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A NOTE ON S^1 -ACTING COBORDISMS

By

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Introduction

Let Y be a connected compact oriented differentiable manifold (without boundary). Suppose that S^1 acts on Y differentiably and effectively without fixed points. This means that no point of Y is left fixed by the whole group, that is $Y^{S^1} = \emptyset$. Let Y' be another manifold which has the properties of Y and let W be a compact differentiable manifold, such that $\partial W = Y \cup (-Y')$ and S^1 -action on ∂W extends differentiably all over W . Suppose also that we have $W^{S^1} = \emptyset$ for the S^1 -action on W . If Y and Y' satisfy the above condition, we say that they are S^1 -acting cobordant or they belong to the same S^1 -acting cobordism class.

Any differentiable principal S^1 -bundle over a differentiable manifold is a manifold on which S^1 acts differentiably and effectively without fixed points by the multiplication of complex numbers in fibre. We shall consider the S^1 -acting cobordism for Y and Y' which have the differential principal S^1 -bundle structures over differentiable structures of a connected compact manifold M and have the S^1 -actions stated in the above. It should be noted that if Y and Y' are S^1 -acting cobordant by the manifold W , the S^1 -action on W is not necessarily free, that is, Y and Y' do not necessarily belong to the same equivariant cobordism class in the usual sense. By imposing some cohomology conditions upon M , Y and Y' have unique spin structures. The purpose of this note is to show, directly by the powerful invariant ρ of M. F. Atiyah and F. Hirzebruch [2], that when W has a spin-structure which is compatible with those of Y and Y' and the dimension of M is $2k$, the principal S^1 -bundle structures of Y and Y' are isomorphic. Finally we shall show several examples.

§ 1. Statement of results

Let M be a connected compact oriented differentiable manifold of dimension $2k$ such that

$$(1) \quad H^1(M; \mathbb{Z}_2) = 0, \quad H^2(M; \mathbb{Z}) = \mathbb{Z}, \quad w_2(M) \neq 0.$$

Suppose that if g is a generator of $H^2(M; \mathbb{Z})$, $g^k \in H^{2k}(M; \mathbb{Z})$ is not zero.

We regard $S^1=SO(2)$ as the multiplicative group which consists of the complex numbers with absolute value 1. In the following, we assume complex line bundles have structural group $U(1)=SO(2)$. The differentiable S^1 -bundles are in 1-1 correspondence with the differentiable complex line bundles over the same base space. A complex line bundle ξ is characterized topologically by its first Chern class $c=c_1(\xi)$ which is of the form ag where a is an integer. We assume that ξ is a differentiable complex line bundle and we denote by Y_ξ the total space of the principal S^1 -bundle corresponding to ξ and denote by X_ξ the total space of the closed unit disk bundle corresponding to ξ . Y_ξ is a closed connected oriented differentiable manifold on which $S^1=\{z||z|=1\}$ acts differentiably and effectively without fixed points by the multiplication of z^m in fibre, for some integer $m \neq 0$. We call this action a *natural S^1 -action* on Y_ξ for m . X_ξ is a connected compact oriented differentiable manifold with the boundary Y_ξ , on which S^1 acts also non-trivially. We notice that if ξ' is the complex line bundle having the first Chern class $c_1(\xi')=-ag$, S^1 -bundles Y_ξ , $Y_{\xi'}$ are isomorphic by the bundle map over the identity of M , which reverses the orientation of fibre.

We assume that $c_1(\xi) \equiv 0 \pmod{2}$. Then from the assumption (1), it follows that

$$H^1(Y_\xi; Z_2) = 0, \quad w_2(Y_\xi) = 0,$$

and

$$H^1(X_\xi; Z_2) = 0, \quad w_2(X_\xi) = 0.$$

Therefore Y_ξ has (for some Riemannian metric) exactly one spin-structure up to isomorphism and X_ξ also does, by the arguments of M.F. Atiyah and R. Bott [1], p. 480. We denote by $\mathcal{S}_M$ the set of S^1 -bundles Y_ξ over differentiable structures of M such that $c_1(\xi)=ag$, $a \equiv 0 \pmod{2}$ and moreover if k is even, a is positive.

Let Y and Y' be connected compact oriented differentiable manifolds such that

$$H^1(Y; Z_2) = H^1(Y'; Z_2) = 0, \quad w_2(Y) = w_2(Y') = 0.$$

And suppose that S^1 acts on them differentiably and effectively without fixed points. We say that Y and Y' are *S^1 -acting spin-cobordant* or they belong to the same *S^1 -acting spin-cobordism class*, if there exists a compact differentiable manifold W on which S^1 acts differentiably and effectively without fixed points and

$$H^1(W; Z_2) = 0, \quad w_2(W) = 0,$$

$$\partial W = Y \cup (-Y').$$

This definition of the S^1 -acting spin-cobordism is essentially due to M.F.

Atiyah and F. Hirzebruch [2].

Now we state the following main theorem.

Theorem 1.1. *Any two S^1 -bundles of $\mathcal{S}_M$ are isomorphic if and only if they belong to the same S^1 -acting spin-cobordism class up to topological equivalences, with respect to some natural S^1 -actions.*

Remark (1) For the S^1 -equivariant oriented cobordism class, the theorem is a direct consequence of the invariance of the Chern number $(c_1)^k[M]$, but we are dealing with the S^1 -acting spin-cobordism class. (2) Suppose that k is even. Let ξ and ξ' be differentiable complex line bundles such that $c_1(\xi) = ag$, $w_2(M) = ag \pmod{2}$, $a < 0$ and $c_1(\xi') = a'g$, $w_2(M) = a'g \pmod{2}$, $a' > 0$. Reversing the orientation of fibre of $Y_{\xi'}$, we obtain the differentiable S^1 -bundle $Y_{\xi''}$ such that $c_1(\xi'') = (-a')g$. By the above theorem Y_{ξ} and $Y_{\xi''}$ are S^1 -bundle isomorphic if and only if they belong to the same S^1 -acting spin-cobordism class with respect to natural S^1 -actions.

To prove the Theorem 1.1, we obtain, in §2, some results about values of local spin-numbers for the S^1 -action on X_{ξ} . We shall give a proof of the main theorem and some examples in §3.

§2. The invariant $\rho(z, Y)$

Let Y be the connected compact oriented differential manifold (without boundary) of dimension $2k+1$ such that $H^1(Y; \mathbb{Z}_2) = 0$, $w_2(Y) = 0$. Suppose that S^1 acts on Y differentiably and effectively without fixed points and suppose that Y bounds a connected compact oriented differentiable S^1 -manifold X such that $H^1(X; \mathbb{Z}_2) = 0$, $w_2(X) = 0$. We denote by $X_{\nu}^{S^1}$ a component of the fixed point set for the S^1 -action on X . By the definition of M.F. Atiyah and F. Hirzebruch [2] or M.F. Atiyah and I.M. Singer [3], we have

$$(2) \quad \rho(z, Y) = \sum_{\nu} \text{spin}(z, X_{\nu}^{S^1}),$$

which does not depend on the choice of X .

For the S^1 -manifold X_{ξ} in the preceding section, it is easily verified that the fixed point set $(X_{\xi})^{S^1}$ is the zero section of the complex line bundle ξ . Hence it is diffeomorphic to the manifold M of base space and is connected. The normal bundle $N((X_{\xi})^{S^1})$ of $(X_{\xi})^{S^1}$ in X_{ξ} is isomorphic to ξ and $z \in S^1$ operates as a multiplication of a complex number z^m in fibre, for the integer m . The eigenvalue of the operation of z in the fibre is z^m and for $z \in S^1$ but z not a root of unity, we have

$$(3) \quad \text{spin}(z, (X_{\xi})^{S^1}) = (-1)^{(k+1)} \mathfrak{A}(M) (z^{-m/2} e^{c/2} - z^{m/2} e^{-c/2})^{-1} [M]$$

by M. F. Atiyah and F. Hirzebruch [2], where $\hat{\mathfrak{U}}$ is the multiplicative sequence with the characteristic series

$$\frac{x/2}{\sinh x/2} = \frac{x}{e^{x/2} - e^{-x/2}}$$

and $c = c_1(\xi)$ is the first Chern class of ξ . c can be written as ag where g is a generator of $H^2(M; \mathbb{Z})$ and $a \in \mathbb{Z}$. By the definition of $\hat{\mathfrak{U}}$, we have

$$(4) \quad \hat{\mathfrak{U}}(M) = \sum_{r=0}^{\infty} \hat{A}_r(p_1(M), \dots, p_r(M))$$

where $\hat{A}_r(p_1(M), \dots, p_r(M)) \in H^{4r}(M; \mathbb{Q})$ are polynomials in the Pontrjagin classes $p_i(M)$. We notice that the right hand side of (3) is a rational function on the complex number plane C if m is even and it is a function on the double branched covering of C if m is odd.

To distinguish values of $\rho(z, Y_i)$, we need the following lemma.

Lemma 2.1. *We have*

$$(5) \quad (z^{-m/2}e^{c/2} - z^{m/2}e^{-c/2})^{-1} = (z^{-m/2} - z^{m/2})^{-1} (1 + \alpha_1 c + \dots + \alpha_k c^k),$$

where α_i is a polynomial of order i , in the variable

$$z' = (z^{-m/2} + z^{m/2})(z^{-m/2} - z^{m/2})^{-1}.$$

Proof. From direct computations, it follows that

$$\begin{aligned} & (z^{-m/2}e^{c/2} - z^{m/2}e^{-c/2})^{-1} \\ &= (z^{-m/2} - z^{m/2})^{-1} \left(1 + \frac{z'}{2}c + \frac{1}{2^2 2!}c^2 + \dots + \frac{z'^{(k-2\lfloor k/2 \rfloor)}}{2^k k!}c^k \right)^{-1}. \end{aligned}$$

We put

$$\left(1 + \frac{z'}{2}c + \frac{1}{2^2 2!}c^2 + \dots + \frac{z'^{(k-2\lfloor k/2 \rfloor)}}{2^k k!}c^k \right)^{-1} = 1 + \alpha_1 c + \alpha_2 c^2 + \dots + \alpha_k c^k.$$

Then α_i is determined uniquely by the following formulas:

$$\begin{aligned} \alpha_1 + \frac{z'}{2} &= 0, \\ \alpha_2 + \alpha_1 \left(\frac{z'}{2} \right) + \frac{1}{2^2 2!} &= 0, \\ &\dots\dots\dots \\ \alpha_i + \alpha_{i-1} \left(\frac{z'}{2} \right) + \dots + \alpha_{i-j} \frac{z'^{(j-2\lfloor j/2 \rfloor)}}{2^j j!} + \dots + \frac{z'^{(i-2\lfloor i/2 \rfloor)}}{2^i i!} &= 0, \\ &\dots\dots\dots \\ \alpha_k + \alpha_{k-1} \left(\frac{z'}{2} \right) + \dots + \frac{z'^{(k-2\lfloor k/2 \rfloor)}}{2^k k!} &= 0. \end{aligned}$$

By an obvious induction, the term of the highest power of z' in α_ξ is

$$(-1)^\xi (z')^\xi / 2^\xi.$$

Thus the lemma is proved.

Theorem 2.2. *It follows that*

$$\begin{aligned} \rho(z, Y_\xi) &= \text{spin}(z, (X_\xi)^{S^1}) \\ &= (-1)^{k+1} (z^{-m/2} - z^{m/2})^{-1} F, \end{aligned}$$

where F is a polynomial of order k , in $z' = (z^{-m/2} + z^{m/2})(z^{-m/2} - z^{m/2})^{-1}$. If $c = c_\xi(\xi) = ag$, then the coefficient of $(z')^k$ in the polynomial F is

$$(-1)^k (a^k / 2^k) g^k [M].$$

Proof. From (3), (4) and the above lemma, it follows that

$$\begin{aligned} (-1)^{k+1} \rho(z, Y_\xi) &= \hat{\mathfrak{U}}(M) (z^{-m/2} - z^{m/2})^{-1} (1 + \alpha_1 c + \cdots + \alpha_k c^k) [M] \\ &= (z^{-m/2} - z^{m/2})^{-1} \left(\sum_{\xi=0}^{\infty} \hat{A}_\xi(M) \right) (1 + \alpha_1 c + \cdots + \alpha_k c^k) [M] \\ &= (z^{-m/2} - z^{m/2})^{-1} (\alpha_k \hat{A}_0(M) c^k + \alpha_{(k-2)} \hat{A}_1(M) c^{(k-2)} \\ &\quad + \cdots + \alpha^{(k-2\lfloor k/2 \rfloor)} \hat{A}_{\lfloor k/2 \rfloor}(M) c^{(k-2\lfloor k/2 \rfloor)}) [M] \\ &= (z^{-m/2} - z^{m/2})^{-1} (a^k \alpha_k g^k [M] + a^{(k-2)} \alpha_{(k-2)} \hat{A}_1(M) g^{(k-2)} [M] \\ &\quad + \cdots + a^{(k-2\lfloor k/2 \rfloor)} \alpha_{(k-2\lfloor k/2 \rfloor)} \hat{A}_{\lfloor k/2 \rfloor}(M) g^{(k-2\lfloor k/2 \rfloor)} [M]). \end{aligned}$$

The formula

$$\begin{aligned} a^k \alpha_k g^k [M] &+ a^{k-2} \alpha_{(k-2)} \hat{A}_1(M) g^{k-2} [M] + \\ &+ \cdots + a^{(k-2\lfloor k/2 \rfloor)} \alpha_{(k-2\lfloor k/2 \rfloor)} \hat{A}_{\lfloor k/2 \rfloor}(M) g^{(k-2\lfloor k/2 \rfloor)} [M] \end{aligned}$$

is a polynomial in z' with rational coefficients. The term of the highest order (k th order) of z' in the polynomial is in $a^k \alpha_k g^k [M]$ and it is

$$(-1)^k (a^k / 2^k) g^k [M].$$

Thus the proof of the theorem is completed.

§ 3. S^1 -acting cobordism of Y_ξ and examples

Using results of the previous section, we prove our main theorem stated in §1.

Proof of Theorem 1.1. Let ξ_1 and ξ_2 be differentiable complex line bundles with structural group $U(1)$, over differentiable structures of M . Y_{ξ_1} and Y_{ξ_2} denote the S^1 -bundles corresponding to ξ_1 and ξ_2 respectively.

Suppose that Y_{ξ_1} and $Y_{\xi_2} \in \mathcal{S}_M$ are topologically isomorphic differentiable S^1 -bundles. Natural S^1 -actions on them for each m are isomorphic. If we take

$$W = Y_{\xi_1} \times I,$$

then we have

$$\begin{aligned} H^1(W; Z_2) &= H^1(Y_{\xi_1}; Z_2) = 0, \\ \partial W &= Y_{\xi_1} \cup (-Y_{\xi_1}) \text{ and } Y_{\xi_1} \cong Y_{\xi_2}. \end{aligned}$$

Let $\pi_1: Y_{\xi_1} \times I \rightarrow Y_{\xi_1}$ be the projection to the first factor. It follows that

$$\begin{aligned} w_2(W) &= w_2(Y_{\xi_1} \times I) \\ &= \pi_1^* w_2(Y_{\xi_1}) \\ &= 0. \end{aligned}$$

Thus Y_{ξ_1} and Y_{ξ_2} belong to the same S^1 -acting spin-cobordism class up to topological equivalences, with respect to natural S^1 -actions for the same m .

Conversely, suppose that Y_{ξ} and $Y_{\xi'}$ with natural S^1 -actions for m_1 and m_2 respectively, belong to the same S^1 -acting spin-cobordism class. Then by M.F. Atiyah and F. Hirzebruch [2], we have

$$\rho(z, Y_{\xi_1}) = \rho(z, Y_{\xi_2})$$

and hence

$$\text{spin}(z, Y_{\xi_1}) = \text{spin}(z, Y_{\xi_2}).$$

Let $c_1(\xi_1) = a_1 g$, $c_1(\xi_2) = a_2 g$ where $a_1, a_2 > 0$.

By Theorem 2.2, we have

$$m_1 = m_2 \text{ and } a_1^k = a_2^k.$$

Since a_1, a_2 are positive integers, it follows that

$$a_1 = a_2.$$

Thus we have $c_1(\xi_1) = c_1(\xi_2)$ and hence

$$\xi_1 \cong \xi_2.$$

The proof of Theorem 1.1 is completed.

For the complex line bundle ξ over M with $c_1(\xi) = ag$, the manifold Y_{ξ} is obviously spin-cobordant to S^{2k+1} in the following sense: There exists a differentiable manifold W such that

$$\begin{aligned} H^1(W; Z_2) &= 0, \quad w_2(W) = 0, \\ \partial W &= Y_{\xi} \cup (-S^{2k+1}). \end{aligned}$$

But from Theorem 2.2, we obtain,

Proposition 3.1. *If $c_1(\xi) \neq \pm g$, Y is not S^1 -acting spin-cobordant to S^{2k+1} with the standard circle action.*

We consider some examples of S^1 -bundles satisfying the conditions of Theorem 1. 1.

Example 1. Let M be a complex projective space CP^{2n} . We take, as $\mathcal{S}_M = \mathcal{S}_{CP^{2n}}$, the set of differentiable S^1 -bundles Y_ξ corresponding to differentiable complex line bundles ξ over differentiable structures of CP^{2n} such that $c_1(\xi) = ag$, $a \equiv 1 \pmod{2}$ and $a > 0$, where $g \in H^2(CP^{2n}; \mathbb{Z})$ is the canonical generator. The condition (1) for $M = CP^{2n}$ are easily verified. By Theorem 1. 1, any two S^1 -bundles of $\mathcal{S}_{CP^{2n}}$ are isomorphic if and only if they are S^1 -acting spin-cobordant upto topological equivalences with respect to natural S^1 -actions.

Example 2. Let HP^n denote a quaternionic projective space of quaternionic dimension n . We can choose $M = CP^{2n} \# HP^n$ ($n > 2$) which is the connected sum of CP^{2n} and HP^n . The condition (1) for this manifold are satisfied. By similar arguments, one can also take, as M ,

$$CP^{2n} \# HP^n \# \dots \# HP^n,$$

$$CP^{2n} \# (HP^{k_1} \times HP^{k_2} \times \dots \times HP^{k_r}) (\sum k_i = n).$$

References

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