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A remark on doubly transitive groups

To Professor Yoshie Katsurada on the occasion of her 60th birthday

By Hiroshi KIMURA

1. This note is a continuation of [12]. We shall use the same notation. The purpose of this note is to prove the following.

THEOREM. *Let $\mathfrak{G}$ be a doubly transitive permutation group of odd degree satisfying the following conditions.*

- (1) $\mathfrak{G}_{1,2}$ is of even order,
- (2) All Sylow subgroups of $\mathfrak{G}_{1,2}$ are cyclic,
- (3) $\chi(\tau)$ contains a regular normal subgroup,
- (4) $\mathfrak{G}$ has one class of involutions,
- (5) $\mathfrak{G}_{1,2}$ has unique involution.

Then $\mathfrak{G}$ contains a regular normal subgroup.

From this and [12, Theorem] we obtain the following.

COROLLARY. *Let $\mathfrak{G}$ be a doubly transitive permutation group of odd degree satisfying the above conditions (1), (2) and (3). Then $\mathfrak{G}$ contains a regular normal subgroup or it is isomorphic to one of the groups S_5 with $n=5$ and $\text{PSL}(2, 11)$ with $n=11$.*

2. Assume $\mathfrak{G}$ does not contain a regular normal subgroup. By [12, Theorem 1] we may assume that $|\mathfrak{R}| > 2$ and $\mathfrak{R}_0 = \langle \tau \rangle$. Thus $d/2$ is odd. From the condition (4) a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$ is also a Sylow 2-subgroup of $\mathfrak{G}$.

LEMMA 1. *A Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$ is not metacyclic.*

PROOF. Let $\mathfrak{S}$ be a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$ containing $\langle \mathfrak{R}, I \rangle$ and let $\mathfrak{S}'$ be a cyclic normal subgroup of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{S}'$ is cyclic. If $|\mathfrak{S}/\mathfrak{S}'| > 2$, then $\mathfrak{G}$ is solvable by [11]. Therefore $\mathfrak{S} = \langle I, \mathfrak{S}' \rangle$. Since $\mathfrak{R} \neq \langle \tau \rangle$, $|\mathfrak{S}'| > 2$. If $\mathfrak{S}$ is abelian, then $\mathfrak{G}$ is solvable by the Burnside's splitting theorem. If $\mathfrak{S}$ is dihedral or semi-dihedral, then $\mathfrak{R}_0 \neq \langle \tau \rangle$, which is a contradiction. If $S' = S\tau$ for a generator S of $\mathfrak{S}'$, $\mathfrak{G}$ is solvable by [13]. Thus $\mathfrak{S}$ is not metacyclic.

LEMMA 2. $\chi_1(\tau)$ is contained in $C_{\mathfrak{G}}(I)$.

PROOF. Assume that there exists a Sylow q -subgroup $\mathfrak{H}'_q$ of $\chi_1(\tau)$ such that $\langle \mathfrak{H}'_q, I \rangle$ is dihedral. Let $\mathfrak{S}'$ be a Sylow 2-subgroup of $C_{\mathfrak{G}}(\mathfrak{H}'_q)$ containing

τ and $\mathfrak{S}$ a Sylow 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{H}'_q)$ containing $\mathfrak{S}'$. By the Frattini argument it may be assumed that $\mathfrak{S}$ is a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$. Since $\text{Aut}(\mathfrak{H}'_q)$ is cyclic, so is $\mathfrak{S}/\mathfrak{S}'$. Assume $\mathfrak{S}'$ is not cyclic. If $\mathfrak{S}'$ contains an involution $\eta(\neq \tau)$, then take η instead of I , and $d=2d(\eta)$ is not divisible by q , which is a contradiction. Thus $\mathfrak{S}'$ is a generalized quaternion. Let ζ and ξ be elements of $\mathfrak{S}'$ such that $\zeta^2=\xi^2=\tau$ and $\xi\zeta=\xi^{-1}$. If $\alpha(\xi)\geq 2$, then let a and b be two points in $\mathfrak{F}(\xi)$ such that ζ has the cycle structure $(a, b)\cdots$. a and b are contained in $\mathfrak{F}(\tau)$. Let I' be an involution with cycle structure $(a, b)\cdots$. Then ζ is an element in $I'\mathfrak{G}_{a,b}$. Therefore $\langle \zeta, \xi \rangle$ is a subgroup of $\langle I', \mathfrak{G}_{a,b} \rangle$. Since a Sylow 2-subgroup of $\langle I', \mathfrak{G}_{a,b} \rangle$ is conjugate to $\langle \mathfrak{R}, I \rangle$, $\langle \zeta, \xi \rangle$ is not a quaternion group, which is a contradiction. If $\alpha(\zeta)\geq 2$ and $\alpha(\xi)=1$, then ζ acts on the set of transpositions which appear in the cycle structure of ξ . Thus there exist two points a and b in $\mathfrak{F}(\zeta)$ such that ξ has the cycle structure $(a, b)\cdots$. Therefore ξ is an element of $I'\mathfrak{G}_{a,b}$, where I' is an involution with the cycle structure $(a, b)\cdots$. Again we have a contradiction. Next assume $\mathfrak{F}(\zeta)=\mathfrak{F}(\xi)=\mathfrak{F}(\zeta\xi)=\{a\}$. a is a point of $\mathfrak{F}(\tau)$. Consider a homomorphism ρ of $\langle \zeta, \xi \rangle$ into $\chi(\tau)_a=C_{\mathfrak{G}_a(\tau)}/\chi_1(\tau)$. Then $\ker \rho=\langle \tau \rangle$. Since $\chi(\tau)$ contains a regular normal subgroup, $\langle \zeta, \xi \rangle/\langle \tau \rangle \cong \langle \zeta, \xi \rangle\chi_1(\tau)/\chi_1(\tau)$ is a Frobenius complement. Thus it must be cyclic or a (generalized) quaternion, which is a contradiction. Hence $\mathfrak{S}'$ must be cyclic and $\mathfrak{S}$ is metacyclic. This contradicts Lemma 1. This proves the lemma.

LEMMA 3. *If $\mathfrak{R}$ is not contained in $\chi_1(\tau)$, then p is prime to d and $d-1$.*

PROOF. Since $\chi_1(\tau)$ does not contain $\mathfrak{R}$, $\chi(\tau)$ satisfies the conditions (1), (2) and (3) in Theorem. $\chi(\tau)$ has two classes of involutions since $\mathfrak{R}$ is cyclic. Let K be an element of $\mathfrak{R}$ not contained in $\chi_1(\tau)$ such that K^2 is contained in $\chi_1(\tau)$. Apply [12, Lemma 8] to $\chi(\tau)$. If $\langle I, \mathfrak{H}_q \rangle \chi_1(\tau)$ and $\langle K, \mathfrak{H}_q \rangle \chi_1(\tau)$ is dihedral, then $\alpha(\langle \mathfrak{H}_q, \tau \rangle)$ is even and $p=q$. On the other hand, since $\chi(\tau)$ contains a regular normal subgroup, $\alpha(\langle \mathfrak{H}_q, \tau \rangle)$ must be equal to a power of p , which is a contradiction. Thus if $\langle I, \mathfrak{H}_q \rangle \chi_1(\tau)$ is dihedral, then $\langle K, \mathfrak{H}_q \rangle \chi_1(\tau)$ is abelian and if $\langle K, \mathfrak{H}_q \rangle \chi_1(\tau)$ is dihedral, then $\langle I, \mathfrak{H}_q \rangle \chi_1(\tau)$ is abelian. Hence $i=i'(\beta'(i'-1)/r'+1)$, where $i'=\alpha(K)$ and $d/2=\beta'/r'$. Since $i=p^m$, $d/2-1$ is divisible by p and hence p is prime to d and $d-1$.

LEMMA 4. *If $\mathfrak{R}$ is contained in $\chi_1(\tau)$ and $d\neq 2$, then d is a factor of $i-1$.*

PROOF. Let $\mathfrak{S}$ be a Sylow 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{R})$ containing I . Since $\bar{\mathfrak{S}}=\mathfrak{S}/\mathfrak{R}\cong\mathfrak{S}\chi_1(\mathfrak{R})/\chi_1(\mathfrak{R})$ is a Frobenius complement, it is cyclic or a (generalized) quaternion group. Let q be a prime factor of $d/2$ which is prime to $i-1$. By Lemma 2 and the Frattini argument $\langle \mathfrak{H}_q, K \rangle$ is abelian. As in the proof of [8, Lemma 3. 9] we may prove that $\bar{\mathfrak{S}}$ is cyclic. That is, $\bar{\mathfrak{S}}$

is metacyclic. This contradicts Lemma 1. This proves the lemma.

COROLLARY 5. d is prime to p .

PROOF. If d is divisible by p , then $\mathfrak{R}$ is not contained in $\chi_1(\tau)$ by Lemma 4. This contradicts Lemma 3.

By Corollary 5 if $\mathfrak{H}_p \neq 1$, then $\langle \mathfrak{H}_p, I \rangle$ is abelian.

LEMMA 6. $(n, |\mathfrak{H}|)$ is a power of p .

PROOF. Assume $(n, |\mathfrak{H}|)$ is not a power of p . Let q be a prime factor ($\neq p$) of $(n, |\mathfrak{H}|)$. Assume that $|\chi_1(\tau)|$ is divisible by q . Let $\mathfrak{H}'_q$ be a Sylow q -subgroup of $\chi_1(\tau)$ contained in $\mathfrak{H}_q$. If $\mathfrak{F}(\tau)$ is proper subset of $\mathfrak{F}(\mathfrak{H}'_q)$, then $\alpha(\mathfrak{H}'_q) = i(\beta'(i-1)+1)$, where β' is some integer. By inductive hypothesis $\chi(\mathfrak{H}'_q)$ contains a regular normal subgroup. In particular $\alpha(\mathfrak{H}'_q)$ is a power of p . Since q is a factor of $n - \alpha(\mathfrak{H}'_q)$, $q = p$. If $\alpha(\tau) = \alpha(\mathfrak{H}'_q)$, then $q = p$ since q is a factor of $n - i$, which is a contradiction. Thus $\mathfrak{H}'_q = 1$. Set $\mathfrak{X} = \langle \tau, \mathfrak{H}_q \rangle$. Then $\mathfrak{F}(\mathfrak{X})$ is a proper subset of $\mathfrak{F}(\tau)$. If $\mathfrak{F}(\mathfrak{X}) = \mathfrak{F}(\mathfrak{H}_q)$, then $q = p$ since $n - i$ is divisible by q . If $\mathfrak{F}(\mathfrak{X})$ is proper subset of $\mathfrak{F}(\mathfrak{H}_q)$, then, as above, $\alpha(\mathfrak{H}_q) = \alpha(\mathfrak{X})(\beta'(\alpha(\mathfrak{X})-1)+1)$ and $\alpha(\mathfrak{H}_q)$ is a power of $\alpha(\mathfrak{X})$. Since $\chi(\tau)$ contains a regular normal subgroup, $\alpha(\mathfrak{X})$ is a power of p and so is $\alpha(\mathfrak{H}_q)$. Since $n - \alpha(\mathfrak{H}_q)$ is divisible by q , $p = q$, which is a contradiction. This proves the lemma.

As in [8, 4-2] we may prove that $\mathfrak{H}_p = 1$. Similarly by using Lemma 6 we may prove that if $\mathfrak{H}_p = 1$, then d has a prime factor which is prime to $i-1$ and $d-1$ is divisible by p (see [8, 4-1]). By Lemma 3 $\mathfrak{R}$ is contained in $\chi_1(\tau)$. This contradicts Lemma 4.

This complete a proof of Theorem.

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