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IMMERSIONS OF TOPOLOGICAL MANIFOLDS

BY

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1. Introduction, Definitions and Notation.

Let M and Q be topological manifolds, with $\dim M \leq \dim Q$. In [2] Gauld proved the classification theorem of M' immersions for M , when topological manifold M' is given such as M a locally flat submanifold of the interior of M' and $\dim M' = \dim Q$. An M' immersion means an immersion of V in Q , where V is a neighbourhood of M in M' and two such are identified if they agree over a common neighbourhood of M . cf. Lashof [6], Less [7] and Kirby [5].

Considering an induced neighbourhood of an immersion $I^r \times M \rightarrow I^r \times Q$ with uniformity, we obtain the classification theorem for immersions of topological manifolds instead of M' immersion, where topological manifold is paracompact Hausdorff. Similar result was proven by Hirsh [4] in differentiable case, and by Haefliger and Poenaru [3] in PL case.

Theorem 1. *Let M^m, Q^n be manifolds with $\dim M < \dim Q$. Let N be a closed subset of M . Suppose*

$$\theta : U \longrightarrow Q$$

be an immersion of a neighbourhood of N in M .

Then the correspondance which assigns to an immersion

$$f : M \longrightarrow Q$$

its differential

$$df : TM \longrightarrow TQ$$

induces a bijection between the regular homotopy classes relative to N of immersions of M in Q and the homotopy classes relative to N of representations from TM to TQ .

We define two (complete) semi-simplicial complexes $\mathfrak{S}_\theta(M, Q)$ and $\mathfrak{R}_\theta(M, Q)$.

Definition of $\mathfrak{S}_\theta(M, Q)$. A typical k -simplex of $\mathfrak{S}_\theta(M, Q)$ is a map

$$f : \Delta^k \times M \longrightarrow \Delta^k \times Q$$

which has the uniform immersion property with respect to Δ^k , i.e. for each $(t, x) \in \Delta^k \times M$, there exists a neighbourhood U of t in Δ^k and local charts

$$g : U \times R^m \longrightarrow U \times M$$

$$h : U \times R^q \longrightarrow U \times Q$$

such that g and h commute with the projection on U . Furthermore f must agree with $id_{\Delta^k} \times \theta$ on some neighbourhood of N in M .

Definition of $\mathfrak{R}_\theta(M, Q)$. This is an s. s. complex of representative germs of TM in TQ , a k -simplex of $\mathfrak{R}_\theta(M, Q)$ is represented by a pair (Φ, φ) such that the following diagram commutes.

$$\begin{array}{ccc}
 \Delta^k \times M & \xrightarrow{\varphi} & \Delta^k \times Q \\
 \downarrow id \times \Delta & & \downarrow id \times \Delta \\
 \Delta^k \times V & \xrightarrow{\Phi} & \Delta^k \times Q \times Q \\
 \downarrow id \times pr_1 & & \downarrow id \times pr_1 \\
 \Delta^k \times M & \xrightarrow{\varphi} & \Delta^k \times Q
 \end{array}$$

where $\Delta : M \rightarrow M \times M$ is the diagonal map, and V is a neighbourhood of $\Delta(M)$ in $M \times M$. φ and Φ must satisfy the following properties.

They commute with the projection on the Δ^k , φ agrees with $id \times \theta$ on $\Delta^k \times$ (a neighbourhood of N in M), Φ agrees with $id \times \theta \times \theta$ on $\Delta^k \times$ (a neighbourhood of $\Delta(N)$ in $M \times M$), and the map: $\Delta^k \times V \rightarrow \Delta^k \times M \times Q$ given by $(t, x, x') \rightarrow (t, x, pr_3 \Phi(t, x, x'))$ has the uniform immersion property with respect to Δ^k .

Face operations are defined by restrictions to a particular face. Each of the above s. s. complexes is a Kan complex without degeneracies. cf. Rourke and Sanderson [10].

A 0-simplex of $\mathfrak{R}_\theta(M, Q)$ is called an immersion. Two immersions $f, f' : M \rightarrow Q$ are regularly homotopic relative to N if and only if they determine vertices of the same 1-simples of $\mathfrak{R}_\theta(M, Q)$. Two representations

$$(\Phi, \varphi), (\Phi', \varphi') : TM \longrightarrow TQ$$

are homotopic relative to N if and only if they determine vertices of the same 1-simplex of $\mathfrak{R}_\theta(M, Q)$. When $N = \emptyset$, we omit the subscript θ .

When M is a manifold with boundary, M is a submanifold of the interior of M' , for some manifold M' with $\dim M = \dim M'$. For instance such a manifold M' is obtained from the disjoint union of M and $\partial M \times I$ by identifying $x \in \partial M$ with $(x, 0) \in \partial M \times I$. Moreover if M'' is other manifold of $\dim M$ containing M in its interior, then there exists a homeomorphism of a neighbourhood of M in M' onto a neighbourhood of M in M''

fixing M . By an immersion of a manifold M with boundary in Q , we mean an immersion

$$f : V \longrightarrow Q$$

where V is a neighbourhood of M in M' , and such two immersions are identified if they agree over a common neighbourhood of M . The tangent bundle TM of M is defined by $TM'|_M$. In this case $\mathfrak{F}_\theta(M, Q)$ and $\mathfrak{R}_\theta(M, Q)$ are independent on the choice of M' .

Definition. The differential

$$d : \mathfrak{F}_\theta(M, Q) \longrightarrow \mathfrak{R}_\theta(M, Q)$$

is an s.s. map as follows. For k -simplex $f \in \mathfrak{F}_\theta(M, Q)$, $d(f) = \Phi$, where

$$\Phi : \Delta^k \times TM \longrightarrow \Delta^k \times TQ$$

is defined by $\Phi(t, x, x') = (t, pr_2 f(t, x), pr_2 f(t, x'))$.

Now the main theorem is the following.

Theorem 2. *Let $\dim M < \dim Q$.*

The map

$$d : \mathfrak{F}_\theta(M, Q) \longrightarrow \mathfrak{R}_\theta(M, Q)$$

is a homotopy equivalence.

Theorem 2 implies Theorem 1 as an immediate corollary. When $\dim M = \dim Q$, the theorem is obtained in Gauld [2]. Therefore we suppose $\dim M < \dim Q$.

2. Proof of Theorem 2.

First we prove when $M - N$ is a handle body, that is, there exists a sequence of manifolds $N = M_0, M_2, \dots, M_l = M$ ($l \leq \infty$) such that M_i is obtained from M_{i-1} by adding a handle. Suppose that M is obtained from M_0 by adding a k -handle and $\dim M = \dim M_0 < \dim Q$.

Lemma 3. *The map*

$$i^* : \mathfrak{F}(M, Q) \longrightarrow \mathfrak{F}(M_0, Q)$$

is a fibration.

Lemma 4. *The map*

$$i^* : \mathfrak{R}(M, Q) \longrightarrow \mathfrak{R}(M_0, Q)$$

is a fibration, where i^ is the restriction map induced by the inclusion*

$$i : M_0 \hookrightarrow M.$$

Lemma 5. *The map*

$$d : \mathfrak{S}_\theta(B^k \times B^{m-k}, Q) \longrightarrow \mathfrak{R}_\theta(B^k \times B^{m-k}, Q)$$

is a homotopy equivalence, where θ is an immersion of $\partial B^k \times B^{m-k}$ in Q .

Lemma 6. *The map*

$$d : \mathfrak{S}(B^m, Q) \longrightarrow \mathfrak{R}(B^m, Q)$$

is a homotopy equivalence.

The fibres of the fibrations

$$i^* : \mathfrak{S}(M, Q) \longrightarrow \mathfrak{S}(M_0, Q)$$

$$i^* : \mathfrak{R}(M, Q) \longrightarrow \mathfrak{R}(M_0, Q)$$

are isomorphic to $\mathfrak{S}_\theta(B^k \times B^{m-k}, Q)$ and $\mathfrak{R}_\theta(B^k \times B^{m-k}, Q)$ respectively.

By Lemma 5 of Gauld [2], a homotopy equivalence

$$d : \mathfrak{S}(M_0, Q) \longrightarrow \mathfrak{R}(M_0, Q)$$

implies the homotopy equivalence

$$d : \mathfrak{S}(M, Q) \longrightarrow \mathfrak{R}(M, Q)$$

By induction, we obtain theorem 2 when $M-N$ is a finite handle body. When $M-N$ is an infinite handle body, consider the projective system of homotopy equivalences given by the above. Taking the limite, the required result is obtained. cf. Philips [9].

In the general case the proof is similar to Gauld [2].

Proofs of Lemma 4-6 are similar to those of Gauld [2]. We need only to prove Lemma 3.

3. Induced neighbourhood by an immersion.

Suppose a map

$$F : I^k \times M \longrightarrow I^k \times Q$$

has an uniform immersion property.

Definition. A neighbourhood of M induced by f is a triple (M', i, Φ) such that M' is a manifold with $\dim M' = \dim Q$, $i : M \rightarrow M'$ is a locally flat embedding into the interior of M' , a map

$$\Phi : I^k \times M' \longrightarrow I^k \times Q$$

has an uniform immersion property with respect to I^k , and the following diagram commutes.

$$\begin{array}{ccc}
 I^k \times M & \xrightarrow{id \times i} & I^k \times M' \\
 & \searrow f & \swarrow \Phi \\
 & I^r \times Q &
 \end{array}$$

In this section we prove the existence and the uniqueness of an induced neighbourhood by f , when M is compact.

Lemma 7 and 8 are due to Haefliger and Poenaru [3].

Lemma 7. *Suppose*

$$F : I^r \times M \longrightarrow I^r \times Q$$

has a uniform immersion property, and X is compact. Then for any $t \in I^r$ there exist a cubic neighbourhood I_t of t in I^r and an induced neighbourhood (M', i, Φ) by $F|I_t \times M$.

Proof. Fix $t \in I^r$. For any $x \in M$, there are a neighbourhood $U(x)$ of t in I^r and $V'(x)$ in M such that $F|U(x) \times V'(x)$ is an embedding of $U(x) \times V'(x)$ in $I^r \times Q$. Define the metric on M .

For the open covering $\{V'(x)\}_{x \in M}$ of M , there is $p > 0$ such that any open ball $V_p(x)$ with the radius p and the center x is contained in some $V'(x')$. We can choose a finite covering $\{V_{p/3}(x_1), \dots, V_{p/3}(x_l)\}$ of M . Let the covering $\{V_1, \dots, V_l\}$ of M be a refinement of $\{V_{p/3}(x_1), \dots, V_{p/3}(x_l)\}$ such that $\bar{V}_i \subset V_{p/3}(x_i)$ and $V_i \cap V_j = \emptyset$ implies $\bar{V}_i \cap \bar{V}_j = \emptyset$.

There are a neighbourhood U_i of t in I^r and an open set $N_i \subset Q$ such that $U_i \subset U(x_i)$, $F(U_i \times V_i) \subset U_i \times N_i$, and if $\bar{V}_i \cap \bar{V}_j = \emptyset$ and $\bar{V}_i \cup \bar{V}_j \subset V'(x)$ for some $x \in M$ then $N_i \cap N_j = \emptyset$. Let $I_t \subset \bigcap_i U_i$ be a cubic neighbourhood of t in I^r . Then the open coverings $\{V_i\}$, $\{N_i\}$ satisfy

1. $F|(I_t \times V_i)$ is an embedding of $I_t \times V_i$ into $I_t \times N_i$.
2. If $V_i \cap V_j = \emptyset$ and $V_i \cap V_l \neq \emptyset$, $V_j \cap V_l \neq \emptyset$ for some V_l , then $N_i \cap N_j = \emptyset$.

Now we construct a q -manifold M' as follows. Consider the disjoint sum $\bigcup_i \{i\} \times N_i$. The relation which identifies (j, x) and (k, y) if $x = y$ and $V_j \cap V_k \neq \emptyset$, is an equivalence relation. The quotient space M' of $\bigcup_i \{i\} \times N_i$ by this relation is q -manifold. The point represented by (j, x) is denoted by $[j, x]$.

The map

$$\varphi : M' \longrightarrow Q$$

where $\varphi([j, x]) = x$ is an immersion.

The map

$$G : I_t \times M \longrightarrow I_t \times M'$$

defined by $G(t', x) = (t', [i, pr_2 F(t', x)])$ with $x \in V_t$ is well-defined and an embedding commuting with the projection on the I_t factor.

Moreover

$$(id_{I_t} \times \varphi) \circ G = F|(I_t \times M).$$

Let the map

$$i : M \longrightarrow M'$$

is the embedding given by $i(x) = pr_2 \circ G(t, x)$. By the Isotopy Extension Theorem of Edward and Kirby [1], there is a cube of homeomorphisms

$$H : I_t \times M' \longrightarrow I_t \times M'$$

such that $G = H \circ (id_{I_t} \times i)$.

Let $\Phi = (id_{I_t} \times \varphi) \circ H$. Then Φ has a uniform immersion property with respect to I_t , and the following diagram commutes.

$$\begin{array}{ccc} I_t \times M & \xrightarrow{id \times i} & I_t \times M' \\ & \searrow F|(I_t \times M) & \swarrow \Phi \\ & I_t \times Q & \end{array}$$

Lemma 8. Suppose M is a compact manifold and a map

$$F : I^r \times M \longrightarrow I^r \times Q$$

has a uniform immersion property with respect to I^r . If (M', i, Φ) and (M'', i', Φ') are neighbourhoods of M induced by F . Then there exist a neighbourhood W of $i(M)$ in M' and a homeomorphism J of $I^r \times W$ onto a neighbourhood of $I^r \times i'(M)$ in $I^r \times M''$ commuting with the projection on X with the following commutative diagram.

$$\begin{array}{ccc} I^r \times M & \xrightarrow{id \times i} & I^r \times W \\ & \searrow F & \swarrow \Phi|I^r \times W \\ & I^r \times Q & \\ & \searrow id \times i' & \swarrow \Phi' \\ & & I^r \times M'' \end{array} \quad \begin{array}{c} \downarrow J \\ I^r \times M'' \end{array}$$

Proof. As in [3, §7. Proposition 2], there exist a open covering $\{W_i\}_{i=1,\dots,l}$ of $I^r \times i(M)$ in $I^r \times M'$ and $\{W'_i\}_{i=1,\dots,l}$ of $I^r \times i'(M)$ in $I^r \times M''$ such that the following conditions are satisfied.

1. $\Phi_i = \Phi|W_i$ and $\Phi'_i = \Phi'|W'_i$ are homeomorphisms of W_i and W'_i into $I^r \times Q$ respectively. And image $\Phi_i \subset \text{Image } \Phi'_i$.

2. If $W_i \cap W_j = \emptyset$, there is an open set V in $I^r \times M''$ such that $W'_i \cup W'_j \subset V$ and $\Phi|V$ is a homeomorphism.

3. $\Phi_i^{-1}\Phi_i(t, i(x)) = (t, i'(x))$ for $x \in M$.

The local homeomorphism J' of $\bigcup_i W_i$ in $I^r \times M''$ given by $J'(t, x) = \Phi_i^{-1}\Phi_i(t, x)$ for $(t, x) \in W_i$ is well defined. J' commutes with the projection on I^r . Because $J'(t, i(x)) = (t, i'(x))$, J' is injective on $I^r \times i(M)$. By Whitehead [11, §4], there is a neighbourhood W of $i(M)$ in M' such that $I^r \times W \subset \bigcup_i W_i$. $J = J'|I^r \times W$ and W satisfy the required conditions.

Lemma 9. *Suppose M is compact.*

A map

$$F : I^r \times M \longrightarrow I^r \times Q$$

has an uniform immersion property with respect to I^r .

Then there is a neighbourhood (M', i, Φ) of M induced by F .

Proof. For any $t \in I^r$, there is a cubic neighbourhood I_t of t in I^r , with a neighbourhood (M_t, i_t, Φ_t) of M induced by $F|I_t \times M$. Let n be a positive integer such that any cube $\bigcap_{k=1}^r [a_k, b_k] \subset I^r$, where $|a_k - b_k| < \frac{2}{n}$, is contained in some I_t . Define the cube $I_i \subset I^r$, $i = 1, \dots, n^r$, as follows.

$$I_i = \bigcap_{k=1}^r \left[\frac{j_k}{n}, \frac{j_k+1}{n} \right]$$

where $i = 1 + \sum_{k=1}^r n^k j_k$, $j_k = 0, \dots, n-1$.

Let

$$X_l = \bigcup_{i \leq l} I_i.$$

Inductively we will construct a neighbourhood X'_l of X_l in I^r and a neighbourhood $(\tilde{M}_l, \tilde{i}_l, \tilde{\Phi}_l)$ of M induced by $F|(X'_l \times M)$. When $l = n^r$, $(\tilde{M}_l, \tilde{i}_l, \tilde{\Phi}_l)$ is the required neighbourhood.

It is clear when $l = 1$.

Suppose we are given X'_l and $(\tilde{M}_l, \tilde{i}_l, \tilde{\Phi}_l)$. X'_{l+1} and $(\tilde{M}_{l+1}, \tilde{i}_{l+1}, \tilde{\Phi}_{l+1})$ are given as follows. Cubes $I_{l,i}$ and $I'_{l,i}$ ($1 \leq i \leq l+1$) are defined by

$$I'_{l,i} = \bigcap_{k=1}^r \left[\frac{j_k}{n} - \delta(j_k) \cdot P_l, \frac{j_k+1}{n} + \delta'(j_k) \cdot P_l \right]$$

$$I'_{l,i} = \bigcap_{k=1}^r \left[\frac{j_k}{n} - \frac{1}{2} \delta(j_k) \cdot P_l, \frac{j_k+1}{n} + \frac{1}{2} \delta'(j_k) \cdot P_l \right]$$

where $\delta(j_k) = 0$ if $j_k = 0$, $\delta(j_k) = 1$ if $j_k \neq 0$
 $\delta'(j_k) = 0$ if $j_k = n-1$, $\delta'(j_k) = 1$ if $j_k \neq n-1$

and P_l is a positive number such that $P_l < \frac{1}{4n}$ and, $I_{l,i} \subset X'_l$ for any $i \leq l$.

$Y_l = \bigcup_{i \leq l} I_{l,i}$ and $Z_l = \bigcup_{i \leq l} I'_{l,i}$ are neighbourhoods of X_l in I^r , and contained in X'_l . For $I_{l,l-1}$, there exists $t_{l+1} \in I^r$ such that $I_{l,l+1} \subset I_{t_{l+1}}$.

Let $\varphi_l : I_{l,l+1} \rightarrow I^r$ and $\varphi' : I^r \rightarrow \left[0, \frac{1}{2}\right] \times I^{r-1}$ be homeomorphisms such that

$$\varphi_l(Y_l \cap I_{l,l+1}) = \left[0, \frac{1}{2}\right] \times I^{r-1}, \quad \varphi_l(Z_l \cap I_{l,l-1}) = \left[0, \frac{1}{4}\right] \times I^{r-1}$$

$$\varphi / \left(\left[0, \frac{1}{4}\right] \times I^{r-1} \right) = id.$$

The homeomorphism defined by

$$\varphi = \varphi_l^{-1} \circ \varphi' \circ \varphi_l : I_{l,l+1} \longrightarrow Y_l \cap I_{t_{l+1}}$$

satisfies $\varphi(Z_l \cap I_{l,l+1}) = id$.

For neighbourhoods $(\tilde{M}_l, i_l, \tilde{\varphi}_l | ((Y_l \cap I_{l,l+1}) \times \tilde{M}_l))$ and $(M_{t_{l+1}}, i_{t_{l+1}}, \Phi_{t_{l+1}} | ((Y_l \cap I_{l,l+1}) \times M_{t_{l+1}}))$ of M induced by $F | ((Y_l \cap I_{l,l+1}) \times M)$, there is a neighbourhood $\tilde{M}_{t_{l+1}}$ of $i_l(M)$ in $\tilde{M}_l$ and a homeomorphism J_l of $(Y_l \cap I_{l,l+1}) \times M_{t_{l+1}}$ into $(Y_l \cap I_{l,l+1}) \times M_{t_{l+1}}$ commuting with the projection on $Y_l \cap I_{l,l+1}$, with the following commutative diagram;

$$\begin{array}{ccc}
 (Y_l \cap I_{l,l+1}) \times M & \xleftarrow{id \times i_l} & (Y_l \cap I_{l,l+1}) \times M_{t_{l+1}} \\
 \searrow F | ((Y_l \cap I_{l,l+1}) \times M) & & \searrow \Phi_l | (Y_l \cap I_{l,l+1}) \times M_{t_{l+1}} \\
 & (Y_l \cap I_{l,l+1}) \times Q & \\
 \swarrow id \times i_{t_{l+1}} & \nwarrow \Phi_{t_{l+1}} | ((Y_l \cap I_{l,l+1}) \times M_{t_{l+1}}) & \downarrow J_l \\
 & (Y_l \cap I_{l,l+1}) \times M_{t_{l+1}} &
 \end{array}$$

Let $X'_{l+1} = \text{Int } Z_l \cup \text{Int } I_{l,l+1}$ and $i_{l+1} = i_l$, where $\text{Int } Z_l$ and $\text{Int } I_{l,l+1}$ are interiors of Z_l and $I_{l,l+1}$ in I^r respectively.

The map

$$\tilde{\varphi}_{l+1} : X'_{l+1} \times \tilde{M}_{l+1} \longrightarrow X_{l+1} \times Q$$

given by

$$\tilde{\varphi}_{l+1}(t, x) = \begin{cases} \tilde{\varphi}_l(t, x) & \text{if } (t, x) \in (\text{Int } Z_l) \times \tilde{M}_{l+1} \\ \Phi_{l_{l+1}} \circ (\varphi^{-1} \times id) \circ J_l \circ (\varphi \times id)(t, x) & \text{if } (t, x) \in (\text{Int } I_{l,l+1}) \times \tilde{M}_{l+1} \end{cases}$$

is well defined because J_l commutes with the projection on the first factor, and $\varphi|(Z_l \cap I_{l,l+1}) = id$. Clearly $\tilde{\varphi}_{l+1}$ has the uniform immersion property, and commutes with the projection on X'_{l+1} .

$J_{l,t} \circ \tilde{i}_l(x) = i_{l_{l+1}}(x)$ for $(t, x) \in (Y_l \cap I_{l,l+1}) \times M$ where $J_{l,t}(x) = pr_2 \circ J_l(t, x)$. Then for $(t, x) \in (\text{Int } I_{l,l+1}) \times M$,

$$\begin{aligned} (\tilde{\varphi}_{l+1} \circ (id \times \tilde{i}_{l+1}))(t, x) &= \Phi_{l_{l+1}}((t, J_{l,\varphi(t)} \circ \tilde{i}_l(x))) \\ &= \Phi_{l_{l+1}}(t, i_{l_{l+1}}(x)) \\ &= F(t, x). \end{aligned}$$

Therefore $(\tilde{M}_{l+1}, \tilde{i}_{l+1}, \tilde{\varphi}_{l+1})$ satisfies the required conditions.

4. Proof of the fibration lemma for immersions.

Now we will prove Lemma 3 when $M - M_0$ has a k -handle when $\dim M < \dim Q$. It suffices to prove the following lifting problem.

$$\begin{array}{ccc} I^r \times 0 & \xrightarrow{H_0} & (B^k \times B^{m-k}, Q) \\ \downarrow & \nearrow H & \downarrow i^* \\ I^r \times I & \xrightarrow{h} & \left(\left(B^k - \frac{3}{4} B_k \right) \times B^{m-k}, Q \right) \end{array}$$

We may assume H_0, h as follows. cf. Gauld [2. Lemma 3]

$$\begin{aligned} h : I^r \times I \times \left(2B^k - \frac{1}{2} \dot{B}^k \right) \times 2B^{m-k} &\longrightarrow I^r \times I \times Q \\ H_0 : I^r \times 0 \times 2B^k \times 2B^{m-k} &\longrightarrow I^r \times 0 \times Q \end{aligned}$$

such that

$$h \left| \left(I^r \times 0 \times \left(2B^k - \frac{1}{2} \dot{B}^k \right) \times 2B^{m-k} \right) \right. = H_0 \left| \left(I^r \times 0 \times \left(2B^k - \frac{1}{2} \dot{B}^k \right) \times 2B^{m-k} \right) \right.$$

and h, H_0 have the uniform immersion property.
Our task is to extend h and H_0 to

$$H : I^r \times I \times 2B^k \times 2B^{m-k} \longrightarrow I^r \times I \times Q$$

By Lemma 9, there are a neighbourhood $(W_1, i_1, \tilde{H}_0)$ of $2B^k \times 2B^{m-k}$ induced by H_0 and a neighbourhood $(W_2, i_2, \tilde{h})$ of $\left(2B^k - \frac{1}{2}\hat{B}^k\right) \times 2B^{m-k}$ induced by h . We may assume that

$$\begin{aligned} W_1 &= 3B^k \times 4B^{n-k} \\ i_1 : 2B^k \times 2B^{m-k} &\hookrightarrow 3B^k \times 4B^{n-k} \end{aligned}$$

By Lemma 8, there exist a neighbourhood W_3 of $i_1\left(\left(2B^k - \frac{1}{2}\hat{B}^k\right) \times 2B^{m-k}\right)$ in $3B^k \times 4B^{n-k}$ and a homeomorphism J of $I^r \times 0 \times W_3$ in $I^r \times 0 \times W_2$ with the following commutative diagram;

$$\begin{array}{ccc} & & I^r \times 0 \times W_3 \\ & \nearrow^{id \times i_1} & \\ I_k \times 0 \times \left(2B^k - \frac{1}{2}\hat{B}^k\right) \times 2B^{m-k} & & \\ & \searrow_{h'} & \\ & I^r \times 0 \times Q & \\ & \nearrow_{h|(I^r \times 0 \times W_2)} & \\ & I^r \times 0 \times W_2 & \\ & \nwarrow_{id \times i_2} & \end{array} \quad \begin{array}{c} \\ \\ \\ \\ \\ J \\ \\ \end{array}$$

where

$$h' = h \left| \left(I^r \times 0 \times \left(2B^k - \frac{1}{2}\hat{B}^k\right) \times 2B^{m-k} \right) = H_0 \left| \left(I^r \times 0 \times \left(2B^k - \frac{1}{2}\hat{B}^k\right) \times 2B^{m-k} \right) \right.$$

We may assume $W_1 = \left(3B^k - \frac{1}{4}\hat{B}^k\right) \times 4B^{n-k}$.

Let $J_t(x) = pr_3 \circ J(t, 0, x)$. The map

$$\tilde{h}' : I^r \times I \times \left(3B^k - \frac{1}{4}\hat{B}^k\right) \times 4B^{n-k} \longrightarrow I^r \times I \times Q$$

given by $\tilde{h}'(t, t', x) = \tilde{h}(t, t', J_t(x))$ has the uniform immersion property, and

$$\begin{aligned} \tilde{h}'(t, t', i_1(x)) &= \tilde{h}(t, t', J_t \circ i_1(x)) \\ &= \tilde{h}(t, t', i_2(x)) \\ &= h(t, t', x) \end{aligned}$$

$$\text{for } (t, t', x) \in I^r \times I \times \left(2B^k - \frac{1}{2} \mathring{B}^k\right) \times 2B^{m-k}.$$

Moreover

$$\tilde{H}_0 \left| \left(I^r \times 0 \times \left(3B^k - \frac{1}{4} \mathring{B}^k \right) \times 4B^{n-k} \right) \right| = \tilde{h}' \left| \left(I^r \times 0 \times \left(3\mathring{B}^k - \frac{1}{4} \mathring{\mathring{B}}^k \right) \times 4B^{m-k} \right) \right|.$$

By Gauld [2], there exists an extension of $\tilde{H}_0$ and

$$\tilde{h}' \left| \left(I^r \times I \times \left(2B^k - \frac{2}{3} \mathring{B}^k \right) \times 2B^{n-k} \right) \right| \text{ with the uniform immersion property;}$$

$$\tilde{H} : I^r \times I \times 3B^k \times 4B^{n-k} \longrightarrow I^r \times I \times Q.$$

The composition

$$H = \tilde{H}_0 \circ (id \times i_1) : I^r \times I \times 2B^k \times 2B^{m-k} \longrightarrow I^r \times I \times Q$$

is a required extension of H_0 and $h \left| \left(I^r \times I \times \left(2B^k - \frac{2}{3} \mathring{B}^k \right) \times 2B^{m-k} \right) \right|$.

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