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Author(s)	Inoue, Jyunji
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A CHARACTERIZATION OF A PARTICULAR CLASS OF HELSON SETS

By

Jyunji INOUE

Let G be a locally compact abelian group, Γ be its dual group, and $M(G)$ be the Banach algebra of all regular complex measures on G . A compact subset P of G is called a *Kronecker set* ($=K$ -set) if every modular continuous function on P is uniformly approximable ($=u$ -approximable) by continuous characters of G . P is called a *Helson set* if for some $\varepsilon > 0$ and each $\mu \in M(P)$

$$(H, \varepsilon) \quad \sup_{\gamma \in \Gamma} \left| \int_G \gamma(-x) d\mu(x) \right| \geq \varepsilon \|\mu\|, \quad \text{where } \|\mu\| = |\mu|(P).$$

If P is a K -set, P has the property $(H, 1)$. W. Rudin asked in [1] whether the property $(H, 1)$ implies that P is a K -set, and I. Wik [2] and R. Kaufman [3] show counter examples.

The purpose of this paper is to prove the following theorem which is a characterization of the set with the property $(H, 1)$.

Theorem. *Let G be a locally compact abelian group, P be a metrizable compact subset of G . Then P satisfies $(H, 1)$ if and only if the following conditions are satisfied.*

- 1) *For every finite subset $\{x_0, x_1, x_2, \dots, x_n\}$ of P , $\{x_0 - x_1, x_0 - x_2, x_0 - x_3, \dots, x_0 - x_n\}$ is a K -set.*
- 2) *For each $\mu \in M(P)$, there exist open subsets C_i in P and $x_0 \in P$ such that*

- a) $C_1 \supset C_2 \supset C_3 \supset C_4 \supset \dots \ni x_0$
- b) $\lim_{i \rightarrow \infty} |\mu|(C_i) = |\mu|(\{x_0\})$
- c) *Every closed subset of $P - C_i$ ($i = 1, 2, 3, \dots$) is a K -set.*

Later we show an example which shows that a closed subset of a K -set is not always a K -set.

Notations. Throughout this paper, G is used to denote a locally compact abelian group. we call a Borel set with the property $(H, 1)$ a $(H, 1)$ -set, and by characters we mean continuous characters of G . $T, C(G)$ is used to denote the group of all complex numbers of absolute value 1, and the set of all continuous functions on G respectively. Let g be a function on G and P be

a subset of G , then by $g|P$, we mean the restriction of g to P .

Definition. We call a compact subset P of G a K' -set, if for each modular continuous function g on P there exists $\beta \in T$ such that βg is u -approximable by characters.

Lemma 1. Let B be a $(H, 1)$ -set in G , and g be a modular continuous function on B . Then for each $\mu \in M(B)$ and $\{\delta_i : \delta_i > 0, i = 1, 2, 3, \dots\}$, there exist $\beta \in T$, and open subsets C_i in B such that

$$\begin{cases} 1) & |\mu|(C_i) < \delta_i, C_1 \supset C_2 \supset C_3 \supset C_4 \supset \dots, \\ 2) & \beta g|B - C_i \text{ is } u\text{-approximable by characters.} \end{cases} \quad (i = 1, 2, 3, \dots)$$

Proof. Since B is a $(H, 1)$ -set, there exist $\gamma_1 \in \Gamma$ and $\alpha_1 \in T$ for which

$$(1.1) \quad \int_B \alpha_1 \gamma_1(-x) g(x) d|\mu|(x) = \left| \int_B \gamma_1(-x) g(x) d|\mu| \right| \geq |\mu|(B) - 1/32$$

and we have

$$(1.2) \quad \int_B \left[1 - \operatorname{Re} \{ \alpha_1 \gamma_1(-x) g(x) \} \right] d|\mu| \leq 1/32$$

Put $A_1 = \{x \in B : 1 - \operatorname{Re} [\alpha_1 \gamma_1(-x) g(x)] > 1/16\}$. Then from the relation $|\mu|(A_1) \cdot (1/16) < 1/32$, we have $|\mu|(A_1) < 1/2$. On the other hand, the relation

$$(1.3) \quad 1 - \operatorname{Re} [\alpha_1 \gamma_1(-x) g(x)] \leq 1/16 \quad (x \in B - A_1)$$

gives

$$(1.4) \quad \left| \operatorname{Im} [\alpha_1 \gamma_1(-x) g(x)] \right| \leq 1/2\sqrt{2} \quad (x \in B - A_1)$$

and hence

$$(1.5) \quad \left| \gamma_1(x) - \alpha_1 g(x) \right| \leq \left| 1 - \operatorname{Re} [\alpha_1 \gamma_1(-x) g(x)] \right| + \left| \operatorname{Im} [\alpha_1 \gamma_1(-x) g(x)] \right| \leq 1/16 + 1/2\sqrt{2} < 1/2 \quad (x \in B - A_1)$$

In the same way, we get open subsets A_i of B , $\gamma_i \in \Gamma$, $\alpha_i \in T$ ($i = 2, 3, 4, \dots$) such that

$$(1.6) \quad \begin{cases} |\gamma_i(x) - \alpha_i g(x)| < 1/i & (x \in B - A_i) \\ |\mu|(A_i) < 1/2^i \end{cases} \quad (i = 1, 2, 3, \dots)$$

Let n_i be positive integers such that

$$(1.7) \quad \begin{cases} n_1 < n_2 < n_3 < n_4 < \dots, \\ |\mu|(\bigcup_{k=n_i}^{\infty} A_k) < \delta_i & (i = 1, 2, 3, \dots) \end{cases}$$

Choose a subsequence $\{\alpha_{j_i}\}$ of $\{\alpha_i\}$ which converges to some $\beta \in T$. Then if we put $\bigcup_{k=n_i}^{\infty} A_k = C_i$ ($i=1, 2, 3, \dots$), $\beta g|B-C_i$ is u -approximable by characters, and the lemma is proved.

Proposition 2. *Let P be a compact metrizable $(H, 1)$ -set in G . Then for each $\mu \in M(P)$, there exist open subsets C_i in P such that*

- 1) $C_1 \supset C_2 \supset C_3 \supset C_4 \supset \dots$,
- 2) $\lim_{i \rightarrow \infty} |\mu|(C_i) = 0$,
- 3) Every closed subset of $P - C_i$ ($i=1, 2, 3, \dots$) is a K' -set.

Proof. Since P is metrizable, $C(P)$ has a countable dense subset with respect to sup-norm, say $\{f_1, f_2, f_3, \dots\}$. Put $B_i = \{x \in P : 1/2 < |f_i(x)| < 3/2\}$ ($i=1, 2, 3, \dots$), and since B_i is also $(H, 1)$ -set, by Lemma 1 there exist open subsets C_{ij} in B_i and $\beta_i \in T$ such that

$$(2.1) \quad \begin{cases} |\mu|(C_{ij}) < 1/2^{i+j}, & C_{i1} \supset C_{i2} \supset C_{i3} \supset C_{i4} \supset \dots, \\ \beta_i f_i / |f_i| |B_i - C_{ij} \text{ is } u\text{-approximable by characters.} \end{cases} \quad (i, j=1, 2, \dots)$$

Put $C_j = \bigcup_{i=1}^{\infty} C_{ij}$, and we have

$$(2.2) \quad |\mu|(C_j) < 1/2^j \quad (j=1, 2, 3, \dots), \quad C_1 \supset C_2 \supset C_3 \supset C_4 \supset \dots.$$

Let D be a closed subset of $P - C_j$, and g be a modular continuous function on D . We can extend g to a continuous function g_1 on P and there exists a sequence of positive integers $n_1, n_2, n_3, \dots$, such that

$$(2.3) \quad \|f_{n_i} - g_1\|_{\infty} < 2^{-i} \quad (i=1, 2, 3, \dots)$$

and $\beta_{n_1}, \beta_{n_2}, \beta_{n_3}, \beta_{n_4}, \dots$, converge to some $\beta \in T$. If $\varepsilon > 0$, then there exist a positive integer k and $\gamma \in \Gamma$ such that

$$(2.4) \quad \begin{cases} |\beta - \beta_{n_k}| < \varepsilon/4, & 2^{-k} < \varepsilon/4 \\ \left| \gamma(x) - \beta_{n_k} f_{n_k}(x) / |f_{n_k}(x)| \right| < \varepsilon/4 \end{cases} \quad (x \in B_{n_k} - C_j)$$

From (2.3), $B_{n_k} \supset D$, and hence $B_{n_k} - C_j \supset D$.

From (2.3) and (2.4), we obtain

$$(2.5) \quad \varepsilon/4 > 2^{-k} > \|f_{n_k} - g_1\|_{\infty} \geq |f_{n_k}(x)| - 1 \quad (x \in D)$$

and hence

$$(2.6) \quad \left| \beta_{n_k} f_{n_k}(x) - \beta_{n_k} f_{n_k}(x) / |f_{n_k}(x)| \right| = \left| |f_{n_k}(x)| - 1 \right| < \varepsilon/4 \quad (x \in D)$$

From (2.3), (2.4) and (2.6) we have

$$\begin{aligned}
 (2.7) \quad & \left| \beta g(x) - \gamma(x) \right| \leq \left| \beta g(x) - \beta_{n_k} g(x) \right| + \left| \beta_{n_k} g(x) - \beta_{n_k} f_{n_k}(x) \right| \\
 & + \left| \beta_{n_k} f_{n_k}(x) - \beta_{n_l} f_{n_k}(x) / |f_{n_k}(x)| \right| + \left| \beta_{n_l} f_{n_k}(x) / |f_{n_k}(x)| - \gamma(x) \right| \\
 & \leq \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon \quad (x \in D)
 \end{aligned}$$

Since $\varepsilon > 0$ can be arbitrary small, this proves that D is a K' -set.

Lemma 3. *Let Q be a compact subset of G , and suppose every closed subset of Q is a K' -set. If every modular constant function on Q is u -approximable by characters, then every closed subset of Q is a K -set.*

Proof. Let Q_1 be a closed subset of Q , and Ω be the set of all continuous functions on Q_1 which are u -approximable by characters. Then Ω form a group with respect to multiplication. Let g be a modular continuous function on Q_1 , then there exists $\alpha \in T$ such that $\alpha g \in \Omega$, and constant function $\alpha \in \Omega$ and so $\alpha^{-1} \alpha g = g \in \Omega$. This proves that Q_1 is a K -set.

Lemma 4. *Let Q be a compact subset of G , and suppose every closed subset of Q is a K' -set. Then for every pair (V, K) , where V is open in Q , K is compact and $Q \supset V \supset K$, either of the following a) or b) is true.*

- a) *Every closed subset of $Q - V$ is a K -set.*
- b) *Every modular continuous function h on $Q - (V - K)$ of the form*

$$(4.1) \quad h(x) = \begin{cases} 1 & : x \in Q - V \\ c & : x \in K, \end{cases} \quad \text{where } c \in T,$$

is u -approximable by characters.

Proof. Let g be a modular continuous function on Q such that

$$(4.2) \quad g(x) = \begin{cases} \alpha & : x \in Q - V \\ 1 & : x \in K \\ * & : x \in V - K, \end{cases} \quad \text{where } \alpha \in T \text{ and } \alpha \text{ is of infinite order.}$$

Then there exists $\beta \in T$ such that βg is u -approximable by characters.

a) If $\alpha\beta$ is of infinite order, then every modular constant function on $Q - V$ is u -approximable by characters, and by lemma 3 every closed subset of $Q - V$ is a K -set.

b) If $\alpha\beta$ is of finite order, then there exists a positive integer m such that $(\alpha\beta)^m = 1$, so the function

$$(4.2) \quad (\beta g)^m \Big|_{Q - (V - K)}(x) = \begin{cases} 1 & : x \in Q - V \\ \beta^m & : x \in K \end{cases}$$

is u -approximable by characters. Since β^m is of infinite order, the function of the form (4.1) is u -approximable by characters, and the lemma is proved.

Proposition 5. *Let P be a compact metrizable $(H, 1)$ -set in G . Then for each $\mu \in M(P)$, either of the following a) or b) is true.*

- a) *There exist open subsets C_i in P and $x_0 \in P$ such that*
- 1) $P \supset C_1 \supset C_2 \supset C_3 \supset C_4 \supset \dots \ni x_0$,
 - 2) $\lim_{i \rightarrow \infty} |\mu|(C_i) = |\mu|\{x_0\}$,
 - 3) *Every closed subset of $P - C_i$ ($i = 1, 2, 3, \dots$) is a K -set.*
- b) *There exist open subsets C_i in P such that*
- 1) $P \supset C_1 \supset C_2 \supset C_3 \supset C_4 \supset \dots$,
 - 2) $\lim_{i \rightarrow \infty} |\mu|(C_i) = 0$,
 - 3) *Every closed subset of $P - C_i$ ($i = 1, 2, 3, \dots$) is a K -set.*

Proof. By Prop. 2 there exist open subsets C'_i of P such that

$$(5.1) \quad \begin{cases} C'_1 \supset C'_2 \supset C'_3 \supset C'_4 \supset \dots, \\ \lim_{i \rightarrow \infty} |\mu|(C'_i) = 0 \\ \text{Every closed subset of } P - C'_i \text{ (} i = 1, 2, 3, \dots \text{) is a } K'\text{-set.} \end{cases}$$

Put $Q'_i = P - C'_i$, then we have $P - \bigcap_{i=1}^{\infty} C'_i = \bigcup_{i=1}^{\infty} Q'_i$, $Q'_1 \subset Q'_2 \subset Q'_3 \subset Q'_4 \subset \dots$, and every closed subset of Q'_i is a K' -set. For each $x \in \bigcup_{i=1}^{\infty} Q'_i$, put

$$(5.2) \quad m_x = \sup \left[\{0\} \cup \{i : x \in Q'_i \text{ and every closed subset of } Q'_i \text{ which don't contain } x \text{ is a } K\text{-set}\} \right].$$

Case 1. Suppose there exists $x_0 \in \bigcup_{i=1}^{\infty} Q'_i$ for which $m_{x_0} = \infty$. Then there exist a sequence of open subsets U_i in P such that

$$(5.3) \quad \begin{cases} U_1 \supset U_2 \supset U_3 \supset U_4 \supset \dots \ni x_0 \\ \lim_{i \rightarrow \infty} |\mu|(U_i) = |\mu|\{x_0\} \end{cases}$$

If we set $C_i = (C'_i \cup U_i)$ ($i = 1, 2, 3, \dots$) we have a).

Case 2. Next consider the case $m_x < \infty$ for all $x \in \bigcup_{i=1}^{\infty} Q'_i$. In this case, we can correspond to each $x \in \bigcup_{j=1}^{\infty} Q'_j$ an open neighborhood U_x of x in P such that for every triple (Q'_j, U, K) , where $j \in \{m_x + 1, m_x + 2, m_x + 3, \dots\}$, U is an open subset in $U_x \cap Q'_j$ and K is a compact subset of U , the case b) of the

Lemma 4. is true.

Case 2. a. Suppose the set $\{m_x : x \in \bigcup_{j=1}^{\infty} Q'_j\}$ is not bounded. Then we can choose $x_i \in \bigcup_{j=1}^{\infty} Q'_j$ and open neighborhoods V_i of x_i in P such that

$$(5.4) \quad \begin{cases} 0 < m_{x_1} < m_{x_2} < m_{x_3} < m_{x_4} < m_{x_5} < \cdots, \\ |\mu|(V_i) < 2^{-i} & (i=1, 2, 3, \cdots). \end{cases}$$

If we put $C_j = C'_j \cup \left(\bigcup_{i=j}^{\infty} V_i \right)$ ($j=1, 2, 3, \cdots$), then we have the case b).

Case 2. b. Finally consider the case $\{m_x : x \in \bigcup_{i=1}^{\infty} Q'_i\}$ is bounded. In this case, we can suppose without loss of generality that $m_x = 0$ for all $x \in \bigcup_{i=1}^{\infty} Q'_i$. Since Q'_i is compact, we can choose a sequence $x_1, x_2, x_3, \cdots \in \bigcup_{i=1}^{\infty} Q'_i$ and an increasing sequence of positive integers $t_1, t_2, t_3, \cdots$, such that

$$(5.5) \quad \begin{cases} \{x_1, x_2, x_3, \cdots, x_{t_k}\} \subset Q'_k \\ \bigcup_{i=1}^{t_k} U_{x_i} \supset Q'_k \end{cases} \quad (k=1, 2, 3, \cdots)$$

Since μ is regular, we can choose compact subsets $K_i^{(n)}$ such that

$$(5.6) \quad \begin{cases} K_i^{(n)} \subset U_{x_i} \cap Q'_n \quad (i=1, 2, \cdots, t_n; n=1, 2, \cdots), \\ Q_1 \subset Q_2 \subset Q_3 \subset Q_4 \subset Q_5 \subset \cdots, \\ \lim_{i \rightarrow \infty} |\mu|(Q_i) = |\mu|(P). \end{cases}$$

where $Q_n = \bigcup_{j=1}^{t_n} \left(K_j^{(n)} - \bigcup_{i=0}^{j-1} U_{x_i} \right)$, $U_{x_0} = \phi$ ($n=1, 2, 3, \cdots$).

By the property of U_x , every modular constant function on Q_n is u -approximable by characters, and by Lemma 3 every closed subset of Q_n is a K -set. If we set $C_i = P - Q_i$, we have the case b) and this completes the proof.

Corollary. Let P be a metrizable $(H, 1)$ -set in G . Then for each $\mu \in M(P)$ there exist open subsets C_i in P and $x_0 \in P$ such that

- 1) $P \supset C_1 \supset C_2 \supset C_3 \supset C_4 \supset \cdots \ni x_0$,
- 2) $\lim_{i \rightarrow \infty} |\mu|(C_i) = |\mu|\{x_0\}$,
- 3) Every closed subset of $P - C_i$ ($i=1, 2, 3, \cdots$) is a K -set.

Proof. By prop. 5, a) or b) of the prop. is true. In the case a) we have nothing to prove. In the case b) we can select $x_0 \in P$ and open neighborhoods W_i of x_0 in P such that

$$\begin{cases} W_1 \supset W_2 \supset W_3 \supset W_4 \supset \dots \ni x_0 \\ |\mu|(W_i) < 1/2^i \end{cases} \quad (i=1, 2, 3, \dots).$$

If we rewrite $C_i \cup W_i$ by C_i , $\{C_i : i=1, 2, 3, \dots\}$ and x_0 satisfies 1), 2), 3).

Lemma 6. *Let Q be a compact subset of G and suppose every closed subset of Q is a K -set. If $G-Q \ni x_0$, $Q \supset V \supset K$, where V is open in Q and K is compact, either of the following a) or b) is true.*

- a) $(Q-V) \cup \{x_0\}$ is a K -set.
- b) Every continuous function of the form

$$(6.1) \quad k(x) = \begin{cases} c : x \in K \\ 1 : x \in (Q-V) \cup \{x_0\}, \text{ where } c \in T, \end{cases}$$

is u -approximable by characters.

Proof. Let h be a modular continuous function on $Q-(V-K)$ of the form

$$(6.2) \quad h(x) = \begin{cases} \alpha : x \in K \\ 1 : x \in Q-V, \text{ where } \alpha \in T \text{ is of infinite order.} \end{cases}$$

Then there exist $\gamma_i \in T$ ($i=1, 2, 3, \dots$) such that

$$(6.3) \quad |\gamma_i(x) - h(x)| < 1/i \quad (x \in Q-(V-K))$$

and we can choose subsequence $\{\gamma_{n_i}(x_0), \gamma_{n_2}(x_0), \dots\}$ of $\{\gamma_1(x_0), \gamma_2(x_0), \dots\}$ which converge to some $\beta \in T$.

a) If β is of infinite order, then $(Q-V) \cup \{x_0\}$ is a K -set.

b) If β is of finite order, there exists a positive integer m such that $\beta^m = 1$ and $\{(\gamma_{n_i})^m | (Q \cup \{x_0\} - (V-K)) : i=1, 2, 3, \dots\}$ converge uniformly to h_1 , where

$$(6.4) \quad h_1(x) = \begin{cases} \alpha^m : x \in K \\ 1 : x \in (Q-V) \cup \{x_0\} \end{cases}$$

and so the function of the form (6.1) is u -approximable by characters.

Proposition 7. *Let P be a compact subset of G . Suppose*

- 1) *Every finite subset of P is a K' -set.*
- 2) *For each $\mu \in M(P)$, there exist open subsets C_i of P and $x_0 \in P$ such that*
 - a) $C_1 \supset C_2 \supset C_3 \supset C_4 \supset \dots \ni x_0$
 - b) $\lim |\mu|(C_i) = |\mu|(\{x_0\})$
 - c) *Every closed subset of $P-C_i$ ($i=1, 2, 3, \dots$) is a K -set.*

Then P is a $(H, 1)$ -set.

Proof. If $\mu \in M(P)$, by condition 2) there exist C_i ($i=1, 2, 3, \dots$) and $x_0 \in P$ for which a), b), c) hold.

(I) *The case when $\mu\{x_0\} = 0$.* In this case, the fact a K -set is a $(H, 1)$ -set and a simple calculation lead us to

$$(7.1) \quad |\mu|(P) = \sup_{r \in I} \left| \int_G r(-x) d\mu(x) \right|$$

(II) *The case when $\mu\{x_0\} \neq 0$.* Fix $\varepsilon > 0$ and positive integer i_0 , such that

$$(7.2) \quad |\mu|(C_{i_0} - x_0) < \varepsilon/2^2.$$

We can choose $\{x_1, x_2, x_3, \dots, x_n\} \subset P - C_{i_0}$, and open neighborhoods U_i of x_i ($i=1, 2, 3, \dots, n$) in $P - C_{i_0}$, such that

$$(7.3) \quad \begin{cases} |\mu|\{x_i\} \geq \varepsilon/2 & (i=1, 2, 3, \dots, n) \\ |\mu|\{x\} < \varepsilon/2 & (x \in P - C_{i_0} \cup \{x_1, x_2, x_3, \dots, x_n\}) \\ |\mu|\left(\bigcup_{i=1}^n U_i - \{x_1, x_2, x_3, \dots, x_n\}\right) < \varepsilon/2^3 \end{cases}$$

We set $\bigcup_{i=1}^n U_i = U$ and $P - C_{i_0} = Q$.

(II) a. At first consider the case there exists an open subset V in Q such that $|\mu|V < \varepsilon/2$ and $(Q - V) \cup \{x_0\}$ is a K -set. In this case, as in the case (I) there exists $r \in I$ for which

$$(7.4) \quad |\mu|(P) - \varepsilon < \left| \int_G r(-x) d\mu(x) \right|$$

(II) b. Next consider the case, that for every open set V in Q with $|\mu|(V) < \varepsilon/2$, $(Q - V) \cup \{x_0\}$ is not a K -set. Then by Lemma 6, for every pair (V, K) , where V is open in Q with $|\mu|(V) < \varepsilon/2$ and K is a compact subset of V , a continuous function of the form (6.1) is u -approximable by characters. Since $\{x_0, x_1, x_2, \dots, x_n\}$ is a K' -set, there exist $d_1 \in T$, $r_1 \in I$, such that

$$(7.5) \quad \sum_{i=0}^n d_1 r_1(-x_i) \mu\{x_i\} > \sum_{i=0}^n |\mu|\{x_i\} - \varepsilon/2^5$$

Also since $Q - U$ is a K -set, there exists $d_2 \in T$, $r_2 \in I$, such that

$$(7.6) \quad \int_{Q-U} d_1 d_2 (r_1 + r_2)(-x) d\mu(x) > |\mu|(Q - U) - \varepsilon/2^6.$$

From (7.3), to each $x \in Q - U$ we can correspond an open neighborhood V_x of x in Q with

$$(7.7) \quad |\mu|(V_x) < \varepsilon/2, \quad |\gamma_2(x) - \gamma_2(y)| < \varepsilon/2^8 \|\mu\| \quad (y \in V_x)$$

and since $Q - U$ is compact and μ is regular, we can choose $\{y_1, y_2, \dots, y_m\} \subset Q - U$ and compact subset K_ε of $V_{y_\varepsilon} \cap (Q - U)$ such that

$$(7.8) \quad \bigcup_{i=1}^m V_{y_i} \supset Q - U, \quad |\mu| \left(\bigcup_{i=1}^m (K_\varepsilon - \bigcup_{k=0}^{i-1} V_{y_k}) \right) > |\mu|(Q - U) - \varepsilon/2^4,$$

where $V_{y_0} = \phi$. We set $\bigcup_{i=1}^m (K_\varepsilon - \bigcup_{k=0}^{i-1} V_{y_k}) = K$. By the properties of V_{y_ε} and K_ε , we can choose $r_3 \in \Gamma$, such that

$$(7.9) \quad \begin{cases} |\gamma_3(-x) - d_2 \gamma_2(-x)| < \varepsilon/2^8 \|\mu\| & (x \in K) \\ |\gamma_3(-x) - 1| < \varepsilon/2^8 \|\mu\| & (x \in \{x_0, x_1, x_2, \dots, x_n\}) \end{cases}$$

Hence from (7.2), (7.3), (7.5), (7.6), (7.8), (7.9), we obtain

$$(7.10) \quad \begin{aligned} \left| \int_G (\gamma_1 + \gamma_3)(-x) d\mu \right| &= \left| \int_G d_1(\gamma_1 + \gamma_3)(-x) d\mu \right| \\ &> \left| \int_{K \cup \{x_0, x_1, \dots, x_n\}} d_1(\gamma_1 + \gamma_3)(-x) d\mu \right| - (\varepsilon/2^2 + \varepsilon/2^3 + \varepsilon/2^4) \\ &\geq \left| \int_{\{x_0, x_1, \dots, x_n\}} d_1 \gamma_1(-x) d\mu \right| + \left| \int_K d_1 d_2(\gamma_1 + \gamma_2)(-x) d\mu \right| \\ &\quad - \left| \int_{\{x_0, x_1, \dots, x_n\}} \{d_1(\gamma_1 + \gamma_3)(-x) - d_1 \gamma_1(-x)\} d\mu \right| \\ &\quad - \left| \int_K \{d_1(\gamma_1 + \gamma_3)(-x) - d_1 d_2(\gamma_1 + \gamma_2)(-x)\} d\mu \right| \\ &\quad - (\varepsilon/2^2 + \varepsilon/2^3 + \varepsilon/2^4) \\ &> |\mu|\{x_0, x_1, x_2, \dots, x_n\} + |\mu|(K) - \sum_{i=2}^6 (\varepsilon/2^i + \varepsilon/2^8 + \varepsilon/2^8) \\ &> |\mu|(P) - \left(\sum_{i=2}^7 \varepsilon/2^i + \varepsilon/2^2 + \varepsilon/2^3 + \varepsilon/2^4 \right) > |\mu|(P) - \varepsilon. \end{aligned}$$

Since in (7.4), (7.10), $\varepsilon > 0$ can be arbitrary small, (7.1) is also holds in the case (II), and this completes the proof.

Theorem. *Let P be a metrizable compact subset of a locally compact abelian group G , and Γ be its dual group. Then following a) and b) are equivalent.*

a) For each $\mu \in M(P)$, $|\mu|(P) = \sup_{\gamma \in \Gamma} \left| \int_G \gamma(-x) d\mu(x) \right|$ holds.

b) 1) For every finite subset $\{x_0, x_1, \dots, x_n\}$ of P , $\{x_0 - x_1, x_0 - x_2, \dots, x_0 - x_n\}$ is a K -set.

2) For each $\mu \in M(P)$, there exist open subsets C_ε in P and $y_0 \in P$

such that

- (i) $P \supset C_1 \supset C_2 \supset C_3 \supset \cdots \ni y_0$,
- (ii) $\lim_{\ell \rightarrow \infty} |\mu|(C_\ell) = |\mu|\{y_0\}$
- (iii) Every closed subset of $P - C_\ell$ ($\ell = 1, 2, 3, \dots$) is a K -set.

Proof. a) implies b): a) implies that P is a $(H, 1)$ -set, and by Cor. of prop. 2, 2) of b) holds. Let $x_i \in P$ ($i = 0, 1, 2, \dots, n$), $c_i \in T$ ($i = 1, 2, 3, \dots, n$) and $\mu \in M(P)$, with $\mu\{x_0\} = 1$, $\mu\{x_i\} = c_i$ ($i = 1, 2, 3, \dots, n$), $\mu(E) = 0$ if $E \cap \{x_0, x_1, \dots, x_n\} = \emptyset$.

For $\varepsilon > 0$, there exists $r \in \Gamma$ for which

$$\left| \sum_{i=0}^n r(-x_i) \mu\{x_i\} \right| \geq \sum_{i=0}^n |\mu|\{x_i\} - \varepsilon$$

and we have

$$\begin{aligned} \left| 1 + \sum_{i=1}^n r(x_0 - x_i) c_i \right| &= \left| \sum_{i=1}^n r(-x_i) c_i + r(-x_0) \right| \\ &= \left| \sum_{i=0}^n r(-x_i) \mu\{x_i\} \right| \geq \sum_{i=0}^n |\mu|\{x_i\} - \varepsilon \end{aligned}$$

Since $c_i \in T$ and $\varepsilon > 0$ is arbitrary, this implies that $\{x_0 - x_1, x_0 - x_2, \dots, x_0 - x_n\}$ is a K -set.

b) implies a): This is true from prop. 7 and the fact a finite subset $\{x_0, x_1, \dots, x_n\}$ of G is K' -set if and only if $\{x_0 - x_1, x_0 - x_2, \dots, x_0 - x_n\}$ is a K -set.

Example. Let S be the closed unit disk with 0 in the center in the complex plane, and Γ be the set of all continuous functions from S into T . Γ form a discrete group with respect to multiplication. Let G be the dual group of Γ , and then S can be naturally imbedded in G (cf. Kaufman [2]). S is a Kronecker set, but $T(\subset S)$ is not so, since the identity function from T to T is not u -approximable by characters, and this shows a closed subset of a Kronecker set is not always a Kronecker set.

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Department of Mathematics,
Hokkaido University

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