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A REMARK ON A STAR-SHAPED HYPERSURFACE WITH CONSTANT REDUCED MEAN CURVATURE

By

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The purpose of the present paper is to give another proof of the following proposition proved by A. Aeppli :

Proposition. *A star-shaped hypersurface F (with respect to a fixed point O) in an $(n+1)$ -dimensional Euclidean space R^{n+1} which has constant reduced mean curvature rH_1 (with respect to O) is a hypersphere around O , where $r=r(p)$ is the distance between O and $p \in F$, and $H_1=H_1(p)$ is the first mean curvature of F .*

Concerning the problem whether a closed hypersurface with $rH_1=\text{const}$ becomes a hypersphere around O or not, Aeppli gave the affirmative answer in the cases

- (1) F is star-shaped (cf [1]¹⁾ Theorem 4 and Footnote 11)),
- (2) $rH_1=1$ (cf [2] Proposition 1),
- (3) F is simple (i. e., without self-intersections) and O lies in the interior of F (cf [2] Proposition 1'').

The author showed already that under a certain condition (which he called "radial convexity") F cannot possess constant reduced mean curvature (cf [5]). In general, the problem of characterizing the hypersphere around O by constant reduced curvatures is not solved. The method of the proof used in this paper is based on "reflection and sliding" due to A. D. Alexandrov (cf [3] and [4]).

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1. Notations and Lemmas. Let F be a connected simple closed hypersurface of class C^2 in an $(n+1)$ -dimensional Euclidean space R^{n+1} , $n \geq 2$, and O be a fixed point in R^{n+1} . Let F be star-shaped with respect to O , i. e., there exists a bijective differentiable central projection from F to the unit hypersphere around O without critical points. Therefore O lies in the

1) Numbers in brackets refer to the references at the end of the paper.

interior of F , and $\mathbf{n}(p) \cdot \mathbf{r}(p) < 0$, where $\mathbf{n}(p)$ is the inner normal of F at p and $\mathbf{r}(p)$ is the radius vector from O to p , and any ray issued from O meets F only once (cf [1] Footnote 10) and [4] §2). In [2], Aepli uses the word "star-shaped" in a slightly weaker sense.

Let $H_\nu = H_\nu(p)$ ($p \in F$) be the ν -th mean curvature $H_\nu = ({}_nC_\nu)^{-1} \sum k_1 \cdots k_\nu$, where k_i 's are the principal curvatures. The ν -th reduced mean curvature with respect to O is $r^\nu H_\nu$. In [2], Aepli showed the following two lemmas:

Lemma 1. (In a local expression of $F: z = x^{n+1} = x^{n+1}(x^1, \dots, x^n)$) (a) $rH_1 = c$ ($c = \text{const}$) is an everywhere absolutely elliptic (partial) differential equation. (b) If $k_i > 0$ for all $i = 1, \dots, n$, then $r^\nu H_\nu = c$ is an absolutely elliptic differential equation.

Lemma 2. Let F_1 and F_2 be two (regular) surfaces in contact at p both of which are solutions of the absolutely elliptic (partial) differential equation (of second order) $\Phi = 0$ (F_1, F_2 of class C^2 ; $\Phi = \Phi(x, z, z_i, z_{ij})$ of class C^0 in all variables $x^1, x^2, \dots, z_{nn}$ and of class C^1 in $z, z_1, \dots, z_{nn}$). Then the intersection of F_1 and F_2 consists of a set N ($p \in N$) such that the contact at p between F_1 and F_2 is not semi-proper, or else F_1 and F_2 coincide in a neighbourhood of p .

Let R_P and E_t be two mappings from R^{n+1} onto itself defined as follows:

(1) R_P is the reflection with respect to a hyperplane P which passes through O ,

(2) $E_t(q)$ is equal to the radius vector $t \mathbf{r}(q)$ for all $q \in R^{n+1}$, that is, E_t is the t -times homothetical extension with respect to O , where t is a positive number.

It is clear that $r^\nu H_\nu$ is invariant under R_P and E_t for all P and t , if we take always the inner normals.

2. Another proof of the proposition. Let P be a hyperplane which passes through O . For sufficiently large positive number t , $(E_t \circ R_P)(F) \cap F = \emptyset$ and F lies in the interior of the hypersurface $(E_t \circ R_P)(F)$. Let t decrease until $(E_t \circ R_P)(F)$ touches F for the first time (consider a continuous function $\rho(p)$ on the compact set F , such that $\rho(p) = r(q)/r(p)$, where q is the unique point on $(E_t \circ R_P)(F)$ which lies on the ray issued from O through p , then the point at which ρ takes the minimum is a first common point such that $(E_t \circ R_P)(F)$ touches F). As we always consider inner normals, and because of the star-shapedness of $(E_t \circ R_P)(F)$ and F , the contact of F and $(E_t \circ R_P)(F)$ is positive and proper. Therefore by Lemma 1(a) and Lemma 2, F and $(E_t \circ R_P)(F)$ coincide in a neighbourhood of a point of contact, i.e., the set of the points of contact is open. By continuity it is also closed. Therefore

$$F = (E_t \circ R_P)(F)^{(2)}.$$

Let q be a point in $F \cap P$. Since $R_P(q) = q$, q and $(E_t \circ R_P)(q)$ lie on the same ray issued from O . The star-shapedness of F yields $q = (E_t \circ R_P)(q)$ because of $F = (E_t \circ R_P)(F)$, i. e., $q = E_t(q)$ and t must coincide with 1, and $E_t = E_1 = \text{identity}$. Therefore $F = R_P(F)$, in other words, F is symmetric with respect to P .

Since we can take P in any direction, F becomes a hypersphere around O . Q.E.D.

The proposition holds good for the ν -th reduced mean curvature $r^\nu H_\nu$, if $k_i > 0$ for all $i = 1, \dots, n$, by virtue of Lemma 1(b):

Let F be the same as in Proposition (except $rH_1 = c$). If $k_i > 0$ for all $i = 1, \dots, n$ and $r^\nu H_\nu = c$ for a fixed ν , then F is a hypersphere around O .

Remark 1. In the proof above we stated that a closed (star-shaped) hypersurface F which is symmetric with respect to any hyperplane through O is a hypersphere around O . For $n=2$ the proof of the analogous assertion is seen in [3]. For higher dimensions we can verify it without difficulty by induction (it is sufficient if we show that the intersection of F and the hyperplane through O is an $(n-1)$ -sphere around O with constant radius).

Remark 2. The more generalization of the proposition is given by A. D. Alexandrov in [4] §2 (Theorem 2). There he showed first that $c=1$ and concluded that F is a hypersphere.

References

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2) The coincidence of F and $(E_t \circ R_P)(F)$ is also obtained by the method based on the integral formula used in [1].