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ON NECESSARY AND SUFFICIENT CONDITIONS FOR L^2 -WELL-POSEDNESS OF MIXED PROBLEMS FOR HYPERBOLIC EQUATIONS

By

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1. Introduction.

Let $\mathbf{R}_+^n$ be the open half space $\{(x, y); x > 0, y \in \mathbf{R}^{n-1}\}$. We consider the mixed problems $(P, B_j; j = 1, \dots, l)$, denote simply it by (P, B_j) , for hyperbolic equations of order m :

$$\begin{aligned} (P(\partial_t, D_x, D_y)u)(t, x, y) &= f(t, x, y) && \text{in } (0, T) \times \mathbf{R}_+^n, \\ (B_j(\partial_t, D_x, D_y)u)(t, 0, y) &= 0 \quad (j = 1, \dots, l) && \text{in } (0, T) \times \mathbf{R}^{n-1}, \\ (\partial_t^k u)(0, x, y) &= 0 \quad (k = 0, 1, \dots, m-1) && \text{in } \mathbf{R}_+^n, \end{aligned}$$

where $\partial_t = \frac{\partial}{\partial t}$, $D_x = -i \frac{\partial}{\partial x}$, $D_y = \left(-i \frac{\partial}{\partial y_1}, \dots, -i \frac{\partial}{\partial y_{n-1}}\right)$ and $i = \sqrt{-1}$.

The purpose of this paper is to determine the necessary and sufficient conditions for well-posedness in the following sense:

Definition. *The mixed problem (P, B_j) is L^2 -well-posed with decreasing order ν (ν ; non negative integer) if and only if there exist positive constants C , T and T' with $T' \leq T$ which satisfy the following condition:*

For every $f \in H^{\nu+1,0}((-\infty, T) \times \mathbf{R}_+^n)^{(1)}$ with $f=0$ ($t < 0$) the mixed problem (P, B_j) has a unique solution $u \in H^m((0, T') \times \mathbf{R}_+^n)$ such that

$$(1.1) \quad \int_0^{T'} \|u(t, \cdot, \cdot)\|_{m-1}^2 dt \leq C \int_0^T \|f(t, \cdot, \cdot)\|_{\nu,0}^2 dt^{(2)}.$$

When $\nu=0$ we call it L^2 -well-posedness (with decreasing order 0).

The contents of this paper are as follows. In Section 2 we give a summary on boundary value problems for elliptic ordinary differential equations depending on parameters. In Section 3 we investigate the zeros of Lopatin-skii's determinant under the L^2 -well-posedness with decreasing order ν . In

1), 2) For the definitions see Section 2. Hereafter we denote various positive constants by C and C' , which are independent of variables in each inequality considered below respectively.

Section 4 we describe a certain necessary and sufficient condition for L^2 -well-posedness with decreasing order ν by the term of the compensating function, and moreover we describe it by the terms of the reflection coefficients in Section 5. In T. Shiota and K. Asano [9] it has been shown by semi-group method that the mixed problem (P, D_x^{2j-1}) ($m=2l$) is well posed in the stronger sense if $P(D)=P(\partial_t, D_x, D_y)$ does not contain the odd order terms in D_x . As one of the applications of the results in Section 5 we show that, in the case of constant coefficients, the property of $P(D)$ mentioned above is essential to be L^2 -well-posed (with decreasing order 0) for the mixed problem (P, D_x^{2j-1}) . This assertion is achieved in Section 6. Finally we present some examples in Section 7.

This paper contains the details of our previous paper [8].

2. Preliminaries.

First of all we state fundamental assumptions. Let $P(D)$ and $B_j(D)$ be *homogeneous* differential operators of order m and m_j ($m_j < m$) with *constant* coefficients respectively. We assume that $P(D)$ is *strictly hyperbolic with respect to t -direction* and *the hyperplane $x=0$ is non-characteristic for $P(D)$* . Then it is easily seen that the number $l(m-l)$ of the roots $\lambda_j^+(\tau, \sigma)$ ($\lambda_j^-(\tau, \sigma)$), located in the upper (lower) half λ -plane, in λ of the characteristic equation $P(\tau, \lambda, \sigma)=0$ is constant for any $(\tau, \sigma) \in \mathbf{C}_+ \times \mathbf{R}^{n-1}$ respectively, where $\mathbf{C}_+ = \{\tau \in \mathbf{C}; \operatorname{Re} \tau > 0\}$. Furthermore we assume that *the hyperplane $x=0$ is non-characteristic for $B_j(D)$ and $m_j > m_k$ if $j > k$* .

Throughout this paper we use the following Fourier-Laplace transforms and norms :

$$\begin{aligned} \hat{u}(\tau, \lambda, \sigma) &= \int_0^\infty dt \int_0^\infty dx \int_{\mathbf{R}^{n-1}} e^{-\tau t - i\lambda x - i\sigma y} u(t, x, y) dy, \\ \hat{u}(\tau, x, \sigma) &= \int_0^\infty dt \int_{\mathbf{R}^{n-1}} e^{-\tau t - i\sigma y} u(t, x, y) dy, \\ \|u(t, \cdot, \cdot)\|_k^2 &= \sum_{j=0}^k \|(\partial_t^j u)(t, \cdot, \cdot)\|_{k-j}^2, \\ \|u(t, \cdot, \cdot)\|_{k,l}^2 &= \sum_{j=0}^l \sum_{h+j+|\alpha|=0}^k \int_{\mathbf{R}^{n-1}} dy \int_0^\infty |(D_x^j \partial_t^h D_y^\alpha u)(t, x, y)|^2 dx, \\ \|\hat{u}(\tau, \cdot, \cdot)\|_k^2 &= \sum_{j=0}^k \int_{\mathbf{R}^{n-1}} (|\tau|^2 + |\sigma|^2)^{k-j} d\sigma \int_0^\infty |(D_x^j \hat{u})(\tau, x, \sigma)|^2 dx, \\ \|\hat{u}(\tau, \cdot, \cdot)\|_{k,l}^2 &= \sum_{j=0}^l \sum_{h=0}^{k-j} \int_{\mathbf{R}^{n-1}} (|\tau|^2 + |\sigma|^2)^{k-h} d\sigma \int_0^\infty |(D_x^j \hat{u})(\tau, x, \sigma)|^2 dx, \end{aligned} \quad (\tau \in \mathbf{C}_+)$$

where $\sigma y = \sigma_1 y_1 + \cdots + \sigma_{n-1} y_{n-1}$, $|\sigma|^2 = \sigma_1^2 + \cdots + \sigma_{n-1}^2$ and $\|\cdot\|_j$ is the norm in

Sobolev space $H^j(\mathbf{R}_+^n)$ (j ; a non negative integer). By $H^{k,l}((-\infty, T) \times \mathbf{R}_+^n)$ (k, l ; non negative integer) we understand the completion of $C_0^\infty((-\infty, T) \times \mathbf{R}_+^n)$ by the norm $\left(\int_{-\infty}^T \|u(t, \cdot, \cdot)\|_{k,l}^2 dt \right)^{\frac{1}{2}}$.

We define Lopatinskiĭ's determinant $R(\tau, \sigma)$ as follows :

$$B(\tau, \sigma) = \det (B_1(\tau, \lambda_1^+(\tau, \sigma), \sigma), \dots, B_l(\tau, \lambda_l^+(\tau, \sigma), \sigma); j \downarrow 1, \dots, l), \\ R(\tau, \sigma) = B(\tau, \sigma) / \prod_{1 \leq j < k \leq l} (\lambda_j^+(\tau, \sigma) - \lambda_k^+(\tau, \sigma)).$$

Then $R(\tau, \sigma)$ is analytic in $\mathbf{C}_+ \times \mathbf{R}^{n-1}$ and can be continuously extended to $\bar{\mathbf{C}}_+ \times \mathbf{R}^{n-1}$. Let V be the zeros of $R(\tau, \sigma)$ in $\mathbf{C}_+ \times \mathbf{R}^{n-1}$ and for every $\tau \in \mathbf{C}_+$ let $S(\tau)$ be the analytic variety $\{\sigma \in \mathbf{R}^{n-1}; (\tau, \sigma) \in V\}$ in $\mathbf{R}^{n-1}$. Then we have $aV = V$ and $aS(\tau) = S(a\tau)$ for every $a > 0$.

Applying now the Fourier-Laplace transform to equations in the mixed problem (P, B_j) we obtain the boundary value problem $(\hat{P}, \hat{B}_j)$ for elliptic ordinary differential equations depending on parameters $(\tau, \sigma) \in \mathbf{C}_+ \times \mathbf{R}^{n-1}$:

$$(P(\tau, D_x, \sigma) \hat{u})(\tau, x, \sigma) = \hat{f}(\tau, x, \sigma) \quad \text{in } \mathbf{R}_+^1, \\ (B_j(\tau, D_x, \sigma) \hat{u})(\tau, 0, \sigma) = 0 \quad (j=1, \dots, l).$$

Let $R_j(\tau, x, \sigma)$ be the determinat replacing j -column in $R(\tau, \sigma)$ by the transposed vector of $(\exp(ix\lambda_1^+(\tau, \sigma)), \dots, \exp(ix\lambda_l^+(\tau, \sigma)))$ and $\Gamma = \Gamma(\tau, \sigma)$ a closed Jordan curve in the lower half λ -plane enclosing all the roots $\lambda_k^-(\tau, \sigma)$ ($k=1, \dots, m-l$). If $R(\tau, \sigma) \neq 0$ for some $(\tau, \sigma) \in \mathbf{C}_+ \times \mathbf{R}^{n-1}$, it is well known that for every $\hat{f}(\tau, \cdot, \sigma) \in C_0^\infty(\mathbf{R}_+^1)$ the boundary value problem $(\hat{P}, \hat{B}_j)$ has a unique solution $\hat{u}(\tau, \cdot, \sigma) \in C^\infty(\bar{\mathbf{R}}_+^1)$, which is written in the following form :

$$(2.1) \quad \hat{u}(\tau, x, \sigma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\lambda x} \hat{f}(\tau, \lambda, \sigma)}{P(\tau, \lambda, \sigma)} d\lambda + \frac{1}{2\pi i} \int_0^\infty G(x, s, \tau, \sigma) \hat{f}(\tau, s, \sigma) ds,$$

$$\text{where } G(x, s, \tau, \sigma) = - \sum_{j=1}^l \frac{R_j(\tau, x, \sigma)}{R(\tau, \sigma)} \int_\Gamma \frac{B_j(\tau, \lambda, \sigma)}{P(\tau, \lambda, \sigma)} e^{-i s \lambda} d\lambda.$$

3. Zeros of Lopatinskiĭ's determinant.

In this section we investigate the zeros of Lopatinskiĭ's determinant $R(\tau, \sigma)$ under L^2 -well-posedness with decreasing order ν . The following theorem shows that if the mixed problem (P, B_j) is L^2 -well-posed with decreasing order ν then the zeros V of $R(\tau, \sigma)$ has the product representation $\mathbf{C}_+ \times S$, where S is the cone surface with vertex at the origin in $\mathbf{R}^{n-1}$.

Theorem 3.1. *If the mixed problem (P, B_j) is L^2 -well-posed with decreasing order ν , then the analytic varieties $S(\tau)$ don't depend on $\tau \in \mathbf{C}_+$.*

Proof. It suffices to prove that if $R(\tau, \sigma_0)$ is identically not zero for some $\sigma_0 \in \mathbf{R}^{n-1}$ then $R(\tau, \sigma_0) \neq 0$ for any $\tau \in \mathbf{C}_+$.

Using Weierstrass preparation theorem, if $R(\tau_0, \sigma_0) = 0$ for some $\tau_0 \in \mathbf{C}_+$ then there are a neighbourhood $U(\sigma_0)$ in $\mathbf{C}^{n-1}$ and a continuous (analytic) function $\tau(\sigma)$ in an open set $D \subset U(\sigma_0)$ such that $R(\tau(\sigma), \sigma) = 0$ and $\tau(\sigma) \in \mathbf{C}_+$ in D . Hence we can find continuous functions $a_j(\sigma)$ ($j=1, \dots, l$) in D such that for any $\sigma \in D$

$$(3.1) \quad (a_1(\sigma), \dots, a_l(\sigma)) \neq 0,$$

$$(3.2) \quad \sum_{j=1}^l a_j(\sigma) \beta_{h,j}(\tau(\sigma), \sigma) = 0 \quad (h=1, \dots, l),$$

where $\beta_{h,1}(\tau, \sigma) = B_h(\tau, \lambda_1^+(\tau, \sigma), \sigma)$,

$$\begin{aligned} \beta_{h,j}(\tau, \sigma) &= \int_0^1 d\theta_1 \cdots d\theta_{j-2} \int_0^1 \theta_1^{j-2} \cdots \theta_{j-2} (D_i^{j-1} B_h)(\tau, g_j(\tau, \sigma; \theta), \sigma) d\theta_{j-1}, \\ g_j(\tau, \sigma; \theta) &= \lambda_1^+(\tau, \sigma) + (\lambda_2^+(\tau, \sigma) - \lambda_1^+(\tau, \sigma)) \theta_1 + \cdots \\ &\quad \cdots + (\lambda_j^+(\tau, \sigma) - \lambda_{j-1}^+(\tau, \sigma)) \theta_1 \cdots \theta_{j-1} \quad (j=2, \dots, l). \end{aligned}$$

Note that we can determine branches λ_j^+ such that $\lambda_j^+(\tau(\sigma), \sigma)$ is continuous (analytic) in $D' \subset D$, because the hyperplane $x=0$ is non-characteristic for $P(D)$.

First we construct smooth solutions of the equations $(P(D)u)(t, x, y) = 0$ and $(B_j(D)u)(t, 0, y) = 0$, which don't satisfy the following estimate

$$(3.3) \quad \int_0^T \|u(t, \cdot, \cdot)\|_{m-1}^2 dt \leq C \|(\partial_t^k u)(0, \cdot, \cdot)\|_h^2,$$

where non negative integers h and k are arbitrarily fixed.

For this purpose we define the function

$$u(t, x, y) = \sum_{j=1}^l \int_{D''} a_j(\sigma) \gamma_j(\tau(\sigma), x, \sigma) \exp(\tau(\sigma)t + i\sigma y) d\sigma,$$

where $D'' = D' \cap \mathbf{R}^{n-1}$ which may be assumed to be not empty, $\gamma_1(\tau, x, \sigma) = \exp(ix\lambda_1^+(\tau, \sigma))$ and

$$\gamma_j(\tau, x, \sigma) = (ix)^{j-1} \int_0^1 d\theta_1 \cdots d\theta_{j-2} \int_0^1 \theta_1^{j-2} \cdots \theta_{j-2} \exp(ixg_j(\tau, \sigma; \theta)) d\theta_{j-1} \quad (j \geq 2).$$

For every positive integer p let us set $u_p(t, x, y) = u(pt, px, py)$. Then, by the homogeneity of $P(D)$ and $B^j(D)$ and (3.2), u_p is a solution of the equations above. From Plancherel theorem we obtain

$$(3.4) \quad \|(\partial_t^k u_p)(0, \cdot, \cdot)\|_h^2 \leq C \sum_{j=0}^h p^{2k+2j-n}.$$

On the other hand, since for each h $(D_x^h \gamma_j)(\tau(\sigma), x, \sigma)$ ($j=1, \dots, l$) are linearly independent as functions in x , it follows from (3.1) that $\sum_{j=1}^l a_j(\sigma) (D_x^h \gamma_j)(\tau(\sigma), x, \sigma) \equiv 0$ for any $\sigma \in D''$. Hence choosing sufficiently small D'' we have, by Plancherel theorem, for any positive T and p

$$(3.5) \quad \int_0^T \|u_p(t, \cdot, \cdot)\|_{m-1}^2 dt \geq C' e^{CTp} \sum_{k=1}^{m-1} p^{2k-n}.$$

By (3.4) and (3.5) u_p does not satisfy the estimate (3.3) for a sufficiently large p .

Next we construct a solution of the mixed problem (P, B_j) which does not satisfy the estimate (1.1). Using the above function u_p let us define for a large K

$$(3.6) \quad v_p(t, x, y) = \sum_{k=0}^K t^k (\partial_t^k u_p)(0, x, y) / k!.$$

Setting $w'_p(t, x, y) = \sum_{q=0}^L x^q f_{p,q}(t, y)$ ($L = \max m_j$), we determine the $f_{p,q}$ such that w'_p is a solution of the following equations:

$$(3.7) \quad \begin{aligned} (B_j(D)(w'_p - v_p))(t, 0, y) &= 0 & (j=1, \dots, l) & \quad \text{in } (0, T) \times \mathbf{R}^{n-1}, \\ (\partial_t^k (w'_p - v_p + u_p))(0, x, y) &= 0 & (k=0, 1, \dots, m-1) & \quad \text{in } \mathbf{R}_+^n. \end{aligned}$$

This is possible if $K > m + L$. In fact, substituting w'_p into the first equation in (3.7), the $f_{p,q}$ are inductively determined by the following forms:

$$(3.8) \quad \begin{aligned} f_{p,q}(t, y) &= 0 \quad \text{if } q \neq m_j \quad (j=1, \dots, l), \\ f_{p,m_1}(t, y) &= (B_1(D)v_p)(t, 0, y) / m_1!, \\ f_{p,m_j}(t, y) &= \left\{ - \sum_{h=1}^{j-1} m_h! A_{m_j-m_h}^j(\partial_t, D_y) f_{p,m_h}(t, y) + (B_j(D)v_p)(t, 0, y) \right\} / m_j! \\ & \quad (j \geq 2), \end{aligned}$$

where $B_j(D) = D_x^{m_j} + \sum_{d=0}^{m_j} A_d^j(\partial_t, D_y) D_x^{m_j-d}$. Note that w'_p satisfies then the second equation in (3.7). Let us set $w_p(t, x, y) = \varphi(x) w'_p(t, x, y)$, where $\varphi(x) \in C_0^\infty(\overline{\mathbf{R}_+^1})$ with $\varphi(x) = 1$ ($0 \leq x \leq \frac{1}{2}$) and $\varphi(x) = 0$ ($x \geq 1$). Then $w_p - v_p + u_p \in H^m((0, T) \times \mathbf{R}_+^n)$ is a solution of the mixed problem (P, B_j) (setting $f = P(D)(w_p - v_p + u_p)$) and satisfies $P(D)(w_p - v_p + u_p)(0, x, y) = 0$ in $\mathbf{R}_+^n$.

If the mixed problem (P, B_j) is L^2 -well-posed with decreasing order ν , then a solution $w_p - v_p + u_p$ must satisfy the estimate (1.1), that is,

$$\int_0^{T'} \| (w_p - v_p + u_p)(t, \cdot, \cdot) \|_{m-1}^2 dt \leq C \int_0^T \| (P(D)(w_p - v_p + u_p))(t, \cdot, \cdot) \|_{\nu,0}^2 dt.$$

Using $P(D)u_p = 0$ we obtain

$$(3.9) \quad \int_0^{T'} \|u_p(t, \cdot, \cdot)\|_{m-1}^2 dt \leq C \left(\int_0^T \|w_p(t, \cdot, \cdot)\|_{m+\nu}^2 dt + \int_0^T \|v_p(t, \cdot, \cdot)\|_{m+\nu}^2 dt \right).$$

Since we have from (3.8)

$$\|w_p(t, \cdot, \cdot)\|_{m+\nu}^2 \leq C \|v_p(t, \cdot, \cdot)\|_{m+L+1+\nu}^2,$$

it follows from (3.6) and (3.9) that

$$(3.10) \quad \int_0^{T'} \|u_p(t, \cdot, \cdot)\|_{m-1}^2 dt \leq C \sum_{k=0}^K \|(\partial_t^k u_p)(0, \cdot, \cdot)\|_{m+L+1+\nu}^2.$$

The right hand in (3.10) is polynomial order in p and the left hand in (3.10) is exponential order in p . Then, for a sufficiently large p , u_p is a required function. Thus the proof is complete.

Remark. The construction of the function $w_p - v_p$ is used in Section 4.

Corollary 3.2. *Suppose that $B_j(D)$ don't contain the terms in ∂_ν . If the mixed problem (P, B_j) is L^2 -well-posed with decreasing order ν , then $S(\tau)$ is empty for any $\tau \in \mathbf{C}_+$.*

Proof. Assume that $R(\tau_0, \sigma_0) = 0$ for some $(\tau_0, \sigma_0) \in \mathbf{C}_+ \times \mathbf{R}^{n-1}$. By Theorem 3.1 and the homogeneity of $R(\tau, \sigma)$ we obtain that $R(\tau, 0) = 0$ for any $\tau \in \mathbf{C}_+$. On the other hand, $R(\tau, 0)$ is written in the following form:

$$R(\tau, 0) = \frac{\det((ia_k \tau)^{m_1}, \dots, (ia_k \tau)^{m_l}; k \downarrow 1, \dots, l)}{\det((ia_k \tau)^0, \dots, (ia_k \tau)^{l-1}; k \downarrow 1, \dots, l)}$$

where the $ia_k \tau (\tau \in \mathbf{C}_+)$ are roots of $P(\tau, \lambda, 0) = 0$ with positive imaginary part. By the theory of characters of unitary groups³⁾ we obtain that $R(\tau, 0) \neq 0$ for every $\tau \neq 0$, here we use the assumption that $m_1 < \dots < m_l$. (see [10]). Thus the proof is complete.

By the first part of the proof of Corollary 3.2 we have

Corollary 3.3. *Suppose that $R(\tau_0, 0) \neq 0$ for some $\tau_0 \in \mathbf{C}_+$. If the mixed problem (P, B_j) is L^2 -well-posed with decreasing order ν , then V is empty.*

4. A certain necessary and sufficient condition for L^2 -well-posedness (I).

In this section we describe a certain necessary and sufficient condition for L^2 -well-posedness with decreasing order ν by the term of the compensating function $G(x, s, \tau, \sigma)$.

Let Σ_+ be the set $\{(\tau', \sigma') \in \mathbf{C}_+ \times \mathbf{R}^{n-1}; |\tau'|^2 + |\sigma'|^2 = 1\}$ and $\bar{\Sigma}_+$ its closure.

3) This method is suggested by Prof. L. Gårding.

Let $V' = V \cap \Sigma_+$, where V is the zeros of $R(\tau, \sigma)$ in $\mathbf{C}_+ \times \mathbf{R}^{n-1}$. When A is a subset in Σ_+ we denote the complement of A in Σ_+ by A^c .

Then we obtain the following main

Theorem 4.1. *Let $S(\tau) \equiv \mathbf{R}^{n-1}$ and $S(\tau) = S$ be independent of $\tau \in \mathbf{C}_+$. The mixed problem (P, B_j) is L^2 -well-posed with decreasing order ν if and only if the following condition (I) is satisfied:*

(I). *For every $(\tau'_0, \sigma'_0) \in (\bar{\Sigma}_+ - \Sigma_+) \cup V'$ there exist a neighbourhood $U(\tau'_0, \sigma'_0)$ and a constant $C(\tau'_0, \sigma'_0)$ such that for any $(\tau', \sigma') \in U(\tau'_0, \sigma'_0) \cap \Sigma_+ \cap V^c$*

$$(4.1) \quad \|(D_x^k G)(x, s, \tau', \sigma')\|_{\mathfrak{L}(L^2(s>0), L^2(x>0))} \leq C(\tau'_0, \sigma'_0) (\operatorname{Re} \tau')^{-\nu-1} \\ (k = 0, 1, \dots, m-1)$$

where $\|\cdot\|_{\mathfrak{L}(L^2(s>0), L^2(x>0))}$ is the operator norm from $L^2(s>0)$ to $L^2(x>0)$.

To prove Theorem 4.1 we need the following lemmas. In particular, Lemma 4.3 is essential. First we state an elementary lemma on strictly hyperbolic polynomial without proof.

Lemma 4.2. *If a homogeneous polynomial $P(\tau, \xi)$ of degree m is strictly hyperbolic with respect to τ , then we have for any $(\tau, \xi) \in \mathbf{C}_+ \times \mathbf{R}^n$*

$$|P(\tau, \xi)|^2 \geq C(\operatorname{Re} \tau)^2 (|\tau|^2 + |\xi|^2)^{m-1}.$$

Lemma 4.3. *If the assumption in Theorem 4.1 and the condition (I) are satisfied, then for every $(\tau, \sigma) \in \mathbf{C}_+ \times \mathbf{R}^{n-1}$ with $\sigma \notin S$ and $f \in H^{\nu+h+1, h}((-\infty, \infty) \times \mathbf{R}_+^n)$ (h ; a non negative integer) with $f=0$ ($t < 0$) the boundary value problem $(\hat{P}, \hat{B}_j)$ has a unique solution $\hat{u}(\tau, \cdot, \sigma) \in H^{m+h}(\mathbf{R}_+^1)$, which satisfies the following estimate*

$$(4.2) \quad (\operatorname{Re} \tau)^{2(\nu+1)} \|\hat{u}(\tau, \cdot, \cdot)\|_{m-1}^2 \leq C \|\hat{f}(\tau, \cdot, \cdot)\|_{\nu, 0}^2, \\ (\operatorname{Re} \tau)^{2(\nu+1)} \|\hat{u}(\tau, \cdot, \cdot)\|_{m+k}^2 \leq C \|\hat{f}(\tau, \cdot, \cdot)\|_{\nu+k+1, k}^2 \quad (k=0, 1, \dots, h)$$

for any $\tau \in \mathbf{C}_+$

Proof. Note that the notation of the norms has no ambiguity, because the measure of S is zero. Using the limit process and the relation $\hat{P}(\tau, D_x, \sigma) \hat{u}(\tau, x, \sigma) = \hat{f}(\tau, x, \sigma)$, it suffices to prove the first estimate of (4.2) for $f \in C_0^\infty((0, \infty) \times \mathbf{R}_+^n)$ and $\hat{u}$ defined in (2.1).

First let $\hat{u}_1$ be the first term in the right hand of (2.1). Then it follows from Lemma 4.2 and Plancherel theorem that

$$(\operatorname{Re} \tau)^2 \|\hat{u}_1(\tau, \cdot, \cdot)\|_{m-1}^2 \leq C \|\hat{f}(\tau, \cdot, \cdot)\|_0^2,$$

which implies the first estimate of (4.2) for $\hat{u}_1$.

Let U' be the set of all points (τ', σ') for which (4.1) is valid and U the set $\{(\tau, \sigma); (\tau', \sigma') \in U'\}$ where $\tau' = \tau \rho^{-1}$, $\sigma' = \sigma \rho^{-1}$ and $\rho = (|\tau|^2 + |\sigma|^2)^{\frac{1}{2}}$. We de-

compose $\hat{u} = \hat{u}_1 + \hat{u}_2 + \hat{u}_3$, where $\hat{u}_3$ is the function multiplying the characteristic function of U to the second term in the right hand of (2.1).

Next to prove the first estimate of (4.2) for $\hat{u}_2$ we remark the following facts. By the homogeneity of $P(D)$ and $B_j(D)$ we have, for every integer $k \geq 0$ and $(\tau, \sigma) \in U$,

$$(4.3) \quad \int_0^\infty \left| \frac{(D_x^k R_j)(\tau, x, \sigma)}{R(\tau, \sigma)} \right|^2 dx = \rho^{2k-2m_j-1} \int_0^\infty \left| \frac{(D_x^k R_j)(\tau', x, \sigma')}{R(\tau', \sigma')} \right|^2 dx.$$

If $K = K(\tau', \sigma')$ is the convex hull of the roots $\lambda_j^+(\tau', \sigma')$ ($j=1, \dots, l$), then we have

$$(4.4) \quad |(D_x^k R_j)(\tau', x, \sigma')| \leq \left\{ \sum_{h=0}^{l-1} \sup_{\lambda \in K} |D_i^h(\lambda^k e^{ix\lambda})|/h! \right\} \prod_{p \neq j} \left\{ \sum_{h=0}^{l-1} \sup_{\lambda \in K} |(D_i^h B_p)(\tau', \lambda, \sigma')|/h! \right\}.$$

By the fact that $R(\tau', \sigma') \geq C$ and $\mathbf{Im} \lambda_j^+(\tau', \sigma') \geq C$ for any $(\tau', \sigma') \in U'$, it follows from (4.3) and (4.4) that

$$(4.5) \quad \int_0^\infty \left| \frac{(D_x^k R_j)(\tau, x, \sigma)}{R(\tau, \sigma)} \right|^2 dx \leq C \rho^{2k-2m_j-1} \quad \text{for any } (\tau, \sigma) \in U.$$

Since $|P(\tau, \lambda, \sigma)|^2 \geq C(|\tau|^2 + |\lambda|^2 + |\sigma|^2)^m$ for any $(\tau, \sigma) \in U$ and $\lambda \in \mathbf{R}^1$, we have

$$(4.6) \quad \int_{-\infty}^\infty \left| \frac{B_j(\tau, \lambda, \sigma)}{P(\tau, \lambda, \sigma)} \right|^2 d\lambda \leq C \rho^{2m_j-2m+1} \quad \text{for any } (\tau, \sigma) \in U.$$

By Plancherel theorem and Schwarz inequality it follows from (4.5) and (4.6) that

$$\|\hat{u}_2(\tau, \cdot, \cdot)\|_m^2 \leq C \|\hat{f}(\tau, \cdot, \cdot)\|_0^2,$$

which implies the first estimate of (4.2) for $\hat{u}_2$.

Finally note that $(\bar{\Sigma}_+ - \Sigma_+) \cup V'$ is compact and

$$(4.7) \quad (D_x^k G)(x, s, \tau, \sigma) = \rho^{k-m+1} (D_x^k G)(\rho x, \rho s, \tau', \sigma') \quad (k=0, 1, \dots, m-1).$$

Using the relation (4.7), the condition (I) and the change of variables we obtain

$$\begin{aligned} \|\hat{u}_3(\tau, \cdot, \cdot)\|_{m-1}^2 &= \sum_{k=0}^{m-1} \int_{D(\tau)} \rho^{2(m-1-k)} d\sigma \int_0^\infty dx \left| \int_0^\infty (D_x^k G)(x, s, \tau, \sigma) \hat{f}(\tau, s, \sigma) ds \right|^2 \\ &= \sum_{k=0}^{m-1} \int_{D(\tau)} \rho^{-3} d\sigma \int_0^\infty dx \left| \int_0^\infty (D_x^k G)(x, s, \tau', \sigma') \hat{f}(\tau, s\rho^{-1}, \sigma) ds \right|^2 \\ &\leq C(\mathbf{Re} \tau)^{-2(\nu+1)} \int_{D(\tau)} \rho^{2\nu-1} d\sigma \int_0^\infty |\hat{f}(\tau, s\rho^{-1}, \sigma)|^2 ds \\ &\leq C(\mathbf{Re} \tau)^{-2(\nu+1)} \|\hat{f}(\tau, \cdot, \cdot)\|_{\nu,0}^2, \end{aligned}$$

where $D(\tau) = \{\sigma \in \mathbf{R}^{n-1}; (\tau, \sigma) \in U\}$. Thus the proof is complete.

By Lemma 4.3 and Paley-Wiener theorem (e.g. Theorem 7.1 in [2]) we obtain the following

Lemma 4.4. *If the assumption in Theorem 4.1 and the condition (I) is satisfied, then for every $a > 0$ and f with $e^{-at}f \in H^{\nu+h+1, h}((-\infty, \infty) \times \mathbf{R}_+^n)$ (h ; a non negative integer) and $f=0$ ($t < 0$) the mixed problem (P, B_j) (with $T=\infty$) has a unique solution u with $e^{-at}u \in H^{m+h}((0, \infty) \times \mathbf{R}_+^n)$ such that*

$$(4.8) \quad \begin{aligned} \int_0^\infty e^{-2at} \|u(t, \cdot, \cdot)\|_{m-1}^2 dt &\leq (C/a^{2(\nu+1)}) \int_0^\infty e^{-2at} \|f(t, \cdot, \cdot)\|_{\nu,0}^2 dt. \\ \int_0^\infty e^{-2at} \|u(t, \cdot, \cdot)\|_{m+k}^2 dt &\leq (C/a^{2(\nu+1)}) \int_0^\infty e^{-2at} \|f(t, \cdot, \cdot)\|_{\nu+k+1,k}^2 dt \\ &\quad (k=0, 1, \dots, h). \end{aligned}$$

The following lemma used in proof of the necessity of Theorem 4.1 is due to T. Okubo. The proof is easy.

Lemma 4.5. *Let f be a function in $H^{\nu,0}((-\infty, \infty) \times \mathbf{R}_+^n)$ with its support in $(0, T) \times \mathbf{R}_+^n$ and u a function satisfying $e^{-a't}u \in H^m((0, \infty) \times \mathbf{R}_+^n)$ for some $a' > 0$ and $(\partial_t^k u)(0, x, y) = 0$ ($k=0, 1, \dots, m-1$). Then the estimate*

$$(4.9) \quad \int_0^\infty e^{-2a't} \|u(t, \cdot, \cdot)\|_{m-1}^2 dt \leq C \int_0^\infty \|f(t, \cdot, \cdot)\|_{\nu,0}^2 dt$$

implies that

$$(4.10) \quad \|\hat{u}(\tau, \cdot, \cdot)\|_{m-1}^2 \leq C_0 C \int_{-\infty}^{+\infty} \|\hat{f}(a+i\eta, \cdot, \cdot)\|_{\nu,0}^2 d\eta$$

for any τ with $\operatorname{Re} \tau = a$ ($a > a'$), where a constant C_0 depends only on a, a' and the support of f .

Remark. If the mixed problem (P, B_j) is L^2 -well-posed with decreasing order ν , then for every $f \in H^{\nu+1,0}((-\infty, \infty) \times \mathbf{R}_+^n)$ with $f=0$ ($t < 0$) the mixed problem (P, B_j) (with $T=\infty$) has a unique solution u satisfying $e^{-at}u \in H^m((0, \infty) \times \mathbf{R}_+^n)$ for some $a > 0$ and the estimate (4.9). This follows from the fact that $P(D)$ and $B_j(D)$ are homogeneous and of constant coefficients.

Now we ready for the proof of Theorem 4.1. Our proof of the necessity is inspired by M. Ikawa [4].

Proof of Theorem 4.1. 1). Sufficiency. The existence follows immediately from Lemm 4.4. To prove the uniqueness we use an extension $\tilde{u}$ of u to $(0, \infty) \times \mathbf{R}_+^n$ which satisfies $e^{-at}\tilde{u} \in H^m((0, \infty) \times \mathbf{R}_+^n)$ for some $a > 0$ and $(B_j(D)\tilde{u})(t, 0, y) = 0$ in $(0, \infty) \times \mathbf{R}^{n-1}$. (For the existence of such an extension see Remark after the proof of Theorem 3.1. Here we use the fact that u may be considered as a sufficiently smooth function in (t, y) by the method of convolution). Then the uniqueness follows from Lemma 4.4 and Paley-

Wiener theorem.

2). Necessity. Suppose that the mixed problem (P, B_j) is L^2 -well-posed with decreasing order ν . Let f be a smooth function with its support in $(0, T) \times \mathbf{R}_+^n$. Then it follows from the Remark above that the estimate (4.10) is valid for the Fourier-Laplace transform $\hat{u}$ of a unique solution u of the mixed problem (P, B_j) (with $T = \infty$). We use notations in the proof of Lemma 4.3. Since the inverse Fourier-Laplace transform u_1 of $\hat{u}_1$ is a solution for Cauchy problem, by Lemma 4.5, the estimate (4.10) is valid for $\hat{u}_1$. Hence $\hat{u}_2 + \hat{u}_3$ must also satisfy (4.10), that is,

$$(4.11) \quad \|(\hat{u}_2 + \hat{u}_3)(\tau, \cdot, \cdot)\|_{m-1}^2 \leq C \int_{-\infty}^{+\infty} \|\hat{f}(a + i\eta, \cdot, \cdot)\|_{\nu,0}^2 d\eta$$

for any τ with $\operatorname{Re} \tau = a$.

If the condition (I) is not satisfied, then there exist a point $(\tau'_0, \sigma'_0) \in (\bar{\Sigma}_+ - \Sigma_+) \cup V'$, an integer $k_0 (0 \leq k_0 \leq m-1)$ and a sequence $\{\tau'_p, \sigma'_p\}$ ($p=1, 2, 3, \dots$) in $\Sigma_+ \cap V^c$ which converges to (τ'_0, σ'_0) such that

$$(4.12) \quad C_p = (\operatorname{Re} \tau'_p)^{\nu+1} \|(D_x^{k_0} G)(x, s, \tau'_p, \sigma'_p)\|_{\mathfrak{L}(L^2(s>0), L^2(x>0))}$$

tends to infinity if p does so.

First we take $g_p, \varphi_p \in C_0^\infty(R_+^1)$ which are identically not zero and satisfy

$$\begin{aligned} & \left| \int_0^\infty \overline{\varphi_p(x)} dx \int_0^\infty (D_x^{k_0} G)(x, s, \tau'_p, \sigma'_p) g_p(s) ds \right| \\ & \geq \frac{1}{2} \|\varphi_p\|_{L^2(\mathbf{R}_+^1)} \|g_p\|_{L^2(\mathbf{R}_+^1)} \|(D_x^{k_0} G)(x, s, \tau'_p, \sigma'_p)\|_{\mathfrak{L}(L^2(s>0), L^2(x>0))}. \end{aligned}$$

Since $\int_0^\infty \overline{\varphi_p(x)} dx \int_0^\infty (D_x^{k_0} G)(x, s, \tau', \sigma') g_p(s) ds$ is continuous at $(\tau'_p, \sigma'_p)^{(4)}$, there is $\theta_p > 0$ such that

$$(4.13) \quad \begin{aligned} & \left| \int_0^\infty \overline{\varphi_p(x)} dx \int_0^\infty (D_x^{k_0} G)(x, s, \tau', \sigma') g_p(s) ds \right| \\ & \geq \frac{1}{4} \|\varphi_p\|_{L^2(\mathbf{R}_+^1)} \|g_p\|_{L^2(\mathbf{R}_+^1)} \|(D_x^{k_0} G)(x, s, \tau'_p, \sigma'_p)\|_{\mathfrak{L}(L^2(s>0), L^2(x>0))} \end{aligned}$$

for any (τ', σ') with $|\sigma' - \sigma'_p| < \theta_p$ and $|\tau' - \tau'_p| < \theta_p$.

Let (τ_p, σ_p) be a point $(a/\operatorname{Re} \tau'_p)(\tau'_p, \sigma'_p)$ and J_p a set $\{(a/\operatorname{Re} \tau'_p)(\sigma'_p + \delta); |\delta| < \alpha_p < \theta_p\}$. Take a sufficiently small α_p such that if $\sigma \in J_p$ then $|\rho_p^{-1} \sigma - \sigma'_p| < \theta_p$, $|\rho_p^{-1} \tau_p - \tau'_p| < \theta_p$ and $C \leq (\operatorname{Re} \tau'_p) \rho_p^{-1} \leq C'$, where $\rho_p = (|\tau_p|^2 + |\sigma|^2)^{\frac{1}{2}}$.

Next take $\phi_p \in C_0^\infty(J_p)$ which is identically not zero. We choose $f \in C_0^\infty$

4) Regarding the continuity of the roots $\lambda_j(\tau, \sigma)$ we use the following classical fact: For a fixed point (τ_0, σ_0) there is a labelling of the roots $\lambda_j(\tau, \sigma)$ such that $\lambda_j(\tau, \sigma)$ is continuous at (τ_0, σ_0) .

$((0, T))$ such that $\hat{f}_p(\tau_p) = \int_0^\infty e^{-at} f(t) dt \neq 0$ if we set $f_p(t) = f(t) \exp(it \mathbf{Im} \tau_p)$. Then we have

$$\int_{-\infty}^{+\infty} |(a + i\eta)^j \hat{f}_p(a + i\eta)|^2 d\eta \leq \sum_{h=0}^j |\mathbf{Im} \tau_p|^{2h} \int_0^\infty e^{-2at} |(\partial_t^{j-h} f)(t)|^2 dt.$$

Replacing τ and $\hat{f}(a + i\eta, x, \sigma)$ in (4.11) by τ_p and $\hat{f}_p(a + i\eta) g_p(\rho_p x) \overline{\phi_p(\sigma)}$ respectively and multiplying $\left(\int_{J_p} d\sigma \int_0^\infty |\phi_p(\sigma) \varphi_p(\rho_p x)|^2 dx \right)^{\frac{1}{2}}$ to (4.11), it follows from Schwarz inequality that

$$\begin{aligned} & \left(\int_{-\infty}^{+\infty} d\eta \int_{J_p} d\sigma \int_0^\infty \sum_{j=0}^v (|a + i\eta|^2 + |\sigma|^2)^j |\hat{f}_p(a + i\eta) g_p(\rho_p x) \phi_p(\sigma)|^2 dx \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_{J_p} d\sigma \int_0^\infty |\phi_p(\sigma) \varphi_p(\rho_p x)|^2 dx \right)^{\frac{1}{2}} \\ & \geq C \left| \int_{J_p} \phi_p(\sigma) \rho_p^{m-1-k_0} d\sigma \int_0^\infty \varphi_p(\rho_p x) dx \right. \\ & \quad \left. \times \int_0^\infty (D_x^{k_0} G)(x, s, \tau_p, \sigma) \hat{f}_p(\tau_p) g_p(\rho_p s) \overline{\phi_p(\sigma)} ds \right|, \end{aligned}$$

which implies by the relation (4.7) and the change of variables that

$$\begin{aligned} & \|g_p\|_{L^2(\mathbf{R}_+^1)} \|\varphi_p\|_{L^2(\mathbf{R}_+^1)} \left(\sum_{j=0}^v \int_{-\infty}^\infty d\eta \int_{J_p} \rho_p^{-1} (|a + i\eta|^2 \right. \\ & \quad \left. + |\sigma|^2)^j |\hat{f}_p(a + i\eta) \phi_p(\sigma)|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_{J_p} \rho_p^{-1} |\phi_p(\sigma)|^2 d\sigma \right)^{\frac{1}{2}} \\ & \geq C \left| \hat{f}_p(\tau_p) \right| \left| \int_{J_p} \rho_p^{-2} |\phi_p(\sigma)|^2 d\sigma \int_0^\infty \varphi_p(x) dx \int_0^\infty (D_x^{k_0} G)(x, s, \tau_p \rho_p^{-1}, \sigma \rho_p^{-1}) g_p(s) ds \right|. \end{aligned}$$

From this and the choice of J_p we have

$$\begin{aligned} & \|g_p\|_{L^2(\mathbf{R}_+^1)} \|\varphi_p\|_{L^2(\mathbf{R}_+^1)} \|\phi_p\|_{L^2(\mathbf{R}^{n-1})}^2 \left(\sum_{j=0}^v \int_{-\infty}^\infty (|a + i\eta|^2 + (\mathbf{Re} \tau_p')^{-2})^j |\hat{f}_p(a + i\eta)|^2 d\eta \right)^{\frac{1}{2}} \\ (4.14) \quad & \geq C(\mathbf{Re} \tau_p') \left| \int_{J_p} |\phi_p(\sigma)|^2 d\sigma \int_0^\infty \varphi_p(x) dx \int_0^\infty (D_x^{k_0} G)(x, s, \tau_p \rho_p^{-1}, \sigma \rho_p^{-1}) g_p(s) ds \right|. \end{aligned}$$

By the choice of f_p and $\mathbf{Im} \tau_p = (\mathbf{Im} \tau_p') / \mathbf{Re} \tau_p'$ the left hand in (4.14) is estimated by

$$\leq C(\mathbf{Re} \tau_p')^{-\nu} \|g_p\|_{L^2(\mathbf{R}_+^1)} \|\varphi_p\|_{L^2(\mathbf{R}_+^1)} \|\phi_p\|_{L^2(\mathbf{R}^{n-1})}^2.$$

By the choice of g_p and φ_p the integrand in the right hand of (4.14) is not zero at (τ_p', σ_p') . Hence if the diameter of J_p is sufficiently small the right hand of (4.14) is estimated by

$$\geq (C/\sqrt{2})(\mathbf{Re} \tau'_p) \int_{J_p} |\phi_p(\sigma)|^2 d\sigma \left| \int_0^\infty \varphi_p(x) dx \int_0^\infty (D_x^k G)(x, s, \tau_p \rho_p^{-1}, \sigma \rho_p^{-1}) g_p(s) ds \right|,$$

where the constant C is same in (4.14). Therefore it follows from them, (4.12) and (4.13) that $1 \geq C C_p$. But this inequality is not valid for a sufficiently large p . Thus the proof is complete.

5. A certain necessary and sufficient condition for L^2 -well-posedness (II).

In this section we describe a certain necessary and sufficient condition for L^2 -well-posedness with decreasing order ν by the terms of the reflection coefficients. To achieve this purpose we first state the following condition introduced by S. Agmon [1].

Condition (#). *The multiplicity of a real root $\lambda(\tau, \sigma)$ in λ of the characteristic equation $P(\tau, \lambda, \sigma) = 0$ is at most double for every non zero (τ, σ) with $\mathbf{Re} \tau = 0$ and $\sigma \in \mathbf{R}^{n-1}$.*

To define the reflection coefficients, for every $(\tau'_0, \sigma'_0) \in (\bar{\Sigma}_+ - \Sigma_+) \cup V'$ we arrange the roots $\lambda_j^+(\tau', \sigma')$ into q groups $\{\lambda_{k,h}^+(\tau', \sigma') ; h=1, \dots, k'\}$ ($k=1, \dots, q$) in a sufficiently small neighbourhood $U(\tau'_0, \sigma'_0) \cap \Sigma_+$ such that $\{\lambda_{k,h}^+(\tau'_0, \sigma'_0) ; h=1, \dots, k'\}$ is k' -multiple root. Let $R_{j,k}(\tau', x, \sigma')$ be the determinant replacing the j -column in $R_j(\tau', x, \sigma')$ by the transposed vector of $(0, \dots, 0, \mathbf{exp}(ix\lambda_{k,1}^+(\tau', \sigma')), \dots, \mathbf{exp}(ix\lambda_{k,k'}^+(\tau', \sigma')), 0, \dots, 0)$. Since $R_j(\tau', x, \sigma') = \sum_{k=1}^q R_{j,k}(\tau', x, \sigma')$, we can define the generalized reflection coefficients $C_{k,h}(\tau', \lambda, \sigma')$ ($k=1, \dots, q ; h=1, \dots, k'$) by the following equality

$$(5.1) \quad \sum_{j=1}^l \frac{R_j(\tau', x, \sigma')}{R(\tau', \sigma')} B_j(\tau', \lambda, \sigma') = \sum_{k=1}^q \sum_{h=1}^{k'} C_{k,h}(\tau', \lambda, \sigma') \gamma_{k,h}(\tau', x, \sigma'),$$

where $\gamma_{k,1}(\tau', x, \sigma') = \mathbf{exp}(ix\lambda_{k,1}^+(\tau', \sigma'))$,

$$\gamma_{k,h}(\tau', x, \sigma') = (ix)^{h-1} \int_0^1 d\theta_1 \cdots d\theta_{j-2} \int_0^1 \theta_1^{h-2} \cdots \theta_{h-2} \mathbf{exp}(ixg_{k,h}(\tau', \sigma', \theta)) d\theta_{h-1},$$

$$g_{k,h}(\tau', \sigma' ; \theta) = \lambda_{k,1}^+(\tau', \sigma') + (\lambda_{k,2}^+(\tau', \sigma') - \lambda_{k,1}^+(\tau', \sigma')) \theta_1 + \cdots \\ \cdots + (\lambda_{k,h}^+(\tau', \sigma') - \lambda_{k,h-1}^+(\tau', \sigma')) \theta_1 \cdots \theta_{h-1} \quad (h \geq 2).$$

In particular, if $\lambda_j^+(\tau', \sigma')$ is simple in $U(\tau'_0, \sigma'_0) \cap \Sigma_+$, for example $\lambda_j^+(\tau'_0, \sigma'_0)$ ($\mathbf{Re} \tau'_0 = 0$) is real and the condition (#) is satisfied³⁾, then the generalized reflection coefficient is written in the following form:

$$(5.2) \quad C_j(\tau', \lambda, \sigma') = \tilde{B}_j(\tau', \lambda, \sigma') / B(\tau', \sigma'),$$

5) See Lemma 6.1.

where $\tilde{B}_j(\tau', \lambda, \sigma')$ is the determinant replacing $\lambda_j^+(\tau', \sigma')$ in $B(\tau', \sigma')$ by λ .

From Theorem 4.1 we obtain the following

Theorem 5.1. *Suppose that the condition (#) is satisfied, $S(\tau) \neq \mathbf{R}^{n-1}$ and $S=S(\tau)$ is independent of τ . The mixed problem (P, B_j) is L^2 -well-posed with decreasing order ν if and only if the following condition (II) is satisfied:*

(II). *For every $(\tau'_0, \sigma'_0) \in (\bar{\Sigma}_+ - \Sigma_+) \cup V'$ there exist a neighbourhood $U(\tau'_0, \sigma'_0)$ and a constant $C(\tau'_0, \sigma'_0)$ such that*

$$(5.3) \quad \left\| \int_{\Gamma} \frac{C_{k,h}(\tau', \lambda, \sigma')}{P(\tau', \lambda, \sigma')} e^{-i s \lambda} d\lambda \right\|_{L^2(\mathbf{R}_+^1)} \leq C(\tau'_0, \sigma'_0) (\mathbf{Im} \lambda_{k,h}^+(\tau', \sigma'))^{\frac{1}{2}} (\mathbf{Re} \tau')^{-\nu-1} \\ (k=1, \dots, q; h=1, \dots, k')$$

for any $(\tau', \sigma') \in U(\tau'_0, \sigma'_0) \cap \Sigma_+ \cap V'^c$.

Proof. 1). (II) implies (I). By (5.1), (5.3) and the definition of $G(x, s, \tau, \sigma)$ we obtain, for every $f \in L^2(\mathbf{R}_+^1)$,

$$(5.4) \quad \|(D_x^j G)(x, s, \tau', \sigma') f(s) ds\|_{L^2(\mathbf{R}_+^1)} \\ \leq \|f\|_{L^2(\mathbf{R}_+^1)} C(\tau'_0, \sigma'_0) (\mathbf{Re} \tau')^{-\nu-1} \sum_{k=0}^q \sum_{h=1}^{k'} \|(D_x^j \gamma_{k,h})(\tau', x, \sigma')\|_{L^2(\mathbf{R}_+^1)} (\mathbf{Im} \lambda_{k,h}^+(\tau', \sigma'))^{\frac{1}{2}}$$

in a sufficiently small neighbourhood $U(\tau'_0, \sigma'_0) \cap \Sigma_+ \cap V'^c$.

In $U(\tau'_0, \sigma'_0) \cap \Sigma_+ \cap V'^c$, if $\mathbf{Im} \lambda_{k,1}^+(\tau'_0, \sigma'_0) = 0$, by the condition (#), $\lambda_{k,1}^+(\tau', \sigma')$ is simple and hence we have $\|(D_x^j \gamma_{k,1})(\tau', x, \sigma')\|_{L^2(\mathbf{R}_+^1)} \leq C (\mathbf{Im} \lambda_{k,1}^+(\tau', \sigma'))^{-\frac{1}{2}}$, and furthermore if $\mathbf{Im} \lambda_{k,h}^+(\tau', \sigma') > 0$ we have $\|(D_x^j \gamma_{k,h})(\tau', x, \sigma')\|_{L^2(\mathbf{R}_+^1)} \leq C$ and $\mathbf{Im} \lambda_{k,h}^+(\tau', \sigma') \leq C$. Therefore the condition (I) follows from (5.4).

2). (I) implies (II). By the condition (I), the definition of $G(x, s, \tau, \sigma)$ and Schwarz inequality we obtain, for every $g, \varphi \in L^2(\mathbf{R}_+^1)$,

$$(5.5) \quad \|g\|_{L^2(\mathbf{R}_+^1)} \|\varphi\|_{L^2(\mathbf{R}_+^1)} C(\tau'_0, \sigma'_0) (\mathbf{Re} \tau')^{-\nu-1} \\ \geq \left| \sum_{k=1}^q \sum_{h=1}^{k'} \left(\int_0^\infty g(s) ds \int_{\Gamma} \frac{C_{k,h}(\tau', \lambda, \sigma')}{P(\tau', \lambda, \sigma')} e^{-i s \lambda} d\lambda \right) \left(\int_0^\infty \gamma_{k,h}(\tau', \lambda, \sigma') \varphi(x) dx \right) \right|$$

in a sufficiently small neighbourhood $U(\tau'_0, \sigma'_0) \cap \Sigma_+ \cap V'^c$.

If the condition (II) is not satisfied, then there exist a point $(\tau'_0, \sigma'_0) \in (\bar{\Sigma}_+ - \Sigma_+) \cup V'$, a pair of integers (k_0, h_0) ($1 \leq k_0 \leq q; 1 \leq h_0 \leq k'$) and a sequence $\{(\tau'_p, \sigma'_p)\}$ in $\Sigma_+ \cap V'^c$ which converges to (τ'_0, σ'_0) such that $C_p^{k_0, h_0}$ tends to infinity if p does so and

$$(5.6) \quad C_p^{k_0, h_0} / C_p^{k, h} \geq C > 0 \quad \text{for every } (k, h) \neq (k_0, h_0),$$

$$\text{where } C_p^{k, h} = \left\| \int_{\Gamma} \frac{C_{k,h}(\tau'_p, \lambda, \sigma'_p)}{P(\tau'_p, \lambda, \sigma'_p)} e^{-i s \lambda} d\lambda \right\|_{L^2(\mathbf{R}_+^1)} (\mathbf{Im} \lambda_{k,h}^+(\tau'_p, \sigma'_p))^{-\frac{1}{2}} (\mathbf{Re} \tau'_p)^{\nu+1}.$$

First we choose $g_p \in C_0(\mathbf{R}_+^1)$ which is identically not zero and satisfies

$$\begin{aligned}
(5.7) \quad & \left| \int_0^\infty g_p(s) ds \int_r \frac{C_{k,h}(\tau'_p, \lambda, \sigma'_p)}{P(\tau'_p, \lambda, \sigma'_p)} e^{-\epsilon s \lambda} d\lambda \right| \\
& \geq \frac{1}{2} \|g_p\|_{L^2(\mathbf{R}_+^1)} \left\| \int_r \frac{C_{k,h}(\tau'_p, \lambda, \sigma'_p)}{P(\tau'_p, \lambda, \sigma'_p)} e^{-\epsilon s \lambda} d\lambda \right\|_{L^2(\mathbf{R}_+^1)}.
\end{aligned}$$

Replacing τ' , σ' and g in (5.5) by τ'_p , σ'_p and g_p respectively, we obtain from (5.7) and the definition of $C_p^{k,h}$

$$\begin{aligned}
(5.8) \quad & \|\varphi\|_{L^2(\mathbf{R}_+^1)} \geq C \left\{ \frac{1}{2} C_p^{k_0, h_0} (\mathbf{Im} \lambda_{k_0, h_0}^+(\tau'_p, \sigma'_p))^{\frac{1}{2}} \left| \int_0^\infty \gamma_{k_0, h_0}(\tau'_p, x, \sigma'_p) \varphi(x) dx \right| \right. \\
& \quad \left. - \sum_{(k,h) \neq (k_0, h_0)} C_p^{k,h} (\mathbf{Im} \lambda_{k,h}^+(\tau'_p, \sigma'_p))^{\frac{1}{2}} \left| \int_0^\infty \gamma_{k,h}(\tau'_p, x, \sigma'_p) \varphi(x) dx \right| \right\}.
\end{aligned}$$

Next if $\mathbf{Im} \lambda_{k_0, h_0}^+(\tau_0, \sigma_0) = 0$ we take $\varphi_p(x) = \exp(i x \overline{\gamma_{k_0, h_0}(\tau'_p, \sigma'_p)})$. Then we have $\|\varphi_p\|_{L^2(\mathbf{R}_+^1)} = (2 \mathbf{Im} \lambda_{k_0, h_0}^+(\tau'_p, \sigma'_p))^{-\frac{1}{2}}$ and $\left| \int_0^\infty \gamma_{k_0, h_0}(\tau'_p, x, \sigma'_p) \varphi_p(x) dx \right| = (2 \mathbf{Im} \lambda_{k_0, h_0}^+(\tau'_p, \sigma'_p))^{-1}$.

Moreover the second term in the right hand of (5.8) is bounded. Hence it follows from them, (5.6) and (5.8) that $1 \geq C C_p^{k_0, h_0} \left(\frac{1}{2} - C' (\mathbf{Im} \lambda_{k_0, h_0}^+(\tau'_p, \sigma'_p)) \right)$.

By the choice of (k_0, h_0) this inequality is not valid for a sufficiently large p .

If $\mathbf{Im} \lambda_{k_0, h_0}^+(\tau'_0, \sigma'_0) > 0$ we can find $\varphi \in L^2(\mathbf{R}_+^1)$ which has a compact support and satisfies $\int_0^\infty \gamma_{k_0, h_0}(\tau'_0, x, \sigma'_0) \varphi(x) dx = 1$ and $\int_0^\infty \gamma_{k,h}(\tau'_0, x, \sigma'_0) \varphi(x) dx = 0$ for every $(k, h) \neq (k_0, h_0)$, because the $\gamma_{k,h}(\tau'_0, x, \sigma'_0)$ are linearly independent in $L^2(0, 1)$. By the continuity of the functions $\int_0^\infty \gamma_{k,h}(\tau', x, \sigma') \varphi(x) dx$ at (τ'_0, σ'_0) , we obtain, for a sufficiently large p ,

$$\begin{aligned}
(5.9) \quad & \int_0^\infty \gamma_{k_0, h_0}(\tau'_p, x, \sigma'_p) \varphi(x) dx \geq \frac{1}{2}, \\
& \int_0^\infty \gamma_{k,h}(\tau'_p, x, \sigma'_p) \varphi(x) dx \leq \epsilon_p^{k,h} \quad \text{if } (k, h) \neq (k_0, h_0),
\end{aligned}$$

where $\epsilon_p^{k,h}$ is sufficiently small. Therefore it follows from (5.8) and (5.9) that

$$\|\varphi\|_{L^2(\mathbf{R}_+^1)} \geq C C_p^{k_0, h_0} \left\{ \frac{1}{4} (\mathbf{Im} \lambda_{k_0, h_0}^+(\tau'_p, \sigma'_p))^{\frac{1}{2}} - \sum_{(k,h) \neq (k_0, h_0)} \epsilon_p^{k,h} (\mathbf{Im} \lambda_{k,h}^+(\tau'_p, \sigma'_p))^{\frac{1}{2}} \right\}.$$

By the choice of (k_0, h_0) this inequality is not valid for a sufficiently large p . Thus the proof is complete.

6. Applications.

In this section we prove S. Agmon's results in [1] and the interesting results stated in Introduction. First to prove S. Agmon's results we need

the following lemma which is implicitly contained in [7].

Lemma 6.1. *Let the condition (#) be satisfied. Then for every non zero (τ_0, σ_0) with $\operatorname{Re} \tau_0 = 0$ and $\sigma_0 \in \mathbf{R}^{n-1}$ there exists a neighbourhood $U(\tau_0, \sigma_0)$ such that*

1). *if a real root in λ of $P(\tau_0, \lambda, \sigma_0) = 0$ is simple then there is an analytic function $\lambda(\tau, \sigma)$ in $U(\tau_0, \sigma_0)$ which satisfies $P(\tau, \lambda(\tau, \sigma), \sigma) = 0$ and*

$$(6.1) \quad |\operatorname{Im} \lambda(\tau, \sigma)| \geq C(\operatorname{Re} \tau) \quad \text{in } U(\tau_0, \sigma_0).$$

2). *if a real root in λ of $P(\tau_0, \lambda, \sigma_0) = 0$ is strictly double then there are analytic functions $\lambda^\pm(\tau, \sigma)$ in $U(\tau_0, \sigma_0) \cap \Sigma_+$ which satisfy $P(\tau, \lambda^\pm(\tau, \sigma), \sigma) = 0$ and*

$$(6.2) \quad |\operatorname{Im} \lambda^\pm(\tau, \sigma)| \geq C(\operatorname{Re} \tau),$$

$$(6.3) \quad |\operatorname{Im} \lambda^+(\tau, \sigma) \operatorname{Im} \lambda^-(\tau, \sigma)| |\lambda^+(\tau, \sigma) - \lambda^-(\tau, \sigma)|^2 \geq C(\operatorname{Re} \tau)^2 \quad \text{in } U(\tau_0, \sigma_0) \cap \Sigma_+.$$

Then we obtain the following

Theorem 6.2. (*S. Agmon*) *Suppose that the condition (#) is satisfied. If $R(\tau, \sigma) \not\equiv 0$ for every non zero $(\tau, \sigma) \in \bar{C}_+ \times \mathbf{R}^{n-1}$, then the mixed problem (P, B_j) is L^2 -well-posed (with decreasing order 0).*

Proof. By the similar consideration in the proof of Theorem 5.1 and using that $R(\tau, \sigma) \not\equiv 0$ for every non zero $(\tau, \sigma) \in \bar{C}_+ \times \mathbf{R}^{n-1}$, we can show that the mixed problem (P, B_j) is L^2 -well-posed if and only if the following condition is satisfied:

For every $(\tau'_0, \sigma'_0) \in (\bar{\Sigma}_+ - \Sigma_+)$ there exist a neighbourhood $U(\tau'_0, \sigma'_0)$ and a constant $C(\tau'_0, \sigma'_0)$ such that, for any $(\tau', \sigma') \in U(\tau'_0, \sigma'_0) \cap \Sigma_+$,

$$(6.4) \quad |C_j(\tau', \lambda_k^-(\tau', \sigma'), \sigma')|^2 \leq C(\tau'_0, \sigma'_0) |\operatorname{Im} \lambda_j^+(\tau', \sigma') \operatorname{Im} \lambda_k^-(\tau', \sigma')| |(\partial_i P)(\tau', \lambda_k^-(\tau', \sigma'), \sigma')|^2 (\operatorname{Re} \tau')^{-2}$$

where the j and k satisfy the conditions $\operatorname{Im} \lambda_j^+(\tau'_0, \sigma'_0) = 0$ and $\operatorname{Im} \lambda_k^-(\tau'_0, \sigma'_0) = 0$ respectively.

First if $\lambda_k^-(\tau'_0, \sigma'_0)$ is simple then (6.4) is valid by (6.1) and (6.2). When $\lambda_k^-(\tau'_0, \sigma'_0)$ is strictly double we denote another branch by λ_k^+ (c.f. Lemma 6.1). Next if $\lambda_k^-(\tau'_0, \sigma'_0)$ is strictly double and $k \neq j$, by the fact that $|C_j(\tau', \lambda_k^-(\tau', \sigma'), \sigma')| \leq C|\lambda_k^+(\tau', \sigma') - \lambda_k^-(\tau', \sigma')|$ and $|(D_i P)(\tau', \lambda_k^-(\tau', \sigma'), \sigma')| \geq C|\lambda_k^+(\tau', \sigma') - \lambda_k^-(\tau', \sigma')|$, (6.4) is valid. Finally if $\lambda_k^-(\tau'_0, \sigma'_0)$ is strictly double and $j = k$ then (6.4) is valid by (6.3). Thus the proof is complete.

Next we prove the following

Theorem 6.3. *Let $Q(D)$ be a homogeneous differential operator, which does not contain the odd order terms in D_x , of order $m-1$ with constant*

coefficients and let $B_j(D) = D_x^{2j-1} (j=1, \dots, l; m=2l)$. If $P(D)$ satisfies the condition (#) and does not contain the odd order terms in D_x , then the mixed problem $(P(D) + \varepsilon D_x Q(D), B_j(D))$ is not L^2 -well-posed (with decreasing order 0) for a sufficiently small ε with certain fixed sign.

Proof. We may assume that, for a sufficiently small ε , $L_\varepsilon(D) = P(D) + \varepsilon D_x Q(D)$ is strictly hyperbolic and satisfies the condition (#). Furthermore the number of the roots $\lambda_j^+(\tau, \sigma) (\lambda_j^-(\tau, \sigma))$ of $L_\varepsilon(\tau, \lambda, \sigma) = 0$ is a constant l for any $(\tau, \sigma) \in \mathbf{C}_+ \times \mathbf{R}^{n-1}$. Since Lopatinskii's determinant written in the following form :

$$R(\tau, \sigma) = \lambda_1^+(\tau, \sigma) \cdots \lambda_l^+(\tau, \sigma) \prod_{1 \leq j < k \leq l} (\lambda_j^+(\tau, \sigma) + \lambda_k^+(\tau, \sigma)),$$

$S(\tau)$ is empty for any $\tau \in \mathbf{C}_+$.

Since $P(D)$ is strictly hyperbolic and $\deg_x Q < \deg_x P$, there is a point $(\tau'_0, \sigma'_0) \in (\bar{\Sigma}_+ - \Sigma_+)$ such that $P(\tau'_0, 0, \sigma'_0) = 0$ and $Q(\tau'_0, 0, \sigma'_0) \neq 0$. By the assumption of $P(D)$ we have $(\partial_x L_\varepsilon)(\tau'_0, 0, \sigma'_0) = \varepsilon Q(\tau'_0, 0, \sigma'_0) \neq 0$. Hence there exist a neighbourhood $U(\tau'_0, \sigma'_0)$ and a simple root, denote $\lambda_1^+(\tau', \sigma')$, in $U(\tau'_0, \sigma'_0)$ such that $\lambda_1^+(\tau'_0, \sigma'_0) = 0$. Here we may assume that $\text{Im } \lambda_1^+(\tau', \sigma') > 0$ in $U(\tau'_0, \sigma'_0) \cap \Sigma_+$, because one can change ε into $-\varepsilon$.

First we consider the case when there exists a root, denote $\lambda_1^-(\tau', \sigma')$, such that $\lambda_1^-(\tau'_0, \sigma'_0) \neq 0$ and $\text{Im } \lambda_1^-(\tau'_0, \sigma'_0) = 0$. Assume that $\text{Im } \lambda_k^-(\tau'_0, \sigma'_0) = 0 (k=1, \dots, h)$. Since $\lambda_k^-(\tau', \sigma') (k=1, \dots, h)$ are simple in a small neighbourhood $U(\tau'_0, \sigma'_0) \cap \Sigma_+$, we obtain

$$\begin{aligned} (6.5) \quad & \int_{\Gamma_1} \frac{C_1(\tau', \lambda, \sigma')}{L_\varepsilon(\tau', \lambda, \sigma')} e^{-\varepsilon s \lambda} d\lambda \\ &= \sum_{k=1}^h \frac{C_1(\tau', \lambda_k^-(\tau', \sigma'), \sigma')}{(\partial_x L_\varepsilon)(\tau', \lambda_k^-(\tau', \sigma'), \sigma')} e^{-\varepsilon s \lambda_k^-(\tau', \sigma')} + \int_{\Gamma_1} \frac{C_1(\tau', \lambda, \sigma')}{L_\varepsilon(\tau', \lambda, \sigma')} e^{-\varepsilon s \lambda} d\lambda, \end{aligned}$$

where Γ_1 is a closed Jordan curve in the lower half λ -plane enclosing all the roots $\lambda_j^-(\tau', \sigma') (j=h+1, \dots, l)$. Multiplying $\exp(-is\lambda_1^-(\tau', \sigma'))$ to (6.5), it follows from Schwarz inequality that

$$\begin{aligned} (6.6) \quad & \left\| \int_{\Gamma_1} \frac{C_1(\tau', \lambda, \sigma')}{L_\varepsilon(\tau', \lambda, \sigma')} e^{-\varepsilon s \lambda} d\lambda \right\|_{L^2(\mathbf{R}_+^1)} (2 \text{Im } \lambda_1^-(\tau', \sigma'))^{-\frac{1}{2}} \\ & \geq \left| \sum_{k=1}^h \frac{C_1(\tau', \lambda_k^-(\tau', \sigma'), \sigma')}{(\partial_x L_\varepsilon)(\tau', \lambda_k^-(\tau', \sigma'), \sigma')} (\lambda_k^-(\tau', \sigma') - \overline{\lambda_1^-(\tau', \sigma')})^{-1} \right| \\ & \quad - \int_{\Gamma_1} \frac{C_1(\tau', \lambda, \sigma')}{L_\varepsilon(\tau', \lambda, \sigma')} |\lambda - \overline{\lambda_1^-(\tau', \sigma')}|^{-1} d\lambda. \end{aligned}$$

By (5.2) we have

$$(6.7) \quad C_1(\tau', \lambda, \sigma') = \lambda \prod_{j \neq 1} (\lambda^2 - \lambda_j^+(\tau', \sigma')^2) / \left\{ \lambda_1^+(\tau', \sigma') \prod_{j \neq 1} (\lambda_1^+(\tau', \sigma')^2 - \lambda_j^+(\tau', \sigma')^2) \right\}.$$

Then it follows from (6.6) and (6.7) that

$$(6.8) \quad \left\| \int_r \frac{C_1(\tau', \lambda, \sigma')}{L_\epsilon(\tau', \lambda, \sigma')} e^{-\epsilon s \lambda} d\lambda \right\|_{L^2(\mathbb{R}_+^1)} \geq C |\lambda_1^+(\tau', \sigma')|^{-1} \left\{ |\lambda_1^-(\tau', \sigma')| |\mathbf{Im} \lambda_1^-(\tau', \sigma')|^{-\frac{1}{2}} - C' |\mathbf{Im} \lambda_1^-(\tau', \sigma')|^{\frac{1}{2}} \right\}.$$

If (5.3) in Theorem 5.1 is valid for $C_1(\tau', \lambda, \sigma')$ then we have from (6.8)

$$|\mathbf{Im} \lambda_1^+(\tau', \sigma') \mathbf{Im} \lambda_1^-(\tau', \sigma')|^{\frac{1}{2}} (\mathbf{Re} \tau')^{-1} \geq C |\lambda_1^+(\tau', \sigma')|^{-1} (|\lambda_1^-(\tau', \sigma')| - C' |\mathbf{Im} \lambda_1^-(\tau', \sigma')|).$$

Using the fact that $|\lambda_1^+(\tau', \sigma')| \leq C(\mathbf{Re} \tau')$ in a small neighbourhood $U(\tau'_0, \sigma'_0)$, this inequality is not valid for (τ', σ') sufficiently close to (τ'_0, σ'_0) .

Next we consider the case when $\mathbf{Im} \lambda_j^-(\tau'_0, \sigma'_0) < 0$ for all $j = 1, \dots, l$. In a sufficiently small neighbourhood $U(\tau'_0, \sigma'_0) \cap \Sigma_+$, we arrange the roots $\lambda_j^-(\tau', \sigma')$ into r groups $\{\lambda_{k,h}^-(\tau', \sigma') ; h = 1, \dots, k'\}$ ($k = 1, \dots, r$) such that $\{\lambda_{k,h}^-(\tau'_0, \sigma'_0) ; h = 1, \dots, k'\}$ is k' -multiple root. By the condition (#) there is a simple root, denote $\lambda_i^-(\tau', \sigma')$, in a small neighbourhood $U(\tau'_0, \sigma'_0) \cap \Sigma_+$. Let Γ_k ($k = 1, \dots, r$) be a sufficiently small and closed Jordan curve in the lower half λ -plane enclosing all the roots $\lambda_{k,h}^-(\tau', \sigma')$ ($h = 1, \dots, k'$). Then we can find $g \in L^2(\mathbb{R}_+^1)$ such that

$$(6.9) \quad \left| \int_0^\infty e^{-\epsilon s \lambda} g(s) ds \right| \geq \frac{1}{2} \text{ on } \Gamma_1 \text{ and } \left| \int_0^\infty e^{-\epsilon s \lambda} g(s) ds \right| \leq \epsilon_k \text{ on } \Gamma_k \text{ for } k \neq 1.$$

By Schwarz inequality we obtain

$$(6.10) \quad \|g\|_{L^2(\mathbb{R}_+^1)} \left\| \int_r \frac{C_1(\tau', \lambda, \sigma')}{L_\epsilon(\tau', \lambda, \sigma')} e^{-\epsilon s \lambda} d\lambda \right\|_{L^2(\mathbb{R}_+^1)} \geq \left| \int_{\Gamma_1} \frac{C_1(\tau', \lambda, \sigma')}{L_\epsilon(\tau', \lambda, \sigma')} d\lambda \int_0^\infty e^{-\epsilon s \lambda} g(s) ds - \sum_{k \neq 1} \left| \int_{\Gamma_k} \frac{C_1(\tau', \lambda, \sigma')}{L_\epsilon(\tau', \lambda, \sigma')} d\lambda \int_0^\infty e^{-\epsilon s \lambda} g(s) ds \right| \right|.$$

If (5.3) in Theorem 5.1 is valid for $C_1(\tau', \lambda, \sigma')$ then we have, by (6.7), (6.9) and (6.10),

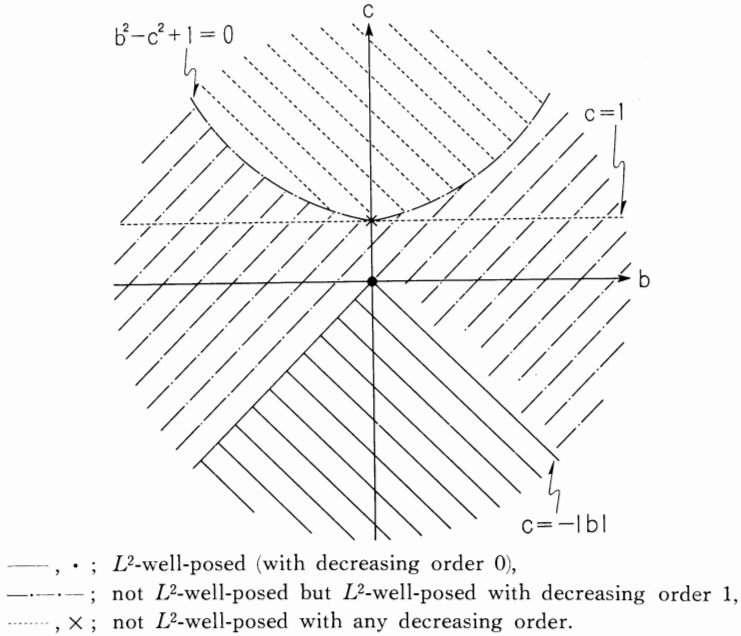
$$\|g\|_{L^2(\mathbb{R}_+^1)} (\mathbf{Im} \lambda_1^+(\tau', \sigma'))^{\frac{1}{2}} (\mathbf{Re} \tau')^{-1} \geq C |\lambda_1^+(\tau', \sigma')|^{-1} (1 - \epsilon_k).$$

By the fact that $|\lambda_1^+(\tau', \sigma')| \leq C(\mathbf{Re} \tau')$ in a small neighbourhood $U(\tau'_0, \sigma'_0)$, this inequality is not valid for (τ', σ') sufficiently close to (τ'_0, σ'_0) . Thus the proof is complete.

7. Examples.

In this section we present some examples.

1). $P(D) = \partial_t^2 - \Delta$ and $B(D) = D_x + b D_y - ic \partial_t$, where Δ is Laplacian in $\mathbb{R}_x^2$ and b and c are real. Then we have the following classification:



Proof. In this case, $\lambda^\pm(\tau, \sigma) = \pm i(\tau^2 + \sigma^2)^{\frac{1}{2}}$ where it is assumed that $(\tau^2 + \sigma^2)^{\frac{1}{2}}$ has a positive real part if $\tau \in \mathbf{C}_+$, Lopatinskii's determinant $R(\tau, \sigma) = i(\tau^2 + \sigma^2)^{\frac{1}{2}} + b\sigma - ic\tau$ and reflection coefficient $C(\tau, \lambda^-(\tau, \sigma), \sigma) = (-i(\tau^2 + \sigma^2)^{\frac{1}{2}} + b\sigma - ic\tau)/R(\tau, \sigma)$. Hence, it is the necessary and sufficient condition for L^2 -well-posedness with decreasing order ν that the following inequality holds in a small neighbourhood in $\Sigma_+ \cap V^c$ of any $(\tau'_0, \sigma'_0) \in (\bar{\Sigma}_+ - \Sigma_+) \cup V'$

$$(7.1) \quad |C(\tau', \lambda^-(\tau', \sigma'), \sigma')|^2 \leq C |\mathbf{Im} \lambda^+(\tau', \sigma')| |\mathbf{Im} \lambda^-(\tau', \sigma')| |(\partial_x P)(\tau', \lambda^-(\tau', \sigma'), \sigma')|^2 (\mathbf{Re} \tau')^{-2\nu-2}.$$

Remark that, by Lemma 6.1, the right hand of (7.1) for $\nu=0$ is bounded below.

If $c < -|b|$, $R(\tau, \sigma) \neq 0$ for $(\tau, \sigma) \in \bar{\mathbf{C}}_+ \times \mathbf{R}^1$. By Theorem 6.2, the mixed problem (P, B) is then L^2 -well-posed. If $c = -|b|$, $R(\tau, \sigma) \neq 0$ for $(\tau, \sigma) \in \mathbf{C}_+ \times \mathbf{R}^1$ and $R(\tau'_0, \sigma'_0) = 0$ for some $(\tau'_0, \sigma'_0) \in (\bar{\Sigma}_+ - \Sigma_+)$ but $C(\tau', \lambda^-(\tau', \sigma'), \sigma')$ is bounded in a neighbourhood of (τ'_0, σ'_0) . By the remark above and (7.1) for $\nu=0$, the mixed problem (P, B) is L^2 -well-posed. If $b^2 - c^2 + 1 < 0$ and $c > 0$, $S(\tau)$ depends on $\tau \in \mathbf{C}_+$. Hence, by Theorem 3.1, the mixed problem (P, B) is not L^2 -well-posed with any decreasing order. If $c = 1$, $S(\tau) = \{0\}$ but (7.1) for any ν is not valid in a neighbourhood of $(\tau'_0, 0)$ ($\tau'_0 \in \mathbf{C}_+$). Hence the mixed problem (P, B) is not L^2 -well-posed with any decreasing order. In the other case, $R(\tau, \sigma) \neq 0$

for $(\tau, \sigma) \in \mathbf{C}_+ \times \mathbf{R}^1$, $R(\tau'_0, \sigma'_0) = 0$ for some $(\tau'_0, \sigma'_0) \in (\bar{\Sigma}_+ - \Sigma_+)$ and the numerator of $C(\tau', \lambda^-(\tau', \sigma'), \sigma')$ is not zero at (τ'_0, σ'_0) . Hence the mixed problem (P, B) is not L^2 -well-posed. But, by the fact that $|R(\tau', \sigma')| \geq C(\mathbf{Re} \tau')$ in a neighbourhood of (τ'_0, σ'_0) , the mixed problem (P, B) is L^2 -well-posed with decreasing order 1.

2). $P(D) = (\partial_t^2 - a_1 \Delta)(\partial_t^2 - a_2 \Delta)$ where Δ is Laplacian in $\mathbf{R}^n$ and a_1 and a_2 are positive and distinct.

If $B_1(D) = I$ and $B_2(D) = D_x$ then, by Theorem 6.2, the mixed problem (P, B_j) is L^2 -well-posed. If $B_1(D) = I$ and $B_2(D) = D_x^2$ or $B_1(D) = D_x$ and $B_2(D) = D_x^3$, by Theorem 5.1 and more precisely taking the residue in (5.3) (c.f. (6.4)), then the mixed problem (P, B_j) is L^2 -well-posed. If $B_1(D) = D_x$ and $B_2(D) = D_x^2$ or $B_1(D) = D_x^2$ and $B_2(D) = D_x^3$, then the mixed problem (P, B_j) is not L^2 -well-posed but L^2 -well-posed with decreasing order 1. Finally if $B_1(D) = I$ and $B_2(D) = D_x^3$, then the mixed problem (P, B_j) is not L^2 -well-posed with any decreasing order by Corollary 3.2.

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