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Author(s)	Tanaka, Hiroshi
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SOME PROPERTIES OF KURAMOCHI BOUNDARIES OF HYPERBOLIC RIEMANN SURFACES

By

Hiroshi TANAKA

1. Constantinescu and Cornea [1] remarked that Kuramochi boundary points share many properties enjoyed by interior points. Let F and K be mutually disjoint compact sets in a hyperbolic Riemann surface and let μ be a positive measure on K . It is known that the balayaged measure (with respect to the Green potential) of μ onto F is supported by the boundary of F . In this paper, we shall prove that a similar result is also valid for a closed set F on the Kuramochi compactification by considering Kuramochi boundary points like interior points (Theorem 1).

As an application, we shall prove that if the set of all non-minimal Kuramochi boundary points is non-empty, then it is uncountable (Theorem 2). A corresponding result for the Martin boundary was proved by Ikegami [2] and Toda [5].

2. Let R be a hyperbolic Riemann surface. We shall use the same notation as in [1], for instance, $\tilde{g}_b$, $\tilde{p}^\mu$, $\tilde{s}_{\tilde{F}}$, f^F , R_N^* , $\mathcal{A}_N$ etc. For a subset A of R , we denote by ∂A the relative boundary of A in $\mathring{R}$ and by $\bar{A}$ the closure of A in R_N^* . Let K_0 be a closed disk in R and $R_0^* = R_N^* - K_0$. The Kuramochi boundary $\mathcal{A}_N$ is decomposed into two mutually disjoint parts, the minimal part $\mathcal{A}_1$ and the non-minimal part $\mathcal{A}_0$. By a measure μ on R_0^* , we always mean a positive measure μ on R_N^* such that $\mu(K_0) = 0$. For a measure μ on R_0^* , we shall denote by $S\mu$ the support of μ . If a measure μ on R_0^* satisfies $\mu(\mathcal{A}_0) = 0$, then it is called canonical. It is known that if μ is a measure on R_0^* , then there exists a unique canonical measure ν such that $\tilde{p}^\mu = \tilde{p}^\nu$. For a closed set F in R and a measure μ , we denote by μ_F the canonical measure associated with $\tilde{p}_{\tilde{F}}^\mu$. We note that $S\mu_F \subset \bar{F}$.

The following properties are known ([1]).

(A) Let F be a non-polar¹⁾ closed set in R and f a Dirichlet function²⁾

1) A subset A of R is called *polar* if there exists a positive superharmonic function s on R such that $s(a) = +\infty$ at every point a of A .

2) This is called eine Dirichletsche Funktion in [1].

on R . If G is a component of $R-F$, then $f^F = f^{\partial G}$ on G .

- (B) Let F be a closed set in R . If s is a Dirichlet function on R , $s=0$ on K_0 and s is a non-negative full-superharmonic function³⁾ on R_0 , then

$$s_{\tilde{F}} = s^{K_0 \cup F} \quad \text{on } R_0 - F.$$

- (C) Let b be any point of $R_0 \cup A_1$. If F is a closed set in R such that $\bar{F}$ is a neighborhood of b in R_N^* , then $(\tilde{g}_b)_{\tilde{F}} = \tilde{g}_b$.
- (D) Let μ be a measure on R_0^* . If F is a closed set in R , then

$$\left(\int \tilde{g}_b d\mu(b) \right)_{\tilde{F}} = \int (\tilde{g}_b)_{\tilde{F}} d\mu(b).$$

- (E) Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of canonical measures on R_0^* . If μ_n converges vaguely to a measure μ as $n \rightarrow \infty$ and each $\tilde{\mu}_n$ is dominated by a fixed full-superharmonic function, then μ is canonical.

We shall prove

Theorem 1. *Let F be a closed subset of R_0 and let μ be a measure on R_0^* such that $S\mu \cap (\bar{F} \cup K_0) = \emptyset$. Then $S\mu_F \subset \bar{F} \cap \overline{R-F}$.*

Proof. (i) First suppose $S\mu$ is compact in R_0 . Since $S\mu_F \subset \bar{F}$, it is sufficient to prove that $S\mu_F \subset \overline{R-F}$. Let b_0 be an arbitrary point of $A_N - \overline{R-F}$. Let U be an open neighborhood of b_0 in R_N^* such that $\overline{U \cap R} \cap \overline{R-F} = \emptyset$ and let $G = U \cap R$. We shall prove that $\mu_F(U) = 0$. Let D be a relatively compact open set in R such that $\bar{D} \cap (K_0 \cup \bar{F}) = \emptyset$ and ∂D consists of a finite number of analytic Jordan curves. We set $s = \tilde{\mu}_{\partial D}^*$ and $f = s$ on R_0 and $= 0$ on K_0 . Then we see that f is a bounded continuous Dirichlet function on R . Since $(\tilde{g}_b)_{\partial D} = \tilde{g}_b$ on $R_0 - \bar{D}$ for $b \in D$, it follows from (D) that $s = \tilde{\mu}_{\partial D}^* = \tilde{\mu}^*$ on $R_0 - \bar{D}$. Hence $\tilde{\mu}_{\tilde{F}}^* = s_{\tilde{F}}$ and $\tilde{\mu}_{\tilde{F}-G}^* = s_{\tilde{F}-G}$. Since the measure associated with s is supported by ∂D and $\partial D \subset R-G$, by (C) and (D), we obtain that $s_{\tilde{R}-G} = s$ on R_0 . By (B) and (A), we have that $s_{\tilde{F}-G} = f^{K_0 \cup (F-G)} = f^{\partial G}$ on G . Thus $\tilde{\mu}_{\tilde{F}-G}^* = s_{\tilde{F}-G} = f^{\partial G} = s_{\tilde{R}-G} = s = \tilde{\mu}^*$ on G . Similarly, by (A) and (B), we obtain that $\tilde{\mu}_{\tilde{F}-G}^* = \tilde{\mu}^*$ on $R_0 - F$. On the other hand, since $\tilde{\mu}_{\tilde{F}-G}^* = \tilde{\mu}_{\tilde{F}}^* = \tilde{\mu}^* q.p.$ ⁴⁾ on $F-G$, we see that $\tilde{\mu}_{\tilde{F}-G}^* = \tilde{\mu}_{\tilde{F}}^* q.p.$ on R_0 . Hence $\tilde{\mu}_{\tilde{F}-G}^* = \tilde{\mu}_{\tilde{F}}^*$ on R_0 . Therefore it follows from the uniqueness of canonical measure that $\mu_F = \mu_{F-G}$. Hence $S\mu_F \subset \overline{F-G} = \bar{F} - U$, so that $\mu_F(U) = 0$.

- (ii) Next suppose $S\mu$ is not necessarily compact in R_0 . By the as-

3) This is called superharmonic by Kuramochi ([3]) and "positive vollsuperharmonisch" in [1].

4) We shall say that a property holds $q.p.$ on a set E if it holds on E except for a polar set; cf. footnote 1).

sumption, we can find a closed subset V of R_0 such that $\bar{V}$ is a neighborhood of $S\mu$ in R_N^* and $\bar{V} \cap (K_0 \cup \bar{F}) = \emptyset$. Let $\{K_n\}_{n=1}^\infty$ be an increasing sequence of compact sets in R such that $\bigcup_{n=1}^\infty K_n = R$. We write $\mu_{V \cap K_n} = \mu_n$ and $(\mu_n)_F = \nu_n$ for simplicity. Since $\tilde{P}^{\nu_n} \leq \tilde{P}^{\nu_{n-1}} \leq \tilde{P}^\nu$ for each n , by the aid of (E), we can show that ν_n converges vaguely to a canonical measure ν . Since $S\nu_n \subset \bar{F} \cap \overline{R-F}$ ($n=1, 2, \dots$) by (i), we see that $S\nu \subset \bar{F} \cap \overline{R-F}$. By Hilfssatz 16.2 in [1], we obtain that $\tilde{P}^{\nu_n}$ increases to $\tilde{P}^\nu$ as $n \rightarrow \infty$. By Satz 15.3, b) in [1], we see that $\tilde{P}^{\nu_n}$ increases to $\tilde{P}^{\nu_{\bar{F}}}$ as $n \rightarrow \infty$. Since $\tilde{P}^{\nu_{\bar{F}}} = \tilde{P}^\nu$ by (C) and (D), it follows that $\tilde{P}^{\nu_n}$ increases to $\tilde{P}^\nu$ as $n \rightarrow \infty$. Furthermore it follows from Satz 15.3, b) in [1] that $\tilde{P}^{\mu_n} = \tilde{P}^{\nu_n}$ increases to $\tilde{P}^{\mu_F}$ as $n \rightarrow \infty$. Hence we have that $\tilde{P}^{\mu_F} = \tilde{P}^\nu$. Since ν is canonical, $\mu_F = \nu$. Therefore $S\mu_F \subset \bar{F} \cap \overline{R-F}$.

3. By a discussion similar to the proof of Théorème 14 in [4], we shall prove

Lemma. *Let F be a closed subset of R_0 and let $b_0 \in \Delta_N - \bar{F}$. Then there exists a measure μ such that $S\mu \subset \bar{F} \cap \overline{R-F}$ and $(\tilde{g}_{b_0})_{\bar{F}} \leq \tilde{P}^\nu \leq \tilde{g}_{b_0}$.*

Proof. First we note the following. Let b be an arbitrary point of R_0 . Let α_0 be a positive real number such that $\{z; \tilde{g}_b(z) \geq \alpha_0\}$ is a compact set in R_0 . For each $\alpha \geq \alpha_0$, $\min(\tilde{g}_b, \alpha)$ is a potential and the associated measure, say λ_α , is supported by $\{z; \tilde{g}_b(z) = \alpha\}$. Then $\tilde{P}^{\lambda_\alpha}$ can be continuously extended over R_0^* . We denote by $\tilde{P}^{\lambda_\alpha} = \min(\tilde{g}_b, \alpha)$ the continuous extension again. Let ν be a measure on R_0^* . Since $\int \tilde{P}^\nu d\lambda_\alpha = \int \min(\tilde{g}_b, \alpha) d\nu$, by letting $\alpha \rightarrow \infty$, we have that $\lim_{\alpha \rightarrow \infty} \int \tilde{P}^\nu d\lambda_\alpha = \tilde{P}^\nu(b)$.

Now we shall prove the lemma. Let $\{b_n\}_{n=1}^\infty$ be a sequence of points in $R_0 - F$ such that $b_n \rightarrow b_0$ as $n \rightarrow \infty$. We denote by μ_n the canonical measure associated with $(\tilde{g}_{b_n})_{\bar{F}}$. By Theorem 1, we see that $S\mu_n$ is contained in $\bar{F} \cap \overline{R-F}$ ($n=1, 2, \dots$). Since $\mu_n(\bar{F} \cap \overline{R-F}) \leq 1$ ($n=1, 2, \dots$), we can find a subsequence $\{\mu_{n_k}\}_{k=1}^\infty$ of $\{\mu_n\}_{n=1}^\infty$ such that μ_{n_k} converges vaguely to a measure μ supported by $\bar{F} \cap \overline{R-F}$ as $k \rightarrow \infty$. Since $\int \tilde{P}^{\lambda_\alpha} d\mu = \int \tilde{P}^\nu d\lambda_\alpha$ and $\int \tilde{P}^{\lambda_\alpha} d\mu_n = \int \tilde{P}^{\mu_n} d\lambda_\alpha$ ($n=1, 2, \dots$), we obtain that $\lim_{k \rightarrow \infty} \int \tilde{P}^{\mu_{n_k}} d\lambda_\alpha = \int \tilde{P}^\nu d\lambda_\alpha$. Since $(\tilde{g}_{b_0})_{\bar{F}} \leq \lim_{n \rightarrow \infty} (\tilde{g}_{b_n})_{\bar{F}} \leq \tilde{g}_{b_0}$, we obtain that

$$\begin{aligned} \int (\tilde{g}_{b_0})_{\bar{F}} d\lambda_\alpha &\leq \int \lim_{k \rightarrow \infty} (\tilde{g}_{b_{n_k}})_{\bar{F}} d\lambda_\alpha \leq \lim_{k \rightarrow \infty} \int (\tilde{g}_{b_{n_k}})_{\bar{F}} d\lambda_\alpha = \lim_{k \rightarrow \infty} \int \tilde{P}^{\mu_{n_k}} d\lambda_\alpha \\ &= \int \tilde{P}^\nu d\lambda_\alpha \leq \int \tilde{g}_{b_0} d\lambda_\alpha. \end{aligned}$$

By letting $\alpha \rightarrow \infty$, we have that $(\tilde{g}_{b_0})_{\bar{F}}(b) \leq \tilde{P}^\nu(b) \leq \tilde{g}_{b_0}(b)$.

Theorem 2. *If the set Δ_0 of all non-minimal Kuramochi boundary points is non-empty, then it is uncountable.*

Proof. Let b_0 be an arbitrary point of Δ_0 . We set $D(r) = \{b \in R_N^*; d(b, b_0) < r\}$ and $C(r) = \{b \in R_N^*; d(b, b_0) = r\}$ for $r > 0$, where d is a metric on R_N^* . Suppose there exists a sequence of positive real numbers $\{r_n\}_{n=1}^\infty$ such that $C(r_n) \cap \Delta_0 = \emptyset$, $r_n > r_{n+1}$ ($n = 1, 2, \dots$) and $\lim_{n \rightarrow \infty} r_n = 0$. For each n , if we apply the Lemma to $F = R - D(r_n) \cap R$ and the above b_0 , then we obtain a measure μ_n supported by $\overline{F} \cap \overline{R - F} \subset C(r_n)$ such that

$$(*) \quad (\tilde{\sigma}_{b_0})_{\overline{F}} \leq \tilde{P}^{\mu_n} \leq \tilde{g}_{b_0}.$$

Since $\mu_n(R_0^*) \leq 1$ and $S\mu_n \subset C(r_n) \subset D(r_1) \cup C(r_1)$ for each n , we can choose a subsequence $\{\mu_{n_k}\}_{k=1}^\infty$ of $\{\mu_n\}_{n=1}^\infty$ such that μ_{n_k} converges vaguely to a measure μ supported by $\bigcap_{n=1}^\infty (D(r_n) \cup C(r_n)) = \{b_0\}$ as $k \rightarrow \infty$. Since b_0 is non-minimal and μ is not identically equal to zero by (*), μ is not canonical. On the other hand, since each μ_n is canonical, it follows from (E) that μ is canonical. This is a contradiction. Hence there exists an $r_0 > 0$ such that $C(r) \cap \Delta_0 \neq \emptyset$ for all r ($0 < r < r_0$). If $r \neq r'$, then $C(r) \cap C(r') = \emptyset$. Thus Δ_0 is uncountable.

Corollary. *If the set Δ_0 is non-empty, then no point of Δ_0 is isolated in Δ .*

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Department of Mathematics,
Hokkaido University

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