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Author(s)	NAKAZI, TAKAHIKO; 中路, 貴彦; OSAWA, TOMOKO
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INVARIANT SUBSPACES OF FINITE CODIMENSION AND UNIFORM ALGEBRAS

TAKAHIKO NAKAZI*

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
e-mail: nakazi@math.sci.hokudai.ac.jp

and TOMOKO OSAWA

Mathematical and Scientific Subjects, Asahikawa National College of Technology,
Asahikawa 071-8142, Japan
e-mail: ohsawa@asahikawa-nct.ac.jp

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Abstract. Let A be a uniform algebra on a compact Hausdorff space X and m a probability measure on X . Let $H^p(m)$ be the norm closure of A in $L^p(m)$ with $1 \leq p < \infty$ and $H^\infty(m)$ the weak $*$ closure of A in $L^\infty(m)$. In this paper, we describe a closed ideal of A and exhibit a closed invariant subspace of $H^p(m)$ for A that is of finite codimension.

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1. Introduction. Let A be a uniform algebra on a compact Hausdorff space X . $M(A)$ denotes the maximal ideal space of A . Let m be a probability measure on X . $H^p(m)$ denotes the norm closure of A in $L^p(m)$ with $1 \leq p < \infty$ and $H^\infty(m)$ denotes the weak $*$ closure of A in $L^\infty(m)$. $H^p(m)$ is called an *abstract Hardy space*. When A is a disc algebra, if m is the normalized Lebesgue measure on the unit circle, $H^p(m)$ is the usual Hardy space and if m is the normalized area measure on the unit disc, $H^p(m)$ is the usual Bergman space.

Let I be a closed ideal of A . In this paper, we are interested in I with $\dim A/I < \infty$. Then A/I is called a *Q-algebra*. Two dimensional Q-algebras can be described easily; that is, $I = \{f \in A; \phi_1(f) = \phi_2(f) = 0\}$, where $\phi_j \in M(A)$ ($j = 1, 2$), or $I = \{f \in A; \phi(f) = D_\phi(f) = 0\}$, where $\phi \in M(A)$ and D_ϕ is a bounded point derivation at ϕ . One of the authors [3] showed that a two dimensional operator algebra on a Hilbert space is a Q-algebra. It seems to be worthwhile to describe a finite dimensional Q-algebra. In Section 2, we describe an ideal I with $\dim A/I < \infty$ using a theorem of T. W. Gamelin [2]. As a result, a finite dimensional Q-algebra is described.

When M is a closed subspace of $H^p(m)$ and $AM \subset M$, M is called an *invariant subspace for A* . In this paper, we are interested in M with $\dim H^p(m)/M < \infty$. When A is the polydisc algebra on T^n and m is the normalized Lebesgue measure on T^n , a finite codimensional invariant subspace M in $H^p(m)$ was described by P. Ahern and D. N. Clark [1] using the ideals in the polynomial ring $\mathcal{C}[z_1, \dots, z_n]$ of finite codimension whose zero sets are contained in the polydisk D^n . In Section 3, for an

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arbitrary uniform algebra A we describe a finite codimensional invariant subspace M in $H^p(m)$ using the result in Section 2.

2. Finite codimensional ideal. Let $\phi \in M(A)$. A closed subalgebra H of A is a (ϕ, k) -subalgebra if there is a sequence of closed subalgebras $A = A_0 \supset A_1 \supset \cdots \supset A_k = H$ such that A_j is the kernel of a continuous point derivation D_j of A_{j-1} at ϕ . If H is a (ϕ, k) -subalgebra of A , then H has finite codimension in A and $M(H) = M(A)$ by [2, Lemma 9.1].

If I is a closed ideal of A and A/I is of finite dimension, $B = \mathcal{C} + I$ is a closed subalgebra of A , and A/B is of finite dimension. By a theorem of T. W. Gamelin [2, Theorem 9.8], we can describe B and so also I . Since B is a special closed subalgebra of A we can describe I more explicitly.

THEOREM 1. *If I is a closed ideal of A and A/I is of finite dimension, then there exists a closed subalgebra $E = E(I)$ of A such that $E = \{f \in A : \phi_1(f) = \cdots = \phi_n(f)\}$, $1 \leq n < \infty$, $\{\phi_j\} \subset M(A)$ and*

$$I = H_\phi^E \cap \ker \phi,$$

where $\phi = \phi_j|_E$, $1 \leq j \leq n$ and H_ϕ^E is a (ϕ, k) -subalgebra with respect to E for some k .

Proof. Put $H = I + \mathcal{C}$; then A/H is of finite dimension. By a theorem of T. W. Gamelin [2, Theorem 9.8], H can be obtained from A in two steps.

(i) There exist pairs of points ψ_j, ψ'_j , $1 \leq j \leq \ell$, in $M(A)$ such that if E consists of the $f \in A$ such that $\psi_j(f) = \psi'_j(f)$, $1 \leq j \leq \ell$, then $H \subset E \subset A$.

(ii) There exist distinct points $\theta_j \in M(E)$ and θ_j -subalgebras H_j of E , $1 \leq j \leq k$, such that $H = H_1 \cap \cdots \cap H_k$.

Put $\tilde{\psi}_j = \psi_j|_E = \psi'_j|_E$ for $1 \leq j \leq \ell$; then $\tilde{\psi}_j$ belongs to $M(E)$. Since I is an ideal of A , $I \subset \bigcap_{j=1}^{\ell} \ker \tilde{\psi}_j$. To see this, let $f \in A$ such that $\psi_j(f) \neq \psi'_j(f)$. If $g \in I$, then $fg \in I$ but $\psi_j(fg) \neq \psi'_j(fg)$ when $\tilde{\psi}_j(g) \neq 0$. This contradicts the fact that $fg \in E$. Thus $\tilde{\psi}_j(g) = 0$. Hence $I \subset \bigcap_{j=1}^{\ell} \ker \tilde{\psi}_j$ and so $H \subseteq \bigcap_{j=1}^{\ell} \ker \tilde{\psi}_j + \mathcal{C}$. By the definition of E , $\tilde{\psi}_1 = \cdots = \tilde{\psi}_\ell$. Therefore E has the form $E = \{f \in A; \phi_1(f) = \cdots = \phi_n(f)\}$, $1 \leq n < \infty$, and $\{\phi_j\} \subset M(A)$.

For each j with $1 \leq j \leq k$, H_j is a θ_j -subalgebra of E for $\theta_j \in M(E)$. Hence there is a sequence of closed subalgebras $E = E_{j0} \supset E_{j1} \supset \cdots \supset E_{j\ell_j} = H_j$ such that E_{jt} is the kernel of a continuous point derivation D_{jt} of $E_{j,t-1}$ at θ_j . We shall write $E_{j\ell_j} = \ker D_{\theta_j}$, where D_{θ_j} is a derivation on $E_{j(\ell_j-1)}$. Then $H = \bigcap_{j=1}^k \ker D_{\theta_j}$ and so $I = \{\bigcap_{j=1}^k \ker D_{\theta_j}\} \cap \ker \theta$, for some $\theta \in M(H)$. Suppose that g is an arbitrary function in I . For any j ($1 \leq j \leq k$), there exists a function $f \in E_{j(\ell_j-1)}$ such that $f \notin E_{j\ell_j} = \ker D_{\theta_j}$. Since $fg \in I$ and $D_{\theta_j}(g) = 0$, $D_{\theta_j}(fg) = \theta_j(g)D_{\theta_j}(f) = 0$ because D_{θ_j} is a derivation on $E_{j(\ell_j-1)}$. This implies that $\theta_j(g) = 0$. Hence $I \subset \bigcap_{j=1}^k \ker \theta_j$. Therefore by the definition of E , $\theta_1 = \cdots = \theta_k \in M(E)$, and so $H_1 = \cdots = H_k$. Thus $\theta_1|_H = \theta$ and $I = (\ker D_{\theta_1}) \cap \ker \theta_1$. Since $I \subset \bigcap_{j=1}^n \ker \phi_j$, $I \subset (\ker \phi_1) \cap (\ker D_{\theta_1}) \cap \ker \theta_1$ and so $\phi_1|_E = \theta_1$.

COROLLARY 1. *If I is a closed ideal of A and A/I is of finite dimension 2, then $I = \{f \in A; \phi_1(f) = \phi_2(f) = 0\}$, where $\phi_j \in M(A)$ ($j = 1, 2$) and $\phi_1 \neq \phi_2$, or $I = \{f \in A; \phi(f) = D_\phi(f) = 0\}$, where $\phi \in M(A)$, and D_ϕ is a bounded point derivation at ϕ .*

Proof. When $\dim A/I = 2$, by Theorem 1, $E = A$ or $E = \{f \in A; \phi_1(f) = \phi_2(f)\}$. If $E = A$, then $H_\phi^E = \{f \in A; D_\phi(f) = 0\}$ and if $E = \{f \in A; \phi_1(f) = \phi_2(f)\}$, then $H_\phi^E = E$, since $\dim A/H_\phi^E = 1$ because $H_\phi^E = I + \mathcal{C}$. This implies the corollary.

COROLLARY 2. *If B is a finite dimensional Q -algebra and $B_0 = \text{rad } B$ is its radical, then there exist subalgebras B_1, B_2, \dots, B_{k+1} in B_0 such that $B_{k+1} = \{0\}$, $\dim B_j/B_{j+1} = 1$ and B_{j+1} is an ideal of B_j for $j = 0, 1, \dots, k$. Hence $\text{rad } B$ has a basis $\{f_0, f_1, \dots, f_k\}$ such that $(f_j)^{2(k+1-j)} = 0$ for $j = 0, 1, \dots, k$.*

Proof. Since B is a Q -algebra, $B = A/I$ for some uniform algebra A and some closed ideal I of A . Also, since B is of finite dimension, we can apply Theorem 1 to A and I . In the notation of Theorem 1, $\text{rad } B = \{f \in E; \phi(f) = 0\}/I$. Since H_ϕ^E is a ϕ -subalgebra with respect to E , there exists a sequence of closed subalgebras $E = E_0 \supset E_1 \supset \dots \supset E_{k+1} = H_\phi^E$ such that E_j is the kernel of a continuous point derivation D_j of E_{j-1} at ϕ . Hence $E_{j+1} \cap \ker \phi$ is an ideal of $E_j \cap \ker \phi$ and $\dim \{E_j \cap \ker \phi / E_{j+1} \cap \ker \phi\} = 1$. Put $B_j = (E_j \cap \ker \phi)/I$. Then $\dim B_j/B_{j+1} = 1$ and B_{j+1} is an ideal of B_j , for $j = 0, 1, \dots, k$, and $B_{k+1} = \{0\}$. For each j , there exists f_j such that $B_j = \langle f_j \rangle + B_{j+1}$ and then $\{f_0, f_1, \dots, f_k\}$ is a basis of $\text{rad } B = B_0$. Observe that f_j^2 belongs to B_{j+1} because $E_{j+1} = \ker D_{j+1}$. Thus $(f_j)^{2(k+1-j)} = 0$.

3. Finite codimensional invariant subspace. For a subset S of $H^p(m)$, $[S]_p$ denotes the closure of S in $H^p(m)$.

THEOREM 2. *If M is an invariant subspace of H^p with $\dim H^p/M = n < \infty$, then there exists a closed ideal of A such that $\dim A/I = n$, $[I]_p = M$ and $I = M \cap A$. If H_ϕ^E is a (ϕ, k) -subalgebra with respect to $E = E(I)$, then $[E_j]_p \supset [E_{j+1}]_p$ for any j ($0 \leq j \leq k-1$) and $\dim H^p/[E]_p = \dim A/E$. Conversely, if $\dim A/I = n < \infty$, then $\dim H^p/[I]_p \leq n$. If $[E_j]_p \supset [E_{j+1}]_p$, for any j with $0 \leq j \leq k-1$ and $\dim H^p/[E]_p = \dim A/E$, then $\dim H^p/[I]_p = n$ and $[I]_p \cap A = I$.*

Proof. Suppose that M is an invariant subspace of $H^p(m)$ and $\dim H^p(m)/M = n < \infty$. Then there exist n linearly independent linear functionals ψ_1, \dots, ψ_n in $(H^p)^*$ such that $\psi_j = 0$ on M for $1 \leq j \leq n$. Put $\phi_j = \psi_j|_A$ for $1 \leq j \leq n$ and $I = M \cap A$. Then $I = \bigcap_{j=1}^n \ker \phi_j$ and so $\dim A/I = n$. For ϕ_1, \dots, ϕ_n are independent linear functionals in A^* because A is dense in $H^p(m)$. If $M \supset [I]_p$, then there exists $\psi_{n+1} \in (H^p)^*$ such that $\psi_{n+1} = 0$ on $[I]_p$ and $\psi_1, \dots, \psi_n, \psi_{n+1}$ are independent linear functionals in $(H^p)^*$. If we put $\phi_{n+1} = \psi_{n+1}|_A$, then $\phi_1, \dots, \phi_n, \phi_{n+1}$ are independent linear functionals in A^* and $I \subseteq \bigcap_{j=1}^{n+1} \ker \phi_j$. This contradiction implies that $M = [I]_p$. Note that $\dim H^p/[E_k] = \dim H^p/[I]_p - 1 = \dim A/I - 1 = \dim A/E_k$. If $\dim H^p/[E_0]_p < \dim A/E_0$ where $E_0 = E$ or $[E_j]_p = [E_{j+1}]_p$, for some j ($0 \leq j \leq k-1$), then this contradicts the fact that $\dim H^p/[E_k]_p = \dim A/E_k$. The converse is clear.

COROLLARY 3. *If M is an invariant subspace of H^p with $\dim H^p/M = 2$, then $M = \{f \in H^p; \Phi_1(f) = \Phi_2(f) = 0\}$, where $\Phi_j \in (H^p)^*$, and $\Phi_j(fg) = \Phi_j(f)\Phi_j(g)$, for $f \in H^p$ and $g \in A$, or $M = \{f \in H^p; \Phi(f) = D_\phi(f) = 0\}$, where $\Phi, D_\phi \in (H^p)^*$, $\Phi(fg) = \Phi(f)\Phi(g)$ and $D_\phi(fg) = \Phi(f)D_\phi(g) + \Phi(g)D_\phi(f)$ for $f \in H^p$ and $g \in A$.*

Proof. This follows from Corollary 1 and Theorem 2.

COROLLARY 4. *If M is an invariant subspace of H^p with $\dim H^p/M = n < \infty$, then there exist f_1, \dots, f_n in A such that $\{f_j + M\}_{j=1}^n$ is a basis in H^p/M .*

Proof. By Theorem 2, if $I = M \cap A$, then $\dim A/I = n$ and $M = [I]_p$. Hence there exist f_1, \dots, f_n in A such that $\{f_j + I\}_{j=1}^n$ is a basis in A/I . If f_j belongs to M , then f_j also belongs to $M \cap A = I$ and so f_j does not belong to M . This proves the corollary.

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