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SOME CHARACTERIZATIONS OF A SUBMANIFOLD WHICH IS ISOMETRIC TO A SPHERE

By

Yoshie KATSURADA

Introduction. H. Liebmann (1900) [1][1], proved the following.

Theorem 0.1. *The only ovaloids with constant mean curvature H in an Euclidean space E^3 are spheres.*

Extension of this theorem to a convex hypersurface in an n-dimensional Euclidean space E^n has been given by W. Süss (1929) [2], (cf. also [3], p. 118, and [4]). Then H. Hopf (1951) [5], and A. D. Alexandrov (1958) [6], have shown the results that the convexity is not necessary for the validity of Liebmann-Süss theorem.

Recently the analogous problem for closed hypersurfaces in an n-dimensional Einstein space R^n admitting a conformal Killing vector field ξ has been discussed by the present author [8], [9], A. D. Alexandrov [7], K. Nomizu [14], [15]. Afterward T. Nagai, H. Kôjyo and the present author [10], [11], [13] have given a certain extension of this problem to an m-dimensional closed submanifold V^m $(1 \leqq m \leqq n-1)$ in R^n. However we have given there a restriction such that at each point on V^m, the vector ξ lies in the vector space spanned by the tangent space of V^m and the Euler-Schouten vector $\underset{E}{n}$.

In the previous paper [12], the present author has proved the following two theorems except this restriction for a closed orientable submanifold V^m in an orientable Riemannian manifold R^n with constant Riemann curvature.

Theorem 0.2. *If in R^n, there exists such a group G of conformal transformations generated by ξ as ρ is positive (or negative) at each point of V^m and if $\underset{\xi}{H_1}$ is constant, then every point of V^m is umbilic with respect to the normal vector $\underset{\xi}{n}$, where ρ denotes $\underset{\xi}{n}_i\xi^i$, $\underset{\xi}{n}$ is the unit vector normal V^m in the vector space spanned by the tangent space of V^m and ξ and $\underset{\xi}{H_\nu}$ means the ν-th mean curvature of V^m with respect to $\underset{\xi}{n}$.*

Thorem 0.3. *If in R^n, there exists such a grop G of conformal transformations generated by ξ as ρ is positive (or negative) at each point of V^m,*

1) Numbers in brackets refer to the references at the end of the paper.

and if the principal curvatures with respect to $\underset{\xi}{n}$, $\underset{\xi}{k_1}, \cdots, \underset{\xi}{k_m}$ *at each point of* V^m, *are positive and* $\underset{\xi}{H_\nu}$ *is constant for any* ν $(1<\nu\leqq m-1)$, *then every point of* V^m *is umbilic with respect to the normal vector* $\underset{\xi}{n}$.

The purpose of this paper is to induce a characteristic property such that V^m is isometric to a sphere from these theorems.

§ 1. Some characterizations of a submanifold to be isometric to a sphere. We suppose an n-dimensional Einstein space R^n $(n\geqq 3)$ of class C^r $(r\geqq 3)$ which has local coordinates x^i and admits a proper conformal Killing vector field ξ^i, that is, ξ^i satisfies an equation:

$$\underset{\xi}{\pounds} g_{ij} \equiv \xi_{i;j}+\xi_{j;i} = 2\phi g_{ij} \qquad \text{([16], p. 32)} \tag{1.1}$$

where g_{ij} is the metric tensor of R^n, $\underset{\xi}{\pounds} g_{ij}$ means the Lie derivative of g_{ij} with respect to ξ and the symbol " ; " the operation of D-symbol due to van der Waerden-Bortlotti ([18], p. 254). Then the Lie derivative of the Riemann curvature tensor R^h_{ijk} of R^n with respect to the conformal Killing vector field ξ is given by

$$\underset{\xi}{\pounds} R^h_{ijk} = \delta^h_j\phi_{i;k}-\delta^h_k\phi_{i;j}+g_{ik}\phi^h_{;j}-g_{ij}\phi^h_{;k} \qquad \text{([16], p. 160)} \tag{1.2}$$

where ϕ_i means $\phi_{;i}$ and δ^h_j is the Kronecker delta.

Since R^n is an Einstein s pace, we have

$$R_{ij} = \frac{R}{n} g_{ij} \qquad (R=\text{Constant}) \tag{1.3}$$

where R_{ij} and R are the Ricci tensor and the curvature scalar of R^n respectively.

Making use of (1.2) and (1.3), after some calculations we obtain the following result

$$\phi_{i;j} = -\frac{R}{n(n-1)}\phi g_{ij}. \tag{1.4}$$

From (1.4) we can see that the Einstein space R^n admitting the conformal Killing vector field ξ with the non-constant scalar field ϕ must always admit the vector field $\phi^i(\phi^i=g^{ij}\phi_j)$, which is the special conformal Killing vector field.

In R^n, we consider a domain M. If the domain M is simply covered by the orbits of transformations generated by ϕ^i, and ϕ^i is everywhere of class C^r and $\neq 0$ in M; then we call M a regular domain with respect to the vector field ϕ^i.

Let us denote by V^m an m-dimensional closed orientable submanifold of class C^3 imbedded in a regular domain M with respect to the vector field ϕ^i, locally given by

$$x^i = x^i(u^\alpha) \qquad i=1, \cdots, n\ ;\ \alpha=1, \cdots, m \tag{1.5}$$

where u^α are local coordinates of V^m. Throghout the present paper Latin indices run from 1 to n and Greek indices from 1 to m. We assume that at any point on V^m the vector ϕ^i is not on its tangent space.

We shall indicate by $\underset{p}{n}{}^i$ $(p=m+1, \cdots, n)$ the contravariont unit vector normal V^m and suppose that they are mutually orthogonal. Let $\underset{\phi}{n}$ be in the vector space spanned by $m+1$ independent vector $\dfrac{\partial x^i}{\partial u^\alpha}$ $(\alpha=1, \cdots, m)$ and ϕ^i and be the unit vector normal V^m. Then we may consider $\underset{\phi}{n}$ as one of the unit normal vectors of V^m, that is, $\underset{m+1}{n}{}^i = \underset{\phi}{n}{}^i$.

If we denote by $\underset{\phi}{k}_1, \cdots, \underset{\phi}{k}_m$ the principal curvatures of V^m with respect to $\underset{\phi}{n}{}^i$ and by $\underset{\phi}{H}_\nu$ the ν-th mean curvature, the υ-th elementary symmetric function of $\underset{\phi}{k}_1, \cdots, \underset{\phi}{k}_m$ divided by the number of terms, *i.e.*,

$$\binom{m}{\upsilon}\underset{\phi}{H}_\nu = \sum_{\alpha_1<\cdots<\alpha_\nu} \underset{\phi}{k}_{\alpha_1}\cdots\underset{\phi}{k}_{\alpha_\nu} \qquad (1\leqq\upsilon\leqq m),$$

then as a special case of Theorem 0.2. and of Theorem 0.3., we have the following two theorems.

Theorem 1.1. *Let ξ^i be a praper conformal Killing vector field in R^n with constant Riemann curvature (i.e., $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$). Then every point of a closed orientable submanifold V^m such that $H_1=$ constant and $\phi_i\underset{\phi}{n}{}^i>0$ (or <0) for all points of V^m is umbilic* with respect to $\underset{\phi}{n}$.

Remark. We remark here that $\phi_i\underset{\phi}{n}{}^i>0$ (or <0) for all points of V^m is equivalent that the vector ϕ^i is not on the tangent space of V^m at any point fo V^m.

Theorem 1.2. *Let ξ^i be a proper conformal Killing vector field in R^n with constant Riemann curvature (i.e., $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$). Then every point of a closed orientable submanifold V^m such that $\underset{\phi}{H}_\nu=$ constant, $\underset{\phi}{k}_1>0, \cdots, \underset{\phi}{k}_m>0$ and $\phi_i\underset{\phi}{n}{}^i>0$ (or <0) for all points of V^m is umbilic* with respect to $\underset{\phi}{n}$.

From the above theorems and the following theorem due to M. Obata [19], we obtain Theorem 1.3 and Theorem 1.4.

Theorem (*Obata*). *Let R^n $(n\geqq 2)$ be a complete Riemann manifold which admits a non-null function φ such that $\varphi_{;i;j}=-C^2\varphi g_{ij}$ ($C=$constant). Then R^n is isometric to a sphere of radius $1/C$.*

Theorem 1.3. *If ξ^i is a proper conformal Killing vector field in R^n with constant Riemann curvature (i.e., $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$), then a closed orientable submanifold V^m such that $\underset{\phi}{H_1}=$constant and $\phi_i\underset{\phi}{n^i}>0$ ($or<0$) for all points of V^m is isometric to a sphere.*

Proof. Let C^i_j be $\sum_{p=m+1}^{n}\underset{p}{n^i}\underset{p}{n_j}$ $(\underset{m+1}{n}=\underset{\phi}{n})$ and $\underset{\lambda}{i}$ $(\lambda=1,\cdots,m)$ mutually orthogonal unit tangent vectors of V^m. Then putting $B^i_\alpha=\dfrac{\partial x^i}{\partial u^\alpha}$, we have

$$\underset{p}{n^i_{;\alpha}}=C^i_{j;k}\underset{p}{n^j}B^k_\alpha=-\sum_{\lambda=1}^{m}(\underset{\lambda}{i}_{j;k}\underset{p}{n^j}B^k_\alpha)\underset{\lambda}{i^i}\,.$$

Therefore we may write as follows

$$\underset{p}{n^i_{;\alpha}}=\underset{p}{\gamma_\alpha^\gamma}B^i_\gamma\,.$$

Since we have

$$g_{ij}B^i_{\beta;\alpha}\underset{p}{n^j}=-g_{ij}B^i_\beta\underset{p}{n^j_{;\alpha}}\,,$$

we obtain

$$\underset{p}{n^i_{;\alpha}}=-\underset{p}{b_\alpha^\gamma}B^i_\gamma \qquad (p=\phi,\ m+2,\cdots,n) \tag{1.6}$$

where $\underset{p}{b_\alpha^\gamma}$ means $g^{\gamma\beta}\underset{p}{b_{\alpha\beta}}$ and $\underset{p}{b_{\alpha\beta}}\overset{\text{def.}}{=}B^i_{\alpha;\beta}\underset{p}{n_i}$, and $g^{\gamma\beta}$ is the contravariant metric tensor of V^m.

As well-known on the theory of a submanifold in a Riemannian manifold, we have

$$B^i_{\alpha;\beta}=\frac{\partial B^i_\alpha}{\partial u^\beta}+\Gamma^i_{jk}B^j_\alpha B^k_\beta-\Gamma'^\gamma_{\alpha\beta}B^i_\gamma\,,$$

$$\underset{p}{n^i_{;\alpha}}=\frac{\partial\underset{p}{n^i}}{\partial u^\alpha}+\Gamma^i_{jk}\underset{p}{n^j}B^k_\alpha-\Gamma'^Q_{p\alpha}\underset{Q}{n^i}\,,$$

where Γ^i_{jk} and $\Gamma'^\gamma_{\alpha\beta}$ are the Christoffel symbols with respect to g_{ij} and $g_{\alpha\beta}$ respectively, and

$$\Gamma'^Q_{p\alpha}\overset{\text{def.}}{=}\frac{\partial\underset{p}{n^i}}{\partial u^\alpha}\underset{Q}{n_i}+\Gamma^i_{jk}\underset{p}{n^j}B^k_\alpha\underset{Q}{n_i}\,.$$

Since $\underset{p}{n^i}\underset{Q}{n_i}=\delta_{PQ}$ (δ_{PQ} means the Kronecker delta, $\Gamma'^Q_{P\alpha}$ is anti-symmetric with

respect to the indices P and Q. Consequently we have

$$\Gamma'''^{P}_{Q\alpha}=0 \qquad \text{for any } P \quad (P=\phi,\ m+2,\cdots,n). \tag{1.7}$$

We now consider the scalar function on V^m given by

$$\varphi=\underset{\phi}{\phi_i n^i},$$

and we assume, moreover, that $\varphi\neq$ constant along V^m, then we have

$$\varphi_\alpha=\underset{\phi}{\phi_{i;j}n^i B^j_\alpha}+\underset{\phi}{\phi_i}\,(\underset{\phi}{n^i_{;\alpha}}+\underset{p}{\Gamma'''^{P}_{\phi\alpha}n^i}).$$

On the other hand the vector ϕ_i is expressed as follows

$$\phi_i=\underset{\phi}{\varphi n_i}+C_\gamma B^\gamma{}_i, \tag{1.8}$$

where $B_i{}^\gamma \overset{\text{def.}}{=} g_{ji}B^j_\beta g^{\beta\gamma}$.

From (1.7) and (1.8), we have

$$\underset{p}{\phi_i\Gamma'''^{P}_{\phi\alpha}n^i}=0.$$

Since R^n with constant Riemann curvature is an Einstein space, we get

$$\underset{\phi}{\phi_{i;j}n^i B_\alpha{}^j}=-\frac{R}{n(n-1)}\underset{\phi}{\phi g_{ij}n^i B^j_\alpha}=0,$$

and also from (1.6), we have

$$\varphi_\alpha=-\underset{\phi}{b_\alpha{}^\gamma B^i_\gamma\phi_i}.$$

Furthermore by virtue of Theorem 1.1, every point of V^m is umbilic with respect to $\underset{\phi}{n^i}$, that is,

$$\underset{\phi}{b_{\alpha\beta}}=\underset{\phi}{H_1}g_{\alpha\beta}. \tag{1.9}$$

So, it follows that

$$\underset{\phi}{b_\alpha{}^\beta}=\underset{\phi}{H_1}\delta_\alpha{}^\beta.$$

Thus we obtain

$$\varphi_\alpha=-\underset{\phi}{H_1}B^i_\alpha\phi_i$$

and

$$\varphi=-\underset{\phi}{H_1}\phi+C \qquad (C=\text{Constant}). \tag{1.10}$$

Since $\underset{\phi}{H_1}=$constant, we get

$$\varphi_{\alpha;\beta} = -\underset{\phi}{H_1}(\phi_{i;j}B^i_\alpha B^j_\beta + \phi_i B^i_{\alpha;\beta}) . \tag{1.11}$$

Consequently from (1.4) and (1.8), because of the relation $B_i{}^r B^i_{\alpha;\beta}=0$, we have

$$\varphi_{\alpha;\beta} = -\underset{\phi}{H_1}\left(-\frac{R}{n(n-1)}\phi g_{\alpha\beta} + \varphi \underset{\phi}{b}_{\alpha\beta}\right),$$

and from (1.9) and (1.10), (1.11) becomes as follows

$$\varphi_{\alpha;\beta} = \left\{-\left(\underset{\phi}{H_1^2} + \frac{R}{n(n-1)}\right)\varphi + \frac{R}{n(n-1)}C\right\} g_{\alpha\beta} \tag{1.12}$$

If $\underset{\phi}{H_1^2} + \dfrac{R}{n(n-1)} = 0$, then (1.12) becomes

$$\varphi_{\alpha;\beta} = \frac{R}{n(n-1)} C g_{\alpha\beta} ,$$

from which $\Delta\varphi = \dfrac{mR}{n(n-1)}C$ (Δ is the Laplacian operator on V^m), which is impossible unless $\varphi=$constant. Therefore $\underset{\phi}{H_1^2} + \dfrac{R}{n(n-1)} \neq 0$, then we have

$$\varphi_{\alpha;\beta} = -\left(\underset{\phi}{H_1^2} + \frac{R}{n(n-1)}\right)\left(\varphi - \frac{\dfrac{R}{n(n-1)}C}{\underset{\phi}{H_1^2} + \dfrac{R}{n(n-1)}}\right) g_{\alpha\beta} .$$

putting

$$\sigma = \varphi - \frac{\dfrac{R}{n(n-1)}C}{\underset{\phi}{H_1^2} + \dfrac{R}{n(n-1)}} ,$$

we obtain

$$\sigma_{\alpha;\beta} = -\left(\underset{\phi}{H_1^2} + \frac{R}{n(n-1)}\right)\sigma g_{\alpha\beta}$$

and

$$\Delta\sigma = -m\left(\underset{\phi}{H_1^2} + \frac{R}{n(n-1)}\right)\sigma ,$$

consequently it follows that $\underset{\phi}{H_1^2} + \dfrac{R}{n(n-1)} > 0$ [17]. Thus V^m is isometric to a sphere according to Obata's theorem.

Theorem 1.4. *If ξ^i is a proper conformal Killing vector field in R^n with constant Riemann curvature (i.e., $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$), then a closed orientable submanifold V^m such that $\underset{\phi}{H_\nu}=const.$ for any ν $(1<\nu\leqq m-1)$, $\underset{\phi}{k_1}>0,\cdots,\underset{\phi}{k_m}>0$ and $\phi_i\underset{\phi}{n^i}>0$ $(or\ <0)$ for all points of V^m is isometric to a sphere.*

Proof. From Theorem 1.2, every point of V^m is umbilic with respect to $\underset{\phi}{n^i}$, that is,

$$\underset{\phi}{k_1}=\cdots=\underset{\phi}{k_m}\,.$$

Accordingly we have

$$\underset{\phi}{H_\nu}=(\underset{\phi}{k_1})^\nu=(\underset{\phi}{H_1})^\nu=\text{const.}$$

Thus we get

$$\underset{\phi}{H_1}=\text{const.}$$

Consquently by virtue of Theorem 1.3, V^m is isometric to a sphere.

§ 2. Some specializations of the preceding theorems. In this section, we shall discuss on some special cases of Theorem 1.3 and Theorem 1.4.

Let R^n be a a Riemannian manifold with constant Riemann curvature and a vector field ξ^i be a vector field given by a concircular scalar field η, that is ξ^i satisfies an equation

$$\xi_{i;j}=\phi g_{ij}$$

where ϕ is not constant, we call such the vector field ξ^i a proper concircular Killing vector field. Then we have

$$\xi_{i;j;k}=\phi_k g_{ij}$$

and

$$\xi_{i;j;k}-\xi_{i;k;j}=\phi_k g_{ij}-\phi_j g_{ik}\,.$$

On the other hand, by means of Ricci identity it follows that

$$\xi_{i;j;k}-\xi_{i;k;j}=-R^l_{ijk}\xi^l\,.$$

Since R^n is of constant Rimann curvature, we obtain

$$R_{lijk}=\kappa(g_{lj}g_{ik}-g_{lk}g_{ij})$$

and so we have

$$R_{lijk}\xi^l=\phi_j g_{ik}-\phi_k g_{ij}\,,$$

$$\kappa(\xi_j g_{ik}-\xi_k g_{ij})=\phi_j g_{ik}-\phi_k g_{ij}$$

and

$$\kappa\xi_j=\phi_j\,. \tag{2.1}$$

In this case we can see easily that the unit normal vector $\underset{\xi}{n}{}^i$ with respect to ξ^i is coincide with the unit normal vector $\underset{\phi}{n}{}^i$ with respect to ϕ^i, that is,

$$\underset{\xi}{n}{}^i=\underset{\phi}{n}{}^i \tag{2.2}$$

Hence we have the following relations

$$\underset{\xi}{k}_\nu=\underset{\phi}{k}_\nu,\qquad \underset{\xi}{H}_\nu=\underset{\phi}{H}_\nu$$

for any ν $(1\leqq\nu\leqq m)$. Consequently we can give

Theorem 2.1. *If ξ^i is a proper concircular Killing vector field in R^n with constant Riemann curvature, then a closed orientable submanifold V^m such that $\underset{\xi}{H}_1=const.$ and $\xi_i\underset{\xi}{n}{}^i>0$ $(or<0)$ for all points of V^m is isometric to a sphere.*

Theorem 2.2. *If ξ^i is a proper concircular Killing vector field in R^n with constant Riemann curvature, then a closed orientable submanifold V^m such that $\underset{\xi}{H}_\nu=const.$ for any ν $(1<\nu\leqq m-1)$, $\underset{\xi}{k}_1>0,\cdots,\underset{\xi}{k}_m>0$ and $\xi_i\underset{\xi}{n}{}^i>0$ $(or<0)$ for all points of V^m is isometric to a sphere.*

If ξ^i is a proper conformal Killing vector field which satisfies $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ in R^n with constant Riemann curvature and if we put a restriction such that at each point on V^m the vector ϕ^i lies in the vector space spanned by the tangent space of V^m and the Eular-Schouten unit vector $\underset{E}{n}{}^i$ of V^m, then we find that the unit normal vector $\underset{\phi}{n}{}^i$ with respect to ϕ^i is coincide with the Euler-Schouten unit vector $\underset{E}{n}{}^i$, that is,

$$\underset{\phi}{n}{}^i=\underset{E}{n}{}^i\,, \tag{2.3}$$

Consequently we have

$$\underset{\phi}{k}_\nu=\underset{E}{k}_\nu\,,\qquad \underset{\phi}{H}_\nu=\underset{E}{H}_\nu$$

for any ν $(1\leqq\nu\leqq m)$, where $\underset{E}{H}_1$ means the first mean curvature H_1 of V^m. Therefore we have

Theorem 2.3. *Let ξ^i be a proper conformal Killing vector field such that $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ in R^n with constant Riemann curvature and V^m a closed*

orientable submanifold such that

(i) $H_1 - constant$,

(ii) $\phi_i \underset{E}{n^i}$ *has fixed sign on* V^m,

(iii) ϕ^i *lies in the vector space spanned by the tangent space of* V^m *and* $\underset{E}{n^i}$ *at each point of* V^m.

Then V^m *is isometric to a sphere.*

Theorem 2.4. *Let* ξ^i *be a proper conformal Killing vector field such that* $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ *in* R^n *with constant Riemann curvature and* V^m *a closed orientable submanifold such that*

(i) $\underset{E}{H_\nu} = constant$ *for any* ν $(1<\nu\leqq m-1)$,

(ii) $\underset{E}{k_1}>0, \cdots, \underset{E}{k_m}>0$ *and* $\phi_i \underset{E}{n^i}>0$ $(or<0)$ *for all points of* V^m,

(iii) ϕ^i *lies in the vector space spanned by the tangent space of* V^m *and* $\underset{E}{n^i}$ *at each point of* V^m.

Then V^m *is isometric to a sphere.*

From these four theorems, we have the following corollaries.

Corollary 2.1. *Let* ξ^i *be a proper concircular Killing vector field in* R^n *with constant Riemann curvature and* V^m *a closed orientable submanifold such that*

(i) $H_1 = constant$,

(ii) $\xi_i \underset{E}{n^i}$ *has fixed sign on* V^m,

(iii) ξ^i *lies in the vector space spanned by the tangent space of* V^m *and* $\underset{E}{n^i}$ *at each point of* V^m.

Then V^m *is isometric to a sphere.*

Corollary 2.2. *Let* ξ^i *be a proper concircular Killing vector field in* R^n *with constant Riemann curvature and* V^m *a closed orientable submanifold such that*

(i) $\underset{E}{H_\nu} = constant$ *for any* ν $(1<\nu\leqq m-1)$,

(ii) $\underset{E}{k_1}>0, \cdots, \underset{E}{k}>0$ *and* $\xi_i \underset{E}{n^i}>0$ $(or<0)$ *for all poinis of* V^m,

(iii) ξ^i *lies in the vector space spanned by the tangent space of* V^m *and* $\underset{E}{n^i}$ *at each point of* V^m.

Then V^m *is isometric to a sphere.*

§ 3. Characteristic properties of a hypersurface to be isometric to

a sphere In this section, we consider a special case such that $m=n-1$, that is, V^m is a hypersurface V^{n-1}.

If n^i, k_ν $(\nu=1,\cdots,n-1)$ and H_ν are the normal unit vector, the principal curvatures and the ν-th mean curvature of V^{n-1} respectively, then we can easily find the following relations

$$\overset{\xi}{n}{}^i=\overset{\phi}{n}{}^i=\overset{E}{n}{}^i=n^i,$$
$$\overset{\xi}{k}_\nu=\overset{\phi}{k}_\nu=\overset{E}{k}_\nu=k_\nu,$$
$$\overset{\xi}{H}_\nu=\overset{\phi}{H}_\nu=\overset{E}{H}_\nu=H_\nu$$

for any ν $(1\leqq\nu\leqq m-1)$. Therefore we have

Theorem 3.1. *Let ξ^i be a proper conformal Killing vector field such that $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ in R^n with constant Riemann curvature and V^{n-1} a closed orientable hypersurface such that*

(i) $H_1=$ *constant,*

(ii) $\phi_i n^i$ *has fixed sign on* V^{n-1}.

Then V^{n-1} is isometric to a sphere.

Theorem 3.2. *Let ξ^i be a proper conformal Killing vector field such that $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ in R^n with constant Riemann curvature and V^{n-1} a closed orientable hypersurface such that*

(i) $H_\nu=$ *constant for any* ν $(1<\nu\leqq n-2)$,

(ii) $k_1>0,\cdots,k_{n-1}>0$ *and* $\phi_i n^i>0$ *(or* <0*) for all points of* V^{n-1}.

Then V^{n-1} is isometric to a sphere.

Moreover from Theorem 3.1 and Theorem 3.2 in the previous paper [8] given by the present author, making use of the Gauss equation for a hypersurface in R^n with constant Riemann curvature, we also have without any difficulty the following theorems.

Theorem 3.3. *Let ξ^i be a proper conformal Killing vector field in R^n with constant Riemann curvature and V^{n-1} a closed orientable hypersurface such that $H_1=$ constant and $\xi_i n^i$ has fixed sign on V^{n-1}. Then V^{n-1} is isometric to a shpere.*

Theorem 3.4. *Let ξ^i be a proper conformal Killing vector field in R^n with constant Riemann curvature and V^{n-1} a closed orientable hypersurface such that*

(i) $H_\nu=$ *constant for any* ν $(1<\nu\leqq n-2)$

(ii) $k_1>0,\cdots,k_{n-1}>0$ *and* $\xi_i n^i>0$ *(or* <0*) for all points of* V^{n-1}.

Then V^{n-1} is isometric to a sphere.

In an Einstein space R^n, the present author gave already the following theorem ([9], p. 170).

Theorem (*Katsurada*). *Let R^n be an Einstein space which admits a proper conformal Killing vector field such that $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ and V^{n-1} a closed orientable hypersurface in R^n such that $H_1=$ constant and $\xi_i n^i>0$ (or <0) for all points of V^{n-1}. Then every point of V^{n-1} is umbilic.*

From the above theorem, we then obtain the following two theorems.

Theorem 3.5. *Let ξ^i be a proper conformal Killing vector field such that $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ in an Einstein space R^n and V^{n-1} a closed orientable hypersurface such that $H_1=$ constant and $\phi_i n^i>0$ (or <0) for all points of V^{n-1}. Then V^{n-1} is isometric to r sphere.*

Theorem 3.6. *Let ξ^i be a proper conformal Killing vector field in an Einstein space R^n and V^{n-1} a closed orientable hypersurface such that $H_1=$ constant and $\xi_i n^i>0$ (or <0) for all points of V^{n-1}. Then V^{n-1} is isometric to a sphere.*

These theorems are proved almost similarly to the proof of Theorem 1.3.

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