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## On the commutativity of fundamental groups of complements to plane curves

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### Introduction

In this paper, we prove the following:

**THEOREM.** *Let  $C \subset \mathbb{P}^3$  be a complex reduced irreducible space curve which is non-degenerate (i.e.,  $C$  is not contained in any plane). Let  $p: C \rightarrow \mathbb{P}^2$  be a general projection. Then the topological fundamental group  $\pi_1(\mathbb{P}^2 \setminus p(C))$  is abelian.*

When  $C$  is non-singular, the image of the general projection has only nodes as its singularities, and hence the complement has an abelian fundamental group thanks to Fulton–Deligne’s theorem on Zariski conjecture ([1, 2]). The point of our theorem is that we make no assumptions on the singularities of the space curve  $C$ .

We apply this theorem to the problem of commutativity of fundamental groups of complements to plane curves of a given degree with prescribed numbers of nodes and cusps. Here a cusp means a germ of curve singularity which is analytically isomorphic to a small neighborhood of the singular point of the affine curve  $x^2 + y^3 = 0$ . The strongest result concerned with this problem is Nori’s result [5, proposition 6.5]. (Nori considered not only plane curves with only nodes and cusps but curves on arbitrary surfaces with any kind of singularities.) Let  $C \subset \mathbb{P}^2$  be a reduced irreducible plane curve of degree  $d$  whose singular locus consists of  $n$  nodes and  $k$  cusps.

**THEOREM (Nori).** *Suppose that  $2n + 6k < d^2$ . Then  $\pi_1(\mathbb{P}^2 \setminus C)$  is abelian.*

We will prove a proposition which enlarges slightly the region of  $(d, n, k)$  on which the commutativity of  $\pi_1(\mathbb{P}^2 \setminus C)$  is guaranteed.

**PROPOSITION 1.** *Suppose that  $2n \geq d^2 - 5d + 8$ . Then  $\pi_1(\mathbb{P}^2 \setminus C)$  is abelian.*

### 1. Proof of Theorem

Let  $G$  denote the Grassmannian variety  $\text{Grass}(\mathbb{P}^1, \mathbb{P}^3)$  of all lines in  $\mathbb{P}^3$ , and let  $(\mathbb{P}^3)^*$  be the dual projective space  $\text{Grass}(\mathbb{P}^2, \mathbb{P}^3)$  of  $\mathbb{P}^3$ . For a plane  $\Pi \in (\mathbb{P}^3)^*$ , we put

$$\Sigma(\Pi) := \{l \in G; l \subset \Pi\}.$$

We also put

$$U := \{l \in G; C \cap l = \emptyset\}.$$

When  $\Pi \in (\mathbb{P}^3)^*$  is general, no three points among  $C \cap \Pi$  are co-linear. Indeed, let  $B \subset (\mathbb{P}^3)^*$  be the Zariski open dense subset consisting of the planes which intersect  $C$

at its non-singular points transversely. Let  $\Pi_0 \in B$  be a plane which serves as a base point of  $B$ . Since  $C$  is irreducible, the monodromy group of the action of  $\pi_1(B, \Pi_0)$  on the set  $C \cap \Pi_0$  is the full symmetric group, because the action is 2-transitive and the monodromy group contains a simple transposition (cf. [4, Uniform Position Lemma]). Hence if there exist co-linear three points among  $C \cap \Pi$  for a general  $\Pi \in (\mathbb{P}^3)^*$ , then all points in  $C \cap \Pi$  are on a line for a general  $\Pi$ . Note that this condition is closed for the choice of  $\Pi$  in  $(\mathbb{P}^3)^*$  because  $C$  is non-degenerate. Thus all points in  $C \cap \Pi$  are on a line for every  $\Pi$ , which is absurd. Consequently,  $\Sigma(\Pi) \setminus U$  is a union of lines on  $\Sigma(\Pi) \cong \mathbb{P}^2$ , which is a normal crossing divisor when the plane  $\Pi$  is generally chosen. Hence  $\pi_1(\Sigma(\Pi) \cap U)$  is abelian for a general  $\Pi$  because of Fulton–Deligne’s theorem ([1–3]). Let  $I \subset (\mathbb{P}^3)^* \times G$  be the closed subvariety  $\{(\Pi, l); \Pi \supset l\}$ . The second projection  $\rho_2: I \rightarrow G$  is a fibre bundle with fibres isomorphic to  $\mathbb{P}^1$ . Hence  $\rho_2$  induces an isomorphism  $\pi_1(\rho_2^{-1}(U)) \cong \pi_1(U)$ . Consider the projection  $\rho_1: \rho_2^{-1}(U) \rightarrow (\mathbb{P}^3)^*$ . The fibre  $\rho_1^{-1}(\Pi) \subset \rho_2^{-1}(U)$  over  $\Pi \in (\mathbb{P}^3)^*$  is isomorphic to  $\Sigma(\Pi) \cap U$ , which is non-empty and non-singular for every  $\Pi$ . We apply Nori’s Lemma [5, lemma 1.5 (C)] to conclude that the inclusion of the fibre induces a surjection  $\pi_1(\Sigma(\Pi) \cap U) \rightarrow \pi_1(\rho_2^{-1}(U))$  for a general  $\Pi$ . Hence  $\pi_1(\rho_2^{-1}(U))$  is abelian, and thus  $\pi_1(U)$  is also abelian.

We fix a plane  $\Pi_0 \subset \mathbb{P}^3$  and denote by  $A$  the complement  $\mathbb{P}^3 \setminus \Pi_0$ . Let  $J_A \subset A \times G$  be the closed subvariety  $\{(P, l); P \in l\}$ . Consider the projection  $\tau_2: J_A \rightarrow G$ . If  $l \not\subset \Pi_0$ , then  $\tau_2^{-1}(l)$  is isomorphic to  $\mathbb{A}^1$ , while if  $l \subset \Pi_0$ , then  $\tau_2^{-1}(l)$  is empty. Thus  $\tau_2$  induces an isomorphism  $\pi_1(\tau_2^{-1}(U)) \cong \pi_1(U \setminus \Sigma(\Pi_0))$ . Since  $\Sigma(\Pi_0)$  is of codimension 2 in  $G$ , we have  $\pi_1(U \setminus \Sigma(\Pi_0)) \cong \pi_1(U)$ . Therefore,  $\pi_1(\tau_2^{-1}(U))$  is isomorphic to  $\pi_1(U)$ . We have an isomorphism  $J_A \cong A \times \Pi_0$  given by  $(P, l) \mapsto (P, l \cap \Pi_0)$ . Let  $\tilde{\tau}_2: A \times \Pi_0 \rightarrow G$  be the morphism corresponding to  $\tau_2: J_A \rightarrow G$  via this isomorphism. Let  $D \subset G$  be the Zariski closed subset  $G \setminus U$ . We provide it with the reduced structure. For  $P \in A$ , consider the scheme theoretic intersection

$$D_P := (\{P\} \times \Pi_0) \cap \tilde{\tau}_2^{-1}(D),$$

which is considered as a sub-scheme of  $\Pi_0$ . If  $P \in C$ , then  $D_P = \Pi_0$ . Suppose that  $P \notin C$ . Let  $p\langle P \rangle: C \rightarrow \Pi_0$  denote the projection with the centre  $P$ . Then we see that the reduced part  $(D_P)_{\text{red}}$  of  $D_P$  coincides with the reduced part  $p\langle P \rangle(C)_{\text{red}}$  of the image of the projection  $p\langle P \rangle$ . Hence  $D_P$  is a reduced divisor of  $\Pi_0$  if  $\deg p\langle P \rangle(C) = \deg C$ .

*Claim.* There are at most finitely many points  $P \in A \setminus C$  such that the degree of  $p\langle P \rangle(C)_{\text{red}}$  is less than the degree of  $C$ .

Since  $C \cap A \subset A$  is of codimension 2, Claim implies that the locus of all points  $P \notin A$  such that the scheme theoretic intersection  $D_P$  is *not* a reduced divisor of  $\Pi_0$  is of codimension 2 in  $A$ . Let  $P \in A$  be generally chosen. Applying [6, theorem 1], we get  $\pi_1(\Pi_0 \setminus p\langle P \rangle(C)) \cong \pi_1(\tilde{\tau}_2^{-1}(U)) \cong \pi_1(U)$ . Therefore  $\pi_1(\Pi_0 \setminus p\langle P \rangle(C))$  is abelian, and Theorem is proved.

*Proof of Claim.* Suppose that there exists an irreducible curve  $\Xi \subset A \setminus C$  such that  $\deg p\langle P \rangle(C)_{\text{red}} < \deg C$  for all  $P \in \Xi$ . Let  $P_0$  be a general point of  $\Xi$  and  $Q_0$  a general point of  $C$ . Since  $P_0$  is not on  $C$ , the projection  $p\langle P_0 \rangle: C \rightarrow \Pi_0$  with the centre  $P_0$  must be of mapping degree  $\geq 2$  onto its image. Thus there exists a point  $R_0 \in C$  such

that  $Q_0 \neq R_0$  and  $p\langle P_0\rangle(Q_0) = p\langle P_0\rangle(R_0)$ . Since  $Q_0$  is chosen generally,  $p\langle P_0\rangle(Q_0)$  is a non-singular point of  $p\langle P_0\rangle(C)_{\text{red}}$ , and the morphism  $p\langle P_0\rangle: C \rightarrow p\langle P_0\rangle(C)_{\text{red}}$  is étale at  $Q_0$  and at  $R_0$ . This implies that the tangent lines  $l(Q_0)$  and  $l(R_0)$  of  $C$  at  $Q_0$  and at  $R_0$ , respectively, intersect each other in  $\mathbb{P}^3$ . Since  $P_0$  is also chosen generally on  $\Xi$ , there exist small open neighbourhoods (in the sense of complex analytic geometry)  $V \subset \Xi$  of  $P_0$  in  $\Xi$  and  $W \subset C$  of  $R_0$  in  $C$ , and an isomorphism  $f: V \xrightarrow{\sim} W$  such that  $p\langle P\rangle(Q_0) = p\langle P\rangle(f(P))$  for all  $P \in V$ . This implies that the tangent line  $l(R)$  of  $C$  at  $R$  intersects  $l(Q_0)$  for all  $R \in W$ . Therefore  $l(R) \cap l(Q_0) \neq \emptyset$  for every non-singular point  $R$  of  $C$ . However, consider the projection  $\lambda: C \rightarrow \mathbb{P}^1$  with the centre  $l(Q_0)$ . Since  $C$  is non-degenerate,  $\lambda$  is surjective, and hence, by Sard's theorem,  $\lambda$  is smooth at a general point  $R \in C$ . This implies that  $l(R) \cap l(Q_0) = \emptyset$  for a general  $R \in C$ , and we get a contradiction.

## 2. Proof of Proposition 1

First note that if  $2n \geq d^2 - 5d + 8$  and  $2n + 2k < d^2 - 4d + 3$ , then  $2n + 6k$  is automatically less than  $d^2$ . Therefore, taking Nori's theorem into account, we may assume that the inequality  $2n + 2k \geq d^2 - 4d + 3$  also holds.

Let  $\nu: \tilde{C} \rightarrow C$  be the normalization of  $C$ . The genus of  $\tilde{C}$  is given by

$$g = (d-1)(d-2)/2 - n - k.$$

The assumed inequalities imply that

$$d \geq \max(2g+1, g+3), \quad \text{and} \quad k \leq d - g - 3. \quad (2.1)$$

Let  $L$  be the pull-back of  $\mathcal{O}_C(1)$  by  $\nu$ , which is an invertible sheaf of degree  $d$  on  $\tilde{C}$ . By the first inequality of (2.1),  $L$  is very ample and  $\dim |L| = d - g$ . Let  $C'$  be the image of the embedding of  $\tilde{C}$  in  $\mathbb{P}^{d-g}$  by  $L$ . Then  $C$  is obtained by a certain projection  $C' \rightarrow \mathbb{P}^2$ . Let  $\Gamma \subset \mathbb{P}^{d-g}$  be the centre of this projection, which is a linear subspace of codimension 3. Let  $T \subset \mathbb{P}^{d-g}$  be the union of all lines tangent to  $C'$ . Then  $\Gamma$  intersects  $T$  at distinct  $k$  points  $P_1, \dots, P_k$ , each of which corresponds to a cusp of  $C$  bijectively. By the second inequality of (2.1), there exists a linear subspace  $\Lambda \subset \Gamma$  of codimension 1 which contains  $P_1, \dots, P_k$ . Let  $p\langle \Lambda\rangle: C' \rightarrow \mathbb{P}^3$  be the projection with the centre  $\Lambda$ . The image  $C'' \subset \mathbb{P}^3$  is a non-degenerate curve, which has at least  $k$  singular points  $Q_1, \dots, Q_k$  corresponding to  $P_1, \dots, P_k \in \Lambda \cap T$ . The projection  $p\langle \Gamma\rangle: C' \rightarrow C$  with the centre  $\Gamma$  factors as  $C' \rightarrow C'' \rightarrow C$ , and  $C'' \rightarrow C$  maps  $Q_1, \dots, Q_k$  to the cusps of  $C$  bijectively. Therefore,  $C''$  has  $k$  cusps and some nodes (possibly none) as its only singularities. Thus the image  $p(C'')$  of the general projection of  $C''$  to  $\mathbb{P}^2$  has the same numbers and types of singular points as the original curve  $C$ . Note that  $C$  and  $p(C'')$  are contained in an irreducible equisingular family of plane curves, every member of which is obtained as an image of a projection from  $C'' \subset \mathbb{P}^3$  to  $\mathbb{P}^2$ . Thus our main theorem implies that  $\pi_1(\mathbb{P}^2 \setminus C)$  is abelian.

## 3. Examples

Let  $\mathcal{A}$  denote the set of triples  $(d, n, k)$  with  $d \in \mathbb{Z}_{>0}$ ,  $n \in \mathbb{Z}_{\geq 0}$  and  $k \in \mathbb{Z}_{\geq 0}$  for which the following hold:

- (i) There exists a reduced irreducible plane curve  $C \subset \mathbb{P}^2$  of degree  $d$  with  $n$  nodes and  $k$  cusps as its only singularities.

(ii) For any plane curve  $C$  as in (i),  $\pi_1(\mathbb{P}^2 \setminus C)$  is abelian.

Nori's result says that if  $(d, n, k)$  satisfies (i) and  $2n + 6k < d^2$ , then  $(d, n, k) \in \mathcal{A}$ . Using Proposition 1, we shall enlarge the known region of  $\mathcal{A}$ .

Let  $C \subset \mathbb{P}^n$  be a non-degenerate irreducible curve of degree  $d$  whose singular locus consists of only nodes and cusps. Let  $k$  be the number of cusps. We write by  $\text{Tan}(C) \subset \mathbb{P}^n$  the union of all lines in  $\mathbb{P}^n$  tangent to  $C$  at its non-singular points. The following proposition is elementary:

**PROPOSITION 2.** *If  $P$  is a general point of  $\text{Tan}(C)$ , then the image of the projection  $C \rightarrow \mathbb{P}^{n-1}$  with the centre  $P$  is of degree  $d$ , has only nodes and cusps as its singularities, and the number of cusps is  $k + 1$ .*

Let  $C$  be a non-singular curve of genus  $g$ , and  $L$  a line bundle on  $C$  of degree  $d \geq \max(2g + 1, g + 3)$ . We embed  $C$  into  $\mathbb{P}^{d-g}$  by  $|L|$ . Applying Proposition 2 to  $C \subset \mathbb{P}^{d-g}$  repeatedly, we obtain a non-degenerate space curve  $C' \subset \mathbb{P}^3$  of degree  $d$  with certain number of nodes (possibly none) and  $d - g - 3$  cusps as its only singularities. Let  $C'' \subset \mathbb{P}^2$  be the image of a general projection  $C' \rightarrow \mathbb{P}^2$ . Then  $C''$  is an irreducible plane curve of degree  $d$  with

$$n = (d^2 - 5d + 8)/2 \text{ nodes} \quad \text{and} \quad k = d - g - 3 \text{ cusps}$$

as its only singularities. Since  $2n \geq d^2 - 5d + 8$ , Proposition 1 implies that  $(d, n, k) \in \mathcal{A}$ . Thus, choosing a line bundle of degree  $\geq 6g + 10$ , we obtain a region of  $\mathcal{A}$  not covered by Nori's theorem.

Note that there is another known region of  $\mathcal{A}$ , which is given by the following theorem due to Zariski [7]. Let  $R(d, k) \subset \mathbb{P}_*(\Gamma(\mathbb{P}^2, \mathcal{O}(d)))$  be the locus of all *rational* plane curves  $C$  of degree  $d$  with  $n$  nodes and  $k$  cusps as its only singularities. Since the genus of the normalization of  $C$  is 0, we have  $n = (d - 1)(d - 2)/2 - k$ .

**THEOREM (Zariski).** (1) *The locus  $R(d, k)$  is non-empty if and only if  $k \leq 3(d - 2)/2$ .*  
 (2) *For a member  $C$  of  $R(d, k)$ ,  $\pi_1(\mathbb{P}^2 \setminus C)$  is abelian unless  $d$  is even and  $k = 3(d - 2)/2$ .*

Thus, if  $n + k = (d - 1)(d - 2)/2$  and  $k \leq (3d - 7)/2$ , we have  $(d, n, k) \in \mathcal{A}$ .

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