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SOME CHARACTERIZATIONS OF A SUBMANIFOLD WHICH IS ISOMETRIC TO A SPHERE

By

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Introduction. H. Liebmann (1900) [1]¹⁾, proved the following.

Theorem 0.1. *The only ovaloids with constant mean curvature H in an Euclidean space E^3 are spheres.*

Extension of this theorem to a convex hypersurface in an n -dimensional Euclidean space E^n has been given by W. Süss (1929) [2], (cf. also [3], p. 118, and [4]). Then H. Hopf (1951) [5], and A. D. Alexandrov (1958) [6], have shown the results that the convexity is not necessary for the validity of Liebmann-Süss theorem.

Recently the analogous problem for closed hypersurfaces in an n -dimensional Einstein space R^n admitting a conformal Killing vector field ξ has been discussed by the present author [8], [9], A. D. Alexandrov [7], K. Nomizu [14], [15]. Afterward T. Nagai, H. Kôjyo and the present author [10], [11], [13] have given a certain extension of this problem to an m -dimensional closed submanifold V^m ($1 \leq m \leq n-1$) in R^n . However we have given there a restriction such that at each point on V^m , the vector ξ lies in the vector space spanned by the tangent space of V^m and the Euler-Schouten vector n .

In the previous paper [12], the present author has proved the following two theorems except this restriction for a closed orientable submanifold V^m in an orientable Riemannian manifold R^n with constant Riemann curvature.

Theorem 0.2. *If in R^n , there exists such a group G of conformal transformations generated by ξ as ρ is positive (or negative) at each point of V^m and if H_1 is constant, then every point of V^m is umbilic with respect to the normal vector n , where ρ denotes $n_\xi \xi^\xi$, n is the unit vector normal to V^m in the vector space spanned by the tangent space of V^m and ξ and H_v means the v -th mean curvature of V^m with respect to n .*

Theorem 0.3. *If in R^n , there exists such a group G of conformal transformations generated by ξ as ρ is positive (or negative) at each point of V^m ,*

1) Numbers in brackets refer to the references at the end of the paper.

and if the principal curvatures with respect to $n, k_1, \dots, k_m$ at each point of V^m , are positive and H_ν is constant for any ν ($1 < \nu \leq m-1$), then every point of V^m is umbilic with respect to the normal vector n .

The purpose of this paper is to induce a characteristic property such that V^m is isometric to a sphere from these theorems.

§ 1. Some characterizations of a submanifold to be isometric to a sphere. We suppose an n -dimensional Einstein space R^n ($n \geq 3$) of class C^r ($r \geq 3$) which has local coordinates x^i and admits a proper conformal Killing vector field ξ^i , that is, ξ^i satisfies an equation:

$$(1.1) \quad \mathfrak{L}_\xi g_{ij} \equiv \xi_{i;j} + \xi_{j;i} = 2\phi g_{ij} \quad ([16], \text{ p. } 32)$$

where g_{ij} is the metric tensor of R^n , $\mathfrak{L}_\xi g_{ij}$ means the Lie derivative of g_{ij} with respect to ξ and the symbol “;” the operation of D -symbol due to van der Waerden-Bortolotti ([18], p. 254). Then the Lie derivative of the Riemann curvature tensor R^h_{ijk} of R^n with respect to the conformal Killing vector field ξ is given by

$$(1.2) \quad \mathfrak{L}_\xi R^h_{ijk} = \delta^h_j \phi_{i;k} - \delta^h_k \phi_{i;j} + g_{ik} \phi^h_{;j} - g_{ij} \phi^h_{;k} \quad ([16], \text{ p. } 160)$$

where ϕ_i means $\phi_{;i}$ and δ^h_j is the Kronecker delta.

Since R^n is an Einstein space, we have

$$(1.3) \quad R_{ij} = \frac{R}{n} g_{ij} \quad (R = \text{Constant})$$

where R_{ij} and R are the Ricci tensor and the curvature scalar of R^n respectively.

Making use of (1.2) and (1.3), after some calculations we obtain the following result

$$(1.4) \quad \phi_{i;j} = -\frac{R}{n(n-1)} \phi g_{ij}.$$

From (1.4) we can see that the Einstein space R^n admitting the conformal Killing vector field ξ with the non-constant scalar field ϕ must always admit the vector field ϕ^i ($\phi^i = g^{ij} \phi_j$), which is the special conformal Killing vector field.

In R^n , we consider a domain M . If the domain M is simply covered by the orbits of transformations generated by ϕ^i , and ϕ^i is everywhere of class C^r and $\neq 0$ in M ; then we call M a regular domain with respect to the vector field ϕ^i .

Let us denote by V^m an m -dimensional closed orientable submanifold of class C^3 imbedded in a regular domain M with respect to the vector field ϕ^i , locally given by

$$(1.5) \quad x^i = x^i(u^\alpha) \quad i=1, \dots, n; \quad \alpha=1, \dots, m$$

where u^α are local coordinates of V^m . Throughout the present paper Latin indices run from 1 to n and Greek indices from 1 to m . We assume that at any point on V^m the vector ϕ^i is not on its tangent space.

We shall indicate by n_{ϕ}^i ($p=m+1, \dots, n$) the contravariant unit vector normal V^m and suppose that they are mutually orthogonal. Let n_{ϕ} be in the vector space spanned by $m+1$ independent vector $\frac{\partial x^i}{\partial u^\alpha}$ ($\alpha=1, \dots, m$) and ϕ^i and be the unit vector normal V^m . Then we may consider n_{ϕ} as one of the unit normal vectors of V^m , that is, $n_{\phi}^i = n_{m+1}^i$.

If we denote by $k_{\phi 1}, \dots, k_{\phi m}$ the principal curvatures of V^m with respect to n_{ϕ}^i and by $H_{\phi \nu}$ the ν -th mean curvature, the ν -th elementary symmetric function of $k_{\phi 1}, \dots, k_{\phi m}$ divided by the number of terms, i. e.,

$$\binom{m}{\nu}_{\phi} H_{\phi \nu} = \sum_{\alpha_1 < \dots < \alpha_{\nu}} k_{\phi \alpha_1} \dots k_{\phi \alpha_{\nu}}, \quad (1 \leq \nu \leq m),$$

then as a special case of Theorem 0.2. and of Theorem 0.3., we have the following two theorems.

Theorem 1.1. *Let ξ^i be a proper conformal Killing vector field in R^n with constant Riemann curvature (i. e., $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$). Then every point of a closed orientable submanifold V^m such that $H_1 = \text{constant}$ and $\phi_i n_{\phi}^i > 0$ (or < 0) for all points of V^m is umbilic with respect to n_{ϕ} .*

Remark. We remark here that $\phi_i n_{\phi}^i > 0$ (or < 0) for all points of V^m is equivalent that the vector ϕ^i is not on the tangent space of V^m at any point of V^m .

Theorem 1.2. *Let ξ^i be a proper conformal Killing vector field in R^n with constant Riemann curvature (i. e., $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$). Then every point of a closed orientable submanifold V^m such that $H_{\phi} = \text{constant}$, $k_{\phi 1} > 0, \dots, k_{\phi m} > 0$ and $\phi_i n_{\phi}^i > 0$ (or < 0) for all points of V^m is umbilic with respect to n_{ϕ} .*

From the above theorems and the following theorem due to M. Obata [19], we obtain Theorem 1.3 and Theorem 1.4.

Theorem (Obata). Let R^n ($n \geq 2$) be a complete Riemann manifold which admits a non-null function φ such that $\varphi_{;i;j} = -C^2 \varphi g_{ij}$ ($C = \text{constant}$). Then R^n is isometric to a sphere of radius $1/C$.

Theorem 1.3. If ξ^i is a proper conformal Killing vector field in R^n with constant Riemann curvature (i.e., $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$), then a closed orientable submanifold V^m such that $H_1 = \text{constant}$ and $\phi_i n^i > 0$ (or < 0) for all points of V^m is isometric to a sphere.

Proof. Let C^i_j be $\sum_{p=m+1}^n n^i_p n_j^p$ ($n = n$) and i ($\lambda = 1, \dots, m$) mutually orthogonal unit tangent vectors of V^m . Then putting $B^i_a = \frac{\partial x^i}{\partial u^a}$, we have

$$n^i_{;a} = C^i_{j;k} n^j_p B^k_a = - \sum_{\lambda=1}^m (i_{j;k} n^j_p B^k_a) i^\lambda_i.$$

Therefore we may write as follows

$$n^i_{;a} = \gamma^i_a B^i_\tau.$$

Since we have

$$g_{ij} B^i_{\beta;a} n^j = -g_{ij} B^i_{\beta} n^j_{;a},$$

we obtain

$$(1.6) \quad n^i_{;a} = -b^i_a B^i_\tau \quad (p = \phi, m+2, \dots, n)$$

where b^i_a means $g^{i\beta} b_{\alpha\beta}$ and $b^i_{\alpha\beta} \stackrel{\text{def}}{=} B^i_{\alpha;\beta} n_i$, and $g^{i\beta}$ is the contravariant metric tensor of V^m .

As well-known on the theory of a submanifold in a Riemannian manifold, we have

$$B^i_{\alpha;\beta} = \frac{\partial B^i_\alpha}{\partial u^\beta} + \Gamma^i_{jk} B^j_\alpha B^k_\beta - \Gamma'^i_{\alpha\beta} B^i_\tau,$$

$$n^i_{;a} = \frac{\partial n^i}{\partial u^a} + \Gamma^i_{jk} n^j_p B^k_a - \Gamma''^i_{pa} n^i_q,$$

where Γ^i_{jk} and $\Gamma'^i_{\alpha\beta}$ are the Christoffel symbols with respect to g_{ij} and $g_{\alpha\beta}$ respectively, and

$$\Gamma''^i_{pa} \stackrel{\text{def}}{=} \frac{\partial n^i}{\partial u^a} n_i + \Gamma^i_{jk} n^j_p B^k_a n_i.$$

Since $n^i_{;a} = \delta_{pa} \delta_{pq}$ (δ_{pq} means the Kronecker delta, Γ''^i_{pa} is anti-symmetric with

respect to the indices P and Q . Consequently we have

$$(1.7) \quad \Gamma''^p_{Pa} = 0 \quad \text{for any } P \quad (P = \phi, m+2, \dots, n).$$

We now consider the scalar function on V^m given by

$$\varphi = \phi_{\phi} n^{\phi},$$

and we assume, moreover, that $\varphi \neq$ constant along V^m , then we have

$$\varphi_{\alpha} = \phi_{\phi; j} n^j B_{\alpha}^j + \phi_{\phi} (n^{\phi}_{; \alpha} + \Gamma''^P_{\phi \alpha} n^{\phi}).$$

On the other hand the vector ϕ_{ϕ} is expressed as follows

$$(1.8) \quad \phi_{\phi} = \varphi n_{\phi} + C_{\gamma} B_{\phi}^{\gamma},$$

where $B_{\phi}^{\gamma} \stackrel{\text{def}}{=} g_{jk} B_{\phi}^j g^{\phi \gamma}$.

From (1.7) and (1.8), we have

$$\phi_{\phi} \Gamma''^P_{\phi \alpha} n^{\phi} = 0.$$

Since R^n with constant Riemann curvature is an Einstein space, we get

$$\phi_{\phi; j} n^j B_{\alpha}^j = - \frac{R}{n(n-1)} \phi g_{\phi j} n^j B_{\alpha}^j = 0,$$

and also from (1.6), we have

$$\varphi_{\alpha} = -b_{\alpha}^{\gamma} B_{\gamma}^{\phi} \phi_{\phi}.$$

Furthermore by virtue of Theorem 1.1, every point of V^m is umbilic with respect to n^{ϕ} , that is,

$$(1.9) \quad b_{\alpha}^{\beta} = H_1 g_{\alpha \beta}.$$

So, it follows that

$$b_{\alpha}^{\beta} = H_1 \delta_{\alpha}^{\beta}.$$

Thus we obtain

$$\varphi_{\alpha} = -H_1 B_{\alpha}^{\phi} \phi_{\phi}$$

and

$$(1.10) \quad \varphi = -H_1 \phi + C \quad (C = \text{Constant}).$$

Since $H_1 = \text{constant}$, we get

$$(1.11) \quad \varphi_{\alpha;\beta} = -H_1(\phi_{\alpha;j} B_{\alpha}^j B_{\beta}^j + \phi_{\alpha} B_{\alpha;\beta}^j).$$

Consequently from (1.4) and (1.8), because of the relation $B_{\alpha}^j B_{\alpha;\beta}^j = 0$, we have

$$\varphi_{\alpha;\beta} = -H_1 \left(-\frac{R}{n(n-1)} \phi g_{\alpha\beta} + \varphi b_{\alpha\beta} \right),$$

and from (1.9) and (1.10), (1.11) becomes as follows

$$(1.12) \quad \varphi_{\alpha;\beta} = \left\{ -\left(H_1^2 + \frac{R}{n(n-1)} \right) \varphi + \frac{R}{n(n-1)} C \right\} g_{\alpha\beta}$$

If $H_1^2 + \frac{R}{n(n-1)} = 0$, then (1.12) becomes

$$\varphi_{\alpha;\beta} = \frac{R}{n(n-1)} C g_{\alpha\beta},$$

from which $\Delta\varphi = \frac{mR}{n(n-1)} C$ (Δ is the Laplacian operator on V^m), which is impossible unless $\varphi = \text{constant}$. Therefore $H_1^2 + \frac{R}{n(n-1)} \neq 0$, then we have

$$\varphi_{\alpha;\beta} = -\left(H_1^2 + \frac{R}{n(n-1)} \right) \left(\varphi - \frac{\frac{R}{n(n-1)} C}{H_1^2 + \frac{R}{n(n-1)}} \right) g_{\alpha\beta}.$$

putting

$$\sigma = \varphi - \frac{\frac{R}{n(n-1)} C}{H_1^2 + \frac{R}{n(n-1)}},$$

we obtain

$$\sigma_{\alpha;\beta} = -\left(H_1^2 + \frac{R}{n(n-1)} \right) \sigma g_{\alpha\beta}$$

and

$$\Delta\sigma = -m \left(H_1^2 + \frac{R}{n(n-1)} \right) \sigma,$$

consequently it follows that $H_1^2 + \frac{R}{n(n-1)} > 0$ [17]. Thus V^m is isometric to a sphere according to Obata's theorem.

Theorem 1.4. *If ξ^i is a proper conformal Killing vector field in R^n with constant Riemann curvature (i.e., $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$), then a closed orientable submanifold V^m such that $H_\nu = \text{const.}$ for any ν ($1 < \nu \leq m-1$), $k_1 > 0, \dots, k_m > 0$ and $\phi_\phi n^\phi > 0$ (or < 0) for all points of V^m is isometric to a sphere.*

Proof. From Theorem 1.2, every point of V^m is umbilic with respect to n^ϕ , that is,

$$k_1 = \dots = k_m.$$

Accordingly we have

$$H_\nu = (k_1)^\nu = (H_1)^\nu = \text{const.}$$

Thus we get

$$H_1 = \text{const.}$$

Consequently by virtue of Theorem 1.3, V^m is isometric to a sphere.

§ 2. Some specializations of the preceding theorems. In this section, we shall discuss on some special cases of Theorem 1.3 and Theorem 1.4.

Let R^n be a Riemannian manifold with constant Riemann curvature and a vector field ξ^i be a vector field given by a concircular scalar field η , that is ξ^i satisfies an equation

$$\xi_{i;j} = \phi g_{ij}$$

where ϕ is not constant, we call such the vector field ξ^i a proper concircular Killing vector field. Then we have

$$\xi_{i;j;k} = \phi_k g_{ij}$$

and

$$\xi_{i;j;k} - \xi_{i;k;j} = \phi_k g_{ij} - \phi_j g_{ik}.$$

On the other hand, by means of Ricci identity it follows that

$$\xi_{i;j;k} - \xi_{i;k;j} = -R_{ijk}^l \xi^l.$$

Since R^n is of constant Riemann curvature, we obtain

$$R_{ijk} = \kappa (g_{ij} g_{ik} - g_{ik} g_{ij})$$

and so we have

$$R_{ijk} \xi^l = \phi_j g_{ik} - \phi_k g_{ij},$$

$$\kappa(\xi_j g_{ik} - \xi_k g_{ij}) = \phi_j g_{ik} - \phi_k g_{ij}$$

and

$$(2.1) \quad \kappa \xi_j = \phi_j.$$

In this case we can see easily that the unit normal vector n^ξ with respect to ξ^ξ is coincide with the unit normal vector n^ϕ with respect to ϕ^ξ , that is,

$$(2.2) \quad n^\xi_\xi = n^\phi_\phi$$

Hence we have the following relations

$$k_\nu = k_\nu, \quad H_\nu = H_\nu$$

for any ν ($1 \leq \nu \leq m$). Consequently we can give

Theorem 2.1. *If ξ^ξ is a proper concircular Killing vector field in R^n with constant Riemann curvature, then a closed orientable submanifold V^m such that $H_1 = \text{const.}$ and $\xi_i n^\xi_i > 0$ (or < 0) for all points of V^m is isometric to a sphere.*

Theorem 2.2. *If ξ^ξ is a proper concircular Killing vector field in R^n with constant Riemann curvature, then a closed orientable submanifold V^m such that $H_\xi = \text{const.}$ for any ν ($1 < \nu \leq m-1$), $k_1 > 0, \dots, k_m > 0$ and $\xi_i n^\xi_i > 0$ (or < 0) for all points of V^m is isometric to a sphere.*

If ξ^ξ is a proper conformal Killing vector field which satisfies $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$ in R^n with constant Riemann curvature and if we put a restriction such that at each point on V^m the vector ϕ^ξ lies in the vector space spanned by the tangent space of V^m and the Euler-Schouten unit vector n^ξ of V^m , then we find that the unit normal vector n^ξ with respect to ϕ^ξ is coincide with the Euler-Schouten unit vector n^ξ , that is,

$$(2.3) \quad n^\xi_\phi = n^\xi_E,$$

Consequently we have

$$k_\nu = k_\nu, \quad H_\nu = H_\nu$$

for any ν ($1 \leq \nu \leq m$), where H_1 means the first mean curvature H_1 of V^m .

Therefore we have

Theorem 2.3. *Let ξ^ξ be a proper conformal Killing vector field such that $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$ in R^n with constant Riemann curvature and V^m a closed*

orientable submanifold such that

- (i) $H_1 = \text{constant}$,
- (ii) $\phi_i n^i_E$ has fixed sign on V^m ,
- (iii) ϕ^i lies in the vector space spanned by the tangent space of V^m and n^i_E at each point of V^m .

Then V^m is isometric to a sphere.

Theorem 2.4. Let ξ^i be a proper conformal Killing vector field such that $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$ in R^n with constant Riemann curvature and V^m a closed orientable submanifold such that

- (i) $H_\nu = \text{constant}$ for any ν ($1 < \nu \leq m-1$),
- (ii) $k_1 > 0, \dots, k_m > 0$ and $\phi_i n^i_E > 0$ (or < 0) for all points of V^m ,
- (iii) ϕ^i lies in the vector space spanned by the tangent space of V^m and n^i_E at each point of V^m .

Then V^m is isometric to a sphere.

From these four theorems, we have the following corollaries.

Corollary 2.1. Let ξ^i be a proper concircular Killing vector field in R^n with constant Riemann curvature and V^m a closed orientable submanifold such that

- (i) $H_1 = \text{constant}$,
- (ii) $\xi_i n^i_E$ has fixed sign on V^m ,
- (iii) ξ^i lies in the vector space spanned by the tangent space of V^m and n^i_E at each point of V^m .

Then V^m is isometric to a sphere.

Corollary 2.2. Let ξ^i be a proper concircular Killing vector field in R^n with constant Riemann curvature and V^m a closed orientable submanifold such that

- (i) $H_\nu = \text{constant}$ for any ν ($1 < \nu \leq m-1$),
- (ii) $k_1 > 0, \dots, k_m > 0$ and $\xi_i n^i_E > 0$ (or < 0) for all points of V^m ,
- (iii) ξ^i lies in the vector space spanned by the tangent space of V^m and n^i_E at each point of V^m .

Then V^m is isometric to a sphere.

§ 3. Characteristic properties of a hypersurface to be isometric to

a sphere In this section, we consider a special case such that $m=n-1$, that is, V^m is a hypersurface V^{n-1} .

If n^ξ, k_ν ($\nu=1, \dots, n-1$) and H_ν are the normal unit vector, the principal curvatures and the ν -th mean curvature of V^{n-1} respectively, then we can easily find the following relations

$$\begin{aligned} n^\xi &= n^\xi = n^\xi = n^\xi, \\ k_\nu &= k_\nu = k_\nu = k_\nu, \\ H_\nu &= H_\nu = H_\nu = H_\nu \end{aligned}$$

for any ν ($1 \leq \nu \leq m-1$). Therefore we have

Theorem 3.1. *Let ξ^ξ be a proper conformal Killing vector field such that $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$ in R^n with constant Riemann curvature and V^{n-1} a closed orientable hypersurface such that*

- (i) $H_1 = \text{constant}$,
- (ii) $\phi_\xi n^\xi$ has fixed sign on V^{n-1} .

Then V^{n-1} is isometric to a sphere.

Theorem 3.2. *Let ξ^ξ be a proper conformal Killing vector field such that $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$ in R^n with constant Riemann curvature and V^{n-1} a closed orientable hypersurface such that*

- (i) $H_\nu = \text{constant}$ for any ν ($1 < \nu \leq n-2$),
- (ii) $k_1 > 0, \dots, k_{n-1} > 0$ and $\phi_\xi n^\xi > 0$ (or < 0) for all points of V^{n-1} .

Then V^{n-1} is isometric to a sphere.

Moreover from Theorem 3.1 and Theorem 3.2 in the previous paper [8] given by the present author, making use of the Gauss equation for a hypersurface in R^n with constant Riemann curvature, we also have without any difficulty the following theorems.

Theorem 3.3. *Let ξ^ξ be a proper conformal Killing vector field in R^n with constant Riemann curvature and V^{n-1} a closed orientable hypersurface such that $H_1 = \text{constant}$ and $\xi_\xi n^\xi$ has fixed sign on V^{n-1} . Then V^{n-1} is isometric to a sphere.*

Theorem 3.4. *Let ξ^ξ be a proper conformal Killing vector field in R^n with constant Riemann curvature and V^{n-1} a closed orientable hypersurface such that*

- (i) $H_\nu = \text{constant}$ for any ν ($1 < \nu \leq n-2$)
- (ii) $k_1 > 0, \dots, k_{n-1} > 0$ and $\xi_\xi n^\xi > 0$ (or < 0) for all points of V^{n-1} .

Then V^{n-1} is isometric to a sphere.

In an Einstein space R^n , the present author gave already the following theorem ([9], p. 170).

Theorem (Katsurada). *Let R^n be an Einstein space which admits a proper conformal Killing vector field such that $\xi_{i,j} + \xi_{j,i} = 2\phi g_{ij}$ and V^{n-1} a closed orientable hypersurface in R^n such that $H_1 = \text{constant}$ and $\xi_i n^i > 0$ (or < 0) for all points of V^{n-1} . Then every point of V^{n-1} is umbilic.*

From the above theorem, we then obtain the following two theorems.

Theorem 3.5. *Let ξ^i be a proper conformal Killing vector field such that $\xi_{i,j} + \xi_{j,i} = 2\phi g_{ij}$ in an Einstein space R^n and V^{n-1} a closed orientable hypersurface such that $H_1 = \text{constant}$ and $\phi_i n^i > 0$ (or < 0) for all points of V^{n-1} . Then V^{n-1} is isometric to a sphere.*

Theorem 3.6. *Let ξ^i be a proper conformal Killing vector field in an Einstein space R^n and V^{n-1} a closed orientable hypersurface such that $H_1 = \text{constant}$ and $\xi_i n^i > 0$ (or < 0) for all points of V^{n-1} . Then V^{n-1} is isometric to a sphere.*

These theorems are proved almost similarly to the proof of Theorem 1.3.

References

- [1] H. Liebmann: *Ueber die Verbiegung der geschlossenen Flächen positive Krümmung*, Math. Ann. 53 (1900), 91–112.
- [2] W. Süss: *Zur relativen Differentialgeometrie V.*, Tôhoku Math. J., 30 (1929), 202–209.
- [3] T. Bonnesen and W. Fenchel: *Theorie der konvexen Körper*, (Springer, Berlin 1934).
- [4] C.C. Hsiung: *Some integral formulas for closed hypersurfaces*, Marth. Scand. 2 (1954), 286–294.
- [5] H. Hopf: *Unber Flächen mit einer Relation zwischen der Hauptkrümmungen*, Math. Nachr. 4 (1951), 232–249.
- [6] A. D. Alexandrov: *Uniqueness theorems for surfaces in the large*, V. Vestnik Leningrad University 13 (1958), 5–8. (Russian, with English summary).
- [7] A. D. Alexandrov: *A characteristic property of spheres*, Ann. di Mat. p. appl., 58 (1962), 303–315.
- [8] Y. Katsurada: *Generalized Minkowski formulas for closed hypersurfaces in Riemann space*, Ann. di Mat. p. appl., 57 (1962), 283–293.
- [9] Y. Katsurada: *On a certain property of closed hypersurfaces in an Einstein space*, Comment. Math. Helv., 38 (1964), 165–171.
- [10] Y. Katsurada and T. Nagai: *On some properties of a submanifold with constant mean curvature in a Riemann space*, Jour. Fac. Sci. Hokkaido Univ., 20 (1968), 79–89.

- [11] Y. Katsurada and H. Kôjyo: *Some integral formulas for closed submanifolds in a Riemann space*, Jour. Fac. Sci. Hokkaido Univ., 20 (1968), 90-100.
- [12] Y. Katsurada: *Closed submanifolds with constant ν -th mean curvature related with a vector field in a Riemann manifold*, Jour. Fac. Sci. Hokkaido Univ., 20 (1968), 171-181.
- [13] T. Nagai: *On certain conditions for a submanifold in a Riemann space to be isometric to a sphere*, Jour. Fac. Sci. Hokkaido Univ. 20 (1968), 135-159.
- [14] K. Nomizu: *I. Hypersurfaces with constant mean curvature in S^n* , to appear.
- [15] K. Nomizu: *II. Surfaces with constant mean curvature in S^3* , to appear.
- [16] K. Yano: *The theory of Lie derivative and its applications*, (North-Holland, Amsterdam, 1957).
- [17] K. Yano: *Differential geometry on complex and almost complex space*, (Pergamon, 49, 1965).
- [18] J. A. Schouten: *Ricci-Calculus*, (Second edition) (Springer, Berlin 1954).
- [19] M. Obata: *Certain conditions for a Riemannian manifold to be isometric with a sphere*, Jour. Math. Soc. Japan, 14 (1962), 333-340.

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