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A NOTE ON PRIMITIVE EXTENSIONS OF RANK 3 OF ALTERNATING GROUPS

By

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1. T. Tsuzuku determined (the degrees of) the primitive extensions of rank 3 of symmetric groups ([4]). In this note we take up alternating groups instead of symmetric groups and prove the following theorem.

Theorem 1. *Let A_n be the alternating group of degree n . If A_n has a primitive extension of rank 3, then $n=1, 3, 5$ or 7 .*

Since our proof is quite similar to Tsuzuku's paper [4], we use the same notations as those in [4] and give a proof in outline only.

S_n : The symmetric group of degree n .

A_n : The alternating group of degree n (on a set Γ).

G : A primitive extension of rank 3 of A_n on a set $\Omega = \{0, 1, 2, \dots, n, \tilde{1}, \tilde{2}, \dots, \tilde{m}\}$ which consists of $1+n+m$ letters.

H : The stabilizer G_0 of a letter, say 0, of Ω . The orbits of H are denoted by $\mathcal{A}_0 = \{0\}$, $\mathcal{A}_1 = \{1, 2, \dots, n\}$ and $\mathcal{A}_2 = \{\tilde{1}, \tilde{2}, \dots, \tilde{m}\}$ and the group $(H, \mathcal{A}_1)$ is isomorphic to (A_n, Γ) .

L : The stabilizer of the subset $\{0, \tilde{1}\}$ of Ω .

$|X|$: The number of elements in a set X .

2. *Proof of Theorem 1.* Clearly A_2 does not have a primitive extension of rank 3 and so $n \neq 2$. In the following we assume that $n \neq 1, 3, 5$ and 7 . By assumption, the group (A_n, Γ) is isomorphic to $(H, \mathcal{A}_1)$ and $|L|$ is equal to $\frac{n!}{2m}$. According to a theorem of Manning ([4], 2. Prop. 1), $|L|$ is divisible by $\frac{(n-2)!}{2}$ and $\frac{(n-1)!}{2} > |L| \geq \frac{(n-2)!}{2}$.

I. The case $|L| > \frac{(n-2)!}{2}$ and L is transitive on $\mathcal{A}_1$.

If L is a primitive subgroup of $(H, \mathcal{A}_1)$, then, in the same way as 3. I in [4], $2n(n-1) > \left\lceil \frac{n+1}{2} \right\rceil!$ and so we have $n=10, 9, 8, 6$, or 4 . In case $n=10, 9$ or 8 , by a theorem of Jordan ([5], th. 13. 9), L is either A_n or S_n and this is a contradiction. For the cases $n=6$ or 4 , and also for the case

L is imprimitive on A_1 , in the same way as 3. I in [4], we obtain contradictions.

II. The case $|L| > \frac{(n-2)!}{2}$ and L is intransitive on A_1 .

Since L is a subgroup of $S_r \times S_{n-r} \cap A_n$ with a positive integer r , $\frac{(n-2)!}{2}$ must be a proper divisor of $r!(n-r)!$. Hence we have the following cases (we may assume $r \leq n-r$): $r=1, 2$ or 3 .

(i) $r=1$: Since L is a subgroup of $S_1 \times S_{n-1} \cap A_n = S_1 \times A_{n-1}$, we may regard L as a subgroup of A_{n-1} (and so S_{n-1}) and we have $|L| = \frac{(n-2)!}{2}t$ where t is a proper divisor of $n-1$. If $t \geq 3$, then $\frac{(n-1)!}{2} > |L| > (n-2)!$ and so $2 < \text{the index of } L \text{ in } S_{n-1} < n-1$, which is impossible. If $t=2$, then the index of L in A_{n-1} is equal to $\frac{n-1}{2}$, which is a contradiction.

(ii) $r=2$: In this case, we have a relation $\frac{(n-2)!}{2} < |L| \leq |S_2 \times S_{n-2} \cap A_n| = (n-2)!$ and hence $L = S_2 \times S_{n-2} \cap A_n$. In the following, for a group Y and a subgroup X of Y , let $\eta|_X$ be the restriction to X of a character η of Y and let θ^Y be the character of Y induced by a character θ of X . Moreover, let 1_X be the principal character of X . Here we remark that, if $Y = X_1 X_2$ where X_1, X_2 are subgroups of a group Y and θ is a character of X_1 , then $\theta^Y|_{X_2} = (\theta|_{X_1 \cap X_2})^{X_2}$. By the structure of G , 1_H^G is equal to $1_G + \varphi_1 + \varphi_2$, where $\varphi_i (i=1, 2)$ is an irreducible character of G with degree f_i , and $1_{H|H}^G$ is equal to $1_H + 1_{A_{n-1}}^A + 1_{A_n}^A$. Since $S_n = (S_2 \times S_{n-2}) A_n$,

$$\begin{aligned} 1_{A_n}^A &= 1_{S_2 \times S_{n-2} \cap A_n}^{A_n} = 1_{S_2 \times S_{n-2}}^{S_n} |_{A_n} \\ &= 1_{A_n} + \chi_{0|A_n}^{0 \cdots 0} + \chi_{00|A_n}^{0 \cdots 0} \\ &\quad (\text{cf. [4], 2. Prop. 5 or [3], 3.3}) \end{aligned}$$

Similarly

$$1_{A_{n-1}}^A = 1_{S_{n-1} \cap A_n}^{S_n} = 1_{A_n} + \chi_{0|A_n}^{0 \cdots 0}.$$

Therefore we have

$$1_{H|H}^G = 3 \cdot 1_{A_n} + 2\chi_{0|A_n}^{0 \cdots 0} + \chi_{00|A_n}^{0 \cdots 0}.$$

If $n \geq 5$, then all the characters appeared in the right-hand side are irreducible and hence we can obtain a contradiction in the same way as 3. II. (ii) in [4]. If $n=4$, then $\chi_{00|A_4}^{00}$ is decomposed into two irreducible characters each of which has degree 1 and so $1_{H|H}^G$ is decomposed into 7 irreducible

characters which have degrees 1, 1, 1, 1, 1, 3 and 3 respectively. But f_1 and f_2 must be partial sums of these integers, which is impossible by a theorem of Frame ([4], 2. Prop. 2).

(iii) $r=3$: In this case $n=8$ or 6 and hence $|L|=3! 5!$ or $3! 3!$ respectively. This is impossible since L is a subgroup of $S_3 \times S_5 \cap A_8$ or $S_3 \times S_3 \cap A_6$ respectively.

III. The case $|L| = \frac{(n-2)!}{2}$. As in 3. III of [4], we have $n=57$. Since L is intransitive on A_1 and $|L| = \frac{55!}{2}$, L must be $S_1 \times S_1 \times A_{55}$. On the other hand, since $S_{57} = (S_1 \times S_1 \times S_{55}) A_{57}$ and $(S_1 \times S_1 \times S_{55}) \cap A_{57} = S_1 \times S_1 \times A_{55}$, we have

$$\begin{aligned} 1_{L^{57}} &= 1_{S_1 \times S_1 \times S_{55} | A_{57}}^{S_{57}} \\ &= 1_{A_{57}} + 2\chi_{0|A_{57}}^{0 \cdots 0} + \chi_{00|A_{57}}^{0 \cdots 0} + \chi_{0|A_{57}}^{0 \cdots 0}. \quad (\text{cf. [3], 3. 3}) \end{aligned}$$

In the same way as 3. III in [4], decomposing $1_{A_{57}} + 1_{A_{55}}^{A_{57}} + 1_{L^{57}}$ into irreducible characters and considering these degrees, we have a desired contradiction.

Thus Theorem 1 is proved.

3. Now we can easily obtain

Theorem 2. *Let G be a primitive extension with degree t of rank 3 of A_n . Then, one of the following holds.*

- (i) $n=1$, $t=3$ and G is isomorphic to the cyclic group of order 3.
- (ii) $n=3$, $t=7$ and G is isomorphic to the Frobenius group of order 21.
- (iii) $n=5$, $t=16$ and G is isomorphic to the semidirect product $A_5 N$, where N is an elementary abelian group of order 16. (see 4. (iv) in [4])
- (iv) $n=7$, $t=50$ and G is isomorphic to $U_3(5)$ (the 3-dimensional projective special unitary group over the finite field consisting of 5^2 elements).

In fact, in the case $n=7$, m must be $7 \cdot 2$, $7 \cdot 3$ or $7 \cdot 6$ by a theorem of Manning ([4], 2. Prop. 1). But it is impossible that m is equal to $7 \cdot 2$ by a theorem of Wielandt ([5], Th. 31. 2.) According to a theorem of Higman ([2], 3. Th.), m cannot be $7 \cdot 3$ and hence m must be $7 \cdot 6$ and G is isomorphic to $U_3(5)$ (see also (6.1) Th. in [2]).

Remark. By a theorem of Higman ([2], 3. Th.), The primitive extensions of rank 3 of S_n are exhausted by the groups listed in 4 of [4].

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