



# HOKKAIDO UNIVERSITY

|                  |                                                                                         |
|------------------|-----------------------------------------------------------------------------------------|
| Title            | On Riemannian Manifolds Satisfying the Condition $R(X,Y)R=0$                            |
| Author(s)        | Fujimura, Shigeyoshi                                                                    |
| Citation         | Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 22(1-2), 1-8 |
| Issue Date       | 1972                                                                                    |
| Doc URL          | <a href="https://hdl.handle.net/2115/58113">https://hdl.handle.net/2115/58113</a>       |
| Type             | departmental bulletin paper                                                             |
| File Information | JFS_HU_v22n1_2-1.pdf                                                                    |



# On Riemannian Manifolds Satisfying the Condition $R(X, Y)R = 0$

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Shigeyoshi FUJIMURA

## § 0. Introduction and Preliminaries.

Let  $(M^n, g)$  ( $n \geq 2$ ) be an  $n$ -dimensional Riemannian manifold<sup>1)</sup> with a positive definite metric tensor  $g$ . We denote by  $R$  and  $\nabla$  the curvature tensor and the Riemannian connection defined by  $g$ , respectively. For tangent vectors  $X$  and  $Y$ , we consider  $R(X, Y)$  as a derivation of the tensor algebra at each point of  $(M^n, g)$ . A conjecture by K. Nomizu [5]<sup>2)</sup> is the following.

CONJECTURE. *Let  $(M^n, g)$  ( $n \geq 3$ ) be a complete irreducible Riemannian manifold. If  $(M^n, g)$  satisfies the condition*

(\*)  $R(X, Y)R = 0$  for all tangent vectors  $X$  and  $Y$ ,  
 $(M^n, g)$  is locally symmetric.

The purpose of this note is to show that under some additional conditions the above conjecture is true. Let  $R$ ,  $Ric$ ,  $S$  and  $C$  be the curvature tensor, the Ricci tensor, the scalar curvature and the Weyl's conformal curvature tensor of  $(M^n, g)$ , respectively. With respect to local coordinates, we denote by  $R^h_{ijk}$ ,  $R_{ij}$  and  $C^h_{ijk}$  the components of  $R$ ,  $Ric$  and  $C$ , respectively.  $(M^n, g)$  is said to be locally symmetric if  $\nabla R = 0$ , Ricci symmetric if  $\nabla Ric = 0$  and conformally symmetric [2] if  $\nabla C = 0$ .

The author wishes to express here his sincere thanks to Professor Yoshie Katsurada and Doctor Tamao Nagai for their kindly guidances and encouragements.

## § 1. Conformally symmetric Riemannian manifolds.

K. Yamauchi [8] proved the following.

THEOREM A. *Let  $(M^4, g)$  be a conformally flat Riemannian manifold with non-zero constant scalar curvature. If  $(M^4, g)$  satisfies the condition (\*),  $(M^4, g)$  is locally symmetric.*

The first purpose of this note is to give a generalization of Theorem A.

THEOREM 1. *Let  $(M^n, g)$  ( $n \geq 4$ ) be a conformally symmetric Riemannian*

1) Throughout the paper, we assume that  $(M^n, g)$  is connected.

2) Numbers in brackets refer to references at the end of this paper.

manifold with constant scalar curvature. If  $(M^n, g)$  satisfies the condition  
 (\*\*)  $R(X, Y)Ric=0$  for all tangent vectors  $X$  and  $Y$ ,  
 $(M^n, g)$  is locally symmetric.

From the definition of  $C$ , we have

$$(1) \quad \nabla_i C^h_{\phantom{h}ij k} = \nabla_i R^h_{\phantom{h}ijk} - \frac{1}{n-2} (\delta^h_k \nabla_i R_{ij} - \delta^h_j \nabla_i R_{ik} + g_{ij} \nabla_i R^h_k - g_{ik} \nabla_i R^h_j) + \frac{1}{(n-1)(n-2)} \nabla_i S (\delta^h_k g_{ij} - \delta^h_j g_{ik}),$$

where  $\delta$  is the Kronecker's delta, and since

$$\nabla_m \nabla_i S - \nabla_i \nabla_m S = 0$$

for a scalar  $S$ , we have

$$(2) \quad \begin{aligned} \nabla_m \nabla_i C^h_{\phantom{h}ij k} - \nabla_i \nabla_m C^h_{\phantom{h}ij k} &= \nabla_m \nabla_i R^h_{\phantom{h}ijk} - \nabla_i \nabla_m R^h_{\phantom{h}ijk} \\ &- \frac{1}{n-2} \left\{ \delta^h_k (\nabla_m \nabla_i R_{ij} - \nabla_i \nabla_m R_{ij}) - \delta^h_j (\nabla_m \nabla_i R_{ik} - \nabla_i \nabla_m R_{ik}) \right. \\ &\left. + g_{ij} (\nabla_m \nabla_i R^h_k - \nabla_i \nabla_m R^h_k) - g_{ik} (\nabla_m \nabla_i R^h_j - \nabla_i \nabla_m R^h_j) \right\}. \end{aligned}$$

If  $\nabla C=0$  and  $\nabla Ric=0$ , from (1), we have  $\nabla R=0$ . From (2), if  $(M^n, g)$  satisfies (\*\*) and  $\nabla C=0$ , we have (\*). Therefore, we obtain the following.

LEMMA 1. Let  $(M^n, g)$  ( $n \geq 3$ ) be a conformally symmetric Riemannian manifold. On  $(M^n, g)$ , (\*\*) is equivalent to (\*), and  $(M^n, g)$  is locally symmetric if and only if  $(M^n, g)$  is Ricci symmetric.

LEMMA 2. Let  $(M^n, g)$ , ( $n \geq 4$ ) be a Riemannian manifold satisfying the following condition

$$(3) \quad \nabla_i C^h_{\phantom{h}ijk} + \nabla_k C^h_{\phantom{h}ilj} + \nabla_j C^h_{\phantom{h}ikl} = 0.$$

Then, a necessary and sufficient condition that

$$(4) \quad \nabla_k R_{ij} - \nabla_j R_{ik} = 0$$

is that  $S$  is constant.

When  $\nabla C=0$ , M. C. Chaki and B. Gupta [2] proved Lemma 2.

PROOF. From the Bianchi's identity and  $C^h_{\phantom{h}ijk}=0$ , it follows that (3) is equivalent to

$$(5) \quad \nabla_k R_{ij} - \nabla_j R_{ik} - \frac{1}{2(n-1)} (g_{ij} \nabla_k S - g_{ik} \nabla_j S) = 0.$$

Substituting  $\nabla_k S=0$  in (5), we have (4). Conversely, contracting (4) by  $g^{ij}$ , we can see that  $S$  is constant.

LEMMA 3. *A necessary and sufficient condition for a Riemannian manifold  $(M^n, g)$  ( $n \geq 2$ ) to be Ricci symmetric is that  $(M^n, g)$  satisfies  $(**)$  and (4).*

PROOF. Applying the Ricci's identity to  $(**)$ , we have

$$(6) \quad R_{hj}R^h_{ikl} + R_{ih}R^h_{jkl} = 0.$$

Derivating (6) by  $\nabla_m$  and contracting by  $g^{ik}g^{jm}$ , we have

$$(7) \quad R^{mj}\nabla_j R_{mi} + R^m_i \nabla^j R_{mj} + R^{mj}_i \nabla_j R_{im} - R^{im} \nabla_j R^j_{mi} = 0.$$

From (4), we have

$$R^{mj}\nabla_j R_{mi} = R^{mj}\nabla_i R_{mj} = \frac{1}{2} \nabla_i (R_{mj}R^{mj}),$$

and from the Bianchi's identity and (4), it follows that

$$\nabla^j R_{mj} = \frac{1}{2} \nabla_m S = 0,$$

and

$$\nabla_j R^j_{mii} = \nabla_i R_{mi} - \nabla_i R_{mi} = 0.$$

Since  $R^{mj}_i$  is skew-symmetric and  $\nabla_j R_{im}$  is symmetric with respect to  $j$  and  $m$ , respectively, we have

$$R^{mj}_i \nabla_j R_{im} = 0.$$

Thus, from (7), we have

$$\nabla_i (R_{mj}R^{mj}) = 0.$$

On the other hand, from  $(**)$ , (4) and the Bianchi's identity, it follows that

$$g^{kl}\nabla_i \nabla_k R_{ij} = g^{kl}\nabla_i \nabla_j R_{ik} = g^{kl}\nabla_j \nabla_i R_{ik} = \frac{1}{2} \nabla_j \nabla_i S = 0.$$

Hence, from the above results and the following equation

$$g^{kl}\nabla_k \nabla_l (R_{ij}R^{ij}) = 2(\nabla_k R_{ij})(\nabla^k R^{ij}) + 2R^{ij}g^{kl}\nabla_l \nabla_k R_{ij},$$

we have that  $(M^n, g)$  is Ricci symmetric. The converse is obvious.

PROOF of Theorem 1. Since  $\nabla C = 0$  and  $S$  is constant, from Lemma 2, we have (4). Hence, using Lemma 3, we have that  $(M^n, g)$  is Ricci symmetric. Therefore, from Lemma 1, our theorem is proved.

Since  $C$  is identically zero for  $n=3$ , we have Lemma 1 without assumption of  $\nabla C = 0$ . Lemma 2 can be proved for 3-dimensional conformally flat Riemannian manifold. Therefore, we have

COROLLARY 1. *Let  $(M^n, g)$  ( $n \geq 3$ ) be a conformally flat Riemannian*

manifold with constant scalar curvature. If  $(M^n, g)$  satisfies  $(**)$ ,  $(M^n, g)$  is locally symmetric.

COROLLARY 2. Let  $(M^n, g) (n \geq 3)$  be an irreducible conformally flat Riemannian manifold with constant scalar curvature. If  $(M^n, g)$  satisfies  $(**)$ ,  $(M^n, g)$  is of constant curvature.

## § 2. Riemannian manifolds admitting projective transformations.

Let  $(M^n, g)$  and  $(\bar{M}^n, \bar{g}) (n \geq 2)$  be two Riemannian manifolds. A necessary and sufficient condition for a correspondence  $f$  from  $(M^n, g)$  to  $(\bar{M}^n, \bar{g})$  to be projective is that there exists a vector field  $\phi_i$  defined on  $(M^n, g)$  such that

$$\left\{ \begin{smallmatrix} \tilde{h} \\ ij \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} h \\ ij \end{smallmatrix} \right\} + \delta_i^k \phi_j + \delta_j^k \phi_i,$$

where  $\left\{ \begin{smallmatrix} h \\ ij \end{smallmatrix} \right\}$  and  $\left\{ \begin{smallmatrix} \tilde{h} \\ ij \end{smallmatrix} \right\}$  are coefficients of connections defined by  $g$  and  $f^* \bar{g}$ , respectively. Particularly, if  $\phi_i$  is non zero vector on  $(M^n, g)$ ,  $f$  is said to be proper, and if  $\phi_{ij}$  is non zero tensor on  $(M^n, g)$ ,  $f$  is said to be strictly proper, where  $\phi_{ij} = \nabla_j \phi_i - \phi_i \phi_j$ .  $f$  is not strictly proper if and only if  $f$  preserves the curvature tensor. Our purpose of this section is to prove the following theorem.

THEOREM 2. Let  $(M^n, g) (n \geq 2)$  be a Riemannian manifold admitting a strictly proper projective transformation. If  $(M^n, g)$  satisfies  $(*)$ ,  $(M^n, g)$  is an Einstein manifold.

We already have

LEMMA 4 (N. S. Sinyukov [7]). Let  $(M^n, g)$  and  $(\bar{M}^n, \bar{g}) (n \geq 2)$  be two Riemannian manifolds such that there exists a projective correspondence between  $(M^n, g)$  and  $(\bar{M}^n, \bar{g})$ . If  $(\bar{M}^n, \bar{g})$  satisfies  $(*)$ , one of the following two conditions must be satisfied.

$$(8) \quad R_{ij} - \frac{1}{n} S g_{ij} = 0.$$

$$(9) \quad \square \phi \cdot g_{ij} - n \phi_{ij} = 0,$$

where  $\square \phi = g^{ij} \phi_{ij}$ .

From LEMMA 4, we have

LEMMA 5. Let  $(M^n, g) (n \geq 2)$  be a Riemannian manifold satisfying  $(**)$  and  $(\bar{M}^n, \bar{g})$  a Riemannian manifold satisfying  $(*)$ . If there exists a projective correspondence  $f$  between  $(M^n, g)$  and  $(\bar{M}^n, \bar{g})$ ,  $(M^n, g)$  is an Einstein manifold, or  $f$  preserves the curvature tensor.

PROOF. Let  $\tilde{g}$  be a metric tensor induced from  $\bar{g}$  by  $f$  and  $\tilde{R}$  the

curvature tensor defined by  $\tilde{g}$ . Then, we have

$$(10) \quad \tilde{R}(X, Y)\tilde{R} = f^*(\bar{R}(\bar{X}, \bar{Y})\bar{R}) = 0,$$

where  $\bar{X}$  and  $\bar{Y}$  are tangent vectors on  $(\bar{M}^n, \bar{g})$  defined by  $f$  from  $X$  and  $Y$ . If we denote by  $\tilde{\nabla}$  the connection defined by  $\tilde{g}$ ,

$$(11) \quad \begin{aligned} \tilde{\nabla}_i \tilde{\nabla}_k \tilde{R}_{ij} - \tilde{\nabla}_k \tilde{\nabla}_i \tilde{R}_{ij} &= \nabla_i \nabla_k R_{ij} - \nabla_k \nabla_i R_{ij} \\ &\quad - (n-1)(\nabla_i \nabla_k \phi_{ij} - \nabla_k \nabla_i \phi_{ij}) \\ &\quad - \phi_{ik} R_{ij} + \phi_{il} R_{kj} - \phi_{jk} R_{il} + \phi_{jl} R_{ik}. \end{aligned}$$

Substituting  $\tilde{R}(X, Y)\tilde{R}ic = R(X, Y)Ric = 0$  and (9) in (11), we have

$$(12) \quad \frac{1}{n} \square \phi (g_{ik} R_{lj} - g_{il} R_{kj} + g_{jk} R_{il} - g_{jl} R_{ik}) = 0.$$

Contracting (12) by  $g^{ik}$ , we have

$$(13) \quad \frac{1}{n} \square \phi \left( R_{ij} - \frac{1}{n} S g_{ij} \right) = 0.$$

Thus, from (8), (9) and (13), we have Lemma 5.

The proof of Theorem 2 is evident from Lemma 5.

**COROLLARY 3.** *Let  $(M^n, g)$  ( $n \geq 2$ ) be a compact Riemannian manifold admitting a strictly proper projective transformation. If  $(M^n, g)$  satisfies (\*),  $(M^n, g)$  is of constant curvature.*

This corollary is proved by Theorem 2 and the following two theorems.

**THEOREM B** (A. Lichnerowicz [4]). *Let  $(M^n, g)$  ( $n \geq 2$ ) be a compact Ricci symmetric Riemannian manifold. If  $(M^n, g)$  satisfies (\*),  $(M^n, g)$  is locally symmetric.*

**THEOREM C** (N. S. Sinyukov [7]). *Let  $(M^n, g)$  ( $n \geq 2$ ) be an arbitrary Riemannian manifold and  $(\bar{M}^n, \bar{g})$  a locally symmetric Riemannian manifold. If there exists a proper projective correspondence between  $(M^n, g)$  and  $(\bar{M}^n, \bar{g})$ ,  $(M^n, g)$  and  $(\bar{M}^n, \bar{g})$  are of constant curvature.*

For an infinitesimal projective transformation, we have Lemma 6 by the same way as Lemma 4.

**LEMMA 6.** *Let  $(M^n, g)$  ( $n \geq 2$ ) be a Riemannian manifold admitting an infinitesimal projective transformation. If  $(M^n, g)$  satisfies (\*), one of the following two conditions must be satisfied*

$$(14) \quad R_{ij} - \frac{1}{n} S g_{ij} = 0.$$

$$(15) \quad g^{ki} \nabla_i \phi_k - n \nabla_j \phi_j = 0,$$

where  $\phi_i$  is the vector field on  $(M^n, g)$  such that  $\mathfrak{L}_X \left\{ \begin{smallmatrix} h \\ ij \end{smallmatrix} \right\} = \delta_i^h \phi_j + \delta_j^h \phi_i$  for an infinitesimal projective transformation  $X$  and  $\mathfrak{L}_X$  denotes the Lie derivative with respect to  $X$ .

From Lemma 6, we can obtain the following.

**THEOREM 3.** *Let  $(M^n, g)$  ( $n \geq 3$ ) be a compact Riemannian manifold admitting a proper infinitesimal projective transformation (i.e.,  $\phi_i \neq 0$ ). If  $(M^n, g)$  satisfies (\*),  $(M^n, g)$  is of constant curvature.*

In order to prove this theorem, we use the following three theorems.

**THEOREM D** (S. Ishihara and Y. Tashiro [3]). *Let  $(M^n, g)$  ( $n \geq 2$ ) be a complete Riemannian manifold. In order for  $(M^n, g)$  to admit a non-constant solution  $\rho$  for the system of partial differential equations*

$$\nabla_j \nabla_i \rho = \frac{1}{n} g^{ki} \nabla_i \nabla_k \rho \cdot g_{ij},$$

*it is necessary and sufficient that  $(M^n, g)$  be conformal to a sphere in the  $(n+1)$ -dimensional Euclidean space.*

**THEOREM E** (K. Sekigawa and H. Takagi [6]). *Let  $(M^n, g)$  ( $n \geq 3$ ) be a complete conformally flat Riemannian manifold. If  $(M^n, g)$  satisfies (\*\*),  $(M^n, g)$  is locally symmetric.*

**THEOREM F** (K. Yano and T. Nagano [9]). *Let  $(M^n, g)$  ( $n \geq 3$ ) be a locally symmetric Riemannian manifold. If  $(M^n, g)$  admits a proper infinitesimal projective transformation,  $(M^n, g)$  is of constant curvature.*

**PROOF** of Theorem 3. Since  $\phi_i$  is a gradient vector, from Theorem D, we have that  $(M^n, g)$  satisfying (15) is conformal to a sphere. Therefore,  $(M^n, g)$  is conformally flat. Applying this fact to Theorem E, it follows that  $(M^n, g)$  satisfying (15) is locally symmetric. Thus,  $(M^n, g)$  is Ricci symmetric all over. From Theorem B and (\*),  $(M^n, g)$  is locally symmetric all over. Then, from Theorem F, our theorem is proved.

### § 3. Conformally symmetric Riemannian manifolds admitting projective transformations.

The following lemma is a generalization of the result obtained by T. Adati and T. Miyazawa [1].

**LEMMA 7.** *Let  $(M^n, g)$  ( $n \geq 4$ ) be a Riemannian manifold satisfying (3). If  $(M^n, g)$  admits a vector field  $v_i$  such that*

$$(16) \quad \nabla_j v_i - \alpha v_i v_j = 0 \quad \text{for a certain constant } \alpha,$$

*then,  $(M^n, g)$  is with constant scalar curvature, or  $v_i$  is a zero vector field.*

Proof. Applying the Ricci's identity to (16), we have

$$(17) \quad \nu_h R^h_{ijk} = 0.$$

Derivating (17) by  $\nabla_i$  and using (16), we get

$$(18) \quad \nu_h \nabla_i R^h_{ijk} = 0, \quad \nu_h \nabla_i R^h_k = 0 \quad \text{and} \quad \nu^h \nabla_h S = 0.$$

Applying the Bianchi's identity to the first equation of (18) and substituting the second into it, we have

$$(19) \quad \nu^h \nabla_h R_{ij} = 0.$$

On the other hand, contracting (5) by  $\nu^h$ , we have

$$(20) \quad \nu^h \nabla_j R_{ih} - \nu^h \nabla_h R_{ij} - \frac{1}{2(n-1)} (\nu_i \nabla_j S - g_{ij} \nu^h \nabla_h S) = 0.$$

Substituting (18) and (19) into (20), we have

$$\nu_i \nabla_j S = 0.$$

Therefore,  $S$  is constant, or  $\nu_i$  is a zero vector field.

**THEOREM 4.** *Let  $(M^n, g)$  ( $n \geq 4$ ) be a conformally symmetric Riemannian manifold satisfying  $(**)$  and  $(\bar{M}^n, \bar{g})$  a Riemannian manifold satisfying  $(*)$ . If there exists a proper projective correspondence between  $(M^n, g)$  and  $(\bar{M}^n, \bar{g})$ ,  $(M^n, g)$  and  $(\bar{M}^n, \bar{g})$  are of constant curvature.*

**PROOF.** From Lemma 5, we have that  $(M^n, g)$  is an Einstein manifold or  $\phi_{ij}=0$ . When  $\phi_{ij}=0$ , from Lemma 7,  $(M^n, g)$  is with constant scalar curvature, because a given correspondence is proper. Therefore, from Theorems C and 1, we have this theorem.

**COROLLARY 4.** *Let  $(M^n, g)$  ( $n \geq 4$ ) be a conformally symmetric Riemannian manifold admitting a proper projective transformation. If  $(M^n, g)$  satisfies  $(**)$ ,  $(M^n, g)$  is of constant curvature.*

Department of Mathematics,  
Hokkaido University.

### References

- [1] T. ADATI and T. MIYAZAWA: On conformally symmetric spaces, Tensor (N.S.), 18 (1967), 335-342.
- [2] M. C. CHAKI and B. GUPTA: On conformally symmetric spaces, Indian J. Math., 5 (1963), 113-122.
- [3] S. ISHIHARA and Y. TASHIRO: On Riemannian manifolds admitting a concircular transformation, Math. J. Okayama Univ., 9 (1959), 19-47.
- [4] A. LICHNEROWICZ: Géométrie des groupes de transformations, Dunod, Paris,

- 1958.
- [5] K. NOMIZU: On hypersurfaces satisfying a certain condition on the curvature tensor, *Tôhoku Math. J.*, 20 (1968), 46-59.
  - [6] K. SEKIGAWA and H. TAKAGI: On the conformally flat spaces satisfying a certain condition on the Ricci tensor, to appear.
  - [7] N. S. SINYUKOV: On the geodesic correspondence of a Riemannian space with a symmetric space, *Dokl. Akad. Nauk, SSSR*, 98 (1954), 21-23.
  - [8] K. YAMAUCHI: On a 4-dimensional Riemannian manifold satisfying a certain condition on the curvature tensor, to appear.
  - [9] K. YANO and T. NAGANO: Some theorems on projective and conformal transformations, *Indag. Math.*, 19 (1957), 451-458.

(Received March 26, 1971)