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A REMARK ON A CLOSED ORIENTABLE HYPERSURFACE WITH CONSTANT REDUCED MEAN CURVATURE

By

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Introduction. The theorem concerning homothety given by A. Aepli (Theorem I in [3]), which brings a relation between the global form and reduced mean curvature rH_1 (with respect to a fixed point O) of a closed orientable hypersurface in $(n+1)$ -dimensional Euclidean space R^{n+1} , suggests us to investigate the global property of a closed orientable hypersurface with constant reduced mean curvature $rH_1=c$. A star-shaped hypersurface (with respect to O) with $rH_1=c$ is a hypersphere around O (cf. footnotes 10), 11) in [3]). So we can confine ourselves to the case of non-star-shaped closed hypersurfaces. In this paper we are going to show that under a certain condition ("radial convexity", defined in § 1) such hypersurfaces cannot have constant reduced mean curvature, and that any ovaloid with $rH_1=c$ is a hypersphere around O :

Theorem. *Let F be a closed orientable hypersurface of class 2 which is radially convex with respect to O in the strict sense. For $p \in F$ we denote by p^* the second point on F which lies on the ray with p . Then there exists at least one point $p \in F$, for which $rH_1(p) \neq rH_1(p^*)$.*

Corollary. *Let F be a closed orientable hypersurface of class 2 which is star-shaped or radially convex in the strict sense with respect to O . Then, if $rH_1(p)=c$ for all $p \in F$, F is a hypersphere around O . Especially any ovaloid with $rH_1=c$ is a hypersphere around O .*

In § 1 we show the existence of a central projection from a radially convex hypersurface onto itself, which reverses the orientation. In § 2 we prove a lemma, which is closely related to the theorem concerning homothety. Thereby the theorem and the corollary are proved immediately in § 3.

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§ 1. Radially Convex Hypersurfaces. We follow the notations in [2] and [3]. Let O be a fixed point in $(n+1)$ -dimensional Euclidean space R^{n+1} , $n \geq 2$.

Definition. We call the hypersurface F radially convex (with respect to O), if $O \in F$ and any half straight line with its endpoint O (i.e., a ray from O) has either two intersecting points with F , only one contact point or no common point. We call F radially convex in the strict sense, if F is radially convex and has an additional condition such that the direction of any ray which is tangent to F is not an asymptotic direction of F at the contact point.

If we put the point O infinitely far away in a certain fixed direction $\mathbf{r}$, radial convexity becomes "convexity in the direction $\mathbf{r}$ " (cf. [1] p. 187, [2] p. 192). We may consider that O is the origin of R^{n-1} . Throughout this section let F be a closed orientable hypersurface of class 2, which is radially convex in the strict sense. We denote by F_0 a subset of F given by

$$F_0 := \{p \in F \mid (\mathbf{r}\mathbf{x})(p) = 0\} \quad (\text{the "Schattengrenze" of } F),$$

where $\mathbf{x} = (x^1, \dots, x^{n-1})$ is the position vector of F and $\mathbf{r}$ is the normal unit vector of F .

We obtain a mapping $p^* = \varphi(p)$ from F onto itself, if for any point p on F we take the ray from O through p and choose the second intersecting point p^* on F (if the ray is tangent to F at p , i.e., $p \in F_0$, let $\varphi(p) = p$). We are to show that φ is a regular centralprojection of class 2, which reverses the orientation.

It is clear that φ is bijective and $\varphi \circ \varphi = id$, that is, $\varphi^{-1} = \varphi$. We have to prove only the continuity in order to show that φ is a homeomorphism (and therefore a centralprojection). For this purpose we consider first a local mapping π_p for $p \in F$.

Let S^n be the unit n -sphere around O and be oriented with the outer normal vector. F and S^n are given by

$$\begin{aligned} \mathbf{x} &= \mathbf{x}(u^1, \dots, u^n) = (x^1(u^1), \dots, x^{n-1}(u^1)), \\ \tilde{\mathbf{x}} &= \tilde{\mathbf{x}}(\tilde{u}^1, \dots, \tilde{u}^n) = (\tilde{x}^1(\tilde{u}^1), \dots, \tilde{x}^{n-1}(\tilde{u}^1)) \end{aligned}$$

respectively. Let π be a mapping $\pi: F \rightarrow S^n$ expressed by

$$\pi((x^1, \dots, x^{n-1})) := (x^1, \dots, x^{n-1})/\rho, \quad \text{i.e., } \tilde{x}^A = \rho^{-1}x^A,$$

where ρ is the function of class 2 on F given by

$$\rho(p) := \|\mathbf{x}(p)\| = \left\{ \sum_{A=1}^{n-1} (x^A)^2 \right\}^{\frac{1}{2}},$$

i.e., the distance between p and O . $\rho(p) \neq 0$ for all $p \in F$, so π is well defined all over F and of class 2. Since $x^A(p) \neq 0$ for at least one A , we can assume

now $x^1(p) > 0$ without losing generality (after a suitable rotation of the coordinate axes, if necessary). It follows that also $\tilde{x}^1(\pi(p)) > 0$, so we can take $(\tilde{x}^2, \dots, \tilde{x}^{n+1})$ as a local coordinate of S^n at $\pi(p)$, preserving the orientation, that is, $\tilde{u}^i = \tilde{x}^{i+1}$ for $i = 1, \dots, n$. Let J be the functional determinant of π .

$$J = \frac{\partial(\tilde{u})}{\partial(u)} = \det \left| \frac{\partial \tilde{x}^{i+1}}{\partial u^j} \right| = |(\rho^{-1} x^{i+1})_j| = \rho^{-n-2} x^1 \sqrt{g}(\mathfrak{u}_g).$$

Therefore the sign of $J(p)$ depends only upon $(\mathfrak{u}_g)(p)$ and, if $(\mathfrak{u}_g)(p) \neq 0$, there exists a neighbourhood U_p of p , such that $\pi_p := \pi|_{U_p}$ is diffeomorphic.

If $p \in F_0$, then it follows that $p^* = \varphi(p) \in F_0$, accordingly there are two local diffeomorphic mappings $\pi_p: U_p \rightarrow S^n$ and $\pi_{p^*}: U_{p^*} \rightarrow S^n$, for which

$$\begin{aligned} \pi_p(p) &= \pi_{p^*}(p^*), & \pi_p(U_p) \cap \pi_{p^*}(U_{p^*}) &\neq \emptyset, \\ \varphi|_W &= \pi_{p^*}^{-1} \pi_p|_W, & \text{where } W &:= \pi_p^{-1}(\pi_p(U_p) \cap \pi_{p^*}(U_{p^*})). \end{aligned}$$

Therefore φ is continuous at p .

If $p \in F_0$, let $\{p_n\}$ be an arbitrary sequence of points on F with $\lim_{n \rightarrow \infty} p_n = p$. The sequence $\{\varphi(p_n)\}$ on F has at least one convergent point p_0 , because F is compact. Accordingly there exists a subsequence $\{p_{i_n}\}$ of $\{p_n\}$ such that $\lim_{n \rightarrow \infty} \varphi(p_{i_n}) = p_0$. Then, because of the continuity of π , the two sequences on S^n $\{\pi(p_{i_n})\}$ and $\{\pi(\varphi(p_{i_n}))\}$ converge to $\pi(p)$ and $\pi(p_0)$ respectively. Since $\pi \circ \varphi = \pi$, these two sequences coincide completely, that means $\pi(p_0) = \pi(p)$, in other words, p_0 and p lie on the same ray. On the other hand since $p \in F_0$, it follows that $p_0 = p$, that is, all convergent points of $\{\varphi(p_n)\}$ coincide with p and $\{\varphi(p_n)\}$ converge to $p = \varphi(p)$. So φ is continuous at p .

The differentiability and the regularity of φ comes out now directly from the radial convexity of F in the strict sense (cf. footnote 2) in [3]. Therefore φ is a regular central projection of class 2 from F onto itself. In order to show that φ reverses the orientation of F , we consider next two connected subsets F_+ and F_- of F , such that $F = F_+ \cup F_0 \cup F_-$.

We construct a continuous function ϕ on F by

$$\phi(p) := \rho(p) - \rho(\varphi(p)).$$

$\phi(p) = 0$ if and only if $p \in F_0$. We put

$$F_+ := \{p \in F \mid \phi(p) > 0\}, \quad F_- := \{p \in F \mid \phi(p) < 0\},$$

and so $F = F_+ \cup F_0 \cup F_-$, $F_0 \cap F_+ = F_0 \cap F_- = F_+ \cap F_- = \emptyset$. Because of the continuity of ϕ , F_+ and F_- are open and mapped by φ mutually onto themselves. And F_0 is not empty. Because, if $F_+ = \emptyset$ (and so $F_- = \emptyset$), it follows that $F_0 = F \neq \emptyset$; if $F_- \neq \emptyset$ (and so $F_+ \neq \emptyset$), ϕ has both positive and negative values

and so because of the continuity of ϕ , $F_0 = \phi^{-1}(\{0\})$ is not empty.

The gradient of the function $\mathfrak{r}\mathfrak{x}$ of class 1 on F is not equal to zero at $p \in F_0$. Because the vector $\mathfrak{x}(p)$ lies in the tangent plane of F at p , i.e., $\mathfrak{x}(p) = \alpha^i \mathfrak{x}_i(p)$, and differs from any asymptotic direction at p , i.e., $l_{jk}(p) \alpha^j \alpha^k \neq 0$. Since the i -th component of $\text{Grad}(\mathfrak{r}\mathfrak{x})$ is $(\mathfrak{r}\mathfrak{x})_i = -l_{ki} \alpha^k$, so $(\mathfrak{r}\mathfrak{x})_i(p) \neq 0$ for at least one i , i.e., $\text{Grad}(\mathfrak{r}\mathfrak{x})(p)$ is not equal to zero. Accordingly any neighbourhood of p possesses a point, at which $\mathfrak{r}\mathfrak{x} \neq 0$, namely the point of $F - F_0$, that is, F_0 has no inner point (and so $F - F_0 \neq \emptyset$ and F_+ and F_- are not empty). We can therefore choose a sequence $\{p_n\}$ on $F - F_0$, that converges to p . Since φ is continuous, the sequence $\{\varphi(p_n)\}$ converges to $\varphi(p) = p$. If we construct a sequence $\{q_n\}$ by $q_{2n-1} := p_n$, $q_{2n} := \varphi(p_n)$, it is convergent to p , and because of $\varphi(q_{2n-1}) = q_{2n}$, one of q_{2n} and q_{2n-1} belongs to F_+ and the other to F_- . Therefore p is a boundary point of both F_+ and F_- , that is, $F_0 \subset \partial F_+ \cap \partial F_-$. Since $\overline{F_+} \cap F_- = \emptyset$, $\partial F_+ = \overline{F_+} - F_+ \subset F - F_- - F_- = F_0$. In the same way $\partial F_- \subset F_0$. Therefore F_0 is the common boundary of F_+ and F_- .

Let $u = (u^i)$ be the local coordinate at $p \in F_0$. Since $\text{Grad}(\mathfrak{r}\mathfrak{x})(p) \neq 0$, we assume now $(\mathfrak{r}\mathfrak{x})_n \neq 0$ at p . Then F_0 is given by $\mathfrak{r}\mathfrak{x}(u^i) = 0$, and because of $(\mathfrak{r}\mathfrak{x})_n(p) \neq 0$, this is equivalent to $u^n = h(u^1, \dots, u^{n-1})$ in a suitable neighbourhood U of p , where h is a function of class 1. We introduce a new parameter $v = (v^i)$ by

$$\begin{aligned} v^n &:= u^n - h(u^1, \dots, u^{n-1}), \\ v^i &:= u^i - u^i(p) \quad \text{for } i = 1, \dots, n-1. \end{aligned}$$

Although v is not a local coordinate since it is not of class 2, v maps U C^1 -diffeomorphic onto a neighbourhood $v(U)$ of the origin of the $(v^1, \dots, v^n)$ -space, $F_0 \cap U$ to the $(v^1, \dots, v^{n-1})$ -plane and p to the origin. The neighbourhood $v(U)$ includes an open ball B around the origin. We put

$$\begin{aligned} B_+ &:= \{(v^i) \in B \mid v^n > 0\}, \quad B_- := \{(v^i) \in B \mid v^n < 0\}, \\ B_0 &:= \{(v^i) \in B \mid v^n = 0\} = v(F_0 \cap U) \cap B \\ V &:= v^{-1}(B). \end{aligned}$$

B_+ does not include points of both $v(F_+ \cap V)$ and $v(F_- \cap V)$ at the same time. Because, if there exist two points $p_0 \in B_+ \cap v(F_- \cap V)$ and $p_1 \in B_- \cap v(F_+ \cap V)$, the segment s from $p_0 = s(0)$ to $p_1 = s(1)$ is included completely in B_+ . We construct a curve in $v^{-1}(B_+)$ by $c := v^{-1} \circ s$. If we put $\lambda(t) := \phi(c(t))$ for $t \in [0, 1]$, it follows that $\lambda(0) > 0$, $\lambda(1) < 0$. Since λ is continuous, there exists a number $t_0 \in [0, 1]$ with $\lambda(t_0) = 0$, i.e., $\phi(c(t_0)) = 0$ hence $c(t_0) \in F_0$. This is contradictory to the fact that $v^{-1}(B_+)$ includes no point of F_0 and that $c([0, 1]) \subset$

$v^{-1}(B_+)$. Therefore either $B_+ = v(F_+ \cap V)$ (accordingly $B_- = v(F_- \cap V)$) or $B_+ = v(F_- \cap V)$ (accordingly $B_- = v(F_+ \cap V)$). Because of the archwise connectedness of B_+ and B_- , $v^{-1}(B_+)$ and $v^{-1}(B_-)$ are archwise connected. Therefore any point p of F_0 possesses a neighbourhood V such that $V_+ := F_+ \cap V$ and $V_- := F_- \cap V$ are archwise connected.

Let p_0 and p_1 be arbitrary points of F_+ . We show that there is a curve in F_+ from p_0 to p_1 and so F_+ is archwise connected (therefore connected). We can choose a curve $\tilde{c}: [0, 1] \rightarrow F$ with $\tilde{c}(0) = p_0$ and $\tilde{c}(1) = p_1$, because F is archwise connected. We consider here a continuous mapping $\tau: F \rightarrow F_+ \cup F_0$ defined by

$$\tau|_{F_+ \cup F_0} := id_{F_+ \cup F_0}, \quad \tau|_{F_-} = \varphi|_{F_-}.$$

Then $c := \tau \circ \tilde{c}$ is a curve in $F_+ \cup F_0$ from p_0 to p_1 .

We construct an open covering of $c([0, 1])$ as follows; for $p = c(t) \in F_+$ let U_p be a neighbourhood of p , which is archwise connected and included in F_+ ; for $q \in F_0$ let V_q be a neighbourhood of q , for which $V_{q,+} := V_q \cap F_+$ is archwise connected, as seen above. Being compact, $c([0, 1])$ is covered by a finite number of U_{p_i} and V_{q_j} , i.e., $\left(\bigcup_{i=1}^n U_{p_i}\right) \cup \left(\bigcup_{j=1}^m V_{q_j}\right) \supset c([0, 1])$. We put $A := \left(\bigcup_{i=1}^n U_{p_i}\right) \cup \left(\bigcup_{j=1}^m V_{q_j}\right)$, and so $A \subset F_+$ and $p_0, p_1 \in A$. We show that A is archwise connected. If $m=0$, it is clear, because $c([0, 1]) \subset A$ and $U_{p_i} \cap c([0, 1]) \neq \emptyset$ for all i . If $m \neq 0$, among U_{p_i} and V_{q_j} there exists at least one neighbourhood which possesses a common point of $c([0, 1])$ with V_{q_i} . In the case that U_{p_i} possesses a common point $c(t_i)$ with V_{q_i} , $U_{p_i} \cap V_{q_i} \neq \emptyset$, and so $U_{p_i} \cup V_{q_i,+}$ is archwise connected. In the case that V_{q_j} possesses a common point $c(t_j)$ with V_{q_i} , we choose a neighbourhood W of $c(t_j)$, which is included in $V_{q_j} \cap V_{q_i}$. Since $c(t_j) \in F_- \cup F_0 = \bar{F}_-$, W possesses a point q of F_- and $q \in V_{q_j} \cap V_{q_i,+}$ and so $V_{q_j} \cup V_{q_i,+}$ is archwise connected. If we continue this process finite times, A becomes at last archwise connected. So we can choose a curve $\bar{c}: [0, 1] \rightarrow A \subset F_+$ with $\bar{c}(0) = p_0$ and $\bar{c}(1) = p_1$. Hence F_+ is connected. Likewise F_- is connected.

Because of the connectedness of F_+ and F_- , and because $(n\kappa)(p) = 0$ if and only if $p \in F_0$, the continuous function $n\kappa$ has a fixed sign on F_+ and F_- respectively. Since $\text{Grad}(n\kappa)(p) \neq 0$ for $p \in F_0$, $n\kappa$ exchanges the sign in a neighbourhood of $p \in F_0$. Therefore it follows that $n\kappa > 0$ on one of F_+ and F_- , and $n\kappa < 0$ on the other.

Let $p \in F - F_0$ and $p^* = \varphi(p)$. φ is given in a suitable neighbourhood W of p by $\varphi = \pi_p^{-1} \circ \pi_p$. Since φ maps F_+ and F_- mutually onto themselves, one of p and p^* belongs to F_- and the other to F_+ , and so $(n\kappa)(p)$ and $(n\kappa)(p^*)$

have different signs. Accordingly the functional determinants $J(p)$ and $J(p^*)$ of π_p and π_{p^*} at p and p^* respectively have different signs, because the sign of $J(q)$ depends only upon $(n\mathfrak{x})(q)$ for $q \in F - F_0$. Therefore the functional determinant of $\pi_p^{-1} \circ \pi_p$ has the negative sign, that is, the sign of the functional determinant of φ is negative on $F - F_0$.

Since φ is everywhere regular, i.e., the functional determinant of φ is nowhere zero, it is negative all over F because of the continuity. Hence φ reverses the orientation of F . So we have proved the following lemma:

Lemma 1. *Let F be a closed orientable hypersurface of class 2 which is radially convex in the strict sense. Then there exists a regular central projection of class 2 from F onto itself, which reverses the orientation.*

§ 2. A Lemma on a Central projection.

Lemma 2. *Let F and $\bar{F}$ be closed orientable hypersurfaces of class 2 which hold a central projection $T: F \rightarrow \bar{F}$. Let T be of class 2 and regular and preserve the orientation. Let the "Schattengrenze" of F ($F_0 := \{p \in F \mid (n\mathfrak{x})(p) = 0\}$) include no inner point. Then there exists at least one point $p \in F$ with $\bar{r}\bar{H}_1(T_p) \neq -rH_1(p)$, where rH_1 and $\bar{r}\bar{H}_1$ are reduced mean curvatures with respect to O of F and $\bar{F}$ respectively.*

Proof. We prove following the proof of Theorem I in [3]. F and $\bar{F}$ are given by

$$\begin{aligned}\mathfrak{x} &= \mathfrak{x}(u^1, \dots, u^n), \\ \bar{\mathfrak{x}} &= \bar{\mathfrak{x}}(u^1, \dots, u^n) = f(u^1, \dots, u^n) \mathfrak{x}(u^1, \dots, u^n) \quad (f > 0) \quad (a)\end{aligned}$$

respectively. If we exchange " $n - \bar{n}$ " of the formula (1.5) in [3] for " $n + \bar{n}$ ", we obtain

$$d(n + \bar{n}, \mathfrak{x}, d\mathfrak{x}, \dots, d\mathfrak{x}) = (n + \bar{n}, d\mathfrak{x}, \dots, d\mathfrak{x}) - (\mathfrak{x}, d\bar{n} + d\bar{n}, d\mathfrak{x}, \dots, d\mathfrak{x}) \quad (b)$$

Then the formulae (1.6) and (1.7) in [3] are exchanged for

$$(n + \bar{n}, d\mathfrak{x}, \dots, d\mathfrak{x}) = \frac{1}{2} n! (n + \bar{n})^2 dA \quad (c)$$

$$-(\mathfrak{x}, d\bar{n} + d\bar{n}, d\mathfrak{x}, \dots, d\mathfrak{x}) = n! (H_1 + f\bar{H}_1) (n\mathfrak{x}) dA \quad (d)$$

respectively. From (b), (c), (d) and the Formula of Stokes we have

$$\frac{1}{n!} \int_{\partial F} (n + \bar{n}, \mathfrak{x}, d\mathfrak{x}, \dots, d\mathfrak{x}) = \frac{1}{2} \int_F (n + \bar{n})^2 dA + \int_F (rH_1 + \bar{r}\bar{H}_1) (n\mathfrak{x}) dA \quad (e)$$

where $\mathfrak{r} = \frac{\mathfrak{x}}{r}$. If we assume here that $rH_1(p) = -\bar{r}\bar{H}_1(T_p)$ for all $p \in F$, then

$$\int_{\mathcal{F}} (\mathfrak{n} + \bar{\mathfrak{n}})^2 dA = 0,$$

so that $\mathfrak{n} = -\bar{\mathfrak{n}}$. Therefore

$$0 = \bar{\mathfrak{x}}_i \bar{\mathfrak{n}} = (f\mathfrak{x})_i (-\mathfrak{n}) = -f_i(\mathfrak{x}\mathfrak{n}) - f(\mathfrak{x}_i \mathfrak{n}) = -f_i(\mathfrak{x}\mathfrak{n})$$

for $i = 1, \dots, n$, and the “Schattengrenze” including no inner point, it follows that $f_i = 0$, i. e., $f = \text{const.}$ Accordingly the hypersurface $\bar{F}$ is given by $f\mathfrak{x}$ with positive constant f , and so $\mathfrak{n} = \bar{\mathfrak{n}}$, which is contradictory to $\mathfrak{n} = -\bar{\mathfrak{n}}$.
Q.E.D.

In the case of parallel mappings this conclusion is not true. Because, if we exchange “ $\bar{\mathfrak{n}} - \mathfrak{n}$ ” of the formula (9.5) in [2] for “ $\bar{\mathfrak{n}} + \mathfrak{n}$ ”, we obtain

$$\begin{aligned} n \int_{\mathcal{F}} (\bar{H}_1 + H_1)(\mathfrak{w}\mathfrak{n}) dA + \frac{1}{2} \int_{\mathcal{F}} (\bar{\mathfrak{n}} + \mathfrak{n})^2 (d\bar{A} - dA) \\ = \frac{1}{(n-1)!} \int_{\partial \mathcal{F}} (\bar{\mathfrak{n}} + \mathfrak{n}, \mathfrak{w}, d\mathfrak{x}, \dots, d\mathfrak{x}). \end{aligned}$$

Assuming $-H_1 = \bar{H}_1$, we obtain

$$\int_{\mathcal{F}} (\bar{\mathfrak{n}} + \mathfrak{n})^2 (d\bar{A} - dA) = 0.$$

Since $d\bar{A} - dA$ does not hold a fixed sign, we cannot conclude $\mathfrak{n} = -\bar{\mathfrak{n}}$.

§ 3. A Closed Hypersurface with $rH_1 = c$.

Theorem. Let F be a closed orientable hypersurface of class 2 which is radially convex with respect to O in the strict sense. For $p \in F$ we denote by p^* the second point on F , which lies on the ray with p . Then there exists at least one point $p \in F$, for which $rH_1(p) \neq rH_1(p^*)$.

Proof. According to Lemma 1 there exists a regular central projection φ of class 2 from F onto itself, which reverses the orientation. Let $\tilde{F}$ be a hypersurface which is identical to F except the orientation, so that the mapping $\varphi: F \rightarrow \tilde{F}$ preserves the orientation. By exchanging the orientation of F $H_1(p) = -\tilde{H}_1(p)$, and so it follows that $rH_1(p) = -\tilde{r}\tilde{H}_1(p)$. Since $\varphi: F \rightarrow \tilde{F}$ satisfies the assumption of Lemma 2, there exists at least one point $p \in F$ such that $rH_1(p) \neq -\tilde{r}\tilde{H}_1(\varphi(p)) = rH_1(\varphi(p)) = rH_1(p^*)$.
Q.E.D.

Corollary. Let F be a closed orientable hypersurface of class 2 which is star-shaped or radially convex in the strict sense with respect to O . Then, if $rH_1(p) = c$ for all $p \in F$, F is a hypersphere around O . Especially any ovaloid with $rH_1 = c$ is a hypersphere around O .

Proof. The conclusion comes out immediately from Theorem above and

the footnotes 10) and 11) in [3]. Since an n -dimensional ovaloid is a closed hypersurface with the positive (or negative) definite second fundamental form, which is the boundary of a bounded convex set in R^{n+1} , the following is verified without difficulty; if an ovaloid has O outer side, it is radially convex with respect to O in the strict sense; if it has O inner side, it is star-shaped with respect to O . If O lies on an ovaloid, then $rH_1=0$ at O and it cannot have constant reduced mean curvature. Q.E.D.

In the case of a closed curve C in R^2 , the analogous definition of radial convexity in the strict sense (*we call C radially convex in the strict sense, if C is radially convex and further $\text{Grad}(\mathfrak{N}\mathfrak{x})(p) \neq 0$ for $(\mathfrak{N}\mathfrak{x})(p) = 0$ $p \in C$*) and the integral formula

$$(\mathfrak{t} + \bar{\mathfrak{t}}) \mathfrak{x} \Big|_{\mathfrak{a}c} = \frac{1}{2} \int_C (\mathfrak{t} + \bar{\mathfrak{t}})^2 ds + \int_C (rk + \bar{r}\bar{k}) (\mathfrak{N}\mathfrak{t}) ds$$

bring the analogous conclusions.

References

- [1] H. HOPF und K. VOSS: Ein Satz aus der Flächentheorie im Großen, Archiv der Math. 3 (1952), 187-192.
- [2] K. VOSS: Einige differentialgeometrische Kongruenzsätze für geschlossene Flächen und Hyperflächen, Math. Ann. 131 (1956), 180-218.
- [3] A. AEPPLI: Einige Ähnlichkeits- und Symmetriesätze für differenzierbare Flächen im Raum, Comment. Math. Helv. 33 (1959), 174-195.

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