4$ [18, 22, 24]. We will also assume existence of a constant $a > 1$, which is independent of d and L , such that $\sum_{s \leq t/2} \hat{\tau}_s \leq aC_1 \|t\|$ (cf., (2.14)) and $K \leq \theta_t \|t\|^\rho \leq aK$ for some $K > 0$, which may depend on d and L . After the proof, we briefly discuss how to remove all these extra assumptions.

The survival probability θ_t is the probability of the event that there is a path of occupied bonds from $(o, 0)$ to $\partial\mathcal{B}_t$. This event can be decomposed into two disjoint events depending on whether or not $(o, 0)$ is *doubly connected* to $\partial\mathcal{B}_t$, denoted by $(o, 0) \rightrightarrows \partial\mathcal{B}_t$, which means that there are at least two bond-disjoint occupied paths from $(o, 0)$ to $\partial\mathcal{B}_t$. If $(o, 0)$ is connected but not doubly connected to $\partial\mathcal{B}_t$, then there is an occupied *pivotal bond* $b = (\underline{b}, \bar{b})$ for $(o, 0) \rightarrow \partial\mathcal{B}_t$ such that $(o, 0) \rightarrow \underline{b}$, $\bar{b} \rightrightarrows \partial\mathcal{B}_t$ and $\mathcal{C}^b(o, 0) \cap \partial\mathcal{B}_t = \emptyset$, where $\mathcal{C}^b(o, 0)$ is the set of sites in $\mathbb{Z}^d \times \mathbb{Z}_+$ connected from $(o, 0)$ without using b . Restricting the location of $\bar{b}$ to $\mathcal{B}_{t/2}$ gives

$$\theta_t \geq \sum_{b: \bar{b} \in \mathcal{B}_{t/2}} \frac{1}{N} \mathbb{P}((o, 0) \rightarrow \underline{b}, \bar{b} \rightrightarrows \partial\mathcal{B}_t, \mathcal{C}^b(o, 0) \cap \partial\mathcal{B}_t = \emptyset), \quad (3.9)$$

where we used $p_c \geq 1$ and the fact that the status of b is independent of the event in $\mathbb{P}$. For an event E and $\mathcal{Z} \subset \mathbb{Z}^d \times \mathbb{Z}_+$, let $\{E \text{ in } \mathcal{Z}\}$ be the set of bond configurations whose restriction on bonds b contained in $\mathcal{Z}$ (i.e., both $\underline{b}$ and $\bar{b}$ are in $\mathcal{Z}$) are in E . Using this notion, we can rewrite the probability in the right-hand side of (3.9) as (see [13, Definition 2.2(c) and Lemma 2.5])

$$\mathbb{P}(\{(o, 0) \rightarrow \underline{b}, \mathcal{C}^b(o, 0) \cap \partial\mathcal{B}_t = \emptyset\} \text{ in } \mathcal{C}^b(o, 0), \{\bar{b} \rightrightarrows \partial\mathcal{B}_t\} \text{ in } \mathcal{C}^b(o, 0)^c). \quad (3.10)$$

By the “conditioning over clusters” technique [2, 12, 13], (3.10) equals

$$\begin{aligned} & \mathbb{E} \left(\mathbb{1}_{\{(o, 0) \rightarrow \underline{b}, \mathcal{C}^b(o, 0) \cap \partial\mathcal{B}_t = \emptyset\}} \mathbb{P}(\bar{b} \rightrightarrows \partial\mathcal{B}_t \text{ in } \mathcal{C}^b(o, 0)^c) \right) \\ &= \mathbb{P}((o, 0) \rightarrow \underline{b}, \mathcal{C}^b(o, 0) \cap \partial\mathcal{B}_t = \emptyset) \mathbb{P}(\bar{b} \rightrightarrows \partial\mathcal{B}_t) \\ & \quad - \mathbb{E} \left(\mathbb{1}_{\{(o, 0) \rightarrow \underline{b}, \mathcal{C}^b(o, 0) \cap \partial\mathcal{B}_t = \emptyset\}} [\mathbb{P}(\bar{b} \rightrightarrows \partial\mathcal{B}_t) - \mathbb{P}(\bar{b} \rightrightarrows \partial\mathcal{B}_t \text{ in } \mathcal{C}^b(o, 0)^c)] \right). \end{aligned} \quad (3.11)$$

In the first term, we use $\{\mathcal{C}^b(o, 0) \cap \partial\mathcal{B}_t = \emptyset\} \supset \{(o, 0) \not\rightarrow \partial\mathcal{B}_t\}$ and $\mathbb{P}(\bar{b} \rightrightarrows \partial\mathcal{B}_t) \geq \mathbb{P}((o, 0) \rightrightarrows \partial\mathcal{B}_t)$, and apply the definition of D in (2.1) and Assumption 2.1. Then, we obtain

$$\begin{aligned} \theta_t &\geq C_1 \|t\| \mathbb{P}((o, 0) \rightrightarrows \partial\mathcal{B}_t) \\ & \quad - \sum_{b: \bar{b} \in \mathcal{B}_{t/2}} \frac{1}{N} \mathbb{E} \left(\mathbb{1}_{\{(o, 0) \rightarrow \underline{b}, \mathcal{C}^b(o, 0) \cap \partial\mathcal{B}_t = \emptyset\}} \mathbb{P}(\{\bar{b} \rightrightarrows \partial\mathcal{B}_t\} \setminus \{\bar{b} \rightrightarrows \partial\mathcal{B}_t \text{ in } \mathcal{C}^b(o, 0)^c\}) \right). \end{aligned} \quad (3.12)$$

This is the starting point for the proof of (3.8).

First, we consider the first term in the right-hand side of (3.12), and show that

$$C_1 \|t\| \mathbb{P}((o, 0) \Rightarrow \partial \mathcal{B}_t) \geq \frac{C_1 K^2}{4e} (1 - 4^{\rho+3} a^2 \bar{\nabla} - c \|t\|^{-(d-4)/2+\epsilon}) \|t\|^{1-2\rho}. \quad (3.13)$$

For the proof of this inequality, it suffices to investigate $\mathbb{P}((o, 0) \Rightarrow \partial \mathcal{B}_t)$. Restricting the number of occupied bonds growing out of $(o, 0)$ to two, we obtain

$$\mathbb{P}((o, 0) \Rightarrow \partial \mathcal{B}_t) \geq \left(\frac{p_c}{N}\right)^2 \left(1 - \frac{p_c}{N}\right)^{N-2} \sum_{\langle x, y \rangle} \mathbb{P} \left(\begin{array}{l} (x, 1) \rightarrow \partial \mathcal{B}_t, (y, 1) \rightarrow \partial \mathcal{B}_t, \\ \mathcal{C}(x, 1) \cap \mathcal{C}(y, 1) = \emptyset \text{ in } \mathcal{B}_t \end{array} \right), \quad (3.14)$$

where $\sum_{\langle x, y \rangle}$ is the sum over all pairs of distinct sites in Ω . By conditioning over the cluster $\mathcal{C}(x, 1)$, the right-hand side of (3.14) equals

$$\left(\frac{p_c}{N}\right)^2 \left(1 - \frac{p_c}{N}\right)^{N-2} \sum_{\langle x, y \rangle} \mathbb{E} \left(\mathbb{1}_{\{(x, 1) \rightarrow \partial \mathcal{B}_t\}} \mathbb{P}((y, 1) \rightarrow \partial \mathcal{B}_t \text{ in } \mathcal{C}(x, 1)^c) \right). \quad (3.15)$$

Note that $p_c^2(1 - \frac{p_c}{N})^{N-2}$ is bounded from above by $36e^{-3/2}$ using $p_c \geq 1$, while it is bounded from below by e^{-1} using $p_c \leq 3/2$. Therefore, if we ignore the condition “in $\mathcal{C}(x, 1)^c$ ” in (3.15), then we obtain the lower bound $\frac{e^{-1}}{N^2} \binom{N}{2} \theta_t^2 \geq \frac{K^2}{4e} \|t\|^{-2\rho}$, which is the main contribution in the right-hand side of (3.13). The correction for this is

$$\left(\frac{p_c}{N}\right)^2 \left(1 - \frac{p_c}{N}\right)^{N-2} \sum_{\langle x, y \rangle} \mathbb{E} \left(\mathbb{1}_{\{(x, 1) \rightarrow \partial \mathcal{B}_t\}} \mathbb{P}(\{(y, 1) \rightarrow \partial \mathcal{B}_t\} \setminus \{(y, 1) \rightarrow \partial \mathcal{B}_t \text{ in } \mathcal{C}(x, 1)^c\}) \right). \quad (3.16)$$

To complete the proof of (3.13), we bound (3.16) from above. Since the event inside $\mathbb{P}$ in (3.16) is the event that all occupied paths from $(y, 1)$ to $\partial \mathcal{B}_t$ go through $\mathcal{C}(x, 1)$, there must be a $(z, s) \in \mathcal{C}(x, 1)$ such that $(y, 1) \rightarrow (z, s) \rightarrow \partial \mathcal{B}_t$. By the Markov property, the expectation in (3.16) is bounded by

$$\begin{aligned} & \mathbb{E} \left(\mathbb{1}_{\{(x, 1) \rightarrow \partial \mathcal{B}_t\}} \sum_{(z, s) \in \mathcal{C}(x, 1)} \tau_{s-1}(z - y) \theta_{t-s} \right) \\ &= \sum_{(z, s)} \mathbb{P}((x, 1) \rightarrow \partial \mathcal{B}_t, (z, s) \in \mathcal{C}(x, 1)) \tau_{s-1}(z - y) \theta_{t-s}. \end{aligned} \quad (3.17)$$

We separate the sum over s into $\sum_{s \leq t/2}$ and $\sum_{s > t/2}$.

1. For $\sum_{s \leq t/2}$, we use the fact that there is a site (z', s') such that $(x, 1) \rightarrow (z', s') \rightarrow \partial \mathcal{B}_t$ and $(z', s') \rightarrow (z, s)$ occur disjointly. By the BK inequality, the contribution to (3.17) is bounded by

$$\sum_{s=2}^{t/2} \sum_{s'=1}^s \sum_{z, z' \in \mathbb{Z}^d} \tau_{s'-1}(z' - x) \tau_{s-s'}(z - z') \tau_{s-1}(z - y) \theta_{t-s} \theta_{t-s'}, \quad (3.18)$$

where $s \geq 2$, because $x \neq y$. Using $t - s' \geq t - s \geq t/2$ and (2.5), we bound (3.18) by $4^\rho (aK)^2 \bar{\nabla} \|t\|^{-2\rho}$. This gives the second term in the right-hand side of (3.13), where we used $(\frac{p_c}{N})^2 (1 - \frac{p_c}{N})^{N-2} \binom{N}{2} \leq 18e^{-3/2} < 4^2 e^{-1}$.

2. For $\sum_{s>t/2}$, we use (3.17) instead of (3.18). By ignoring the event $\{(x, 1) \rightarrow \partial\mathcal{B}_t\}$, (3.17) is bounded by

$$\sum_{s=t/2}^t \hat{\tau}_{s-1} \bar{\tau}_{s-1} \theta_{t-s} \leq c \|t\|^{-d/2} \sum_{s=t/2}^t \|t-s\|^{-\rho}. \quad (3.19)$$

Since $\rho \leq 1$, the right-hand side is bounded by $c \|t\|^{-d/2+1-\rho+\epsilon} \leq c \|t\|^{-2\rho-(d-4)/2+\epsilon}$, which gives the third term in the right-hand side of (3.13).

This completes the proof of (3.13).

Next, we investigate the second term in (3.12), and prove that

$$\begin{aligned} \sum_{b: \bar{b} \in \mathcal{B}_{t/2}} \frac{1}{N} \mathbb{E} \left(\mathbb{1}_{\{(o,0) \rightarrow \underline{b}, \mathcal{C}^b(o,0) \cap \partial\mathcal{B}_t = \emptyset\}} \mathbb{P}(\{\bar{b} \rightrightarrows \partial\mathcal{B}_t\} \setminus \{\bar{b} \rightrightarrows \partial\mathcal{B}_t \text{ in } \mathcal{C}^b(o,0)^c\}) \right) \\ \leq (8^\rho a^3 C_1 K^2 \bar{\nabla} + c \|t\|^{-(d-4)/2+\epsilon}) \|t\|^{1-2\rho}. \end{aligned} \quad (3.20)$$

Note that the event $\{\bar{b} \rightrightarrows \partial\mathcal{B}_t\} \setminus \{\bar{b} \rightrightarrows \partial\mathcal{B}_t \text{ in } \mathcal{C}^b(o,0)^c\}$ implies existence of a $(z, s) \in \mathcal{C}^b(o,0)$ such that $\bar{b} \rightarrow (z, s) \rightarrow \partial\mathcal{B}_t$ and $\bar{b} \rightarrow \partial\mathcal{B}_t$ occur disjointly. Using the BK inequality, we can extract $\tau_{s-s'}(z-v) \theta_{t-s}$ and $\theta_{t-s'}$, where $\bar{b} = (v, s')$. Using (2.1), we bound the left-hand side of (3.20) by

$$\begin{aligned} \sum_{\substack{(z,s),(v,s'): \\ 1 \leq s' < t/2}} \sum_{b: \bar{b}=(v,s')} \frac{1}{N} \mathbb{P}((o,0) \rightarrow \underline{b}, (z,s) \in \mathcal{C}^b(o,0)) \tau_{s-s'}(z-v) \theta_{t-s} \theta_{t-s'} \\ \leq \sum_{\substack{(z,s),(v,s'): \\ 1 \leq s' < t/2}} \sum_{\substack{(y,r),(u,s'-1) \\ 0 \leq r < s'}} \tau_r(y) \tau_{s'-1-r}(u-y) D(v-u) \tau_{s-s'}(z-v) \tau_{s-r}(z-y) \theta_{t-s} \theta_{t-s'} \\ \leq \frac{2^\rho a K}{\|t\|^\rho} \sum_{r=0}^{t/2-1} \hat{\tau}_r \sum_{s=r+1}^t \sum_{s'=r+1}^{(t/2) \wedge s} \sum_{x \in \mathbb{Z}^d} (\tau_{s'-1-r} * D * \tau_{s-s'})(x) \tau_{s-r}(x) \theta_{t-s}, \end{aligned} \quad (3.21)$$

where $(y, r) \in \mathbb{Z}^d \times \mathbb{Z}_+$ in the second expression is a branch point arising from the intersection $\{(o,0) \rightarrow \underline{b}\} \cap \{(z,s) \in \mathcal{C}^b(o,0)\}$, and we used $s' \leq t/2$ and (2.5) to bound $\theta_{t-s'}$ as in the last inequality. We separate the sum over s into $\sum_{s \leq 3t/4}$ and $\sum_{s > 3t/4}$.

1. For $\sum_{s \leq 3t/4}$, we bound θ_{t-s} by $4^\rho a K \|t\|^{-\rho}$ using (2.5), and then bound the remaining term by $\bar{\nabla} \sum_{r=0}^{t/2-1} \hat{\tau}_r \leq a C_1 \bar{\nabla} \|t\|$. This gives the first term in the right-hand side of (3.20).
2. For $\sum_{s > 3t/4}$, we bound $\bar{\tau}_{s-r}$ by $c \|t\|^{-d/2}$ using $r < t/2$, and then bound the remaining term, using $\rho \leq 1$, by

$$c \|t\| \sum_{r=0}^{t/2-1} \sum_{s=3t/4}^t \|t-s\|^{-\rho} \leq c \|t\|^{3-\rho+\epsilon}. \quad (3.22)$$

This gives the second term in the right-hand side of (3.20).

This completes the proof of (3.20).

The proof of (3.8) is completed by (3.13) and (3.20). We obtain (2.16) from (3.8) if $\bar{\nabla} \ll 1$, $t \gg 1$ and $d > 4$. Together with $\rho \leq 1$ proved in Section 3.1, this completes the proof of $\rho = 1$. $\square$

Remark. In the above proof, we exploited the assumptions stated below (3.8). These assumptions can be removed via a *delocalization argument* [4] (or, it is also called *ultraviolet regularization* [2, 3, 12]). In fact, we can prove that there is a $c_\ell > 0$ such that

$$t^{-\rho} \geq c_\ell [1 - O(\nabla_\ell) - O(t^{-(d-4)/2+\epsilon})] t^{1-2\rho}, \quad \text{for } t \gg \ell. \quad (3.23)$$

Recall that $(\alpha, \eta) = (2, 0)$ implies $\lim_{\ell \rightarrow \infty} \nabla_\ell = 0$, as explained below (2.13). Taking ℓ and t in (3.23) sufficiently large, independently of d and L , we obtain (2.16) for the time-oriented models. Therefore, we do not need to control the constants in terms of d and L , as in the extra assumptions stated below (3.8).

We briefly explain the idea for the proof of (3.23). Recall (3.9), where b is the last pivotal bond for $(o, 0) \rightarrow \partial\mathcal{B}_t$. The space-time rectangle $\mathcal{R}_\ell(b)$ is defined as

$$\mathcal{R}_\ell(b) = \{\underline{b} + (re_b, s) \in \mathbb{Z}^d \times \mathbb{Z}_+ : r \in [-\ell, \ell], s \in [0, \ell]\}, \quad (3.24)$$

where $e_b = (v - u)/|v - u|$ for $b = ((u, s), (v, s + 1))$. We may modify the occupation status of bonds contained in $\mathcal{R}_\ell(b)$, in order to thin the connection from $(o, 0)$ to $\partial\mathcal{B}_t$. Let $E_{\mathcal{R}_\ell(b)}$ be such an event that $\underline{b}$ is “minimally” connected, via b , to both $X_\pm \equiv \underline{b} + (\pm \ell e_b, \ell)$. Then, we obtain (cf., (3.9))

$$\theta_t \geq \sum_{b: \bar{b} \in \mathcal{B}_{t/2}} \mathbb{P}(E_{\mathcal{R}_\ell(b)}) \mathbb{P} \left(\begin{array}{l} (o, 0) \rightarrow \underline{b}, \mathcal{C}^{\mathcal{R}_\ell(b)}(o, 0) \cap \partial\mathcal{B}_t = \emptyset, \\ \{X_+ \rightarrow \partial\mathcal{B}_t\} \circ \{X_- \rightarrow \partial\mathcal{B}_t\} \end{array} \right), \quad (3.25)$$

where $E_1 \circ E_2$ is the event that E_1 and E_2 occur disjointly, and $\mathcal{C}^{\mathcal{R}_\ell(b)}(o, 0)$ is the set of sites connected from $(o, 0)$ without using any bonds contained in $\mathcal{R}_\ell(b)$. In (3.25), we used the fact that $E_{\mathcal{R}_\ell(b)}$ is independent of the other three events in $\mathbb{P}$. We choose $c_\ell = \inf_b \mathbb{P}(E_{\mathcal{R}_\ell(b)})$. For the remaining term, we follow the same strategy as in the proof for the case $\bar{\nabla} \ll 1$, except that we do not need an argument around (3.14), which is also a sort of delocalization argument. This leads to (3.23).

It remains to determine $E_{\mathcal{R}_\ell(b)}$. This was well-explained in [4] for the time-discretized contact process. A variant of $E_{\mathcal{R}_\ell(b)}$ in [4] was chosen in such a way that c_ℓ is bounded away from zero uniformly in the discretized-time unit. It is not hard to adapt the idea of [4] to our settings, and we refrain from giving its details. See [4, Figure 1].

Proof for percolation. The strategy is the same as above. We prove below

$$\theta_t \geq c [1 - O(\nabla_0) - O(\|t\|^{-(d-5-\rho\vee 1)+\epsilon})] \|t\|^{2-2\rho}, \quad (3.26)$$

for $t \geq 2L$ (so that $\partial\mathcal{B}_{t/2} \subset \mathcal{B}_t$), and hence Theorem 2.2 for percolation, assuming $\nabla_0 \ll 1$. Similarly to the proof for the time-oriented models, we will also assume that $p_c \leq 3/2$, which is true when $\nabla_0 \ll 1$ and $d > 6$ [11, 18], and that there is a (d, L) -independent constant $a > 1$ such that $\sum_{x \in \mathcal{B}_{3t/2+L}} \tau(x) \leq aC_2 \|t\|^2$ (cf., (2.15)) and $K \leq \theta_t \|t\|^\rho \leq aK$ for some $K > 0$, which may depend on d and L . These assumptions can be removed as discussed in the above remark and as in [2, 3, 12], and we omit its details for simplicity.

The percolation version of (3.12) is

$$\begin{aligned} \theta_t &\geq C_2 \|t\|^2 \mathbb{P}(o \Leftrightarrow \partial\mathcal{B}_{3t/2}) \\ &\quad - \sum_{b: \bar{b} \in \mathcal{B}_{t/2}} \frac{1}{N} \mathbb{E} \left(\mathbb{1}_{\{o \leftrightarrow \underline{b}, \mathcal{C}^b(o) \cap \partial\mathcal{B}_t = \emptyset\}} \mathbb{P}(\{\bar{b} \Leftrightarrow \partial\mathcal{B}_t\} \setminus \{\bar{b} \Leftrightarrow \partial\mathcal{B}_t \text{ in } \mathcal{C}^b(o)^c\}) \right), \end{aligned} \quad (3.27)$$

where “ $\Leftrightarrow$ ” represents a double connection for percolation, and we used $\mathbb{P}(\bar{b} \Leftrightarrow \partial\mathcal{B}_t) \geq \mathbb{P}(o \Leftrightarrow \partial\mathcal{B}_{3t/2})$ and Assumption 2.1. We only discuss below bounding the first term of the right-hand side of (3.27). The second term can be estimated similarly, following the same strategy as explained below (3.20).

To investigate the first term of the right-hand side of (3.27), it thus suffices to consider $\mathbb{P}(o \Leftrightarrow \partial\mathcal{B}_{3t/2})$. To bound this from below, we restrict the number of occupied bonds growing out of $o \in \mathbb{Z}^d$ to two. Let $\tilde{\mathcal{C}}_{3t/2}(x) \subset \mathcal{B}_{3t/2+L}$ be the set of sites to which there is an occupied path from x that includes at most one bond growing from $\partial\mathcal{B}_{3t/2}$ and no bonds growing from $o \in \mathbb{Z}^d$. Then, we have (cf., (3.14))

$$\mathbb{P}(o \Leftrightarrow \partial\mathcal{B}_{3t/2}) \geq \left(\frac{p_c}{N}\right)^2 \left(1 - \frac{p_c}{N}\right)^{N-2} \sum_{\langle x, y \rangle} \mathbb{P} \left(\begin{array}{c} x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c, \ y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c, \\ \tilde{\mathcal{C}}_{3t/2}(x) \cap \tilde{\mathcal{C}}_{3t/2}(y) = \emptyset \end{array} \right). \quad (3.28)$$

By conditioning over the cluster $\tilde{\mathcal{C}}_{3t/2}(x)$, the above expression equals

$$\begin{aligned} & \left(\frac{p_c}{N}\right)^2 \left(1 - \frac{p_c}{N}\right)^{N-2} \sum_{\langle x, y \rangle} \mathbb{E} \left(\mathbb{1}_{\{x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c\}} \mathbb{P}(y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c \cap \tilde{\mathcal{C}}_{3t/2}(x)^c) \right) \\ &= \left(\frac{p_c}{N}\right)^2 \left(1 - \frac{p_c}{N}\right)^{N-2} \sum_{\langle x, y \rangle} \left[\mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c) \mathbb{P}(y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c) \right. \\ & \quad \left. - \mathbb{E} \left(\mathbb{1}_{\{x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c\}} \mathbb{P}(\{y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c\} \setminus \{y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c \cap \tilde{\mathcal{C}}_{3t/2}(x)^c\}) \right) \right]. \end{aligned} \quad (3.29)$$

Here, we have $\mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c)$, instead of $\mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2})$ as explained below (3.15) for the time-oriented models. The correction is the probability of the event that all occupied paths between x and $\partial\mathcal{B}_{3t/2}$ go through the origin, and thus is bounded by the probability of the event that $x \leftrightarrow o$ and $o \leftrightarrow \partial\mathcal{B}_{3t/2}$ occur disjointly. Using the BK inequality and monotonicity, we obtain

$$\mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c) \geq \mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2}) - \tau(x) \theta_{3t/2} \geq \theta_{3t/2+L} - \tau(x) \theta_{3t/2}. \quad (3.30)$$

First, we consider the contribution to (3.29) from $\mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c) \mathbb{P}(y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c)$. By (3.30), the main contribution comes from $\theta_{3t/2+L}^2$ that is bounded from below by $\frac{K^2}{4e} \left\| \frac{3}{2}t + L \right\|^{-2\rho} \geq (4^{\rho+1}e)^{-1} K^2 \|t\|^{-2\rho}$, where we used $p_c \leq 3/2$ (cf., the argument below (3.15)) and $t \geq 2L$ together with (2.5). The contribution from the terms containing $\tau(x) \theta_{3t/2}$ or $\tau(y) \theta_{3t/2}$ is bounded by $K^2 O(\nabla_0) \|t\|^{-2\rho}$, using $N^{-2} = D(x) D(y)$ in (3.29).

Next, we investigate the contribution from the expectation in the right-hand side of (3.29). Since the event inside $\mathbb{P}$ is the event that all occupied paths from y to $\partial\mathcal{B}_{3t/2}$ in $\{o\}^c$ go through $\tilde{\mathcal{C}}_{3t/2}(x) \subset \mathcal{B}_{3t/2+L}$, there must be a $z \in \tilde{\mathcal{C}}_{3t/2}(x)$ such that $y \leftrightarrow z$ and $z \leftrightarrow \partial\mathcal{B}_{3t/2}$ occur disjointly. Therefore, the expectation in the right-hand side of (3.29) is bounded, using the BK inequality, by

$$\begin{aligned} & \mathbb{E} \left(\mathbb{1}_{\{x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c\}} \sum_{z \in \tilde{\mathcal{C}}_{3t/2}(x)} \tau(z - y) \mathbb{P}(z \leftrightarrow \partial\mathcal{B}_{3t/2}) \right) \\ & \leq \sum_{z \in \mathcal{B}_{3t/2+L}} \mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2}, z \in \tilde{\mathcal{C}}_{3t/2}(x)) \tau(z - y) \mathbb{P}(z \leftrightarrow \partial\mathcal{B}_{3t/2}). \end{aligned} \quad (3.31)$$

We separate the sum into $\sum_{z \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}}$ and $\sum_{z \in \mathcal{B}_{t/2}}$.

1. For $\sum_{z \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}}$, we ignore¹ the event $\{x \leftrightarrow \partial \mathcal{B}_{3t/2}\}$ as in (3.19), and use $\mathbb{P}(z \leftrightarrow \partial \mathcal{B}_{3t/2}) \leq \theta_{(3t/2-|z|) \vee 0}$. Then, the contribution to (3.31) is bounded by

$$\begin{aligned} \sum_{z \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}} \tau(z-x) \tau(z-y) \theta_{(3t/2-|z|) \vee 0} &\leq c \|t\|^{(d-1)+2(2-d)} \left(L + \sum_{s=0}^t \|s\|^{-\rho} \right) \\ &\leq c \|t\|^{-2\rho-(d-\rho-3-\rho \vee 1)+\epsilon}, \end{aligned} \quad (3.32)$$

which is bounded by $c \|t\|^{-2\rho-(d-5-\rho \vee 1)+\epsilon}$, using $\rho \leq 2$.

2. For $\sum_{z \in \mathcal{B}_{t/2}}$, we bound $\mathbb{P}(z \leftrightarrow \partial \mathcal{B}_{3t/2})$ by $aK \|t\|^{-\rho}$, and note that the event inside the other $\mathbb{P}$ in (3.31) implies existence of $w \in \mathcal{B}_{3t/2+L}$ such that $x \leftrightarrow w$, $w \leftrightarrow z$ and $w \leftrightarrow \partial \mathcal{B}_{3t/2}$ occur disjointly. Again by the BK inequality, the contribution to (3.31) is bounded by

$$aK \|t\|^{-\rho} \sum_{\substack{z \in \mathcal{B}_{t/2} \\ w \in \mathcal{B}_{3t/2+L}}} \tau(x-w) \tau(w-z) \tau(z-y) \mathbb{P}(w \leftrightarrow \partial \mathcal{B}_{3t/2}). \quad (3.33)$$

We further separate the sum over w into $\sum_{w \in \mathcal{B}_{t/2}}$ and $\sum_{w \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}}$.

- (a) For $\sum_{w \in \mathcal{B}_{t/2}}$, we bound $\mathbb{P}(w \leftrightarrow \partial \mathcal{B}_{3t/2})$ by $aK \|t\|^{-\rho}$, and then bound the remaining term, using $x \neq y$, by $aK \|t\|^{-\rho} (\tau * \tau * \tau)(y-x) \leq aK \nabla_0 \|t\|^{-\rho}$.
- (b) For $\sum_{w \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}}$, we use $\mathbb{P}(w \leftrightarrow \partial \mathcal{B}_{3t/2}) \leq \theta_{(3t/2-|w|) \vee 0}$ and perform the sum over $z \in \mathbb{Z}^d$ using [10, Proposition 1.7(i)]. Since $x, y \in \Omega$, the contribution to (3.33) is bounded by

$$\begin{aligned} c \|t\|^{-\rho} \sum_{w \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}} \| |w| - L \|^{(2-d)+(4-d)} \left\| \frac{3}{2}t - |w| \right\|^{-\rho} \\ \leq c \|t\|^{-\rho+(6-2d)+(d-1)} \left(L + \sum_{s=0}^t \|s\|^{-\rho} \right) \leq c \|t\|^{-2\rho-(d-5-\rho \vee 1)+\epsilon}. \end{aligned} \quad (3.34)$$

Summarizing the above estimates, we conclude that the contribution from the expectation in the right-hand side of (3.29) is bounded from above by $(a^2 K^2 \nabla_0 + c \|t\|^{-(d-5-\rho \vee 1)+\epsilon}) \|t\|^{-2\rho}$. Together with the discussion below (3.30), we complete bounding $C_2 \|t\|^2 \mathbb{P}(o \leftrightarrow \partial \mathcal{B}_{3t/2})$ in (3.27) from below by the same formula in the right-hand side of (3.26), where $c = O(C_2 K^2)$. The sum in (3.27) can be estimated similarly to (3.33), where z in (3.33) corresponds to $\bar{b}$ in (3.27). This completes the proof of (3.26).

We obtain (2.16) from (3.26) if $\nabla_0 \ll 1$, $t \gg 1$ and $d > 5 + \rho \vee 1$, and thus obtain $\rho = 2$ for $d > 7$. This completes the proof. $\square$

¹Some readers might wonder whether the event $\{x \leftrightarrow \partial \mathcal{B}_{3t/2}\}$ could be used to have less power in (3.32). In fact, if we use the inequality

$$\mathbb{P}(x \leftrightarrow \partial \mathcal{B}_{3t/2}, z \in \tilde{\mathcal{C}}_{3t/2}(x)) \leq \sum_{w \in \mathcal{B}_{3t/2+L}} \tau(w-x) \tau(z-w) \theta_{(3t/2-|w|) \vee 0},$$

then the contribution due to $w \in \mathcal{B}_{t/2}$ is bounded by (3.34), while the contribution from $w \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}$ has a worse bound $c \|t\|^{-2\rho+\mu}$, where μ is negative only when $d > 9$.

Remark. The value of ρ for percolation is expected to be 2 as soon as $d > 6$. The main obstacle to going down from $d > 7$ is in (3.32) and (3.34), which correspond respectively to (3.19) and (3.22) for the time-oriented models. In (3.19) and (3.22), the sum over s is fully controlled using $\theta_{t-s} \asymp \|t - s\|^{-\rho}$. On the other hand, the point-to-surface connectivity $\theta_{(3t/2-|v|)\vee 0}$, with $v = z$ in (3.32) and $v = w$ in (3.34), is insufficient to obtain the desired bound, when v is close to the boundary $\partial\mathcal{B}_{3t/2}$. This difficulty is considered to be caused by naively bounding the probability inside $\mathbb{E}$ in (3.29) as in (3.31). Since $\{y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c\} \setminus \{y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^c \cap \tilde{\mathcal{C}}_{3t/2}(x)^c\}$ is the event that all occupied paths from y to $\partial\mathcal{B}_{3t/2}$ (in $\{o\}^c$) have to go through $\tilde{\mathcal{C}}_{3t/2}(x)$ before reaching to the boundary, the approximation by the unrestricted two-point function $\tau(z - y)$ in (3.31) could be very crude when z is close to $\partial\mathcal{B}_{3t/2}$, due to the isotropic property for percolation. If we assume that there is a $\kappa \geq 1$ such that, for $|z| = \ell$,

$$\mathbb{P}(o \leftrightarrow z, o \not\leftrightarrow \partial\mathcal{B}_t) \leq c\|\ell\|^{2-d-\kappa}(\|\ell\| \wedge \|t - \ell\|)^\kappa, \quad (3.35)$$

then we will be able to obtain the desired inequality (2.16) down to $d > 6$. Note that (3.35) contains the factor $\|t - \ell\|$ that decreases as z approaches the boundary $\partial\mathcal{B}_t$, that the sum of the right-hand side over $z \in \mathcal{B}_t$ is bounded by $c\|t\|^2$, and that the limit $t \rightarrow \infty$ of the right-hand side, while ℓ or ℓ/t is fixed, is $c\|\ell\|^{2-d}$. Therefore, (3.35) is a good candidate for the bound on the restricted two-point function, though we have not proved whether (3.35) really holds or does not. (The random walk *restricted* Green's function satisfies (3.35) at $\ell = t$ with $\kappa = 1$.)

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