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ON VALUATIONS OF POLYNOMIAL RINGS OF MANY VARIABLES

Part one

By

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In his paper "A construction for absolute values in polynomial rings", "Trans. Amer. Math. Soc. Vol. 40 (1936)", MacLane developed a valuation theory for a polynomial ring of one variable determining every possible valuation in the ring whose coefficient field has a discrete valuation. In the present paper the writer wants to extend MacLane's theory to polynomial rings of two variables and also to those of many variables.

We shall refer the above MacLane's paper to as M. For example, M. Theorem 8. 1 means Theorem 8. 1 in the MacLane's paper. While the present paper is concerned with polynomial rings of many variables, we shall start for completeness with considerations on valuations of a polynomial ring of one variable.

§ 1. Valuations of $K(x)$

Let K be a field. Suppose K has a discrete non-archimedean valuation V_{00}, namely, V_{00} is a real valued function of K satisfying the following conditions;

If $a \in K$ and $a \neq 0$, then $V_{00}a \neq \pm\infty$ and $V_{00}0 = \infty$,

$V_{00}ab = V_{00}a + V_{00}b$,

$V_{00}(a+b) \geqq \mathrm{Min}\,[V_{00}a,\ V_{00}b]$.

These are called the three conditions of a valuation.

In particular, if $V_{00}a > V_{00}b$, then $V_{00}(a+b)$ must equal $V_{00}b$. We shall sometimes call this inequality the triangle law.

All real numbers which are the values taken by V_{00} in K form an isolated point set Γ_{00} and is a cyclic additive group. Therefore in this paper we may assume that Γ_{00} is a set of all rational integers.

If a non-zero integral multiple of a real number μ belongs to Γ_{00}, then μ is said to be commensurable with Γ_{00}, and if otherwise μ is called incommensurable.

Let R be an integral domain and W a valuation of R. Let a and b be

two elements of R. We shall say that a and b are equivalent in W (notation: $a\sim b$) if $W(a-b)>Wa=Wb$, while b is equivalence divisible by a (notation: $a|b$ in W) if there exists an element c in R such that $ac\sim b$ in W. Generally when a valuation U_1 of $K[x]$ is given, a polynomial $\phi_2(x)$ in $K[x]$ is called an x-key polynomial over U_1, if it satisfies the following three conditions;

(1) if $\phi_2(x)|f(x)g(x)$ *in* U_1, then $\phi_2(x)|f(x)$ *or* $\phi_2(x)|g(x)$ *in* U_1,

(2) if $\phi_2(x)|h(x)$ in U_1 and $h(x)\neq 0$, then $deg_x\phi_2(x)\leqq deg_x h(x)$, where $deg_x h(x)$ means the degree of $h(x)$ with respect to x,

(3) the leading coefficient of $\phi_2(x)$ is 1. (M. Definition 4. 1)

Let $\phi_2(x)$ be an x-key polynomial over U_1 and μ_2 a real number greater than $U_1\phi_2(x)$. Then the function U_2 defined as follows is a valuation of $K[x]$;

$$U_2h(x)=\operatorname*{Min}_i\left[U_1h_i(x)+i\mu_2\right]$$

where $h(x)$ is any polynomial in $K[x]$ and $h(x)=\sum_i h_i(x)(\phi_2(x))^i$, $deg_x h_i(x)<deg_x\phi_2(x)$ for $i=0,1,2,\cdots$, is the expansion of $h(x)$ with respect to $\phi_2(x)$. (M. Theorem 4. 2)

Definition 1. 1. *The above defined function U_2 is called an augmented valuation of U_1 in narrow sense and is denoted as $U_2=[U_1,\ U_2\phi_2=\mu_2]$ and this relation is denoted as $U_1<U_2$.*

Similarly, if $\phi_3(x)$ is an x-key polynomial over U_2, then we define an augmented valuation U_3 of U_2, using $\phi_3(x)$. Continuing in this way, we obtain valuations of $K[x]$ as follows;

$$U_1<U_2<\cdots<U_i<U_{i+1}<\cdots \qquad (s).$$

The x-key polynomials $\phi_i(x)$ corresponding to this series must satisfy the following two conditions besides the above mentioned three conditions (1), (2) and (3);

(4) $deg_x\phi_i(x)\leqq deg_x\phi_{i+1}(x)$,

(5) $\phi_i(x)$ and $\phi_{i+1}(x)$ are not equivalent in U_i,

both for $i=1,2,\cdots$. (M. Definition 6. 1)

Definition 1. 2. *We call the above series (s) a series of x-augmented inductive valuations and when $b>a$, U_b in the series is called an x-augmented valuation of U_a in broad sense and U_a an x-descended valuation of U_b.*

An x-augmented valuation of a valuation V of $K[x]$ in narrow sense

is an x-augmented valuation of V in broad sense. So, hence in this paper whenever we say that W is an x-augmented valuation of V or write as "$W>V$", it means that W is an x-augmented valuation of V in broad sense. Therefore, when $W>V$, W and V induce the same valuation in K and $Wx=Ux$.

Now, let μ_1 be a real number. Then the function V_{10} defined as follows is an extended valuation of V_{00} to $K[x]$;

$$V_{10}f(x)=\operatorname*{Min}_{i}[V_{00}f_i+i\mu_1],$$

where $f(x)=\sum_i f_i x^i$, $f_i\in K$, is any polynomial in $K[x]$. (M. Theorem 3. 1).

If a polynomial $\phi_2(x)$ is an x-key polynomial over V_{10}, we define an x-augmented valuation V_{20} of V_{10} using $\phi_2(x)$ in the same way as we defined U_2 from U_1. Thus we obtain a series of x-augmented inductive valuations;

$$V_{10}<V_{20}<V_{30}<\cdots<V_{i0}<V_{i+1,0}<\cdots. \quad \text{(M. Definition 6. 1)}$$

In this case for every non-zero polynomial $g(x)$

$$V_{i+1,0}g(x)\geqq V_{i0}g(x) \qquad \text{for} \quad i=1,2,3,\cdots.$$

Therefore $V_{i+1,0}$ is called an augmented valuation of V_{i0}.

If the above series lasts infinitively and if for every non-zero polynomial $f(x)$ the sequence $V_{10}f(x)$, $V_{20}f(x),\cdots$ does not tend to ∞, then the limit valuation of this sequence is denoted by $V_{\infty 0}f(x)$. $V_{\infty 0}$ is also a valuation of $K[x]$.

Every discrete extended valuation of V_{00} to $K[x]$ can be represented either as an x-augmented valuation or as a limit valuation. (M. Theorem 8. 1)

All values $V_{k0}f(x)$ form a cyclic additive group Γ_{k0} for $k=0,1,2,\cdots$ and $\mu_{k+1}=V_{k+1,0}\phi_{k+1}(x)$ is always commensurable with Γ_{k0} in this paper, so there is a smallest positive integer τ_{k+1} such that $\tau_{k+1}\mu_{k+1}$ is in Γ_{k0}.

Let $f(x)=\sum_{i=0}^{n}f_i(x)(\phi_k(x))^i$ be the expansion of $f(x)$ with respect to $\phi_k(x)$ which is the x-key polynomial producing the valuation V_{k0}. Then the polynomial $f(x)$ can be an x-key polynomial with which we can produce an augmented valuation $V_{k+1,0}$ of V_{k0}, if and only if the three following conditions hold:

(1) $f_n(x)=1$,

(2) $V_{k0}f(x)=V_{k0}f_0(x)=V_{k0}\left(\phi_k(x)\right)^n=n\mu_k$,

(3) $f(x)$ is equivalence-irreducible in V_{k0}, so owing to (2) $n\equiv 0 \pmod{\tau_k}$. (M. Theorem 9. 4)

Let S_k be a ring which consists of all polynomials $f(x)$ with $V_{k0}f(x)\geqq 0$

and Δ_k the residue class ring of V_{k0} in $K[x]$. Namely two polynomials $f(x)$ and $g(x)$ for which $V_{k0}f(x)=V_{k0}g(x)=0$ belong to the same residue class of Δ_k, if and only if

$$V_{k0}\big(f(x)-g(x)\big)>V_{k0}f(x)=0\,.$$

And the natural homomorphism H_k of S_k on the residue class ring Δ_k has the following properties;

$$H_k\big(f(x)+h(x)\big)=H_kf(x)+H_kh(x)$$

$$H_k\big(f(x)h(x)\big)=\big(H_kf(x)\big)\big(H_kh(x)\big)$$

and $$H_kl(x)=0 \text{ if and only if } V_{k0}l(x)>0\,.$$

A commensurable inductive valuation V_{k0} of $K[x]$ may be conveniently symbolized thus:

$$V_{k0}=[V_{00},\ V_{10}x=\mu_1,\ V_{20}\phi_2=\mu_2,\cdots,V_{k0}\phi_k=\mu_k]$$

or more briefly $V_{k0}=[V_{k-1,0},\ V_{k0}\phi_k=\mu_k]$.

Let F_0 be a residue class field of the original valuation V_{00} of K. Then about a residue class ring Δ_k we have the following theorem (M. Theorem 12. 1):

There is a sequence of fields $F_1=F_0,\ F_2, F_3,\cdots,F_k$, such that each field is an algebraic extension of its preceding field and such that for any $t=1,2\cdots,k$, the V_{t0}-residue class ring Δ_{t0} of $K[x]$ is isomorph to the ring $F_t[y]$ of polynomials in a variable y with coefficients in F_t.

§ 2. Continuous augmented valuations

Definition 2. 1. *Q, V, U and W are elements in an ordered set L. If $Q<V<W$ and $Q<U<W$ implies always $U<V$ or $U>V$ or $U=V$, L is said to be partially linearly ordered.*

We shall show that a set of all descended valuations of a valuation of $K[x]$ is partially linearly ordered, while a set of all descended valuations of a valuation of $K[x, y]$ is not always partially linearly ordered.

Definition 2. 2. *An x-augmented valuation W of a valuation V of $K[x]$ is called a continuous x-augmented valuation of V if there is no valuation T of $K[x]$ such that $V<T<W$.*

Definition 2. 3. *Let $V_{10}<V_{20}<\cdots<V_{p-1,0}<V_{p0}<\cdots<V_{t0}$ be a series of x-augmented inductive valuations of $K[x]$ and suppose, for every k between p and $t-1$, $V_{k+1,0}$ is a continuous x-augmented valuation of V_{k0}. Then the*

series is said to be continuous from V_{p0} to V_{t0}, $t-p$ is called the x-distance between V_{p0} and V_{t0} and $V_{p0}<V_{p+1,0}<\cdots<V_{t0}$ is called a series of continuous x-augmented inductive valuations of $K[x]$.

Theorem 2.4. *Let W be a discrete valuation of $K[x]$ which induces a valuation V_{00} in K and let $Wx=\mu_1$. Let V_{10} be the valuation of $K[x]$ defined as follows;*

$$V_{10}f(x)=\operatorname*{Min}_{i}[V_{00}f_i+i\mu_1],$$

where $f(x)=\sum_i f_i x^i$, $f_i\in K$, is in $K[x]$.

Then W can be represented either as the last member of a series of continuous x-augmented inductive valuations starting from V_{10} or as a limit valuation of it.

Proof. Suppose that $V_{10}<W$. Consider then the set P of all those polynomials $f(x)$ for which $V_{10}f(x)<Wf(x)$. Out of P pick up all those polynomials whose degrees are minimum and whose leading coefficients are 1.

Furthermore out of these polynomials choose a polynomial $\phi_2(x)$ so that $W\phi_2(x)-V_{10}\phi_2(x)$ becomes minimum. Since both W and V_{10} are discrete valuations of $K[x]$, it is evident that such a polynomial $\phi_2(x)$ does exist. Then $\phi_2(x)$ is an x-key polynomial over V_{10} of $K[x]$. Let $h(x)$ be a polynomial, and let the expansion in $\phi_2(x)$ of $h(x)$ be

$$h(x)=\sum_{i=0}^{n}h_i(x)\phi_2^i$$

$$(deg_x h_i(x)<deg_x\phi_2 \qquad \text{for} \quad i=0,1,2,\cdots,n)$$

and $\mu_2=V_{20}\phi_2=W\phi_2>V_{10}\phi_2$, then a function V_{20} defined by

$$V_{20}h(x)=\operatorname*{Min}_{i}\left[V_{10}h_i(x)+i\mu_2\right]$$

is a continuous x-augmented valuation of V_{10} of $K[x]$.

It was proved in M. Theorem 8.1 that V_{20} is an x-augmented valuation of V_{10}. Suppose that there were a valuation T such that $V_{10}<T<V_{20}\leqq W$. T appears in a series of x-augmented inductive valuations which starts from V_{10} and ends at W. If $t(x)$ is the x-key polynomial over V_{10} of $K[x]$ which produces T, it follows according to M. Theorem 6.4

$$Tt(x)=Wt(x)>V_{10}t(x).$$

Then because of the quality of ϕ_2 (x), we have

$$deg\,\phi_2(x)\leqq deg\,t(x).$$

Sometimes we denote $deg_x t(x)$ briefly as $deg\,t(x)$. Since V_{20} is an x-augmented inductive valuation of T, we have

$$deg\, t(x) \leqq deg\, \phi_2(x) \qquad \text{and so}$$
$$deg\, t(x) = deg\, \phi_2(x)\,.$$

By M. Lemma 6. 3

$$V_{10}\phi_2(x) = V_{10}t(x) < Wt(x) = Tt(x) < V_{20}\phi_2(x) = W\phi_2(x)\,.$$

Therefore $$Wt(x) - V_{10}t(x) < W\phi_2(x) - V_{10}\phi_2(x)\,.$$

This contradicts the above conditions imposed on $\phi_2(x)$, and thus V_{20} is a continuous x-augmented valuation of V_{10}.

In case $V_{20} < W$, we can derive V_{30} from V_{20} in the same way as above, so that V_{30} is a continuous x-augmented valuation of V_{20} of $K[x]$. By continuing this method we can establish our theorem in the same way as in M. Theorem 8. 1.

Theorem 2. 5. *The series of continuous x-augmented inductive valuations from V_{10} to W or their limit valuation in Theorem 2. 4 is uniquely determined by W and hence the x-distance between V_{10} and W is also uniquely determined.*

Proof. Consider two series of continuous x-augmented inductive valuations from V_{10} to W;

$$V_{10} < V_{20} < \cdots < V_{\alpha-1,0} < V_{\alpha 0} < \cdots < V_{p0} = W$$
$$V_{10} < V'_{20} < \cdots < V'_{\alpha-1,0} < V'_{\alpha 0} < \cdots < V'_{q0} = W\,,$$

and assume that $V_{20} = V'_{20}$, $V_{30} = V'_{30}, \cdots$, $V_{\alpha-1,0} = V'_{\alpha-1,0}$. We shall then show that $V_{\alpha 0} = V'_{\alpha 0}$.

Both $V_{\alpha 0}$ and $V'_{\alpha 0}$ are continuous x-augmented valuations of $V_{\alpha-1,0}$ of $K[x]$, so let $\phi_\alpha(x)$ and $\phi'_\alpha(x)$ be respectively x-key polynomials over $V_{\alpha-1,0}$ that produce $V_{\alpha 0}$ and $V'_{\alpha 0}$. Then

$$deg\, \phi_\alpha = deg\, \phi'_\alpha > deg\, r(x)$$

where $r(x) = \phi_\alpha(x) - \phi'_\alpha(x)$, and moreover

$$W\phi_\alpha = V_\alpha\phi_\alpha > V_{\alpha-1,0}\phi_\alpha$$
$$W\phi'_\alpha = V'_\alpha\phi'_\alpha > V_{\alpha-1,0}\phi'_\alpha$$

This implies

$$V_{\alpha 0}r(x) = V_{\alpha-1,0}r(x) = Wr(x) \geqq \mathrm{Min}[W\phi_\alpha,\ W\phi'_\alpha]$$
$$> \mathrm{Min}[V_{\alpha-1,0}\phi_\alpha,\ V_{\alpha-1,0}\phi'_\alpha]\,,$$

and so $$V_{\alpha-1,0}r(x) > V_{\alpha-1,0}\phi_\alpha = V_{\alpha-1,0}\phi'_\alpha\,.$$

As both $V_{\alpha 0}$ and $V'_{\alpha 0}$ are continuous x-augmented valuations of $V_{\alpha-1,0}$

$$W\phi_a - V_{a-1,0}\phi_a = W\phi'_a - V_{a-1,0}\phi'_a .$$

Hence we have $$W\phi_a = W\phi'_a .$$

Since $r(x)=\phi_a-\phi'_a$, it follows by the triangle law $Wr(x)\geqq W\phi_a = W\phi'_a$. Namely $V_{a-1,0}r(x)\geqq V_{a0}\phi_a = V_{a0}\phi'_a$.

$$\phi'_a = \phi_a - r(x)$$

$$V_{a0}\phi'_a = \mathrm{Min}\left[V_{a0}\phi_a,\ V_{a-1,0}r(x)\right] = V_{a0}\phi_a = W\phi_a = W\phi'_a = V'_{a0}\phi'_a .$$

Let the expansion in ϕ'_a of an arbitrary polynomial $h(x)$ be $\sum h_i(x)\phi'^i_a$. Then

$$\begin{aligned} V'_{a0}h(x) &= \min_i \left[V_{a-1,0}h_i(x) + iV'_{a0}\phi'_a\right] \\ &= \min_i \left[V_{a-1,0}h_i(x) + iV_{a0}\phi'_a\right] \\ &= \min_i \left[V_{a0}h_i(x)\phi'^i_a\right] \\ &\leqq V_{a0}\left[\sum_i h_i(x)\phi'^i_a\right] \\ &= V_{a0}h(x) . \end{aligned}$$

In the same way, $$V'_{a0}h(x)\geqq V_{a0}h(x) .$$

Thus $$V_{a0}h(x) = V'_a h(x) .$$

Theorem 2.6. *If $V_{10}<V_{20}<\cdots<V_{p0}=W$ is a series of continuous x-augmented inductive valuations and U is a descended valuation of W, then U coincides with one of $V_{10}, V_{20},\cdots V_{p0}$ and x-distance between U and W is uniquely determined by U and W.*

Proof. According to Theorem 2.4 one can build a series of continuous x-augmented inductive valuations

$$V_{10}<V'_{20}<\cdots<V'_{q0}=W$$

which goes through U; say $U=V'_{a0}$ $(0<a\leqq q)$. By Theorem 2.5 this is the same as

$$V_{10}<V_{20}<\cdots<V_{p0}=W .$$

Hence U must coincide with one of $V_{10},\cdots,V_{p0}$ and x-distance between U and W is uniquely determined by U and W.

Corollary 2.7. *If W is an augmented valuation of U of $K[x]$, then we can build a series of continuous x-augmented inductive valuations from U to W.*

This corollary is self-evident from Theorem 2.6.

Theorem 2.8. *A set of all x-descended valuations of a valuation of $K[x]$ is partially linearly ordered.*

Proof. Assume that Q, U and W are valuations of $K[x]$ and $Q<W$ and $U<W$. Then we can build a series of continuous x-augmented inductive valuations from V_{10} to W, according to Theorem 2.4. And owing to Theorem 2.6, Q and U must appear in the series,

therefore $$Q<U \quad \text{or} \quad Q>U \quad \text{or} \quad Q=U.$$

§ 3. The shortest series of x-augmented inductive valuations

Mac Lane gave the following lemma (M. Lemma 15.1); if an inductive valuation

$$V_k=[V_{k-2},\ V_{k-1}\phi_{k-1}=\mu_{k-1},\quad V_k\phi_k=\mu_k]$$

has two key polynomials ϕ_{k-1} and ϕ_k of the same degree, then

$$W=[V_{k-2},\ W\phi_k=\mu_k]$$

is an inductive valuation equal to V_k.

Herein $W\phi_k=\mu_k=V_k\phi_k$, so instead of $W=[V_{k-2},\ W\phi_k=\mu_k]$, we may write this conclusion as $V_k=[V_{k-2},\ V_k\phi_k=\mu_k]$.

According to this lemma, we can shorten every series of continuous x-augmented inductive valuations.

Let $$V_{10}<V_{20}<\cdots<V_{p0}=W$$

be a series of continuous x-augmented inductive valuations and ϕ_k an x-key polynomial which produces V_k, for $k=1,2,\cdots,p$:

$$V_{10}\phi_1=\mu_1,\quad V_{20}\phi_2=\mu_2,\cdots,\quad V_{p0}\phi_p=\mu_p.$$

Suppose

$$1=deg\,\phi_1=deg\,\phi_2=\cdots=deg\,\phi_a<deg\,\phi_{a+1}=\cdots$$
$$\cdots=deg\,\phi_b<deg\,\phi_{b+1}=\cdots=deg\,\phi_l<deg\,\phi_{l+1}=\cdots=deg\,\phi_p.$$

Then according to M. Theorem 15.2

$$W=V_{p0}=[V_{00},\ V_{10}\phi_1=\mu_1,\ V_{a0}\phi_a=\mu_a,\ V_{b0}\phi_b=\mu_b,\cdots,\ V_{l0}\phi_l=\mu_l,$$
$$V_{p0}\phi_p=\mu_p].$$

Definition 3.1. *Let $W=[V_{00},\ V_{10}\phi=\mu_1,\ V_{20}\phi_2=\mu_2,\cdots,V_p\phi_p=\mu_p]$. If $deg\,\phi_2<deg\,\phi_3<\cdots<deg\,\phi_p$, the series $V_{10}<V_{20}<\cdots<V_p$ is called the shortest series of x-augmented inductive valuations from V_{10} to W.*

When a series of continuous augmented inductive valuations of a valuation W is given, construction of the shortest series of augmented inductive valuations of W is very easy, as we have shown above.

Conversely when the shortest series of augmented inductive valuations of a valuation W is given, we consider the method to construct a series of continuous augmented inductive valuations of W.

Let the shortest series of augmented inductive valuations of W be given as follows;

$$V_1 < V_2 < \cdots < V_a < V_{a+1} < \cdots < V_p = W.$$

Now we want to construct a series of continuous augmented valuations between V_a and V_{a+1} as follows;

$$V_a < U_1 < U_2 < \cdots < U_b < V_{a+1} \cdots\cdots\cdots (1).$$

Let $\phi_a, \phi_1, \phi_2, \cdots, \phi_b, \phi_{a+1}$ be respectively the x-key polynomials which produce $V_a, U_1, U_2, \cdots, U_b, V_{a+1}$. Then according to M. Lemma 15. 1, the following relations hold.

$$deg\,\phi_a < deg\,\phi_1 = deg\,\phi_2 = \cdots = deg\,\phi_b = deg\,\phi_{a+1}.$$

Put $V_a\phi_a = \mu_a$, $U_1\phi_1 = \rho_1$, $U_2\phi_2 = \rho_2, \cdots$, $U_b\phi_b = \rho_b$ and $V_{a+1}\phi_{a+1} = \mu_{a+1} (> V_a\phi_{a+1})$.

If Γ_a is a set of all values that V_a takes in $K[x]$, then Γ_a contains $\rho_1, \rho_2 \cdots, \rho_b$ and $V_a\phi_{a+1}$. For, $V_{a+1}\phi_{a+1} > U_i\psi_i = \rho_i$ according to M. Lemma 6. 3 and ψ_i is a key polynomial which produces U_i, for $i = 1, 2, \cdots, b$, so $V_{a+1}\psi_i = U_i\psi_i$ for each i,

Let $\phi_{a+1} = \psi_i + C_i(x)$. Then $deg\,C_i(x) < deg\,\psi_i = deg\,\phi_{a+1}$ so by the definition of V_{a+1}

$$V_{a+1}C_i(x) = V_aC_i(x).$$

Therefore, owing to the triangle law

$$V_{a+1}\phi_{a+1} > V_{a+1}\psi_i = \rho_i = V_aC_i(x),$$

so each $\rho_i = U_i\psi_i$ belongs to Γ_a.

According to M. Lemma 6. 3

$$U_i\phi_{a+1} = U_i\psi_i = \rho_i.$$

Therefore a necessary and sufficient condition for that a series

$$V_a < U_1 < U_2 < \cdots < U_b < V_{a+1}$$

be a series of continuous augmented inductive valuations is that the series

$$V_a\phi_{a+1} < \rho_1 < \rho_2 < \cdots < \rho_b < V_{a+1}\phi_{a+1}$$

is most dense in Γ_a except $V_{a+1}\phi_{a+1}$.
Namely we pick up all those numbers in Γ_a that are between $V_a\phi_{a+1}$ and $V_{a+1}\phi_{a+1}$.
Such numbers exist, for Γ_a is a set of isolated points.

Let $$\phi_{a+1}=(\phi_a^{\tau_a})^m+\cdots+1(x)$$

be an expansion of ϕ_{a+1} with respect to ϕ_a, where $V_a\phi_a^{\tau_a}$ belongs to Γ_{a-1}.
Then according to M. Lemma 9. 3, there exists a polynomial $G_i(x)$ such that

$$V_aG_i(x)=\rho_i$$

and $$deg\,G_i(x)<deg\,\phi_{a+1},$$

where $$i=1,2,\cdots,b.$$

Put $$\psi_1=\phi_{a+1}-\sum_{i=1}^{b}G_i(x).$$

Then ψ_1 is a key polynomial over V_a,

because $$V_a(\sum_i G_i)=\operatorname{Min}_i[\rho_i]=\rho_1>V_a\phi_{a+1}.$$

Therefore $$\psi_1\sim\phi_{a+1}\qquad\text{in}\quad V_a.$$

ϕ_{a+1} is a key polynomial which produces V_{a+1} from V_a, so ψ_1 can be a key polynomial over V_a, for $deg\,\psi_1=deg\,\phi_{a+1}$. On the other hand, ψ_1 produces U_1, then we may define U_1 as follows.

$$U_1\psi_1=V_{a+1}\psi_1=V_{a+1}(\phi_{a+1}-\sum_i G_i)=\rho_1$$

because $$V_{a+1}\phi_{a+1}=\mu_{a+1}>\rho_1.$$

If we put $$\psi_2=\psi_1+G_1(x)$$

then we can prove in the same way that ψ_2 is a key polynomial over U_1 and we may define U_2 as $U_2\psi_2=\rho_2$. Next we put $\psi_3=\psi_2+G_2(x)$, and so on. Thus we can show that the series

$$V_a<U_1<U_2<\cdots<U_b<V_{a+1}$$

is a series of continuous augmented inductive valuations between V_a and V_{a+1}.

§ 4. Valuations of $K(x, y)$

Now we consider valuations of polynomial rings of two variables. A valuation of a ring R can be extended uniquely to its quotient field. (M. Thorem 2. 1)

We assume that a valuation W of a field $K[x, y]$ is given, where x and y are algebrically independent with respect to K. Let W induce a valuation

V_{00} in K and a valuation V_{p0} in $K[x]$.

Since V_{p0} is an extended valuation of V_{00} to $K[x]$, we can build by Theorem 2. 4 a series of continuous x-augmented inductive valuations of $K[x]$ such that

$$V_{p0}=[V_{00},\ V_{10}\ x=\mu_1,\ V_{20}\phi_2=\mu_2,\cdots,V_{p0}\phi_p=\mu_p].$$

If we denote $K(x)=K_x$, then W is an extended valuation of V_{p0} to a ring $K_x[y]$ whose coefficient field K_x has a valuation V_{p0}. Therefore again by Theorem 2. 4, we can build a series of continuous y-augmented inductive valuations of $K_x[y]$ such that

$$V_{pq}=[V_{p0},\ V_{p1}y=\nu_1,\ V_{p2}\phi_2=\nu_2,\cdots,V_{pq}\phi_q=\nu_q],$$

where ϕ_{j+1} is a y-key polynomial over V_{pj} of $K_x[y]$ and

$$V_{p,j+1}\phi_{j+1}=\nu_{j+1}>V_{pj}\phi_{j+1}\qquad\text{for}\quad j=1,2,\cdots,q-1.$$

V_{pq} is denoted briefly as $V_{pq}=[V_{p,q-1},V_{pq}\phi_q=\nu_q]$. V_{pq} is a valuation of $K(x, y)$, so it is equal to W.

Suppose W induces another valuation V_{0r} in $K[y]$. Then we can build in the same way a series of continuous y-augmented inductive valuations of $K[y]$ such that

$$V_{0r}=[V_{00},\ V_{01}y=\nu_1,\ V_{02}\zeta_2=\rho_2,\cdots,V_{0r}\zeta_r=\rho_r].$$

Since V_{0r} is a valuation of a quotient field $K(y)=K_y$, we can obtain in the same way a series of continuous x-augmented inductive valuations of $K_y[x]$ such that

$$\overline{V}_{sr}=[V_{0r},\ \overline{V}_{1r}x=\mu_1,\ \overline{V}_{2r}\xi_2=\sigma_2,\cdots,\overline{V}_{sr}\xi_r=\sigma_s],$$

where ξ_{i+1} is an x-key polynomial over $\overline{V}_{ir}$ in $K_y[x]$

and $$\overline{V}_{i-1,r}\xi_{i+1}=\sigma_{i+1}>\overline{V}_{ir}\xi_{i+1}\qquad\text{for}\quad i=1,2,\cdots,s-1.$$

$\overline{V}_{sr}$ is denoted briefly as $\overline{V}_{sr}=[\overline{V}_{s-1,r},\ \overline{V}_{sr}\xi_r=\sigma_s]$ and we can also conclude that $\overline{V}_{sr}$ is equal to W in the same way.

Thus a valuation W of $K[x, y]$ has two different expressions $V_{pq}=[V_{p,q-1},\ V_{pq}\phi_q(y)=\nu_q]$ and $\overline{V}_{sr}=[\overline{V}_{s-1,r},\ \overline{V}_{sr}\xi_r(x)=\sigma_s]$. Namely V_{pq} is an expression of W in the case when W is regarded as the last member of a series of continuous y-augmented inductive valuations of the ring $K_x[y]$ whose coefficient field K_x has the valuation V_{p0} which W induces in K_x, while $\overline{V}_{rs}$ is an expression of W in the case when W is regarded as the last member of a series of continuous x-augmented inductive valuations of the ring $K_y[x]$ whose coefficient field K_y has the valuation V_{0r} which W induces in K_y.

§ 5. *y*-augmented valuations

Now we want to build a y-augmented valuation of a valuation W of $K[x, y]$ which was given in § 4. In this case we must naturally use V_{pq} as W. First we define a y-key polynomial $\phi_{q+1}(y)$ over V_{pq} in the ring $K_x[y]$ in the same way when we defined an x-key polynomial $\phi_2(x)$ over U_1 in the ring $K[x]$ in § 1. But this time $\phi_{q+1}(y)$ is a polynomial of y with coefficients in K_x.

Let any polynomial $h(y)$ of y with coefficients in K_x have the following expansion with respect to $\phi_{q+1}(y)$;

$$h(x)=\sum_i h_i(y)\left(\phi_{q+1}(y)\right)^i$$

where $h_i(y)$ is a polynomial of y in $K_x[y]$ and

$$deg_y h_i(y)<deg_y \phi_{q+1}(y) \qquad \text{for} \quad i=0,1,2,\cdots .$$

Then the function $V_{p,q+1}$ defined as follows is called a y-augmented valuation of V_{pq}:

$$V_{p,q+1}h(y)=\operatorname*{Min}_i\left[V_{pq}h_i(y)+i\nu_{q+1}\right],$$

where $\nu_{q+1}=V_{p,q+1}\phi_{q+1}(y)$ is an arbitrary real number greater than $V_{pq}\phi_{q+1}(y)$.

$V_{p,q+1}$ is denoted briefly as $V_{p,q+1}=[V_{pq},\ V_{p,q+1}\phi_{q+1}(y)=\nu_{q+1}]$ and this relation is denoted as $V_{pq}\overset{y}{<}V_{p,q+1}$.

$V_{p,q+1}$ is a valuation of $K_x[y]$, so it induces the same valuation in the subring $K[x, y]$ of $K_x[y]$. Then by M. Theorem 5.1, for every polynomial $f(x, y)$ in $K[x, y]$

$$V_{p,q+1}f(x, y)\geqq V_{p,q}f(x\ y)\ ,$$

and owing to the definition of the y-augmented valuation $V_{p,q+1}$, for every polynomial $g(x)$ in $K[x]$

$$V_{p,q+1}g(x)=V_{pq}g(x)\ .$$

Consequently for a polynomial $1(x, y)$ in $K[x, y]$

if $$V_{p,q+1}1(x, y)>V_{pq}1(x, y)$$

then y actually appears in the polynomial $1(x, y)$.

Next we define a y-key polynomial $\phi_{q+2}(y)$ over $V_{p,q+1}$ in $K_x[y]$ and with it we define a y-augmented valuation $V_{p,q+2}$ of $V_{p,q+1}$. Thus we obtain a series of y-augmented inductive valuations in $K_x[y]$ as follows;

$$V_{pq}\overset{y}{<}V_{p,q+1}\overset{y}{<}V_{p,q+2}\overset{y}{<}\cdots .$$

In this series, when $b>a$, V_{pb} is called a y-augmented valuation of V_{pa} and V_{pa} a y-descended valuation of V_{pb} and this relation is denoted as $V_{pa} \overset{y}{<} V_{pb}$. Therefore when $V_{pa} \overset{y}{<} V_{pb}$, V_{pa} and V_{pb} induce the same valuation V_{p0} in K_x and $V_{pa}y = V_{pb}y$.

And completely in the same way when we established Definition 2. 2. and Definition 2. 3., we define a continuous y-augmented valuation in $K[x, y]$ and the y-distance between two valuations in $K[x, y]$.

Definition 5. 1. *If W_2 is a y-augmented valuation of W_1 in $K[x, y]$ and there exists a polynomial $f(y)$ in $K[y]$ such that $W_2 f(y) > W_1 f(y)$, then W_2 is called a y-simply augmented valuation of W_1 and this relation is denoted as*

$$W_1 \overset{ys}{<} W_2 .$$

In the following series of y-simply augmented inductive valuations of $K[x, y]$

$$W_a \overset{ys}{<} W_{a+1} \overset{ys}{<} \cdots \overset{ys}{<} W_b \overset{ys}{<} \cdots ,$$

W_b is called a y-simply augmented valuation of W_a and W_a a y-simply descended valuation of W_b.

Definition 5. 2. *If W_2 is a y-augmented valuation of W_1 in $K[x, y]$ but is not a y-simply augmented valuation of W_1, then W_2 is called an xy-doubly augmented valuation of W_1 and W_1 an xy-doubly descended valuation of W_2 and is denoted as*

$$W_1 \overset{xy}{<} W_2 .$$

Namely when W_2 is an xy-doubly augmented valuation of W_1, if $W_2 f(x, y) > W_1 f(x, y)$, then both x and y actually appear in the polynomial $f(x, y)$.

§ 6. x-augmented valuations

Next we want to build an x-augmented valuation of the valuation W given in § 4. In this case naturally we must use $\overline{V}_{sr}$ as W. Completely in the same way when we built a y-augmented valuation of V_{pq} of the ring $K_x[y]$, we define an x-key polynomial $\xi_{s+1}(x)$ with coefficients in K_y over $\overline{V}_{sr}$ and with $\xi_{s+1}(x)$ we define an x-augmented valuation $\overline{V}_{s+1,r}$ of $\overline{V}_{sr}$ in the ring $K_y[x]$ whose coefficient field K_y has the valuation V_{0r}. And furthermore we define a series of x-augmented inductive valuations as follows:

$$\overline{V}_{sr} \overset{x}{<} \overline{V}_{s+1,r} \overset{x}{<} \cdots \overset{x}{<} \overline{V}_{ar} \overset{x}{<} \cdots \overset{x}{<} \overline{V}_{br} \overset{x}{<} \cdots .$$

We call $\overline{V}_{br}$ in this series an x-augmented valuation of $\overline{V}_{ar}$ and also in the same way as in § 5 we define a continuous x-augmented valuation of $\overline{V}_{sr}$ and the x-distance between two valuations.

$\overline{V}_{s+1,r}$ is an x-augmented valuation of $\overline{V}_{sr}$ in $K[x, y]$, so again by M. Theorem 5. 1, for every polynomial $f(x, y)$

$$\overline{V}_{s+1,r}f(x, y) \geqq \overline{V}_{sr}f(x, y)\,.$$

And owing to the definition of an x-augmented valuation for every polynomial $g(y)$ in $K[y]$

$$\overline{V}_{s+1,r}g(y) = \overline{V}_{sr}g(y)\,,$$

consequently if $$\overline{V}_{s+1,r}1(x, y) > \overline{V}_{sr}1(x, y)$$

then x actually appears in the polynomial $1(x, y)$.

Definition 6. 1. *If W_2 is an x-augmented valuation of a valuation W_1 in $K[x, y]$ and there exists a polynomial $f(x)$ in $K[x]$ such that $W_2f(x) > W_1f(x)$, then W_2 is called an x-simply augmented valuation of W_1 and is denoted as $W_1 \overset{xs}{<} W_2$.*

Definition 6. 2. *When U is an x-augmented valuation of V or a y-augmented valuation of V, U is called an augmented valuation of V and is denoted as $V < U$.*

Definition 6. 3. *When U is an x-simply augmented valuation of V in $K[x, y]$ or a y-simply augmented valuation of V, U is called a simply augmented valuation of V and is denoted as*

$$V \overset{s}{<} U\,.$$

Definition 6. 4. *In the following series of augmented inductive valuations in $K[x, y]$*

$$U_1 < U_2 < \cdots < U_a < \cdots < U_b < \cdots$$

U_b is called an augmented valuation of U_a and U_a an descended valuation of U_b.

Definition 6. 5. *In the following series of simply augmented valuations in $K[x, y]$*

$$U_1 \overset{s}{<} U_2 \overset{s}{<} \cdots \overset{s}{<} U_a \overset{s}{<} \cdots \overset{s}{<} U_b \overset{s}{<} \cdots$$

U_b is called a simply augmented valuation of U_a and U_a a simply descended valuation of U_b.

When U and V are two valuations of a ring R, if for every element r

in R $Ur=Vr$, then we say that U equals V and denote as $U=V$. We say that the valuation W given in §4 has two expressions

$$V_{pq}=[V_{p,q-1},\ V_{pq}\phi_q(y)=\nu_q] \quad \text{and} \quad \overline{V}_{sr}=[\overline{V}_{s-1,r},\ \overline{V}_{sr}\xi_r(x)=\sigma_s].$$

Theorem 6.6. *No y-simply augmented valuation of a valuation W in $K[x, y]$ equals an x-augmented valuation of W.*

Proof. This is evident according to their definitions.

Similarly no x-simply augmented valuation of a valuation W in $K[x, y]$ equals an y-augmented valuation of W.

Theorem 6.7. *If W_2 is an x-augmented valuation of a valuation W_1 in $K[x, y]$ but not an x-simply augmented valuation of W_1, then W_2 is an xy-doubly augmented valuation of W_1 in $K[x, y]$.*

Proof. W_2 is an x-augmented valuation of W_1 but not an x-simply augmented valuation of W_1, so for every polynomial $f(x)$ in $K[x]$ $W_2f(x)=W_1f(x)$.
Namely W_2 and W_1 induce the same valuation V_{p0} in the field K_x and $W_2\ y=W_1\ y$. So regarding W_2 as a y-augmented valuation of W_1 in the ring $K_x[y]$, we can build the following series of continuous y-augmented inductive valuations by Corollary 2.7.;

$$W_1=V_{pq}\overset{y}{<}V_{p,q+1}\overset{y}{<}\cdots\overset{y}{<}V_{pl}=W_2.$$

But W_2 is an x-augmented valuation of W_1 in $K[x, y]$, so for every polynomial $g(y)$ in $K[y]$ $W_2g(y)=W_1g(y)$. Therefore no valuation in this series is a y-simply augmented valuation of its preceding valuation. So W_2 is an xy-doubly augmented valuation of W_1.

§7. Doubly augmented valuations

Theorem 7.1. *Let U be an x-augmented valuation of a valuation Q of $K[x, y]$ and W also an x-augmented valuation of U.*
Then, for a polynomial $f(x, y)$,

$$Uf(x, y)=Q(x,\ y) \quad \textit{implies} \qquad Wf(x, y)=Uf(x, y).$$

Proof. Since U is an x-augmented valuation of Q in $K[x, y]$, $Ua(y)=Qa(y)$ for every polynomial of y.
Also since W is an x-augmented one of U,

$$Wa(y)=Ua(y)=Qa(y).$$

Therefore W, U and Q induce the same valuation V_{0r} in the ring $K[y]$, and

$$Wx = Ux = Qx .$$

Owing to Corollary 2. 7 we can build a series of continuous x-augmented inductive valuations of a ring $K_y[x]$ from Q to W through U:

$$Q = V'_{1r} \overset{x}{<} V'_{2r} \overset{x}{<} \cdots \overset{x}{<} V'_{pr} = U \overset{x}{<} V'_{p+1,r} \overset{x}{<} \cdots \overset{x}{<} V'_{qr} = W ,$$
$$V'_{1r} f(x, y) = Qf(x, y) = Uf(x, y) = V'_{pr} f(x, y) .$$

Hence it follows from M. Theorem 6. 5 that

$$V'_{pr} f(x, y) = V'_{qr} f(x, y) = Wf(x, y).$$

Theorem 7. 2. *If U is an xy-doubly augmented valuation of a valuation Q of $K[x, y]$ and W is an augmented valuation of U, then W, U and Q induce the same valuations both in $K[x]$ and in $K[y]$ and W is also an xy-doubly augmented valuation of U.*

Proof. Since U is an x-augmented valuation of Q and also a y-augmented one of Q, U and Q induce the same valuation V_{0r} in $K[y]$ and also the same valuation V_{p0} in $K[x]$. So, for every polynomial $a(x)$ of x and $b(y)$ of y, we have

$$Ua(x) = Qa(x) \quad \text{and} \quad Ub(y) = Qb(y) .$$

Assume that W is an x-simply augmented valuation of U. Then there exists a polynomial $l(x)$ of x such that

$$W1(x) > Ul(x) .$$

But W is an x-simple augmented valuation of U, so W is an x-augmented valuation of U. Hence

$$Wl(x) > Ul(x) = Ql(x) .$$

This is however impossible owing to Theorem 7. 1.
Thus W is not an x-simply augmented valuation of U. It is evident that Theorem 7. 1 holds when we interchange the letter "x" with the letter "y".

So W is not a y-simply augmented valuation of U, even if W is a y-augmented valuation of U. But W is an augmented valuation of U, hence W must be an xy-doubly augmented valuation of U.

Theorem 7. 3. *The set of all xy-doubly descended valuations of a valuation of $K[x, y]$ is partially linearly ordered.*

Proof. Let V, U, and U' be xy-doubly descended valuations of W and let $V \overset{xy}{<} U \overset{xy}{<} W$ and $V \overset{xy}{<} U' \overset{xy}{<} W$.
Then according to Theorem 7. 2 they all induce the same valuation V_{0r} in the ring $K[y]$ and besides

$$Wx = Ux = U'x = Vx.$$

Therefore they may be considered x-descended valuations of W in $K_y[x]$ whose coefficient field K_y has a valuation V_{0r}. Hence owing to Theorem 2.8 they are partially linearly ordered.

Theorem 7.4. *If U is an xy-doubly augmented valuation of a valuation Q of $K[x, y]$ and W is an augmented valuation of U, then there exists only one series of continuous xy-doubly augmented inductive valuations of $K[x, y]$ from Q to W through U and the x-distance between U and W is equal to the y-distance between U and W.*

Proof. Since W, U and Q induce the same valuation V_{0r} in K_y and $Wx = Ux = Qx$, we can build according to Corollary 2.7 a series of continuous x-augmented inductive valuations of $K_y[x]$ from Q to W through U:

$$(1) \qquad Q = V_1' \overset{x}{<} V_2' \overset{x}{<} \cdots \overset{x}{<} V_l' = W.$$

Since W and U are xy-doubly augmented valuations of Q, W, U and Q induce the same valuation V_{p0} in the ring $K[x]$ and W and U may be considered as y-augmented valuations of Q of $K_x[y]$ whose coefficient field K_x has the valuation V_{p0} because $Wy = Uy = Qy$.

Therefore again by Corollary 2.7 we can build a series of continuous y-augmented valuations from Q to W through U:

$$(2) \qquad Q = V_1'' \overset{y}{<} V_2'' \overset{y}{<} \cdots \overset{y}{<} V_{q'}' = W.$$

W is an xy-doubly augmented valuation of Q, so V_2', $V_3', \cdots, V_p'$ are all xy-doubly augmented valuations of Q in $K[x, y]$. But an xy-doubly augmented valuation of Q is a y-augmented valuation of Q. Therefore V_2', $\cdots, V_l'$ may be considered as a series of y-augmented inductive valuations of Q in $K[x, y]$.

Hence owing to Theorem 2.6 all V_1', $V_2', \cdots, V_{q'}'$ must appear in the series (2). Similarly all V_1'', $V_2'', \cdots, V_q''$ must appear in the series (1).

Therefore both the series (1) and (2) coincide. Since U appears in the series, the x-distance between U and W is equal to the y-distance between U and W.

We shall later show that a set of all simply descended valuations of a valuation of $K[x, y]$ is not always partially linearly ordered.

§ 8. Series of augmented inductive valuations in $K[x, y]$

The following theorem can be proved in the same way as M. Theorem 3.1.

Theorem 8. 1. *If R is an integral domain and has a valuation V_{00}, then for any polynomial $f(x)=\sum_i f_i x^i$ in $R[x]$, the function V_{10} defined as follows is a valuation of the ring $R[x]$;*

$$V_{10}f(x) = \operatorname{Min}_i [V_{00}f_i + i\mu_1]$$

where $\mu_1 = V_{10}x$ is an arbitrary real number. The valuation is denoted by

$$V_{10} = [V_{00},\ V_{10}x = \mu_1] .$$

When a valuation W of $K[x, y]$ is given, in Theorem 8. 1 we substitute Wx for $V_{10}x=\mu_1$, and we obtain

$$V_{10} = [V_{00},\ Wx=\mu_1] .$$

Generally a polynomial $f(x, y)$ of x and y can be expressed as $\sum_i f_i(x)y^i$, where $f_i(x)$ are polynomials of x. Then the following theorem is an immediate consequence of Theorem 8. 1.

Theorem 8. 2. *If W is a valuation of $K[x, y]$ and $V_{10} = [V_{00},\ Wx = \mu_1]$ is a valuation defined above, then for any polynomial $f(x, y) = \sum_i f_i(x)y^i$, where $f_i(x)$ are polynomials of x, the function V_{11} defined as follows is a valuation of $K[x, y]$;*

$$V_{11}f(x, y) = \operatorname{Min}_i [V_{10}f_i(x) + i\nu_1]$$

where $\nu_1 = V_{11}y = Wy$.

Now we want to build a series of augmented inductive valuations of $K[x, y]$ from V_{11} to W.

Theorem 8. 3. *If a valuation W of $K[x, y]$ induces in $K[x]$ a series of continuous x-augmented inductive valuations*

$$V_{10} \overset{x}{<} V_{20} \overset{x}{<} \cdots \overset{x}{<} V_{p0} ,$$

then for any polynomial $f(x, y) = \sum_i f_i(x)y^i$, the functions V_{t1} defined as follows are x-simply augmented valuations of V_{11} in $K[x, y]$, where t ranges between 1 and p;

$$V_{t1}f(x, y) = \operatorname{Min}_i \left[V_{t0}f_i(x) + i\nu_1 \right]$$

where $\nu_1 = Wy$, and moreover

(1) $\quad V_{11} \overset{xs}{<} V_{21} \overset{xs}{<} \cdots \overset{xs}{<} V_{p1}$

gives a series of continuous x-simply augmented inductive valuation of $K[x, y]$.

Proof. Owing to Theorem 8. 1 it is evident that each V_{t1} is an x-simply augmented valuation of V_{11} for $t=1, 2, \cdots, p$. Assume that the series (1) is not continuous x-simply augmented inductiove valuations. Then there exists a valuation U of $K[x, y]$ such that

$$V_{11} \overset{xs}{<} V_{21} \overset{xs}{<} \cdots \overset{xs}{<} V_{a1} \overset{xs}{<} U \overset{xs}{<} V_{a+1,1} \overset{xs}{<} \cdots \overset{xs}{<} V_{p1}.$$

So for suitable polynomials $g(x)$ and $h(x)$

$$V_{a1}g(x) < Ug(x) \quad \text{and} \quad Uh(x) < V_{a+1,1}h(x).$$

In $K[x]$ V_{a1}, U and $V_{a+1,1}$ induce respectively valuations V_{a0}, U' and $V_{a+1,0}$ and $V_{at}g(x) = V_{a0}g(x)$, so

$$V_{a0}g(x) < U'g(x) \quad \text{and} \quad U'h(x) < V_{a+1,0}h(x).$$

Namely $V_{a0} \overset{x}{<} U' \overset{x}{<} V_{a+1,0}$.

But this is impossible because $V_{a+1,0}$ is a continuous x-augmented valuation of V_{a0} in $K[x]$. Thus (1) is a series of continuous x-simply augmented inductive valuations.

W and V_{p1} induce in $K[x]$ the same valuation V_{p0}, so owing to Theorem 2.4 we can build a series of continuous y-augmented inductive valuations of $K_x[y]$ from V_{p1} to W;

$$(2) \qquad V_{p1} \overset{y}{<} V_{p2} \overset{y}{<} \cdots \overset{y}{<} V_{pq} \overset{y}{<} \cdots \overset{y}{<} V_{pt} = W.$$

In this series, according to Theorem 7. 2, there must exist a valuation V_{pq} such that the subseries between V_{p1} and V_{pq} is a series of continuous y-simply augmented inductive valuations and the subseries between V_{pq} and W is a series of continuous xy-doubly augmented inductive valuations.

§ 9. Last simply augmented valuation

When W is a valuation, we define a valuation V_{11} of $K[x, y]$ as in § 8. But by interchanging x with y in construction of the series, we obtain another series;

$$V_{11} \overset{ys}{<} V_{12} \overset{ys}{<} \cdots \overset{ys}{<} V_{1q} \overset{xs}{<} V_{2q} \overset{xs}{<} \cdots \overset{xs}{<} V_{pq} \overset{xy}{<} V_{p+1,q} \overset{xy}{<} \cdots \overset{xy}{<} V_{sp} = W.$$

Namely, the beginning part of this series between V_{11} and V_{1q} is a series of continuous y-simply augmented inductive valuations of $K[x, y]$, the middle part of the series between V_{1q} and V_{pq} is a series of continuous x-simply

augmented inductive valuations of $K[x, y]$ and the last part of the series between V_{pq} and W is a series of continuous xy-doubly augmented inductive valuations.

Thus we can build the two different series of augmented inductive valuations from V_{11} to W. We shall however show in Part 2 that we can build many other series of continuous augmented inductive valuations which connect V_{11} and W.

Theorem 9.1. *Let W be a valuation of $K[x, y]$ and V_{11} a valuation of $K[x, y]$ defined in §8. Then in every series of continuous augmented inductive valuations which begins at V_{11} and comes to end at W, there appears a valuation U such that the beginning part of the series between V_{11} and U is a series of continuous simply augmented inductive valuations of $K[x, y]$ and the last part of the series between U and W is a series of continuous xy-doubly augmented inductive valuations of $K[x, y]$; and such a valuation U is determined uniquely by W.*

Definition 9.2. *The valuation U in Theorem* 9.1 *is called the last simply augmented valuation of W.*

Proof of Theorem 9.1. We prove first the uniqueness.

Let the following series be as in our Theorem;

$$V_{11} \overset{s}{<} \cdots \overset{s}{<} U \overset{xy}{<} \cdots \overset{xy}{<} W. \tag{3}$$

By connecting the series (1) in Theorem 8.3 with the other series (2) in §8, we obtain the following series;

$$V_{11} \overset{xs}{<} V_{21} \overset{xs}{<} \cdots \overset{xs}{<} V_{p1} \overset{ys}{<} V_{p2} \overset{ys}{<} \cdots \overset{ys}{<} V_{pq} \overset{xy}{<} V_{p,q+1} \overset{xy}{<} \cdots \overset{xy}{<} V_{pt} = W. \tag{4}$$

Namely, W is an xy-doubly augmented valuation of V_{pq}, so W and V_{pq} induce the same valuation V_{p0} in the ring $K[x]$ and W is also an xy-doubly augmented valuation of U. Hence W, U and V_{pq} induce the same valuation V_{p0} in the field K_x. Besides

$$Wy = Uy = V_{pq}y\,.$$

Therefore according to Theorem 2.6, U must coincide with one of valuations in the series (4) between V_{p1} and W. Assume that $U \neq V_{pq}$. Then the part of the series (4) between V_{pq} and U is a series of continuous simply augmented inductive valuations of $K[x, y]$ due to its construction, while the part of the series (4) between V_{pq} and U is simultaneously a series of continuous xy-doubly augmented inductive valuations of $K[x, y]$ according to the construction of the other series. This is, however, impossible, so that we have

$U=V_{pq}$.

Therefore the last simply augmented valuation of W is determined uniquely by W and it is independent of choice of the series.

According to Theorem 7.4 there is only one series of continuous xy-doubly augmented inductive valuations which connects W with its last simply augmented valuation V_{pq}.

The proof of the existence of a series of continuous simply augmented inductive valuations binding V_{11} and V_{pq} and the method to construct the series is not so simple. We shall show them in Part 2 of this paper, together with other topics, such as key-polynomials of simply augmented valuations and those of xy-doubly augmented valuations, residue class rings of a valuation of $K[x, y]$ and valuations of $K[x, y, z]$.

§ 10. Some examples of valuations of $K[x, y]$

Let K be the field of all rational numbers with the 3-adic valuation. Namely, for any rational number k expressed as $k=3^n\frac{b}{c}$, where b, c and n are integers with $(b, c)=(c, 3)=(3, b)=1$, V_{00} is defined by

$$V_{00}k=n\,.$$

The residue class field F_0 of K with respect to V_{00} is a Galois field which consists of only three elements 1, 0 and -1.

A valuation V_{10} of $K[x]$ is defined as follows, for any polynomial $f(x)=\sum_i f_i x^i$, $f_i\in K$;

$$V_{10}f(x)=\operatorname*{Min}_i[V_{00}f_i]\,,$$

V_{10} is denoted as $V_{10}=[V_{00},\ V_{10}x=0]$.

Example 10.1. Let $K[x]$ have the above defined valuation V_{10}. Then a valuation V_{11} of $K[x, y]$ is defined as follows, for any polynomial

$$f(x, y)=\sum_i f_i(x)y^i=\sum_{i,j}a_{ij}x^iy^j,\ a_{ij}\in K\,;$$

$$V_{11}f(x, y)=\operatorname*{Min}_i\left[V_{10}f_i(x)\right]=\operatorname*{Min}_{i,j}[V_{00}a_{ij}]\,.$$

V_{11} is denoted as $V_{11}=[V_{10},\ V_{11}y=0]$.

In the ring $K_x[y]$ with the valuation V_{11}, y^2+1 can be a y-key polynomial over V_{11}.

Since $V_{10}x=0$, the residue class ring Δ_{10} of $K[x]$ is $F_0[X]$, where $X=H_{10}x$ is the residue class of x; X can be a variable with respect to F_0. (M. Theorem 10.2). So a residue class field Λ_{11} of a field K_x is $F_0(X)={}_0F_x$. Since $V_{11}y=0$, the residue class ring Δ_{11} of $K_x[y]$ is a ring ${}_0F_x[Y]$, where $Y=H_{11}y$

is the residue class of y, and Y and X are variables algebrically independent with respect to F_0.

The class $H_{11}(y^2+1)=(H_{11}y)^2+H_{11}1=Y^2+1$. Y^2+1 is an irreducible polynomial in $F_0[Y]$ and also is irreducible in ${}_0F_x[Y]$. Therefore y^2+1 is equivalence irreducible over V_{11} in $K_x[y]$ due to M. Lemma 11. 2.

Besides, y^2+1 satisfies all the conditions for a y-key polynomial over V_{11} (M. Theorem 9. 4). So y^2+1 is a y-key polynomial over V_{11} in $K_x[y]$. Any polynomial $f(x, y)$ can be expressed as

$$f(x, y)=\sum_i f_i(x, y)(y^2+1)^i$$

where the degree of each $f_i(x, y)$ with respect to y is less than that of y^2+1 for $i=0, 1, 2, \cdots$. We describe it briefly as $deg_y f_i(x, y)<deg_y(y^2+1)$, $i=0, 1, 2, \cdots$. Then the function defined as follows is an augmented valuation of V_{11} in the ring $K[x, y]$;

$$V_{12}f(x, y)=\operatorname{Min}_i\left[V_{11}f_i(x, y)+i\right],$$

where $V_{12}(y^2+1)=1>V_{11}(y^2+1)$.

V_{12} is a y-simply augmented valuation of V_{11}. V_{12} induces in the ring $K[y]$ the following valuation V_{02};

$$V_{02}=\left[V_{00},\ V_{01}y=0,\ V_{02}(y^2+1)=1\right].$$

Then it is evident that for any polynomial $f(x, y)=\sum_i f_i(y)x^i$ *in* $K[x, y]$, the function $\overline{V}_{12}$ defined as follows is also a valuation of $K_y[x]$ whose coefficient field K_y has the valuation V_{02} and $\overline{V}_{12}$ is equal to V_{12} in $K[x, y]$;

$$\overline{V}_{12}f(x, y)=\operatorname{Min}_i\left[V_{02}f_i(y)\right]$$

where $\overline{V}_{12}x=0$.

Example 10. 2. Let $K[x, y]$ have the same valuation $V_{11}=[V_{10},\ V_{11}y=0]$ defined as in Example 10.1. Then $y-x$ is a y-key polynomial over V_{11} in the ring $K_x[y]$, because

$$V_{11}y=0=V_{11}x \qquad \text{(M. Corollary 13. 2)}.$$

Therefore for any polynomial $f(x, y)=\sum_i f_i(x)(y-x)^i$ the function W_{12} defined as follows is a y-augmented valuation of V_{11} in $K[x, y]$;

$$W_{12}f(x, y)=\operatorname{Min}_i\left[V_{10}f_i(x)+i\right],$$

where $$W_{12}(y-x)=1>V_{11}(y-x).$$

But in this case W_{12} is an xy-doubly augmented valuation of V_{11} of

$K[x, y]$. The reason is as follows. By the remainder theorem, we have for any polynomial $g(y)$

$$g(y) = (y-x)\, l(x, y) + g(x)\,.$$

According to the definition of V_{11}, $V_{11}g(x) = V_{11}g(y)$ and therfore $W_{12}g(y) = V_{11}g(y)$ by M. Lemma 4. 3, so W_{12} is not a y-simply augmented valuation of V_{11} and consequently W_{12} is an xy-doubly augmented valuation of V_{11}.

Example 10. 3. Let $K[x, y]$ have the same valuation V_{11} as in Example 10. 1. Then y^2+1 is a y-key polynomial over V_{11} in $K_x[y]$. So it can be seen in the same way that x^2+1 is an x-key polynomial over V_{11} in $K_y[x]$, where K_y has the valuation $V_{01} = [V_{00},\ V_{01}y = 0]$ and $V_{11} = [V_{01},\ V_{11}x = 0]$.

But $K[x, y]$ having the valuation $\overline{V}_{12} = V_{12}$ defined as in Example 10. 1, x^2+1 can not be an x-key polynomial over $\overline{V}_{12}$ in $K_y[x]$. For, since

$$\overline{V}_{12}\Big((x^2+1)-(x^2-y^2)\Big) = \overline{V}_{12}(y^2+1) = 1 > 0 = \overline{V}_{12}(x^2+1)\,,$$

it follows

$$x^2+1 \sim x^2-y^2 \sim (x+y)(x-y) \quad \text{in} \quad \overline{V}_{12}\,.$$

Hence x^2+1 is an x-key polynomial in $\overline{V}_{12}$ in $K_y[x]$, then $x+y$ or $x-y$ must be equivalence-divisible by x^2+1 in $\overline{V}_{12}$ in $K_y[x]$. But this is impossible because

$$deg_x(x^2+1) > deg_x(x+y) = deg_x(x-y)\,.$$

Example 10. 4. Let $V_{20} = \Big[V_{00},\ V_{10}x = 0,\ V_{20}(x^2+1) = 1\Big]$ be a valuation of $K[x]$ and let $K[x, y]$ have the valuation $V_{11} = [V_{10},\ V_{11}y = 0]$.

Then $y-x$ is a y-key polynomial over V_{11} in $K_x[y]$ and $W_{12} = \Big[V_{11},\ W_{12}(y-x) = 1\Big]$ is an xy-doubly augmented valuation of V_{11} of $K_x[y]$. But $K_x[y]$ having the valuation $V_{21} = [V_{20},\ V_{21}y = 0]$, $y-x$ can be a y-key polynomial over V_{21} in $K_x[y]$, while the valuation $V_{22} = \Big[V_{21},\ V_{22}(y-x) = 1\Big]$ of $K_x[y]$ whose coefficient field has the valuation V_{20} is not an xy-doubly augmented valuation of V_{21}, but is a y-simply augmented valuation of V_{21}.

Any polynomial $f(x)$ can be expressed as $\sum_i f_i(x)(x^2+1)^i$,

where $$deg_x f_i(x) < deg_x(x^2+1) = 2\,.$$

Then V_{20} is defined as follows;

$$V_{20}f(x) = \operatorname*{Min}_i \Big[V_{10}f_i(x) + i\Big]\,,$$

where $$V_{20}(x^2+1)=1>V_{10}(x^2+1).$$

And for any polynomial $f(x, y)=\sum_i f_i(x)y^i$, V_{21} is defined as

$$V_{21}f(x, y)=\operatorname{Min}_i\left[V_{20}f_i(x)\right]$$

where $$V_{21}y=0.$$

So V_{21} can be considered as a valuation of $K_x[y]$. Since $V_{21}y=0=V_{21}x$, again by M. Corollary 13. 2 $y-x$ can be a y-key polynomial over V_{21} in $K_x[y]$. Therefore for any polynomial $f(x, y)=\sum_i f_i(x)(y-x)^i$ the function V_{22} defined as follows is a y-augmented valuation of V_{21} in $K_x[y]$.

$$V_{22}f(x, y)=\operatorname{Min}_i\left[V_{20}f_i(x)+i\right]$$

where $$V_{22}(y-x)=1>V_{21}(y-x).$$

But V_{22} is a y-simply augmented valuation of V_{21}, because

$$\begin{aligned}y^2+1&=(y-x)^2+2x(y-x)+(x^2+1)\\ V_{22}(y^2+1)&=\operatorname{Min}\left[V_{22}(y-x)^2,\ V_{22}2x(y-x),\ V_{22}(x^2+1)\right]\\ &=\operatorname{Min}[2,\ 1,\ 1]=1>0=V_{21}(y^2+1).\end{aligned}$$

Thus it is shown that the y-key polynomial over V which produces U is not always a polynomial of y, even if U is a y-simply augmented valuation of V in $K_x[y]$.

Example 10. 5. Next we show that V_{22} defined as in Example 10. 4 induces in $K[y]$ the valuation $V_{02}=\left[V_{00},\ V_{01}y=0,\ V_{02}(y^2+1)=1\right]$ defined as in Example 10. 1.

$$\begin{aligned}V_{22}y&=V_{22}[y-x+x]=\operatorname{Min}\left[V_{22}(y-x),\ V_{22}x\right]=0=V_{01}y\\ y^2+1&=(y-x)^2+2x(y-x)+x^2+1.\end{aligned}$$

Therefore

$$V_{22}(y^2+1)=\operatorname{Min}[2,\ 1,\ 1]=1=V_{02}(y^2+1)$$

and $$y^2+1\sim 2x(y-x)+x^2+1 \quad \text{in } V_{22}.$$

Now assume that in $K[y]$ V_{22} induces a valuation greater than V_{02}, i. e. there exists a polynomial $f(y)$ such that

$$V_{22}f(y)>V_{02}f(y).$$

Among such polynomials $f(y)$, we take a polynomial $\phi(y)$ whose degree is

minimum and whose leading coefficient is 1. Then $\phi(y)$ is a y-key polynomial over V_{02} and

$$V_{22}\phi(y) > V_{02}\phi(y).$$

(1) Let $\phi(y) = (y^2+1)^m + (ay+b)(y^2+1)^{m-1} + \cdots + (cy+d)$.

Then owing to M. Theorem 9. 4,

$$V_{02}\phi(y) = V_{02}(y^2+1)^m = m = V_{02}(cy+d) \leqq V_{02}(ay+b)(y^2+1)^{m-1}.$$

Namely

$$m \leqq V_{02}(ay+b)+m-1$$
$$1 \leqq \mathrm{Min}[V_{01}a y,\ V_{00}b].$$

Consequently (2) $$1 \leqq V_{00}a.$$

(3) $$V_{22}\phi(y) > V_{02}\phi(y) = m.$$

Subsitituting $2x(y-x)+x^2+1$ for y^2+1 in (1), we have

$$h(x, y) = \left[2x(y-x)+x^2+1\right]^m + (ay+b)\left[2x(y-x)+x^2+1\right]^{m-1} + \cdots + [cy+d]$$

and $$\phi(y) \sim h(x, y) \quad \text{in} \quad V_{22} \quad (4).$$

Then the coefficient of $(y-x)^m$ of $h(x, y)$ is

$$(2x)^m + a(2x)^{m-1} = (2x)^{m-1}[2x+a].$$
$$V_{22}\left[(2x)^{m-1}(2x+a)\right] = V_{20}\left[(2x)^{m-1}(2x+a)\right] = \mathrm{Min}[V_{20}\,2x,\ V_{00}a] = 0$$

because of (2).

$$V_{22}h(x, y) \leqq V_{22}(2x)^{m-1}(2x+a)(y-x)^m = m.$$

But by (4)

$$V_{22}h(x, y) = V_{22}\phi(y) > V_{02}\phi(y) = m.$$

This is a contradition, therefore V_{22} induces V_{02} in $K[y]$.

Example 10. 6. Let $V_{11}=[V_{00},\ V_{10}x=0,\ V_{11}y=0]$ be the valuation of $K[x, y]$ defined as in Example 10. 1 and $\overline{V}_{21}=[V_{11},\ \overline{V}_{21}(x^2+1)=1]$ be a valuation of $K_y[x]$. Then $\overline{V}_{21}$ is an x-simply augmented valuation of V_{11} and is equal to another valuation $V_{21}=[V_{20},\ V_{21}y=0]$ of $K_x[y]$ whose coefficient field K_x has the valuation $V_{20}=[V_{00},\ V_{10}x=0,\ V_{20}(x^2+1)=1]$. Moreover $V_{22}=[V_{21},\ V_{22}(y-x)=1]$ defined as in Example 10. 4 is a y-simply augmented valuation of V_{21}. Namely

$$V_{11} \xrightarrow{x\text{-simply aug.}} \overline{V}_{21} = V_{21} \xrightarrow{y\text{-simply aug.}} V_{22}.$$

Next by interchanging x and y, it follows that $\overline{V}_{12}=[V_{11},\ \overline{V}_{12}(y^2+1)=1]$ is

a y-simply augmented valuation of V_{11} of $K_x[y]$ and it is equal to another valuation $V_{12}=[V_{02},\ V_{12}x=0]$ of $K_y[x]$ whose coefficient field K_y has the valuation $V_{02}=[V_{00},\ V_{01}y=0,\ V_{02}(y^2+1)=1]$ and moreover $\overline{V}_{22}=[V_{12},\ \overline{V}_{22}(x-y)=1]$ is an x-simply augmented valuation of V_{12} of $K_y[x]$. Namely

$$V_{11} \quad \underset{\rightarrow}{y\text{-simply aug.}} \quad \overline{V}_{12}=V_{12} \quad \underset{\rightarrow}{x\text{-simply aug.}} \quad \overline{V}_{22}.$$

It is evident that $V_{21}\neq V_{12}$ and $V_{21}\not> V_{12}$ and $V_{21}\not< V_{12}$ because

$$V_{12}(x^2+1)=\mathrm{Min}[V_{11}x^2,\ V_{00}\,1]=0<1=V_{21}(x^2+1)\ .$$
$$V_{21}(y^2+1)=\mathrm{Min}[V_{11}y^2,\ V_{00}\,1]=0<1=V_{12}(y^2+1)\ .$$

Therefore next proving that $V_{22}=\overline{V}_{22}$, it follows that this example is that of the case when a set of simply descended valuations of a valuation of $K[x, y]$ is not partially linearly ordered.

We proved that V_{22} induces in $K[y]$ $V_{02}=[V_{00},\ V_{01}y=0,\ V_{02}(y^2+1)=1]$ in Example 10. 5. And $\overline{V}_{22}$ also induces V_{02} in $K[y]$ owing to its definition.

$$V_{22}(x-y)=V_{22}(-1)(y-x)=V_{22}(y-x)=\overline{V}_{22}(x-y)=1\ .$$

Let be $f(x, y)=\sum_i f_i(y)(x-y)^i$ any polynomial of $K[x, y]$, then

$$\begin{aligned}\overline{V}_{22}f(x,\ y)&=\overline{V}_{22}\Big(\sum_i f_i(y)(x-y)^i\Big)=\min_i\Big[V_{02}f_i(y)+i\overline{V}_{22}(x-y)\Big]\\&=\min_i\Big[V_{02}f_i(y)+iV_{22}(x-y)\Big]=\min_i\Big[V_{22}f_i(y)(x-y)^i\Big]\\&\leqq V_{22}\Big(\sum_i f_i(y)(x-y)^i\Big)=V_{22}f(x,\ y)\ .\end{aligned}$$

Again by interchanging x and y, in the same way it follows that

$$\overline{V}_{22}f(x,\ y)\geqq V_{22}f(x,\ y)\ .$$

So, consequently $$V_{22}=\overline{V}_{22}\ .$$

Example 10. 7. Let $K[x, y]$ have the valuation V_{22} defined as in Example 10. 4, then $y-x-\dfrac{9}{x^2+1}$ is a y-key polynomial over V_{22} in $K_x[y]$, because

$V_{22}(y-x)=1=V_{22}\left(\dfrac{-9}{x^2+1}\right)$. Therefore it follows again from M. Corollary 13.2 that $y-x-\dfrac{9}{x^2+1}$ is a y-key polynomial over V_{22} in $K_x[y]$.

Any polynomial $f(x, y)$ can be expressed as an expansion with respect to $\left(y-x-\dfrac{9}{x^2+1}\right)$;

$$f(x,\ y)=\sum_i \frac{a_i(x)}{b_i(x)}\left(y-x-\frac{9}{x^2+1}\right)^i\ .$$

Then the function W_{23} defined as follows is a y-augmented valuation of V_{22};

$$W_{23}f(x, y) = \operatorname*{Min}_{i}\left[V_{22}\frac{a_i(x)}{b_i(x)} + 2i\right],$$

where $W_{23}\left(y-x-\frac{9}{x^2+1}\right)=2>1=V_{22}\left(y-x-\frac{9}{x^2+1}\right)$. Moreover W_{23} is an xy-doubly augmented valuation of V_{22}. The reason is as follows. Again by the remainder theorem any polynomial $f(y)$ can be expressed as follows;

$$f(y)=\left(y-x-\frac{9}{x^2+1}\right)\frac{1(x, y)}{b(x)}+f\left(x+\frac{9}{x^2+1}\right)$$

Let $f(y)=\sum_i f_i y^i$, $f_i \in K$. Then

$$f\left(x+\frac{9}{x^2+1}\right)=\sum_i f_i\left(x+\frac{9}{x^2+1}\right)^i$$

$$V_{22}f_i\left(x+\frac{9}{x^2+1}\right)^i=\operatorname*{Min}_{i}\left[V_{22}f_i+iV_{22}\left(x+\frac{9}{x^2+1}\right)\right]$$

$$=\operatorname*{Min}_{i}[V_{00}f_i],$$

because $V_{22}x=0<1=V_{22}\frac{9}{x^2+1}$ and so

$$V_{22}\left(x+\frac{9}{x^2+1}\right)=0.$$

On the other hand,

$$V_{22}f(y)=\operatorname*{Min}_{i}[V_{00}f_i+iV_{01}y]=\operatorname*{Min}_{i}[V_{00}f_i]$$

$$=V_{22}f\left(x+\frac{9}{x^2+1}\right).$$

Hence again by M. Lemma 4. 3 it follows

$$W_{23}f(y)=V_{22}f(y).$$

Thus W_{23} is an xy-doubly augmented valuation of V_{22}. Furthermore,

$$V_{11}<V_{21}<V_{22}<W_{23}, \quad \text{where}$$

V_{11} x-simply aug. V_{21} y-simply aug. V_{22} xy-doubly aug. W_{23}.

So V_{22} is the last simply augmented valuation of W_{23}.

Remark 1.

Let V_{10} and $V_{20}=\left[V_{10}, V_{20}(x^2+1)=1\right]$ be the two valuations of $K[x]$

defined in Example 10. 4.

$$V_{10}\big((x^2+1)-(x^2-2)\big) = V_{00}3 = 1 > 0 = V_{10}(x^2+1)$$

so $x^2+1 \sim x^2-2$ *in* V_{10} and $deg_x(x^2+1) = deg_x(x^2-2)$.
Then x^2-2 can be an x-key polynomial over V_{10} and the function $V_{20}^* = [V_{10}, V_{20}^*(x-2)=1]$ is also an x-augmented valuation of V_{10} and $V_{20}^* = V_{20}$ owing to M. Theorem 15. 3.

$$V_{20}^*(x^2-2) = 1 = V_{20}^*3\,.$$

Then $x^2+1=(x^2-2)+3$ can be an x-key polynomial over V_{20}^* by M. Corollary 13. 2. and the function V_{30}^* defined as follows in an x-augmented valuation of V_{20}^*;

$$V_{30}^* = \Big[V_{20}^*,\ V_{30}^*(x^2+1) = 2\Big],$$

because $$V_{30}^*(x^2+1) > V_{20}^*(x^2+1)\,.$$

But V_{30}^* cannot be an x-augmented valuation of V_{20}. For, even if we build forcibly a function V_{30} as follows;

$$V_{30} = \Big[V_{20},\ V_{30}(x^2+1) = 2\Big] = \Big[V_{00},\ V_{10}x = 0,\ V_{20}(x^2+1) = 1,\ V_{30}(x+1) = 2\Big],$$

it follows that the polynomial x^2+1 is used in V_{30} twice as an x-key polynomial. But this breaks the condition (5) for x-key polynomials in **§1**.

Namely V_{30}^* is an x-augmented valuation of V_{20}^* and $V_{20} = V_{20}^*$, but in spite of it, V_{30}^* is not an x-augmented valuation of V_{20}. In this case we adopt another method of definition. When valuations $T_1, T_2, \cdots$ of $K[x, y]$ are equal to each other, we consider that $T_1, T_2, \cdots$ induce in $K[x, y]$ the same valuation W which is defined abstractly as a function satisfying only the three conditions of a valuation in **§1** and we regard each T_i as an expression of W, namely, a concrete representation of W and if a valuation U is an augmented valuation of one T_i, then we call U an augmented valuation of W. In this paper we adopt this method of definition like W at the end of §4. Following this method of definition, the above valuation V_{30}^* is an x-augmented valuation of the valuation whose expressions are V_{20}^* and V_{20}.

Remark 2.

Let U and V be two valuations of $K[x]$ and $U \neq V$. If for every polynomial $f(x)$ in $K[x]$ $Uf(x) \geqq Vf(x)$, then U is called to be greater than V.

Owing to M. Theorem 5. 1. an x-augmented valuation W of a valuation Q in $K[x]$ is greater than Q. But even if U is a valuation greater than a valuation V in $K[x]$, U is not always an x-augmented valuation of V.

The following is an example of this case.

Let be V_{10} and $V_{20}=\left[V_{10}, V_{20}(x^2+1)=1\right]$ the two valuations of $K[x]$ defined in Example 10.4. x^2+1 is an x-key polynomial over V_{10} and $V_{10}(x^2+1)=0$. Then the function $V^*=\left[V_{10}, V^*(x^2+1)=\frac{1}{2}\right]$ is also an x-augmented valuation of V_{10}, and according to their definitions it is evident that V_{20} is greater than V^*.

But V_{20} cannot be an x-augmented valuation of V^*. Because, if V_{20} is an x-augmented valuation of V^*, then a set Γ_2 of all values that V_{20} takes in $K[x]$ must include a set Γ_* of all values that V^* takes in $K[x]$. Γ_2 is a set of all integers, so Γ_2 does not include $\frac{1}{2}=V^*(x^2+1)$. All valuations that are equal to each other have the same set of values. Then V_{20} cannot be an x-augmented valuation of V^*, even if we adopt the method of definition in Remark 1.

In this paper when we write as "$Ug(x)>Vfx$)", it means that a value of the polynomial $g(x)$ by a valuation U is greater than a value of $f(x)$ by V. But when we write as "$W>Q$", it means that W is an augmented valuation of Q and it does not mean that the valuation W is greater than the valuation Q.

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