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ON SOME PROPERTIES OF A SUBMANIFOLD WITH CONSTANT MEAN CURVATURE IN A RIEMANN SPACE

By

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Introduction. It has been proved by H. Liebmann [1]¹⁾ that the only ovaloids with constant mean curvature in an Euclidean space E^3 are the spheres. The same problem for closed hypersurfaces in an n -dimensional Euclidean space E^n has been investigated by W. Süss [2], T. Bonnesen and W. Fenchel [3], (cf. p. 118), H. Hopf [4], C. C. Hsiung [5] and A. D. Alexandrov [7]. The analogous problem for closed hypersurfaces in an n -dimensional Riemann space R^n has been discussed by C. C. Hsiung [6], A. D. Alexandrov [8], Y. Katsurada [9], [10], [11], K. Yano [12] and T. Ôtsuki [15].

It is the aim of the present authors to investigate the analogous problem for an m -dimensional closed submanifold V^m in the n -dimensional Riemann space R^n . The generalized Minkowski formulas for V^m in R^n are given in §1. In §2 and §3, we derive the second and the third integral formulas which are valid for V^m in R^n under some conditions. Making use of those integral formulas, certain property of V^m in constant Riemann curvature space is proved in §4. Also, we prove a theorem for V^m in R^n which admits an one-parameter group of homothetic transformations.

§1. Generalized Minkowski formulas for a submanifold. We consider a Riemann space R^n ($n \geq 3$) of class C^ν ($\nu \geq 3$) which admits an one-parameter continuous group G of transformations generated by an infinitesimal transformation

$$(1.1) \quad \bar{x}^i = x^i + \xi^i(x) \delta\tau,$$

where x^i are local coordinates in R^n and ξ^i are the components of a contra-variant vector ξ . We suppose that the paths of these transformations cover R^n simply and that ξ is everywhere continuous and $\neq 0$. If the vector ξ is a Killing vector, a homothetic Killing, a conformal Killing etc. ([13], p. 32),

1) Numbers in brackets refer to the references at the end of the paper.

then the group is called isometric, homothetic, conformal etc. respectively.

We now consider a closed orientable submanifold V^m of class C^3 imbedded in R^n , locally given by

$$x^i = x^i(u^a)^{(2)}.$$

Let the contravariant unit vectors i_{λ}^i ($\lambda = 1, 2, \dots, m$) span the tangent vector space at each point of V^m and they be orthogonal to one another. We shall indicate by n_P^i ($P = m+1, m+2, \dots, n$) the contravariant unit vectors normal to V^m and suppose that they are mutually orthogonal.

Putting

$$(1.2) \quad B_j^i = \sum_{\lambda=1}^m i_{\lambda}^i i_{\lambda}^j, \quad C_j^i = \sum_{P=m+1}^n n_P^i n_P^j$$

we have

$$(1.3) \quad B_j^i + C_j^i = \delta_j^i.$$

The first fundamental tensor $g_{\alpha\beta}$ of V^m is given by

$$g_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial u^{\alpha}} \frac{\partial x^j}{\partial u^{\beta}}$$

and $g^{\alpha\beta}$ are defined by $g^{\alpha\beta} g_{\beta\gamma} = \delta_{\gamma}^{\alpha}$, where g_{ij} denotes the first fundamental tensor of R^n .

Denoting by “;” the operation of D -symbol due to van der Waerden-Bortolotti ([16], p. 254), we have

$$(1.4) \quad \left(\frac{\partial x^i}{\partial u^{\alpha}} \right)_{;\beta} = H_{\alpha\beta}^i,$$

where $H_{\alpha\beta}^i$ means the Euler-Schouten curvature tensor ([16], p. 256). Then, putting $H_{\alpha\beta}^i n_P^i = b_{\alpha\beta}^P$, we have

$$(1.5) \quad H_{\alpha\beta}^i = \sum_{P=m+1}^n b_{\alpha\beta}^P n_P^i.$$

Multiplying (1.5) by $g^{\alpha\beta}$ and contracting, we get

$$(1.6) \quad g^{\alpha\beta} H_{\alpha\beta}^i = \sum_{P=m+1}^n m H_1 n_P^i,$$

where we put $H_1 = \frac{1}{m} g^{\alpha\beta} b_{\alpha\beta}^P$ and H_1 is the first mean curvature of V^m for the normal direction n_P^i .

2) Throughout the present paper the Latin indices run from 1 to n and the Greek indices from 1 to m ($m \leq n-1$).

Let $n_{\substack{E \\ m+1}}^i$ be the unit vector of the same direction to the vector $g^{\alpha\beta}H_{\alpha\beta}^i$. Then, we may consider $n_{\substack{E \\ m+1}}^i$ as one of the unit normal vectors of V^m , that is, $n_{\substack{m+1 \\ E}}^i = n_{\substack{E \\ m+1}}^i$. In this case, we obtain from (1.6)

$$(1.7) \quad g^{\alpha\beta}H_{\alpha\beta}^i = mH_1n_{\substack{E \\ m+1}}^i,$$

where H_1 is the first mean curvature of V^m .

To the vector ξ , there belongs a covariant vector $\bar{\xi}$ of V^m with the components

$$(1.8) \quad \bar{\xi}_\alpha = \frac{\partial x^i}{\partial u^\alpha} \xi_i.$$

Covariantly differentiating³⁾ the vector $\bar{\xi}_\alpha$, by means of (1.4) we have

$$(1.9) \quad \bar{\xi}_{\alpha;\beta} = H_{\alpha\beta}^i \xi_i + \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \xi_{i;j}.$$

Multiplying (1.9) by $g^{\alpha\beta}$ and contracting, we get by (1.7)

$$(1.10) \quad g^{\alpha\beta} \bar{\xi}_{\alpha;\beta} = mH_1n_{\substack{E \\ m+1}}^i \xi_i + \frac{1}{2} g^{\alpha\beta} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \mathfrak{L}_\xi g_{ij},$$

where $\mathfrak{L}_\xi g_{ij}$ is the Lie derivative of g_{ij} with respect to the infinitesimal transformation (1.1) ([13], p. 5). If we put

$$\frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \mathfrak{L}_\xi g_{ij} = \mathfrak{L}_\xi g_{\alpha\beta},$$

then (1.10) is rewritten as follows:

$$\frac{1}{m} \bar{\xi}^a_{;a} = H_1n_{\substack{E \\ m+1}}^i \xi_i + \frac{1}{2m} g^{\alpha\beta} \mathfrak{L}_\xi g_{\alpha\beta}.$$

Since V^m is closed and orientable, we have

$$\int \cdots \int_{V^m} \bar{\xi}^a_{;a} dA = 0,$$

where dA is the area element of V^m ([14], p. 31). Thus we obtain the following integral formula:

$$\int \cdots \int_{V^m} H_1n_{\substack{E \\ m+1}}^i \xi_i dA + \frac{1}{2m} \int \cdots \int_{V^m} g^{\alpha\beta} \mathfrak{L}_\xi g_{\alpha\beta} dA = 0. \quad (I')$$

3) In this paper, covariant differentiation means always the operation of D -symbol.

Let the group G be conformal, that is, ξ^i satisfy the equation

$$(1.11) \quad \mathfrak{L}_{\xi} g_{ij} \equiv \xi_{i;j} + \xi_{j;i} = 2\Phi g_{ij}$$

(cf. [13], p. 32). Then (I') becomes

$$\int \cdots \int_{V^m} H_1 n^i \xi_i dA + \int \cdots \int_{V^m} \Phi dA = 0. \quad (I')_c$$

Let G be homothetic, that is, $\Phi \equiv c = \text{const.}$ Then

$$\int \cdots \int_{V^m} H_1 n^i \xi_i dA + c \int \cdots \int_{V^m} dA = 0. \quad (I')_h$$

Let G be isometric, that is, $c=0$. Then

$$\int \cdots \int_{V^m} H_1 n^i \xi_i dA = 0. \quad (I')_i$$

§ 2. The second integral formula. By virtue of (1.2) and (1.3) it follows that

$$(2.1) \quad \begin{aligned} n^i_{;a} &= C^i_{j;k} n^j \frac{\partial x^k}{\partial u^a} \\ &= - \sum_{\lambda=1}^m \left(i_{j;k} n^j \frac{\partial x^k}{\partial u^a} \right) i^i_{\lambda}. \end{aligned}$$

Then we may put

$$(2.2) \quad n^i_{;a} = r^r_{\alpha} \frac{\partial x^i}{\partial u^r}.$$

Multiplying (2.2) by $g_{ij} \frac{\partial x^j}{\partial u^{\delta}}$ and summing for i and j , we have

$$(2.3) \quad g_{ij} \frac{\partial x^j}{\partial u^{\delta}} n^i_{;a} = r^r_{\alpha} g_{r\delta}.$$

Since we have

$$(2.4) \quad b_{\delta a} = g_{ij} \left(\frac{\partial x^j}{\partial u^{\delta}} \right)_{;a} n^i = -g_{ij} \frac{\partial x^j}{\partial u^{\delta}} n^i_{;a},$$

we obtain from (2.2), (2.3) and (2.4)

$$(2.5) \quad n^i_{;a} = -b^r_{\alpha} \frac{\partial x^i}{\partial u^r},$$

where $b^r_{\alpha} = g^{\delta r} b_{\alpha \delta}$.

To the vector ξ given in §1, we put

$$(2.6) \quad \eta_\alpha = n_{E;\alpha}^i \xi_i.$$

Covariantly differentiating (2.6), by means of (1.4) and (2.5) we get

$$\eta_{\alpha;\bar{\beta}} = - \left(b_{E;\bar{\beta}}^\gamma \frac{\partial x^i}{\partial u^\gamma} \xi_i + b_{E;\bar{\beta}}^\gamma H_{\gamma\bar{\beta}}^i \xi_i + b_{E;\bar{\beta}}^\gamma \frac{\partial x^i}{\partial u^\gamma} \frac{\partial x^j}{\partial u^{\bar{\beta}}} \xi_{i;j} - F_{\alpha\bar{\beta}} \right),$$

where $F_{\alpha\bar{\beta}} \stackrel{\text{def.}}{=} \Gamma_{E;\bar{\beta}}^{\gamma\delta} n_{\gamma;\alpha}^i \xi_i$ and $\Gamma_{E;\bar{\beta}}^{\gamma\delta}$ means $\nabla_j n_{\gamma;\alpha}^i B_{\alpha}^j n_k^{\bar{\beta}}$.

Multiplying the above equation by $g^{\alpha\bar{\beta}}$ and contracting, we obtain

$$(2.7) \quad g^{\alpha\bar{\beta}} \eta_{\alpha;\bar{\beta}} = - \left(g^{\alpha\bar{\beta}} b_{E;\bar{\beta}}^\gamma \frac{\partial x^i}{\partial u^\gamma} \xi_i + g^{\alpha\bar{\beta}} b_{E;\bar{\beta}}^\gamma H_{\gamma\bar{\beta}}^i \xi_i + g^{\alpha\bar{\beta}} b_{E;\bar{\beta}}^\gamma \frac{\partial x^i}{\partial u^\gamma} \frac{\partial x^j}{\partial u^{\bar{\beta}}} \xi_{i;j} - g^{\alpha\bar{\beta}} F_{\alpha\bar{\beta}} \right).$$

We shall first calculate the first term of the right hand side of (2.7):

$$(2.8) \quad g^{\alpha\bar{\beta}} b_{E;\bar{\beta}}^\gamma \frac{\partial x^i}{\partial u^\gamma} \xi_i = g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} b_{E;\bar{\delta}}^\gamma \frac{\partial x^i}{\partial u^\gamma} \xi_i.$$

Since the Codazzi equations hold good for the submanifold V^m in a Riemann space R^n , we have

$$b_{E;\bar{\delta}}^\gamma - b_{E;\bar{\beta}}^\gamma g_{\alpha\bar{\beta}}^{\gamma\bar{\delta}} = -R_{ikjl} n_{E;\bar{\beta}}^i \frac{\partial x^k}{\partial u^\alpha} \frac{\partial x^j}{\partial u^{\bar{\delta}}} \frac{\partial x^l}{\partial u^{\bar{\beta}}} \quad ([16], \text{ p. 266}),$$

where R_{ikjl} is the curvature tensor of R^n .

Then, from (2.8) we get

$$(2.9) \quad \begin{aligned} g^{\alpha\bar{\beta}} b_{E;\bar{\beta}}^\gamma \frac{\partial x^i}{\partial u^\gamma} \xi_i &= g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left(b_{E;\bar{\delta}}^\gamma - R_{ikjl} n_{E;\bar{\beta}}^i \frac{\partial x^k}{\partial u^\alpha} \frac{\partial x^j}{\partial u^{\bar{\delta}}} \frac{\partial x^l}{\partial u^{\bar{\beta}}} \right) \frac{\partial x^h}{\partial u^\gamma} \xi_h \\ &= \left(m H_{1;\bar{\delta}} - R_{ikjl} n_{E;\bar{\beta}}^i \frac{\partial x^k}{\partial u^\alpha} \frac{\partial x^j}{\partial u^{\bar{\delta}}} \frac{\partial x^l}{\partial u^{\bar{\beta}}} g^{\alpha\bar{\beta}} \right) g^{\gamma\bar{\delta}} \frac{\partial x^h}{\partial u^\gamma} \xi_h. \end{aligned}$$

Next, we calculate the second term of the right hand side of (2.7). By means of (1.5), it follows that

$$(2.10) \quad g^{\alpha\bar{\beta}} b_{E;\bar{\beta}}^\gamma H_{\gamma\bar{\beta}}^i \xi_i = g^{\alpha\bar{\beta}} b_{E;\bar{\beta}}^\gamma \left(b_{\gamma\bar{\beta}}^i n_{E;\bar{\beta}}^i + \sum_{P=m+2}^n b_{\gamma\bar{\beta}}^i n_P^i \right) \xi_i.$$

Now, we assume that at each point on V^m the contravariant vector ξ^i is contained in the vector space spanned by $m+1$ independent vectors $\frac{\partial x^i}{\partial u^\alpha}$ ($\alpha=1, 2, \dots, m$) and $n_{E;\bar{\beta}}^i$. This assumption is always satisfied for the case $m=n-1$, that is, V^m is a hypersurface in R^n . Especially, if we consider a closed curve in 3-dimensional Euclidean space, the above condition for the vector ξ^i means that ξ^i is contained in the osculating plane at each point on the curve. Now, we consider a closed plane curve and take a point in the

interior of the curve as the origin of an euclidean coordinates x^i . Attaching to each point x the vector $\xi(x)$ with components $\xi^i = x^i$, the vector ξ satisfies our assumption.

From our assumption for ξ^i and (2.10), it follows that

$$g^{\alpha\beta} b_{\alpha}^{\gamma} H_{\gamma\beta}^i \xi_i = g^{\alpha\beta} b_{\alpha}^{\gamma} b_{\gamma\beta}^i n^i \xi_i.$$

Then, we have

$$(2.11) \quad g^{\alpha\beta} b_{\alpha}^{\gamma} H_{\gamma\beta}^i \xi_i = m \left\{ m H_1^2 - (m-1) H_2 \right\} n^i \xi_i.$$

By means of (2.9) and (2.11), we get from (2.7)

$$(2.12) \quad \eta_{i;\alpha}^{\alpha} = - \left(m H_{1;\delta} - R_{ikjl} n^i \frac{\partial x^k}{\partial u^{\alpha}} \frac{\partial x^j}{\partial u^{\delta}} \frac{\partial x^l}{\partial u^{\beta}} g^{\alpha\beta} \right) g^{\gamma\delta} \frac{\partial x^h}{\partial u^{\gamma}} \xi_h \\ - m \left\{ m H_1^2 - (m-1) H_2 \right\} n^i \xi_i - \frac{1}{2} H_{\xi}^{\beta\gamma} \mathfrak{L} g_{\beta\gamma} + g^{\alpha\beta} F_{\alpha\beta},$$

where $H_{\xi}^{\beta\gamma} = g^{\alpha\beta} g^{\gamma\delta} b_{\alpha\delta}$ and H_2 denotes the second mean curvature of V^m for the normal direction n^i . By means of (2.12) we have the following integral formula :

$$\int \cdots \int_{V^m} \left(m H_{1;\delta} - R_{ikjl} n^i \frac{\partial x^k}{\partial u^{\alpha}} \frac{\partial x^j}{\partial u^{\delta}} \frac{\partial x^l}{\partial u^{\beta}} g^{\alpha\beta} \right) g^{\gamma\delta} \frac{\partial x^h}{\partial u^{\gamma}} \xi_h dA \\ = - \int \cdots \int_{V^m} \left[m \left\{ (m H_1^2 - (m-1) H_2) n^i \xi_i + \frac{1}{2m} H_{\xi}^{\beta\gamma} \mathfrak{L} g_{\beta\gamma} \right\} - g^{\alpha\beta} F_{\alpha\beta} \right] dA. \quad (II')$$

If the group G of transformations is conformal, (II') becomes

$$\int \cdots \int_{V^m} \left(m H_{1;\delta} - R_{ikjl} n^i \frac{\partial x^k}{\partial u^{\alpha}} \frac{\partial x^j}{\partial u^{\delta}} \frac{\partial x^l}{\partial u^{\beta}} g^{\alpha\beta} \right) g^{\gamma\delta} \frac{\partial x^h}{\partial u^{\gamma}} \xi_h dA \\ = - \int \cdots \int_{V^m} \left[m \left\{ (m H_1^2 - (m-1) H_2) n^i \xi_i + \Phi H_1 \right\} - g^{\alpha\beta} F_{\alpha\beta} \right] dA. \quad (II')_c$$

§3. The third integral formula. Putting

$$(3.1) \quad \rho = n^i \xi_i$$

by means of (2.6) we have

$$(3.2) \quad \rho_{i;\alpha} = \eta_{\alpha} + n^i \xi_{i;j} \frac{\partial x^j}{\partial u^{\alpha}}.$$

By covariant differentiation of (3.2), it follows that

$$(3.3) \quad \rho_{;a;\beta} = \eta_{a;\beta} + n_{E;\beta}^i \xi_{i;j} \frac{\partial x^j}{\partial u^a} - F_{\alpha\beta} + n_{E;\beta}^i \xi_{i;j;k} \frac{\partial x^j}{\partial u^a} \frac{\partial x^k}{\partial u^\beta} + n_{E;\beta}^i \xi_{i;j} H_{\alpha\beta}{}^j.$$

Multiplying (3.3) by $g^{\alpha\beta}$ and contracting, by virtue of (1.7) and (2.5) we get

$$(3.4) \quad \begin{aligned} g^{\alpha\beta} \rho_{;a;\beta} &= \eta_{;a}^\alpha - g^{\alpha\beta} b_{\beta}^i \xi_{i;j} \frac{\partial x^j}{\partial u^a} \frac{\partial x^j}{\partial u^\alpha} - g^{\alpha\beta} F_{\alpha\beta} \\ &\quad + n_{E;\beta}^i \xi_{i;j;k} \frac{\partial x^j}{\partial u^a} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta} + m H_1 n_{E;\beta}^i n^j \xi_{i;j}. \end{aligned}$$

Therefore we have the following integral formula :

$$\int_{V^m} \left\{ -\frac{1}{2} H_{E;\xi}^{\alpha\gamma} \mathfrak{L} g_{\alpha\gamma} + n_{E;\beta}^i \xi_{i;j;k} \frac{\partial x^j}{\partial u^a} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta} + \frac{m}{2} H_1 n_{E;\beta}^i n^j \xi_{i;j} - g^{\alpha\beta} F_{\alpha\beta} \right\} dA = 0. \quad (III')$$

Let the group G be conformal. Then we have

$$(3.5) \quad \xi_{;i;j;k}^i = \mathfrak{L}_{\xi} \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} - R_{jkh}^i \xi^h,$$

$$(3.6) \quad \mathfrak{L}_{\xi} \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \delta_j^i \Phi_k + \delta_k^i \Phi_j - \Phi^i g_{jk},$$

where $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ are the Christoffel symbols and $\Phi_k = \Phi_{;k}$, $\Phi^i = g^{ij} \Phi_j$ ([13], p. 160).

By means of (1.11), the first term of the left hand side of (III') becomes

$$(3.7) \quad -\frac{1}{2} H_{E;\xi}^{\alpha\gamma} \mathfrak{L} g_{\alpha\gamma} = -m \Phi H_1.$$

Now, we calculate the second term of the left hand side of (III'). By means of (3.5) and (3.6) it follows that

$$(3.8) \quad n_{E;\beta}^i \xi_{i;j;k} \frac{\partial x^j}{\partial u^a} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta} = -m n_{E;\beta}^i \Phi_i - R_{ijhk} n_{E;\beta}^i \xi^h \frac{\partial x^j}{\partial u^a} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta}.$$

We consider that the vector ξ^i satisfy the assumption stated in §2. Then, we may put

$$(3.9) \quad \xi^h = \varphi^r \frac{\partial x^h}{\partial u^r} + \rho n_{E;\beta}^h.$$

From (3.9), we get

$$(3.10) \quad \varphi^r = g_{lm} \xi^l \frac{\partial x^m}{\partial u^r} g^{r\delta}.$$

Making use of (3.9) and (3.10), it follows that

$$\begin{aligned}
 (3.11) \quad & R_{ijhk} n^i \xi^h \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta} \\
 &= R_{ijhk} n^i \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^h}{\partial u^r} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta} g^{r\delta} \xi_m \frac{\partial x^m}{\partial u^\delta} \\
 &\quad + \rho R_{ijhk} n^i n^h \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta}.
 \end{aligned}$$

By means of (1.11), the third term of the left hand side of (III') becomes

$$(3.12) \quad \frac{m}{2} H_1 n^i n^j \mathfrak{L} g_{ij} = m \Phi H_1.$$

Therefore, by virtue of (3.7), (3.8), (3.11) and (3.12), (III') is rewritten as follows :

$$\begin{aligned}
 (3.13) \quad & \int \cdots \int_{V^m} \left\{ m n^i \Phi_i + R_{ijhk} n^i \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^h}{\partial u^r} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta} g^{r\delta} \xi_m \frac{\partial x^m}{\partial u^\delta} \right. \\
 & \quad \left. + \rho R_{ijhk} n^i n^h \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta} + g^{\alpha\beta} F_{\alpha\beta} \right\} d\Lambda = 0.
 \end{aligned}$$

By means of (II')_e in §2 and (3.13), finally we obtain the following integral formula :

$$\begin{aligned}
 & \int \cdots \int_{V^m} \left\{ m H_{1;\delta} + \rho R_{ijhk} n^i n^h \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta} \right\} d\Lambda \\
 &= - \int \cdots \int_{V^m} m \left\{ (m H_1^2 - (m-1) H_2) n^i \xi_i + n^i \Phi_i + \Phi H_1 \right\} d\Lambda. \quad (III')_e
 \end{aligned}$$

If the group G is homothetic, we have

$$\begin{aligned}
 & \int \cdots \int_{V^m} \left\{ m H_{1;\delta} + \rho R_{ijhk} n^i n^h \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta} \right\} d\Lambda \\
 &= - \int \cdots \int_{V^m} m \left\{ (m H_1^2 - (m-1) H_2) n^i \xi_i + c H_1 \right\} d\Lambda. \quad (III')_h
 \end{aligned}$$

§4. Some properties of a closed orientable submanifold. If R^n is the constant Riemann curvature space and if V^m has the property $H_1 = \text{const.}$, then the left hand side of (II') in §2 vanishes and we have

$$\int \cdots \int_{V^m} \left[\left\{ m H_1^2 - (m-1) H_2 \right\} n^i \xi_i + \frac{1}{2m} H_{\beta r} \mathfrak{L} g_{\beta r} - g^{\alpha\beta} F_{\alpha\beta} \right] d\Lambda = 0. \quad (II'')$$

If the group G of transformations is concircular (II'') becomes

$$\int \cdots \int_{V^m} \{mH_1^2 - (m-1)H_2\} n^i \xi_i dA + \int \cdots \int_{V^m} \Phi H_1 dA = 0. \quad (II'')_c$$

If the group G is homothetic,

$$\int \cdots \int_{V^m} \{mH_1^2 - (m-1)H_2\} n^i \xi_i dA + c \int \cdots \int_{V^m} H_1 dA = 0, \quad (II'')_h$$

and if G is isometric

$$\int \cdots \int_{V^m} \{mH_1^2 - (m-1)H_2\} n^i \xi_i dA = 0. \quad (II'')_i$$

Now, we shall prove the following theorem:

Theorem 4.1. *Let R^n be a constant Riemann curvature space and V^m a closed orientable submanifold with $H_1 = \text{const.}$ We suppose that there exists a continuous one-parameter group G of concircular transformations generated by a vector ξ^i of R^n such that the scalar product $\rho = n^i \xi_i$ does not change the sign (and is not $\equiv 0$) on V^m , where the vector ξ is contained in the vector space spanned by $m+1$ vectors $\frac{\partial x^i}{\partial u^\alpha}$ ($\alpha = 1, 2, \dots, m$) and n^i . Then every point of V^m is umbilic with respect to Euler-Schouten vector n .*

Proof. Multiplying the formula $(I')_c$ in §1 by H_1 ($=\text{const.}$), we obtain

$$\int \cdots \int_{V^m} H_1^2 \rho dA + \int \cdots \int_{V^m} \Phi H_1 dA = 0.$$

Therefore, subtracting this formula from the formula $(II'')_c$ in §4, we find

$$(4.1) \quad \int \cdots \int_{V^m} (m-1)(H_1^2 - H_2) \rho dA = 0.$$

By means of

$$mH_1 = \sum_{\alpha} k_{\alpha}, \quad \binom{m}{2} H_2 = \sum_{\alpha < \beta} k_{\alpha} k_{\beta}$$

we get

$$(4.2) \quad H_1^2 - H_2 = \frac{1}{m^2(m-1)} \sum_{\alpha < \beta} (k_{\alpha} - k_{\beta})^2,$$

where $k_1, k_2, \dots, k_m$ denote the principal curvatures for n .

Then we see that

$$(4.3) \quad H_1^2 - H_2 \underset{E}{\geq} 0.$$

Therefore, because of (4.2), it follows that

$$k_1 \underset{E}{=} k_2 \underset{E}{=} \cdots \underset{E}{=} k_m$$

hold good at each point of V^m .

Theorem 4.2. *Let R^n be an n -dimensional Riemann space and V^m an m -dimensional closed orientable submanifold with $H_1 = \text{const.}$ We suppose that there exists a continuous one-parameter group G of homothetic transformations generated by a vector ξ^i of R^n such that the scalar product $\rho = n^i \xi_i$ does not change the sign (and is not $\equiv 0$) on V^m , where the vector ξ is contained in the vector space spanned by $m+1$ vectors $\frac{\partial x^i}{\partial u^\alpha}$ ($\alpha = 1, 2, \dots, m$) and n^i . If the relation*

$$R_{ijhk} n^i n^h \sum_{\lambda=1}^m i^j i^k \underset{E}{\geq} 0$$

holds good on V^m , then every point of V^m is umbilic with respect to Euler-Schouten vector n .

Proof. Multiplying the formula (I')_h in §1 by H_1 ($= \text{const.}$), we have

$$(4.4) \quad \int \cdots \int_{V^m} H_1^2 n^i \xi_i dA + \int \cdots \int_{V^m} c H_1 dA = 0.$$

From our assumption $H_1 = \text{const.}$, (III')_h becomes

$$(4.5) \quad \begin{aligned} & \frac{1}{m} \int \cdots \int_{V^m} \rho R_{ijhk} n^i n^h \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta} dA \\ &= - \int \cdots \int_{V^m} \{m H_1^2 - (m-1) H_2\} n^i \xi_i dA - \int \cdots \int_{V^m} c H_1 dA. \end{aligned}$$

By means of (3.1), (4.4) and (4.5), we have

$$(4.6) \quad \int \cdots \int_{V^m} \rho \left\{ (m-1) (H_1^2 - H_2) + \frac{1}{m} R_{ijhk} n^i n^h \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta} \right\} dA = 0.$$

Since we have

$$g^{\alpha\beta} \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} = \sum_{\lambda=1}^m i^j i^k,$$

if $R_{ijhk} n^i n^h \sum_{\lambda=1}^m i^j i^k \underset{E}{\geq} 0$ hold good on V^m , we get from (4.6) the relation

$$H_1^2 - H_2 = 0 \text{ on } E.$$

This means $k_1 = k_2 = \dots = k_m$ on E .

References

- [1] H. LIEBMANN: *Über die Verbiegung der geschlossenen Flächen positiver Krümmung*, Math. Ann., 53 (1900), 91-112.
- [2] W. SÜSS: *Zur relativen Differentialgeometrie V.*, Tôhoku Math. J., 30 (1929), 202-209.
- [3] T. BONNESEN und W. FENCHEL: *Theorie der konvexen Körper* (Springer, Berlin 1934).
- [4] H. HOPF: *Über Flächen mit einer Relation zwischen den Hauptkrümmungen*, Math. Nachr., 4 (1951), 232-249.
- [5] C. C. HSIUNG: *Some integral formulas for closed hypersurfaces*, Math. Scand., 2 (1954), 286-294.
- [6] C. C. HSIUNG: *Some integral formulas for closed hypersurface in Riemann space*, Pacific Jour. of Math., 6 (1956), 291-299.
- [7] A. D. ALEXANDROV: *Uniqueness theorems for surfaces in the large, V.* Vestnik Leningrad University, 13 (1958), 5-8 (Russian, with English summary).
- [8] A. D. ALEXANDROV: *A Characteristic property of spheres*, Ann. di. Mat. (Serie IV), 58 (1962), 303-315.
- [9] Y. KATSURADA: *Generalized Minkowski formulas for closed hypersurfaces in Riemann Space*, Ann. Mat. p. appl., 57 (1962), 283-293.
- [10] Y. KATSURADA: *On a certain property of closed hypersurfaces in an Einstein space*, Comment. Math. Helv., 38 (1964), 165-171.
- [11] Y. KATSURADA: *On the isoperimetric problem in a Riemann space*, Comment. Math. Helv., 41 (1966-67), 18-29.
- [12] K. YANO: *Closed hypersurfaces with constant mean curvature in a Riemannian manifold*, J. Math. Soc. Japan, 17 (1965), 333-340.
- [13] K. YANO: *The theory of Lie derivatives and its applications* (Amsterdam, 1957).
- [14] K. YANO and S. BOCHNER: *Curvature and Betti numbers* (Princeton, Annals of Math. Studies, 1953).
- [15] T. ÔTSUKI: *Integral formulas for hypersurfaces in a Riemannian manifold and their applications*, Tôhoku Math. J., (2) 17 (1965), 335-348.
- [16] J. A. SCHOUTEN: *Ricci-Calculus* (Springer, Berlin 1954).

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