A remark on the Steenrod representation of $B(Z_p \times Z_p)$

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction

For a topological space X, $z \in H_n(X; Z)$ is Steenrod representable if there exists a closed oriented smooth n-manifold M and a continuous map $f: M \to X$ such that $f_*(\sigma)=z$, where σ is a fundamental homology class of M. In [4], Thom showed that for a finite polyhedron X any $z \in H_n(X; Z)$ is representable if $n \leqq 6$, but if $n \geqq 7$ not everything is representable. He exhibited a class in $H_7(L^7(3) \times L^7(3); Z)$ which was not, where $L^7(3)$ is 7-dimesional lens space mod 3. Moreover Burdick [1] extended to $B(Z_3 \times Z_3)$, classifying space of $Z_3 \times Z_3$, and computed all representable elements. He dermined E^∞ terms of bordism spectral sequence of $B(Z_3 \times Z_3)$ and used necessary condition of representability of Thom [4].

In this note we show the case $p=2$ and any odd prime p. Latter case we use the same methods as Burdick's.

We have

THEOREM 1.

(*a*) *Every elements of* $H_*(B(Z_2 \times Z_2); Z)$ *are Steenrod representable.*

(*b*) *For p an odd prime the elements of* $H_*(B(Z_p \times Z_p); Z)$ *which are Steenrod representable are generated by* $e_0 \otimes e_0$, $e_{2i-1} \otimes e_{2j-1}$, $e_0 \otimes e_{2j-1}$, $e_{2i-1} \otimes e_0$, $\{(e_2 \otimes e_{2j-1} + e_1 \otimes e_{2j}) + (e_6 \otimes e_{2j-5} + e_5 \otimes e_{2j-4}) + \cdots\}$, *and* $\{(e_4 \otimes e_{2j-3} + e_3 \otimes e_{2j-2}) + (e_8 \otimes e_{2j-7} + e_7 \otimes e_{2j-6}) + \cdots\}$.

The author whishes to express his thanks to Professors H. Suzuki and F. Uchida for their many valuable suggestions.

§ 2. Homology groups of $B(Z_p \times Z_p)$

Let $X=B(Z_p \times Z_p)$, $Y=B(Z_p)$.

Case (a): $p=2$.

Let RP^n be the n dimensional real projective space, RP^∞ be the direct limit of it. Then we can consider $Y=RP^\infty$, and so $X=Y \times Y$. The cell structure of RP^n and its boundary operations are given as follows:

$$RP^n = e_0 \cup e_1 \cup \cdots \cup e_n,$$

$$\partial e_{2i} = 2e_{2i-1}\,, \qquad \partial e_{2i+1} = 0\,, \tag{1.1}$$

where e_i is the i dimensional cell. Y is a CW complex with one cell e_i in each dimension. We will use the same symbol e_i for the homology class containing e_i.

Let $C_*(X)$ and $C_*(Y)$ be the chain complexes as CW complex X and Y respectively. $C_*(X)\cong C_*(Y)\otimes C_*(Y)$ by cross product, thus $C_n(X)=H_n(X^n, X^{n-1}; Z)\cong \sum_{i+j=n} H(Y^i, Y^{i-1}; Z)\otimes H_j(Y^j, Y^{j-1}; Z)$, where X^n and Y^n are n-skeleton of X and Y respectively. Therefore $C_n(X)$ is generated by $e_i\otimes e_{n-i}$ for $i=0, 1, \cdots, n$ and $\partial_n : C_n(X)\to C_{n-1}(X)$ is given as follows:

$$\begin{aligned}
&\partial_n(e_{2i-1}\otimes e_{2j-1}) = 0\,,\\
&\partial_n(e_{2i}\otimes e_{2j-1}) = 2e_{2i-1}\otimes e_{2j-1}\,,\\
&\partial_n(e_{2i-1}\otimes e_{2j}) = -2e_{2i-1}\otimes e_{2j-1}\,,\\
&\partial_n(e_{2i}\otimes e_{2j}) = 2e_{2i-1}\otimes e_{2j} + 2e_{2i}\otimes e_{2j-1}\,.
\end{aligned} \tag{1.2}$$

Then we have

(1.3) *$H_{2n}(X; Z)$ is generated by $e_{2i-1}\otimes e_{2n-2i+1}$ for $i=1, \cdots, n$ and $H_{2n-1}(X; Z)$ is generated by $e_{2i-1}\otimes e_{2n-2i}+e_{2i}\otimes e_{2n-1-2i}$ for $i=0, 1, \cdots, n$ and every elements are order 2. $H_0(X; Z)\cong Z$ generated by $e_0\otimes e_0$.*

Case (b): p is the odd prime.

Let S^{2n+1} be the unit $(2n+1)$-sphere. A point of S^{2n+1} is represented by a $(n+1)$-tuple of complex numbers $(z_0, z_1, \cdots, z_n)$ with $\sum_{i=0}^{n}|z_i|^2=1$. Let T be the rotation of S^{2n+1} defined by $T(z_0, z_1, \cdots, z_n)=(\lambda z_0, \lambda z_1, \cdots, \lambda z_n)$, where $\lambda=\exp(2\pi i/p)$. T generates a fixed point free topological transformation group of S^{2n+1} of order p, so we will say it Z_p action on S^{2n+1}. Then the lens space mod p is defined to be the orbit space $L^{2n+1}(p)=S^{2n+1}/Z_p$. This is the closed orientable $2n+1$ smooth manifold. For $m<n$ consider S^{2m+1} as contained in S^{2n+1} with $(z_0, \cdots, z_m)=(z_0, \cdots, z_m, 0, 0, \cdots)$. Then $L^1(p)\subset L^3(p)\subset\cdots$. Let $L^\infty(p)$ be the direct limit of this sequence, then we can consider $Y=L^\infty(p)$, and so $X=Y\times Y$. The cell structure of $L^{2n+1}(p)$, and its boundary relations are given as follows:

$$L^{2n+1}(p) = e_0\cup e_1\cup\cdots\cup e_{2n+1}\,,$$

$$\partial e_{2i} = pe_{2i-1}\,, \qquad \partial e_{2i+1} = 0\,. \tag{1.4}$$

Y is a CW complex with one cell e_i in each dimension and the $(2n+1)$-skeleton is $L^{2n+1}(p)$. $C_n(X)$ is generated by $e_i\otimes e_{n-i}$ $(i=0, 1, \cdots, n)$ and ∂_n: $C_n(X)\to C_{n-1}(X)$ is given as follows:

(1. 5)
$$\begin{aligned}\partial_n(e_{2i-1}\otimes e_{2j-1})&=0\,,\\ \partial_n(e_{2i}\otimes e_{2j-1})&=p\,e_{2i-1}\otimes e_{2j-1}\,,\\ \partial_n(e_{2i-1}\otimes e_{2j})&=-p\,e_{2i-1}\otimes e_{2j-1}\,,\\ \partial_n(e_{2i}\otimes e_{2j})&=p\,e_{2i-1}\otimes e_{2j}+p\,e_{2i}\otimes e_{2j-1}\,.\end{aligned}$$

Then we have

(1. 6) *$H_{2n}(X;Z)$ is generated by $e_{2i-1}\otimes e_{2n-2i+1}$ $(i=1,\cdots,n)$, $H_{2n-1}(X;Z)$ is generated by $e_{2i-1}\otimes e_{2n-2i}+e_{2i}\otimes e_{2n-2i-1}$ $(i=0,1,\cdots,n)$ and every elements are order p. $H_0(X;Z)\cong Z$ generated by $e_0\otimes e_0$.*

§ 3. Theorems

Let $\Omega_n(X, A)$ be n-dimensional oriented bordism group of (X, A). There is a natural homomorphism $\mu:\Omega_n(X, A)\rightarrow H_n(X, A;Z)$. Given $[B^n, f]\in\Omega_n(X, A)$, let $\sigma_n\in H_n(B^n, \partial B^n;Z)$ denote the fundamental homology class of B^n. Then μ is defined $\mu[B^n, f]=f_*(\sigma_n)\in H_n(X, A;Z)$. The image of μ is the subgroup of integral homology classes representable in the sense of Steenrod. μ has following properties which are proved by Conner-Floyd.

THEOREM 2. (Conner-Floyd) ([2], (7. 2))

The edge homomorphism $\Omega_n(X, A)=J_{n,0}\rightarrow E^\infty_{n,0}\rightarrow E^2_{n,0}=H_n(X, A;Z)$ of the bordism spectral sequence coincides with the homomorphism $\mu:\Omega_n(X, A)\rightarrow H_n(X, A;Z)$.

THEOREM 3. (Conner-Floyd) ([2], (15. 1))

If (X, A) is a CW pair then the bordism spectral sequence is trivial if and only if $\mu:\Omega_n(X, A)\rightarrow H_n(X, A;Z)$ is an epimorphism for all $n\geqq 0$.

THEOREM 4. (Conner-Floyd) ([2], (15. 2))

If (X, A) is a CW pair such that each $H_n(X, A;Z)$ is finitely generated and has no odd torsion, then the bordism spectral sequence is trivial.

Next theorem is useful to obtain the manifold with Z_p action.

THEOREM 5. (Conner-Floyd) ([2], (46. 1))

Consider the generating set α_{2k-1}; $k=1,2,\cdots$ for $\Omega_*(Z_p)$, p an odd prime, where $\alpha_{2k-1}=[T, S^{2k-1}]$. Then there exist closed oriented manifolds M^{4k}, $k=1,2,\cdots$, such that for each k, $p\alpha_{2k-1}+[M^4]\alpha_{2k-5}+[M^8]\alpha_{2k-9}+\cdots=0$ in $\Omega_*(Z_p)$.

§ 4. Proof of Theorem 1.

Case (a): $p=2$.

This case follows immediately from Theorems 3 and 4. Because each

$H_n(B(Z_2\times Z_2); Z)$ is finitely generated and has no odd torsion from (1. 3).

REMARK. $e_0\otimes e_0$, $e_{2i-1}\otimes e_0$, $e_0\otimes e_{2j-1}$ and $e_{2i-1}\otimes e_{2j-1}$ are explicitly represented by $RP^0\times RP^0$, $RP^{2i-1}\times RP^0$, $RP^0\times RP^{2j-1}$ and $RP^{2i-1}\times RP^{2j-1}$ respectively. $e_{2i-1}\otimes e_{2n-2i}+e_{2i}\otimes e_{2n-1-2i}$ is represented by $H_{2i,2n-2i}$ which is the subset in $RP^{2i}\times RP^{2n-2i}$ defined by the equation

$$x_0y_0+x_1y_1+\cdots+x_my_m=0\,,$$

where $m=\min(2i, 2n-2i)$, and $(x_0, \cdots, x_{2i})$ and $(y_0, \cdots, y_{2n-2i})$ are the standard homogeneous coordinates in RP^{2i} and RP^{2n-2i} respectively. It is a smooth submanifold of codimension 1, and orientable because its first *Stiefel-Whitney* class $w_1=0$. Consider the intersection of $H_{2i,2n-2i}$ and 1 cycles of $RP^{2i}\times RP^{2n-2i}$ we can see that $i_*: H_{2n-1}(H_{2i,2n-2i}; Z)\to H_{2n-1}(RP^{2i}\times RP^{2n-2i}; Z)$ is non-trivial, that is onto.

Case (b): p an odd prime.

By Theorem 5 there exists compact orientable $2n$ dimensional manifold V^{2n} with $\partial V^{2n}=pS^{2n-1}\cup M^4\times S^{2n-5}\cup M^8\times S^{2n-9}\cup\cdots$ and an action of Z_p restricted to $M^{4k}\times S^{2n-4k-1}$ is $id\times T$. We can chose following classifying maps from the property of classifying space:

$$f_{2n}: V^{2n}/Z_p\longrightarrow Y=B(Z_p)\qquad \text{such that}$$
$$f_{2n}(V^{2n}/Z_p)\subseteq Y^{2n}, \quad f_{2n}(M^{4k}\times S^{2n-4k-1}/Z_p)\subseteq Y^{2n-4k-1}$$

and $f_{2n*}(\sigma_{2n})=e_{2n}$, where σ_{2n} is fundamental homology class of V^{2n}/Z_p.

Let $f_0: V^0/Z_p\to Y^0$ and let $f_{2n-1}: S^{2n-1}/Z_p\to Y^{2n-1}$ be inclusion, then $f_{2n-1*}(\sigma'_{2n-1})=e_{2n-1}$, where σ'_{2n-1} is fundamental class of S^{2n-1}/Z_p.

Next let $G=Z_p\times Z_p$ and choose classifying maps

$$\begin{aligned} g_j &: S^{2j-1}\times S^{2n-2j+1}/G\longrightarrow X^{2n}\,,\\ h_j &: V^{2j}\times S^{2n-2j-1}/G\longrightarrow X^{2n-1}\,,\\ k_j &: S^{2j-1}\times V^{2n-2j}/G\longrightarrow X^{2n-1}\,,\\ l_j &: V^{2j}\times V^{2n-2j}/G\longrightarrow X^{2n}\end{aligned}$$

such that
$$h_j(M^{4k}\times S^{2j-4k-1}\times S^{2n-2j-1}/G)\subseteq X^{2n-4k-2}\,,$$
$$k_j(M^{4k}\times S^{2j-1}\times S^{2n-2j-4k-1}/G)\subseteq X^{2n-4k-2}\,,$$

and
$$l_j\Big(\Big\{(M^{4k}\times V^{2j}\times S^{2n-2j-4k-1}/G)\cup(M^{4k}\times S^{2j-4k-1}\times V^{2n-2j}/G)\Big\}\Big)\subseteq X^{2n-4k-1}.$$

Then each fundamental class is mapped onto $e_{2j-1}\otimes e_{2n-2j+1}$, $e_{2j}\otimes e_{2n-2j-1}$, $e_{2j-1}\otimes e_{2n-2j}$ and $e_{2j}\otimes e_{2n-2j}$ by g_{j*}, h_{j*}, k_{j*} and l_{j*} respectively.

Let
$$\begin{aligned} \alpha_j^{2n} &= [g_j,\ S^{2j-1}\times S^{2n-2j+1}/G]\,, && j=1,\cdots,n,\\ \delta_j^{2n} &= [l_j,\ V^{2j}\times V^{2n-2j}/G]\,, && j=0,\cdots,n,\\ \beta_j^{2n-1} &= [h_j,\ V^{2j}\times S^{2n-2j-1}/G]\,, && j=0,\cdots,n-1,\\ \gamma_j^{2n-1} &= [k_j,\ S^{2j-1}\times V^{2n-2j}/G]\,, && j=1,\cdots,n. \end{aligned}$$

Then α_j^{2n} and δ_j^{2n} generate $\Omega_*(X^{2n}, X^{2n-1})$ freely over Ω_*, and β_j^{2n-1} and γ_j^{2n-1} generate $\Omega_*(X^{2n-1},X^{2n-2})$ freely over Ω_*, because $\mu : \Omega_*(X^r, X^{r-1})\to H_*(X^r, X^{r-1}; \Omega_*)$ is an Ω_* isomorphism.

LEMMA.

C^2-term of bordism spectral sequence of $X=B(Z_p\times Z_p)$ is generated over Ω_* by δ_0^0, α_i^{2n}, β_0^{2n-1}, γ_n^{2n-1}, and $(\beta_j^{2n-1}+\gamma_j^{2n-1})$, $(n=1,2,\cdots;\ i=1,\cdots,n;\ j=1,\cdots,n-1)$ and B^2-term is generated over Ω_* by $p\alpha_i^{2n}$, $p\beta_0^{2n-1}$, $p\gamma_n^{2n-1}$ and $p(\beta_j^{2n-1}+\gamma_j^{2n-1})$, $(n=1,2,\cdots;\ i=1,\cdots,n;\ j=1,\cdots,n-1)$.

PROOF.

$$\begin{aligned} C_*^2 &= Ker(\partial : \Omega_*(X^r, X^{r-1})\longrightarrow \Omega_*(X^{r-1}, X^{r-2}))\\ &= \mu^{-1}(Ker\ \partial : H_*(X^r, X^{r-1}; \Omega_*)\longrightarrow H_*(X^{r-1}, X^{r-2}; \Omega_*)) \end{aligned}$$

and

$$\partial\mu(\delta_j^{2n})=pe_{2j-1}\otimes e_{2n-2j}+pe_{2j}\otimes e_{2n-2j-1}\,,$$
$$\partial\mu(\alpha_j^{2n})=0\,,\quad \partial\mu(\beta_j^{2n-1})=pe_{2j-1}\otimes e_{2n-2j-1}$$

and

$\partial\mu(\gamma_j^{2n-1})=-pe_{2j-1}\otimes e_{2n-2j-1}$ therefore C^2-term follows.

$$\begin{aligned} B_*^2 &= Im\,(\partial : \Omega_*(X^{r+1}, X^r)\longrightarrow \Omega_*(X^r, X^{r-1}))\\ &= \mu^{-1}Im(\partial : H_*(X^{r+1}, X^r; \Omega_*)\longrightarrow H_*(X^r, X^{r-1}; \Omega_*)) \end{aligned}$$

and $\mu^{-1}\partial(e_{2j}\otimes e_{2n+1-2j})=p\alpha_j^{2n}$, $\mu^{-1}\partial(e_{2j-1}\otimes e_{2n-2j+2})=-p\alpha_j^{2n}$, $\mu^{-1}\partial(e_{2j}\otimes e_{2n-2j})=p(\gamma_j^{2n-1}+\beta_j^{2n-1})$, $\mu^{-1}\partial(e_0\otimes e_{2n})=p\beta_0^{2n-1}$, $\mu^{-1}\partial(e_{2n}\otimes e_0)=p\gamma_n^{2n-1}$ and $\mu^{-1}\partial(e_{2j-1}\otimes e_{2n+1-2j})=0$, so we have B^2-term.

Next theorem essentially is the same as the case $p=3$ proved by Burdick [1].

THEOREM 6. *The bordism spectral sequence of $X=B(Z_p\times Z_p)$ is as follows:*

$$E^2\cong\cdots\cong E^5;\quad E^6\cong\cdots\cong E^\infty$$

E^∞ is generated by δ_0^0, α_i^{2n}, β_0^{2n-1}, γ_n^{2n-1} $(n=1,2,\cdots,\ i=1,2,\cdots,n)$, $\{(\beta_1^{2n-1}+\gamma_1^{2n-1})+(\beta_3^{2n-1}+\gamma_3^{2n-1})+\cdots\}$, and $\{(\beta_2^{2n-1}+\gamma_2^{2n-1})+(\beta_4^{2n-1}+\gamma_4^{2n-1})+\cdots\}$ with relations $[M^4][\alpha_i^{2n}-\alpha_{i-2}^{2n}]=0$, and every element except δ_0^0 has order p.

PROOF.

Every elements of $H_m(X; Z)$ have order p (odd prime). Ω_n is free group if $n\equiv 0$ $(mod\ 4)$ and 2-torsion groups if $n\not\equiv 0$ $(mod\ 4)$.

$$E^2_{m,n}=H_m(X;\Omega_n)=H_m(X;Z)\otimes\Omega_n+H_{m-1}(X;Z)*\Omega_n\,.$$

Therefore we have $d^2=d^3=d^4=0$, so $E^2\cong\cdots\cong E^5$. Now recall the definition of $d^r_{m,n}$:

$$\begin{tikzcd}
 & & \Omega_{m+n}(X^{m-1},X^{m-r}) \arrow[dl, dashed, "i''_*"'] \arrow[d, "\partial_3"] \\
 & \Omega_{m+n}(X^m,X^{m-r}) \arrow[r, "\partial_2"] \arrow[d, "j_*"'] \arrow[dr, dashed, "\Psi"] & \Omega_{m+n-1}(X^{m-r},X^{m-r-1}) \arrow[d, "i'_*"] \\
\Omega_{m+n}(X^m,X^{m-r-1}) \arrow[ur, dashed, "i_*"] \arrow[r, "j'_*"'] & \Omega_{m+n}(X^m,X^{m-1}) \arrow[r, "\partial_1"'] & \Omega_{m+n-1}(X^{m-1},X^{m-r-1}),
\end{tikzcd}$$

where i, j, i', j' are inclusion maps and ∂_1, ∂_2, ∂_3 are boundary homomorphisms of triple, then there exist homomorphism Ψ such that $\Psi=\partial_1\cdot j_*=i'_*\cdot\partial_2$, every triangles are commutative.

Let $C^r_{m,n}=Im\,j_*$, $C^{r+1}_{m,n}=Im\,j'_*$, $B^{r+1}_{m-r,n+r-1}=Im\,\partial_2$ and $B^r_{m-r,n+r-1}=Im\,\partial_3$. Then the definition of $d^r_{m,n}$ is composition of $d^r_{m,n}: E^r_{m,n}=C^r_{m,n}/B^r_{m,n}\xrightarrow[p_r]{}C^r_{m,n}/C^{r+1}_{m,n}\xrightarrow[\cong]{\partial_1}$ $Im\,\Psi\xrightarrow[\cong]{i_*^{-1}}B^{r+1}_{m-r,n+r-1}/B^r_{m-r,n+r-1}\longrightarrow C^r_{m-r,n+r-1}/B^r_{m-r,n+r-1}=E^r_{m-r,n+r-1}$. Here let $r=5$ then $d^5(\delta^0_0)=d^5(\alpha^{2n}_i)=d^5(\beta^{2n-1}_0)=d^5(\gamma^{2n-1}_n)=0$. Because δ^0_0, α^{2n}_i, β^{2n-1}_0, γ^{2n-1}_n are represented by closed manifolds ∂_1 will kill them.

For $j=1,\cdots,n-1$ let N^{2n-1}_j be the manifold obtained from $V^{2j}\times S^{2n-2j-1}\cup S^{2j-1}\times V^{2n-2j}$ by joining $pS^{2j-1}\times S^{2n-2j-1}$ in $\partial(V^{2j}\times S^{2n-2j-1})$ to $-pS^{2j-1}\times S^{2n-2j-1}$ in $\partial(S^{2j-1}\times V^{2n-2j})$.

Then $\partial N^{2n-1}_j=M^4\times S^{2j-5}\times S^{2n-2j-1}\cup -M^4\times S^{2j-1}\times S^{2n-2j-5}$
$\cup M^8\times S^{2j-9}\times S^{2n-2j-1}\cup -M^8\times S^{2j-1}\times S^{2n-2j-9}\cup\cdots$.

There is an induced action of $G=Z_p\times Z_p$ on N^{2n-1}_j. Choose classifying maps $\phi_j: N^{2n-1}_j/G\longrightarrow X^{2n-1}$ such that $\phi_j(\partial(N^{2n-1}_j/G))\subseteq X^{2n-6}$ and such that

$$\begin{tikzcd}
N^{2n-1}_j/G \arrow[rr, "\phi_j"] & & X^{2n-1} \\
 & (V^{2j}\times S^{2n-2j-1}/G)\cup(S^{2j-1}\times V^{2n-2j}/G) \arrow[ul] \arrow[ur, "h_j\cup k_j"'] &
\end{tikzcd}$$

commutes up to homotopy.

Then $\phi_{j*}(\sigma)=e_{2j}\otimes e_{2n-2j-1}+e_{2j-1}\otimes e_{2n-2j}$, where σ is a fundamental class of

N_j^{2n-1}/G. Thus $[\phi_j, N_j^{2n-1}/G]=\beta_j^{2n-1}+\gamma_j^{2n-1}$ in $\Omega_*(X^{2n-1}, X^{2n-2})$.
By the definition of d^5, $d^5(\beta_j^{2n-1}+\gamma_j^{2n-1})=[M^4][\alpha_{j-2}^{2n-6}-\alpha_j^{2n-6}]$. Therefore Ker d^5 is generated by δ_0^0, α_i^{2n}, β_0^{2n-1}, γ_n^{2n-1} and $\{(\beta_1^{2n-1}+\gamma_1^{2n-1})+(\beta_3^{2n-1}+\gamma_3^{2n-1})+\cdots\}$ and $\{(\beta_2^{2n-1}+\gamma_2^{2n-1})+(\beta_4^{2n-1}+\gamma_4^{2n-1})+\cdots\}$.
Let K_1^{2n-1} be the identification manifold obtained from

$$V^2\times S^{2n-3}\cup S^1\times V^{2n-2}\cup V^6\times S^{2n-7}\cup S^5\times V^{2n-6}\cup V^{10}\times S^{2n-11}\cup\cdots$$

by identifying pair-wise of boundary components of this manifold.

Then K_1^{2n-1} is an orientable closed $(2n-1)$-manifold with induced natural action of $G=Z_p\times Z_p$.
Let $\Psi_1 : K_1^{2n-1}/G\to X^{2n-1}$ be a classifying map, then $[\Psi_1, K_1^{2n-1}/G]=(\beta_1^{2n-1}+\gamma_1^{2n-1})+(\beta_3^{2n-1}+\gamma_3^{2n-1})+\cdots$ in $\Omega_*(X^{2n-1}, X^{2n-2})$. Likewise construct K_2^{2n-1} from

$$V^4\times S^{2n-5}\cup S^3\times V^{2n-4}\cup V^8\times S^{2n-9}\cup S^7\times V^{2n-8}\cup\cdots$$

and $\Psi_2 : K_2^{2n-1}/G\to X^{2n-1}$ with

$$[\Psi_2, K_2^{2n-1}/G]=(\beta_2^{2n-1}+\gamma_2^{2n-1})+(\beta_4^{2n-1}+\gamma_4^{2n-1})+\cdots$$

Therefore every generator of E^6 can be represented by a closed manifolds, so $d^6=d^7=\cdots=0$ and hence $E^6\cong\cdots\cong E^\infty$.

PROOF of THEOREM 1.

The classes listed in Theorem 6 really belong to $E^\infty_{*,0}$. Therefore from Theorems 2 and 6 $e_0\otimes e_0$, $e_0\otimes e_{2j-1}$, $e_{2i-1}\otimes e_0$ and $e_{2i-1}\otimes e_{2j-1}$ are represented by $V^0\times V^0/G$, $V^0\times S^{2j-1}/G$, $S^{2i-1}\times V^0/G$ and $V^{2i-1}\times V^{2j-1}/G$ respectively.

$$(e_2\otimes e_{2j-1}+e_1\otimes e_{2j})+(e_6\otimes e_{2j-5}+e_5\otimes e_{2j-4})+\cdots \quad \text{and}$$
$$(e_4\otimes e_{2j-3}+e_3\otimes e_{2j-2})+(e_8\otimes e_{2j-7}+e_7\otimes e_{2j-6})+\cdots \quad \text{are}$$

represented by K_1^{2j+1}/G and K_2^{2j+1}/G respectively.

The proof of Theorem 1 is completed.

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(Received, August 2, 1971)