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ON A CERTAIN PROPERTY OF A CLOSED HYPERSURFACE IN A RIEMANN SPACE

By

Tsunehira KOYANAGI

Introduction. Recently the properties of closed hypersurfaces in a Riemann space have been investigated by Y. Katsurada [1]¹⁾, [2], K. Yano [3], T. Ōtsuki [4] and others. First Y. Katsurada proved the following theorem with respect to the ν -th mean curvature $H_\nu = \text{constant}$ ($\nu = 1, 2, \dots, m-1$):

Theorem 0.1. *Let R^{m-1} be a Riemann space with constant Riemann curvature, V^m a closed orientable hypersurface in R^{m-1} . If there exists a one-parameter group G of conformal transformations of R^{m-1} such that the scalar product $n^i \xi_i$ of the normal vector n of V^m and the generating vector ξ of G does not change the sign on V^m and is not identically zero, and if the principal curvatures $k_1, k_2, \dots, k_m$ on V^m are positive and H_ν is constant for any ν ($1 \leq \nu \leq m-1$), then every point of V^m is umbilic. ([1], p. 291)*

The purpose of the present paper, especially in the only case of $\nu=2$, is to investigate an analogous property in a more general Riemann space admitting a one-parameter group of conformal transformations. In §1 the integral formula of Minkowski type given by Y. Katsurada [2] which holds in arbitrary Riemann space is explained. In §2, we give an integral formula for a closed orientable hypersurface with certain special second fundamental tensor in a Riemann space admitting a one-parameter group of conformal transformations. And in §3 a property of a closed orientable hypersurface with such the second fundamental tensor is discussed.

The author wishes to express to Prof. Y. Katsurada his very sincere thanks for her kind guidance.

§ 1. The generalized Minkowski formula (I). We consider a Riemann space R^{m-1} ($m \geq 3$) of class C^ν ($\nu \geq 3$) which admits a one-parameter continuous group G of transformations generated by an infinitesimal transformation

$$(1.1) \quad \bar{x}^i = x^i + \xi^i(x) \delta\tau$$

(where x^i are local coordinates in R^{m-1} and ξ^i are the components of a con-

1) Numbers in brackets refer to the references at the end of paper.

travariant vector ξ). We suppose that the paths of these transformations cover R^{m+1} simply and that ξ is everywhere continuous and $\neq 0$. If ξ is a conformal Killing vector, then the group G is called conformal.

We now consider a closed orientable hypersurface V^m of class C^3 imbedded in R^{m+1} , locally given by

$$x^i = x^i(u^a);$$

here and henceforth, Latin indices run from 1 to $m+1$ and Greek indices from 1 to m .

To the vector ξ introduced above, there is a covariant vector $\bar{\xi}$ of V^m with the components

$$\bar{\xi}_a = B_a^i \xi_i,$$

where B_a^i mean $\partial x^i / \partial u^a$ and ξ_i the covariant components of ξ ; we shall compute its covariant derivatives along V^m : by virtue of the fact that the covariant derivatives of B_a^i are

$$\frac{\delta}{\delta u^{\bar{\beta}}} (B_a^i) = b_{a\bar{\beta}} n^i$$

where $b_{a\bar{\beta}}$ is the second fundamental tensor and n^i the unit normal vector of V^m , we find

$$(1.2) \quad \bar{\xi}_{a;\bar{\beta}} = b_{a\bar{\beta}} n^i \xi_i + B_a^i B_{\bar{\beta}}^j \mathfrak{L}_{\xi} g_{ij}$$

(the symbol “ $\cdot$ ” always means the covariant derivative). Multiplying (1.2) by the contravariant metric tensor $g^{a\bar{\beta}}$ of V^m and summing for α and β , we get

$$(1.3) \quad g^{a\bar{\beta}} \bar{\xi}_{a;\bar{\beta}} = m H_1 n^i \xi_i + \frac{1}{2} g^{a\bar{\beta}} B_a^i B_{\bar{\beta}}^j \mathfrak{L}_{\xi} g_{ij},$$

where H_1 is the first mean curvature $\frac{1}{m} g^{a\bar{\beta}} b_{a\bar{\beta}}$ of V^m and $\mathfrak{L}_{\xi} g_{ij}$ is the Lie derivative of the fundamental tensor g_{ij} of R^{m+1} with respect to the infinitesimal transformation (1.1). If we put

$$(1.4) \quad \mathfrak{L}_{\xi} g_{a\bar{\beta}} = B_a^i B_{\bar{\beta}}^j \mathfrak{L}_{\xi} g_{ij},$$

then (1.3) is rewritten as follows:

$$\frac{1}{m} \bar{\xi}_{;\alpha}^{\alpha} \equiv \frac{1}{m} g^{a\bar{\beta}} \bar{\xi}_{a;\bar{\beta}} = H_1 n^i \xi_i + \frac{1}{2m} g^{a\bar{\beta}} \mathfrak{L}_{\xi} g_{a\bar{\beta}}.$$

dA being the area element of V^m , we have

$$\int \cdots \int_{V^m} \xi_{;a}^a dA = 0, \quad ([5], \text{ p. 31})$$

because V^m is closed and orientable. Thus we obtain the integral formula

$$(I) \quad \int \cdots \int_{V^m} H_1 n^i \xi_i dA + \frac{1}{2m} \int \cdots \int_{V^m} g^{a\beta} \mathfrak{L}_{\xi} g_{a\beta} dA = 0.$$

Let the group G be conformal, that is, ξ satisfy the equation

$$\mathfrak{L}_{\xi} g_{ij} = \xi_{i;j} + \xi_{j;i} = 2\phi g_{ij},$$

then (I) becomes

$$(I)_c \quad \int \cdots \int_{V^m} H_1 n^i \xi_i dA + \int \cdots \int_{V^m} \phi dA = 0 \quad ([2], \text{ p. 167}).$$

§ 2. An integral formula for a closed orientable hypersurface with certain special second fundamental tensor. We first take the covariant vector ${}_1\eta$ of V^m , whose components are given by

$${}_1\eta_{\bar{r}} = (b_a^\alpha)^2 B_{\bar{r}}^i \xi_i$$

and calculate its covariant derivatives along V^m , we have

$${}_1\eta_{\bar{r};\theta} = (b_a^\alpha)^2_{;\theta} B_{\bar{r}}^i \xi_i + (b_a^\alpha)^2 B_{\bar{r};\theta}^i \xi_i + (b_a^\alpha)^2 B_{\bar{r}}^i B_{\theta}^j \xi_{i;j}.$$

Remembering the following Gauss' formulas for a hypersurface

$$B_{\bar{\beta};\bar{r}}^i = b_{\bar{\beta}\bar{r}} n^i,$$

we find that

$${}_1\eta_{\bar{r};\theta} = (b_a^\alpha)^2_{;\theta} B_{\bar{r}}^i \xi_i + (b_a^\alpha)^2 b_{\bar{r}\theta} n^i \xi_i + (b_a^\alpha)^2 B_{\bar{r}}^i B_{\theta}^j \xi_{i;j}.$$

Multiplying by $g^{r\theta}$ and summing for $\bar{r}$ and θ , we obtain

$$(2.1) \quad g^{r\theta} {}_1\eta_{\bar{r};\theta} = (b_a^\alpha)^2_{;\theta} g^{r\theta} B_{\bar{r}}^i \xi_i + (b_a^\alpha)^3 n^i \xi_i + (b_a^\alpha)^2 g^{r\theta} B_{\bar{r}}^i B_{\theta}^j \xi_{i;j}.$$

Moreover we consider the three covariant vectors ${}_2\eta$, ${}_3\eta$, ${}_4\eta$ of V^m whose components are given by

$${}_2\eta_{\alpha} = b_{\alpha}^i b_{\bar{\beta}}^j B_{\bar{r}}^i \xi_i,$$

$${}_3\eta_{\bar{r}} = (b_{\alpha}^i b_{\bar{\beta}}^j) B_{\bar{r}}^i \xi_i,$$

$${}_4\eta_{\bar{\beta}} = (b_{\alpha}^i) b_{\bar{\beta}}^j B_{\bar{r}}^i \xi_i$$

respectively. And by making use of the similar calculation mentioned above, we obtain

$$(2.2) \quad g^{\alpha\theta} {}_2\eta_{\alpha;\theta} = (b_\alpha^\beta b_\beta^\gamma)_{;\theta} g^{\alpha\theta} B_\gamma^i \xi_i + b_\alpha^\beta b_\beta^\gamma b_\gamma^\alpha n^i \xi_i + b_\alpha^\beta b_\beta^\gamma g^{\alpha\theta} B_\gamma^i B_\theta^j \xi_{i;j},$$

$$(2.3) \quad g^{r\theta} {}_3\eta_{r;\theta} = (b_\alpha^\beta b_\beta^\alpha)_{;\theta} g^{r\theta} B_\gamma^i \xi_i + b_\gamma^\alpha (b_\alpha^\beta b_\beta^\alpha) n^i \xi_i + (b_\alpha^\beta b_\beta^\alpha) g^{r\theta} B_\gamma^i B_\theta^j \xi_{i;j}$$

and

$$(2.4) \quad g^{\beta\theta} {}_4\eta_{\beta;\theta} = (b_\alpha^\alpha b_\beta^\beta)_{;\theta} g^{\beta\theta} B_\gamma^i \xi_i + b_\alpha^\alpha (b_\beta^\beta b_\beta^\alpha) n^i \xi_i + b_\alpha^\alpha b_\beta^\beta g^{\beta\theta} B_\gamma^i B_\theta^j \xi_{i;j}.$$

Accordingly putting $\bar{\eta}_{r;\theta} = {}_1\eta_{r;\theta} + 2{}_2\eta_{r;\theta} - {}_3\eta_{r;\theta} - 2{}_4\eta_{r;\theta}$, from (2.1), (2.2), (2.3) and (2.4), we get

$$(2.5) \quad g^{r\theta} \bar{\eta}_{r;\theta} = \left\{ (b_\alpha^\alpha)^2 - b_\beta^\alpha b_\alpha^\beta \right\}_{;\theta} g^{r\theta} B_\gamma^i \xi_i - 2 \left\{ b_\alpha^\alpha b_\beta^\gamma - b_\beta^\alpha b_\gamma^\alpha \right\}_{;\theta} g^{\beta\theta} B_\gamma^i \xi_i \\ + \left\{ (b_\alpha^\alpha)^3 + 2b_\alpha^\beta b_\beta^\gamma b_\gamma^\alpha - 3b_\alpha^\alpha (b_\beta^\beta b_\beta^\alpha) \right\} n^i \xi_i + \left\{ (b_\alpha^\alpha)^2 - b_\beta^\alpha b_\alpha^\beta \right\} g^{r\theta} B_\gamma^i B_\theta^j \xi_{i;j} \\ - 2 \left\{ b_\alpha^\alpha b_\beta^\gamma - b_\beta^\alpha b_\gamma^\alpha \right\} g^{\beta\theta} B_\gamma^i B_\theta^j \xi_{i;j}.$$

On the other hand, if we denote by $k_1, k_2, \dots, k_m$ the principal curvatures of V^m , then the second mean curvature H_2 and the third mean curvature H_3 of V^m are respectively given by

$$(2.6) \quad \binom{m}{2} H_2 = \sum_{\alpha < \beta} k_\alpha k_\beta = \frac{1}{2} \left\{ (b_\alpha^\alpha)^2 - b_\beta^\alpha b_\alpha^\beta \right\}$$

and

$$(2.7) \quad \binom{m}{3} H_3 = \sum_{\alpha < \beta < \gamma} k_\alpha k_\beta k_\gamma = \frac{1}{3!} \left\{ (b_\alpha^\alpha)^3 + 2b_\alpha^\beta b_\beta^\gamma b_\gamma^\alpha - 3b_\alpha^\alpha (b_\beta^\beta b_\beta^\alpha) \right\},$$

where $b_{\alpha\beta}$ means the second fundamental tensor of V^m , $b_\alpha^\beta = b_{\alpha\gamma} g^{\gamma\beta}$ and $b_\alpha^\alpha = b_{\alpha\beta} g^{\beta\alpha}$. Therefore (2.5) may be written as follows:

$$(2.8) \quad g^{r\theta} \bar{\eta}_{r;\theta} = m(m-1) H_{2;\theta} g^{r\theta} B_\gamma^i \xi_i - 2C_{\beta;\theta}^\gamma g^{\beta\theta} B_\gamma^i \xi_i \\ + m(m-1)(m-2) H_3 n^i \xi_i + m(m-1) H_2 g^{r\theta} B_\gamma^i B_\theta^j \xi_{i;j} \\ - 2C_{\beta}^\gamma g^{\beta\theta} B_\gamma^i B_\theta^j \xi_{i;j},$$

where $C_{\beta\gamma}^\alpha$ means $b_\alpha^\beta b_{\beta\gamma} - b_\beta^\alpha b_{\alpha\gamma}$ and $C_{\beta}^\gamma = C_{\beta\alpha}^\gamma g^{\alpha\gamma}$. In (2.8), from the definition of $C_{\beta\gamma}^\alpha$, we see easily that the $C_{\beta}^\gamma g^{\beta\theta} B_\gamma^i B_\theta^j$ of the final term of the right-hand member are symmetric in i and j . Consequently we obtain

$$(2.9) \quad g^{r\theta} \bar{\eta}_{r;\theta} = m(m-1) H_{2;\theta} g^{r\theta} B_\gamma^i \xi_i - 2C_{\beta;\theta}^\gamma g^{\beta\theta} B_\gamma^i \xi_i \\ + m(m-1)(m-2) H_3 n^i \xi_i + \frac{1}{2} m(m-1) H_2 g^{r\theta} \mathfrak{L}_{\xi} g_{r\theta} \\ - C_{\beta}^\gamma g^{\beta\theta} \mathfrak{L}_{\xi} g_{r\theta},$$

by virtue of (1.4).

We now assume that the group G of transformations is conformal, that is, $\mathfrak{L}g_{ij}=2\Phi g_{ij}$, then the last two terms of the right-hand member of (2.9) become in the form

$$\frac{1}{2} m(m-1) H_2 g^{i\theta} \mathfrak{L} g_{i\theta} - C_{\beta}^{\gamma} g^{\beta\theta} \mathfrak{L} g_{i\theta} = m(m-1)(m-2) \Phi H_2,$$

on making use of (1.4) and $C_{\beta}^{\gamma} g^{\beta\theta} = m(m-1) H_2$. In this case (2.9) can be written as follows:

$$(2.10) \quad g^{i\theta} \tilde{\gamma}_{i;\theta} = m(m-1) H_2 g^{i\theta} B_i^{\xi} \xi_{\theta} - 2C_{\beta}^{\gamma} g^{\beta\theta} B_i^{\xi} \xi_{\theta} \\ + m(m-1)(m-2)(H_3 n^i \xi_i + \Phi H_2).$$

Moreover if V^m is a closed orientable hypersurface with the second fundamental tensor $b_{\alpha\beta}$ such that $C_{\alpha\beta}$ are covariantly constant, both the first term and the second of the right-hand member of (2.10) vanish, because of $C_{\alpha\beta} g^{\alpha\beta} = m(m-1) H_2$; it follows that (2.10) becomes

$$(2.11) \quad \frac{1}{m(m-1)(m-2)} \tilde{\gamma}_{i;\alpha}^{\alpha} \equiv \frac{1}{m(m-1)(m-2)} g^{i\theta} \tilde{\gamma}_{i;\theta} \\ = H_3 n^i \xi_i + \Phi H_2,$$

by virtue of dividing the two sides of (2.10) by $m(m-1)(m-2)$. Since V^m is closed and orientable, we have

$$\int \cdots \int_{V^m} \tilde{\gamma}_{i;\alpha}^{\alpha} dA = 0.$$

Thus we finally reach the integral formula

$$(II)_c \quad \int \cdots \int_{V^m} H_3 n^i \xi_i dA + \int \cdots \int_{V^m} \Phi H_2 dA = 0.$$

§ 3. A property of a closed orientable hypersurface with the special second fundamental tensor in a Riemann space. In this section we shall prove the following two theorems.

Theorem 3.1. *Let R^{m+1} be a Riemann space and V^m a closed orientable hypersurface in R^{m+1} . If there exists a one-parameter group G of conformal transformations of R^{m+1} such that the scalar product $n^i \xi_i$ of the normal vector n of V^m and the generating vector ξ of G does not change the sign (and $\neq 0$) on V^m , and if the principal curvatures $k_1, k_2, \dots, k_m$ on V^m are positive and $C_{\alpha\beta}$ is covariantly constant, then every point of V^m is umbilic.*

Proof. From $C_{\alpha\beta} g^{\alpha\beta} = m(m-1) H_2$, H_2 is constant under our conditions.

Multiplying by H_2 the formula (I)_c in § 1, we obtain

$$\int \cdots \int_{V^m} H_1 H_2 n^i \xi_i dA + \int \cdots \int_{V^m} \Phi H_2 dA = 0,$$

and subtracting from this the formula (II)_c in § 2, we find

$$(3.1) \quad \int \cdots \int_{V^m} (H_1 H_2 - H_3) n^i \xi_i dA = 0.$$

On the other hand, from

$$(3.2) \quad H_1 H_2 - H_3 = \frac{2}{m^2(m-1)} \sum_a k_a \sum_{\beta < \gamma} k_\beta k_\gamma - \frac{3!}{m(m-1)(m-2)} \sum_{\alpha < \beta < \gamma} k_\alpha k_\beta k_\gamma \\ = \frac{3!}{m^2(m-1)(m-2)} \sum_{\alpha < \beta < \gamma} k_\alpha (k_\beta - k_\gamma)^2,$$

we see that

$$(3.3) \quad H_1 H_2 - H_3 \geq 0 \quad ([1], \text{ p. } 292).$$

From (3.1), (3.3) and the fact that $n^i \xi_i$ has a fixed sign we conclude that

$$H_1 H_2 - H_3 = 0,$$

from which, because of (3.2), $k_1 = k_2 = \cdots = k_m$ at each point of V^m . This means that every point of V^m is umbilic.

We next assume that the Riemann space R^{m+1} under consideration is a space of constant curvature K : $R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$ where R_{ijkl} is the curvature tensor of R^{m+1} . As well-known, a hypersurface in a Riemann space has the following property

$$(3.4) \quad R_{\alpha\beta\gamma\delta} = b_{\alpha\delta}b_{\beta\gamma} - b_{\alpha\gamma}b_{\beta\delta} + R_{ijkl}B_\alpha^iB_\beta^jB_\gamma^kB_\delta^l,$$

where $R_{\alpha\beta\gamma\delta}$ is the curvature tensor of V^m . Because of a space of constant curvature, (3.4) becomes

$$R_{\alpha\beta\gamma\delta} = b_{\alpha\delta}b_{\beta\gamma} - b_{\alpha\gamma}b_{\beta\delta} + K(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}).$$

Multiplying by $g^{\alpha\delta}$ and summing for α and δ , we have

$$R_{\beta\gamma} \equiv g^{\alpha\delta} R_{\alpha\beta\gamma\delta} = b_\alpha^\alpha b_{\beta\gamma} - b_\beta^\alpha b_{\alpha\gamma} - (m-1)K g_{\beta\gamma},$$

from which, by covariant differentiation along V^m ,

$$R_{\beta\gamma;\theta} = (b_\alpha^\alpha b_{\beta\gamma} - b_\beta^\alpha b_{\alpha\gamma})_{;\theta}, \quad \text{that is, } R_{\beta\gamma;\theta} = C_{\beta\gamma;\theta}.$$

Thus, we have

Corollary. *Let R^{m+1} be a space of constant curvature and V^m a closed orientable Ricci symmetric hypersurface in R^{m+1} . We suppose that R^{m+1} admits a one-parameter group G of conformal transformations such that the scalar product $n^i \xi_i$ of the normal vector n of V^m and the generating vector ξ of G does not change the sign and is not identically zero on V^m , and the principal curvatures $k_1, k_2, \dots, k_m$ on V^m are positive, then every point of V^m is umbilic.*

Finally we consider the hypersurface with $H_2 = \text{constant}$ and $C_{a\theta;\beta} g^{a\beta} = 0$. Taking account of $C_{a\beta} g^{a\beta} = m(m-1)H_2$, it is evident that these conditions are weaker than $C_{a\theta;\beta} = 0$, and also we get (II)_c from (2.10). Thus we have

Theorem 3.2. *Let R^{m+1} be a Riemann space and V^m a closed orientable hypersurface in R^{m+1} . If there exists a one-parameter group G of conformal transformations of R^{m+1} such that the scalar product $n^i \xi_i$ does not change the sign and is not identically zero on V^m , and if the principal curvatures $k_1, k_2, \dots, k_m$ on V^m are positive, $H_2 = \text{constant}$ and $C_{a\theta;\beta} g^{a\beta} = 0$, then every point of V^m is umbilic.*

Remark. Now we leave us the following question: "which is more general, Theorem 0.1. given by Y. Katsurada or Theorem 3.2. in this paper?"

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Department of Mathematics,
Hokkaido University

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