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On homotopically trivial links

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction.

Throughout this paper we shall only be concerned with the combinatorial category, consisting of simplicial complexes and piecewise linear maps (for the combinatorial category see [3]). Zeeman [3] shows that if $(n, k, 2)$ -link $L = (S^n \supset K_1^k \cup K_2^k)$ is homotopically trivial and if $2n \geq 3k + 4$, then L is geometrically trivial. And if $n = k + 2$, it is well known that there exists a homotopically trivial $(n, n-2, 2)$ -link which is not geometrically trivial (for example, in $(3, 1, 2)$ -link). For the case $n \geq k + 3$ and $2n \leq 3k + 3$ Zeeman says that there exist homotopically trivial links which are not geometrically trivial same as $(3, 1, 2)$ -link. But there is no proof for these links be geometrically non-trivial. So we consider the relation between homotopically trivial links and geometrically trivial links under $n \geq k + 3$ and $2n \leq 3k + 3$. In this paper we obtain a geometrical sufficient condition for a homotopically trivial $(n, k, 2)$ -link be geometrically trivial. I should like to express my sincere gratitude to the members of Kōbe and Hokkaidō topology seminars for many discussion of this problem.

§ 2. Notations and Definitions

S^n is a standard n -sphere and D^n is a standard n -cell. An $(n, k, 2)$ -link L is a pair $(S^n \supset K_1^k \cup K_2^k)$ of an n -sphere S^n and a disjoint union of locally flatly embedded k -spheres K_i^k , $i=1, 2$ in S^n . ∂X and $\text{Int } X$ mean the boundary and the interior of a manifold X . $X * Y$ denote the join of spaces X and Y . $\cong$ means "homeomorphic to". $\bigvee_{i=1}^m S_i^p$ means one point join of p -spheres $S_1^p, \dots, S_m^p$ and $I = [0, 1]$. For any manifolds X, Y such that X is a submanifold of Y , $U(X, Y)$ means a regular neighborhood of X in Y . And we always take $U(X, Y)$ to be a second derived neighborhood of X in Y for a suitable subdivision of X and Y unless otherwise stated.

DEFINITION 1. Let X, Y be subsets in an n -manifold Z . We say that X and Y *split each other* in Z if there exists an n -cell B^n in Z such that either $X \subset \text{Int } B^n$, $Y \cap B^n = \phi$ or $Y \subset \text{Int } B^n$, $X \cap B^n = \phi$.

DEFINITION 2. We say that an $(n, k, 2)$ -link $L = (S^n \supset K_1^k K_2^k)$ is *geometrically trivial* or briefly *G-trivial* if there exist locally flatly embedded $(k+1)$ -cells B_1^{k+1}, B_2^{k+1} in S^n such that $\partial B_i^{k+1} = K_i^k$, $i=1, 2$ and $B_1^{k+1} \cap B_2^{k+1} = \emptyset$.

REMARK 1. When $n \geq k+3$ it is sufficient that there exists a locally flatly embedded $(k+1)$ -cell B^{k+1} which is bounded by K_1^k or K_2^k in $S^n - K_2^k$ or $S^n - K_1^k$. For if $\partial B^{k+1} = K_1^k$ and $B^{k+1} \subset S^n - K_2^k$, by *Zeeman's Unknotting theorem* [4] we can find a locally flatly embedded $(k+1)$ -cell B_2^{k+1} bounded by K_2^k in $S^n - U(B_1^{k+1}, S^n - K_2^k) \cong D^n$.

DEFINITION 3. We say that an $(n, k, 2)$ -link $L = (S^n \supset K_1^k \cup K_2^k)$ is *homotopically trivial* or briefly *H-trivial* if K_1 is contractible in $S^n - \text{Int } U(K_2, S^n)$ and if K_2 is contractible in $S^n - \text{Int } U(K_1, S^n)$.

DEFINITION 4. We say that an $(n, k, 2)$ -link $L = (S^n \supset K_1^k \cup K_2^k)$ is *weak H-trivial* if either K_1^k or K_2^k is contractible in the complementary of the other or equivalently if there exists a map $F : (S_1^k \cup S_2^k) \times I \rightarrow S^n$ such that

- (1) $F_0(S_i^k) = K_i^k$, $i=1, 2$,
- (2) $(S^n \supset F_1(S_1^k) \cup F_1(S_2^k))$ is a G-trivial link,
- (3) $F_t(S_1^k) \cap F_t(S_2^k) = \emptyset$ for any $t \in [0, 1]$.

REMARK 2. $G\text{-trivial} \iff H\text{-trivial} \iff \text{weak } H\text{-trivial}$.

REMARK 3. If $2n \geq 3k+4$

weak H -trivial $\iff H$ -trivial $\iff G$ -trivial (Zeeman [3]).

REMARK 4. If $n = k+2$

weak H -trivial $\iff H$ -trivial $\iff G$ -trivial



Fig. 1.

REMARK 5. Let $n = k+2$ and $(S^n \supset K_i^k)$, $i=1, 2$ be both trivial knots. And if $k \geq 2$, any such $(k+2, k, 2)$ -link is H -trivial. If $k=1$, for any such $(3, 1, 2)$ -link, homologically trivial link $\iff$ weak H -trivial link $\iff H$ -trivial.

We assume $n \geq k+3$ throughout the remaining of the paper unless otherwise is stated. Then the *knots* $(S^n \supset K_i^k)$, $i=1, 2$ are both trivial by [4]. Hence $K_1^k \subset S^n - \text{Int } U(K_2^k, S^n) \cong S^{n-k-1} \times D^{k+1}$.

Condition $(n-k-1, k, m, \mu)$ (briefly *Cond. (m, μ)*). For any link $L =$

$(S^n \supset K_1^k \cup K_2^k)$ ($n \geq k+3$) since $K_1^k \subset S^n - \text{Int } U(K_2^k, S^n) \cong S^{n-k-1} \times D^{k+1}$ we identify $S^n - \text{Int } U(K_2^k, S^n)$ and $S^{n-k-1} \times D^{k+1}$ and so we consider $S^{n-k-1} \times D^{k+1} \supset K_1^k$. We say that L satisfies *Cond. (m, μ)* if there exist a point $p \in S^{n-k-1}$ and a neighborhood U_p of p in S^{n-k-1} satisfying the following conditions;

- (1) $U_p \times D^{k+1} \cap K_1^k \cong \bigcup_{i=1}^m D_i^k$ for some m ,
- (2) $\bigcup_{i=1}^m D_i^k$ properly embedded in $U_p \times D^{k+1}$,
- (3) $(\partial D_1^k \cup \cdots \cup \partial D_\mu^k)$ and $(p \times \partial D^{k+1}) \cup (\partial D_{\mu+1}^k \cup \cdots \cup \partial D_m^k)$

split each other in $\partial(U_p \times D^{k+1})$ for some $\mu \leq m$.

PROPOSITION. If a link $L = (S^n \supset K_1^k \cup K_2^k)$ satisfies *Cond. (m, m)* , L is weak H -trivial.

PROOF. Since $\partial D_1^k \cup \cdots \cup \partial D_m^k$ splits from $p \times \partial D^{k+1}$ in $\partial(U_p \times D^{k+1})$, $\bigcup_{i=1}^m D_i^k$ contracts into $\partial U_p \times D^{k+1}$. Next we push $\bigcup_{i=1}^m D_i^k$ into $(S^{n-k-1} - \text{Int } U_p) \times D^{k+1} \cong D^n$ using a collar of $\partial U_p \times D^{k+1}$ in $(S^{n-k-1} - \text{Int } U_p) \times D^{k+1}$. Hence K_1^k contracts into $(S^{n-k-1} - \text{Int } U_p) \times D^{k+1} \cong D^n$. So K_1^k is contractible in $S^{n-k-1} \times D^{k+1} = S^n - \text{Int } U(K_2^k, S^n)$. Therefore L is weak H -trivial.

THEOREM. If an $(n, k, 2)$ -link $L = (S^n \supset K_1^k \cup K_2^k)$ ($n \geq k+3$) satisfies *Cond. (m, m)* , L is G -trivial.

§ 3. Lemmas and Proof of Theorem.

LEMMA 1. For a link $L = (S^n \supset K_1^k \cup K_2^k)$, L satisfies (1), (2) of *Cond. (m, μ)* and satisfies that $\bigcup_{i=1}^m \partial D_i^k$ bound a manifold M_1 in $\partial U_p \times D^{k+1}$ which is homeomorphic to $S^k - \bigcup_{i=1}^\mu \text{Int } D_i^k$, then L is G -trivial.

PROOF. Let $M_2 = (S^{n-k-1} - \text{Int } U_p) \times D^{k+1} \cap K_1^k$. Then $(S^{n-k-1} - \text{Int } U_p) \times D^{k+1} \cap K_1^k \cong S^k - \bigcup_{i=1}^\mu \text{Int } D_i^k$ by (1), (2) of *Cond. (m, μ)* . Hence $M_1 \cong M_2$. Since $(S^{n-k-1} - \text{Int } U_p) \times D^{k+1}$ has a collar [3], we push homeomorphically M_1 into $\text{Int}((S^{n-k-1} - \text{Int } U_p) \times D^{k+1})$ keeping ∂M_1 fixed using the collar. We denote it M_0 . Then the followings hold for M_0 ;

- 1) $n \geq k+3$,
- 2) M_0 is homeomorphic to M_2 and $\partial M_0 = \partial M_1 = \partial M_2$,
- 3) M_0 is $(k-2)$ -connected because $M_0 \cong S^k - \bigcup_{i=1}^\mu \text{Int } D_i^k$ has a homotopy

type of $\bigvee_{i=1}^{n-1} S_i^{k-1}$,

- 4) M_0 is homotopic to M_2 keeping the boundary fixed because $(S^{n-k-1}$

$-\text{Int } U_p) \times D^{k+1} \cong D^n$.

So by *Zeeman's Unknotting Theorem* [3. chap. 8] M_0 and M_2 are ambient isotopic by an ambient isotopy of $S^{n-k-1} \times D^{k+1}$ keeping $\partial(S^{n-k-1} \times D^{k+1}) \cup U_p \times D^{k+1}$ fixed. Therefore conversely we can consider M_2 ambient isotopic to M_1 through M_0 by an ambient isotopy of $S^{n-k-1} \times D^{k+1}$ keeping $\partial(S^{n-k-1} \times D^{k+1}) \cup W_p \times D^{k+1}$ where $W_p = U_p - (\text{collar of } \partial U_p)$. Next we push homeomorphically M_1 into $U_p \times D^{k+1} \cong D^n$ using a collar of $\partial U_p \times D^{k+1}$ in $U_p \times D^{k+1}$. Hence we can ambient isotop K_1^k into $\text{Int}(U_p \times D^{k+1}) \cong \text{Int } D^n$ and so K_1^k bound a k -cell in $\text{Int}(U_p \times D^{k+1}) \subset S^{n-k-1} \times D^{k+1} = S^n - \text{Int } U(K_2^k, S^n)$ because $n \geq k+3$. Therefore L is G -trivial (see Remark 1).

LEMMA 2. For any link $L = (S^n \supset K^k \cup K_1^{k-1} \cup \dots \cup K_m^{k-1})$ (codimension is unrestricted) if $(S^n \supset K_1^{k-1} \cup \dots \cup K_\mu^{k-1})$ ($\mu \leq m$) is G -trivial and if $K_1^{k-1} \cup \dots \cup K_\mu^{k-1}$ and $K^k \cup K_{\mu+1}^{k-1} \cup \dots \cup K_m^{k-1}$ split each other in S^n , K_i^{k-1} , $1 \leq i \leq \mu$ bound disjoint locally flat k -cells in $S^n - (K^k \cup K_{\mu+1}^{k-1} \cup \dots \cup K_m^{k-1})$.

PROOF. Since $(S^n \supset K_1^{k-1} \cup \dots \cup K_\mu^{k-1})$ is G -trivial K_i^{k-1} , $1 \leq i \leq \mu$ bound locally flat disjoint k -cells D_i^k , $1 \leq i \leq \mu$. And we may suppose that

$\bigcup_{i=1}^m U D_i^k \subset S^n - \text{Int} |St(v, S^n)|$ for some vertex v . Since

$K_1^{k-1} \cup \dots \cup K_\mu^{k-1}$ and $K^k \cup K_{\mu+1}^{k-1} \cup \dots \cup K_m^{k-1}$ split each other, there exists an n -cell B_1^n such that $K^k \cup K_{\mu+1}^{k-1} \cup \dots \cup K_m^{k-1} \subset \text{Int } B_1^n$ and $\bigcup_{i=1}^\mu U K_i^{k-1} \cap B_1^n = \emptyset$.

Let $B_2^n = S^n - \text{Int } U(B_1^n, S^n)$. Then by [1] there exists a PL homeomorphism $h : S^n \rightarrow S^n$ which is isotopic to the identity and $h(B_1^n) = |St(v, S^n)|$. Since h is isotopic to the identity and since $(S^n \supset K_1^{k-1} \cup \dots \cup K_\mu^{k-1})$ is G -trivial, $(S^n \supset h(K_1^{k-1}) \cup \dots \cup h(K_\mu^{k-1}))$ is also G -trivial. Hence $h(K_i^{k-1})$, $1 \leq i \leq \mu$ bound locally flat disjoint k -cells in $S^n - \text{Int} |St(v, S^n)| \subset S^n - \bigcup_{i=\mu+1}^m U h(K_i^{k-1}) \cup h(K^k)$.

Hence K_i^{k-1} , $1 \leq i \leq \mu$ bound locally flat disjoint k -cells in $S^n - K^k \cup K_{\mu+1}^{k-1} \cup \dots \cup K_m^{k-1}$.

LAMMA 3. If a link $L = (S^n \supset K_1^k \cup K_2^k)$ satisfies *Cond. (m, m)* , ∂D_i^k , $1 \leq i \leq m$ bound locally flat disjoint k -cells $B_1^k, \dots, B_m^k$ in $\partial U_p \times D^{k+1}$.

PROOF. Since ∂D_i^k , $1 \leq i \leq m$ bound locally flat disjoint k -cells $D_1^k, \dots, D_m^k$ in $U_p \times D^{k+1} \cong D^n$, $(\partial(U_p \times D^{k+1}) \supset \partial D_1^k \cup \dots \cup \partial D_m^k)$ is G -trivial by [2. Th. 7 and Remark 1]. And since $\bigcup_{i=1}^m U \partial D_i^k$ and $p \times \partial D^{k+1}$ split each other in $\partial(U_p \times D^{k+1})$ by *Cond. (m, m)* , ∂D_i^k , $1 \leq i \leq m$ bound locally flat disjoint k -cells in $\partial(U_p \times D^{k+1}) - p \times \partial D^{k+1}$ by Lemma 2. But since $U_p \times \partial D^{k+1}$ is a regular neighborhood of $p \times \partial D^{k+1}$ in $\partial(U_p \times D^{k+1})$ and since the complement of $U_p \times \partial D^{k+1}$ in $\partial(U_p \times D^{k+1})$ is $\partial U_p \times D^{k+1}$, hence ∂D_i^k , $1 \leq i \leq m$ bound locally flat disjoint k -cells $B_1^k, \dots, B_m^k$ in $\partial U_p \times D^{k+1}$.

PROOF OF THEOREM. Since the link $L = (S^n \supset K_1^k \cup K_2^k)$ satisfies *Cond.* (m, m) , ∂D_i^k , $1 \leq i \leq m$ bound locally flat disjoint k -cells $B_1^k, \dots, B_m^k$ in $\partial U_p \times D^{k+1}$ by Lemma 3. Since $n \geq k+3$, there exist embeddings

$$h_i : I \times D^{k-1} \rightarrow \partial U_p \times D^{k+1}, \quad 1 \leq i \leq m-1$$

such that

$$(1) \quad h_i(I \times D^{k-1}) \cap h_j(I \times D^{k-1}) = \phi,$$

$$(2) \quad \text{for all } i, \quad h_i(I \times D^{k-1}) \cap \bigcup_{j=1}^m \bar{U} B_j^k = h_i(\{0\} \times D^{k-1} \cup \{1\} \times D^{k-1}) \subset B_1^k \cup B_i^k$$

$$(3) \quad h_i(\{0\} \times D^{k-1}) \subset B_1^k, \quad h_i(\{1\} \times D^{k-1}) \subset B_i^k.$$

Let $M = \left(\bigcup_{j=1}^m \bar{U} B_j^k \right) \cup \left(\bigcup_{i=1}^m \bar{U} h_i(I \times D^{k-1}) \right) - \bigcup_{i=1}^{m-1} \bar{U} h_i(I \times (\text{Int } D^{k-1}))$. Then M is a manifold homeomorphic to $S^k - \bigcup_{i=1}^m \bar{U} D_i^k$. Hence $\bigcup_{i=1}^m \bar{U} \partial D_i^k$ bounds a manifold homeomorphic to $S^k - \bigcup_{i=1}^m \bar{U} D_i^k$ in $\partial U_p \times D^{k+1}$ and L is G -trivial by Lemma 1.

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