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On the sectional curvatures and the Euler-Poincaré characteristic of a Riemannian manifold

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

By Izumi HASEGAWA

§ 1. Introduction.

In a compact oriented Riemannian manifold M of even dimension n , the generalized Gauss-Bonnet formula ([1], [2]) is expressed by the following integral:

$$(1) \quad \chi(M) = (2/c_n) \int_M K_n dV$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M , c_n is the volume of the Euclidean unit n -sphere, K_n denotes the Lipschitz-Killing curvature of M , and dV is the volume element of M .

It is well known that the following is conjectured from the generalized Gauss-Bonnet formula.

CONJECTURE. *Let M be a compact oriented Riemannian manifold of even dimension $n (= 2m)$. If the sectional curvatures are all non-negative, then M has the non-negative Euler-Poincaré characteristic $\chi(M)$. If the sectional curvatures are all non-positive, then $(-1)^m \chi(M) \geq 0$.*

In 2-dimensional case, the sign of the Gaussian curvatures, i.e., of the sectional curvatures determines the sign of $\chi(M)$. In 4-dimensional case, the conjecture is resolved by J. Milnor, and its proof was provided by S. S. Chern [3]. S. S. Chern [3], J. A. Thorpe [4], and Y. K. Cheung and C. C. Hsiung [7] gave certain answers under the curvature conditions respectively for determining the sign of $\chi(M)$.

The purpose of the present paper is to give an answer to the conjecture on a Riemannian manifold with some other curvature conditions.

I would like to express my deep appreciation to Professor Yoshie Katsurada for her constant guidance and criticism.

§ 2. Preliminaries.

Let M be a compact oriented Riemannian manifold of even dimension n . Let $A^p(M)$ denote the bundle of p -vectors of M and let $A^p(x)$ be the

fiber over $x \in M$. The inner product on the fiber $\Lambda^p(x)$ related to the inner product on the tangent space M_x of M at x is given by

$$(2) \quad \langle u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p \rangle = \det[\langle u_i, v_j \rangle] \quad (u_i, v_j \in M_x).$$

We can consider the Grassmann bundle $G_p(M)$ of tangent p -planes of M as a subbundle of the unit sphere bundle of $\Lambda^p(M)$ by identifying $P \in G_p(M)$ with $e_1 \wedge \cdots \wedge e_p \in \Lambda^p(M)$ where $\{e_1, \dots, e_p\}$ is any orthonormal basis for P .

Let R be the curvature operator and R_p the p -th curvature operator which is used in [6]. Explicitly, this curvature operator R_p is given by

$$(3) \quad \begin{aligned} \langle R_p(u_1 \wedge \cdots \wedge u_p), v_1 \wedge \cdots \wedge v_p \rangle \\ \equiv \frac{1}{2^{\frac{p}{2}} p!} \mathcal{E}^{i_1 \cdots i_p} \mathcal{E}^{j_1 \cdots j_p} R(u_{i_1}, u_{i_2}, v_{j_1}, v_{j_2}) \cdots R(u_{i_{p-1}}, u_{i_p}, v_{j_{p-1}}, v_{j_p}) \end{aligned} \quad (u_i, v_j \in M_x)$$

where $\mathcal{E}^{i_1 \cdots i_p}$ denotes the sign of permutation $(i_1, \dots, i_p)$ of $(1, \dots, p)$. Clearly $R_2 = R$. Furthermore, for $x \in M$,

$$(4) \quad K_n(x) = \langle R_n(e_1 \wedge \cdots \wedge e_n), e_1 \wedge \cdots \wedge e_n \rangle$$

where $\{e_1, \dots, e_n\}$ is any orthonormal basis for M_x .

§ 3. The p -th sectional curvatures K_p .

Now for any positive even integer $p \leq n$, we define a smooth function K_p on the Grassmann bundle $G_p(M)$. The function K_p , called the p -th sectional curvature of M , means the Lipschitz-Killing curvature of the geodesic p -dimensional submanifolds of M . Clearly, the p -th sectional curvature of M is the real valued function $K_p: G_p(M) \rightarrow \mathbf{R}$ given by

$$(5) \quad K_p(P) = \langle R_p(P), P \rangle \quad (P \in G_p(M)).$$

For $p=2$, K_p is the usual sectional curvature K of M . We can easily get the following

PROPOSITION. Let M be 4-dimensional Einstein space with $K_4 \equiv 0$. Then M is flat.

PROOF. In 4-dimensional case, there exists an orthonormal basis $\{e_1, \dots, e_4\}$ of M_x such that

$$(6) \quad K_4(x) = \frac{1}{3} \{K_{12}K_{34} + K_{13}K_{42} + K_{14}K_{23} + (R_{1234})^2 + (R_{1342})^2 + (R_{1423})^2\} \quad (x \in M)$$

where $K_{ij} = K(e_i \wedge e_j)$. It is well known, that $K_{12} = K_{34}$, $K_{13} = K_{42}$, and $K_{14} = K_{23}$ in 4-dimensional Einstein space. Consequently M is flat ($K \equiv 0$). q.e.d.

We consider the following *curvature condition* (C):

When the sectional curvatures are always non-negative (or, non-positive), there exists the following property

$$K(P) \cdot K_{p-2}(P^\perp) \leq \langle R(P) \wedge R_{p-2}(P^\perp), W \rangle$$

$$\left(\text{or, } (-1)^{\frac{p}{2}} K(P) \cdot K_{p-2}(P^\perp) \leq (-1)^{\frac{p}{2}} \langle R(P) \wedge R_{p-2}(P^\perp), W \rangle \right)$$

for any even $p(\leq n)$ and all p -plane $W = P \wedge P^\perp \in G_p(M)$ where P is 2-plane and $P^\perp$ is the orthogonal complement of P in W .

REMARK 1. It is clear that the *curvature condition* (C) is satisfied for the constant curvature. At this juncture, an equality is preserved.

REMARK 2. It follows from the simple calculation that the *curvature condition* (C) is satisfied for $n=4$.

§ 4. The main theorems.

We have the following Lemma in [5]:

LEMMA. Suppose that p and q are positive even integers with $p+q \leq n$. The form $\Theta_{i_1 \dots i_{p+q}}^{(p+q)}$ (see [4] and [5]) can be expressed as

$$\Theta_{i_1 \dots i_{p+q}}^{(p+q)} = \frac{p! q!}{(p+q)!} \sum_A \Theta_{k_1 \dots k_p}^{(p)} \vee \Theta_{k_{p+1} \dots k_{p+q}}^{(q)}$$

where the sum ranges over all partitions $A = (A_1, A_2)$ of $\{i_1, \dots, i_{p+q}\}$ into sets A_1 of p elements and A_2 of q elements, and where $(k_1, \dots, k_{p+q})$ is, for each A , an even permutation of $(i_1, \dots, i_{p+q})$ such that

$$k_1, \dots, k_p \in A_1 \quad \text{and} \quad k_{p+1}, \dots, k_{p+q} \in A_2.$$

If we reform the above lemma, we have the following

LEMMA*. Suppose that p and q are even integers with $p+q \leq n$. For $Q \in G_{p+q}(M)$, let $\{e_1, \dots, e_{p+q}\}$ be an orthonormal basis for Q and let

$$\beta = \{e_{i_1} \wedge \dots \wedge e_{i_p} \mid 1 \leq i_1 < \dots < i_p \leq p+q\}.$$

Then $\beta \subset G_p(M)$ and

$$(7) \quad R_{p+q}(Q) = \frac{p! q!}{(p+q)!} \sum_{P \in \beta} R_p(P) \wedge R_q(P^\perp)$$

where $P^\perp$ is the oriented orthogonal complement of P in Q .

THEOREM 1. Let M be a compact oriented Riemannian manifold of even dimension $n=2m$. Suppose that M has the curvature condition (C). If the sectional curvatures are all non-negative, then the Euler-Poincaré

characteristic is non-negative, and if the sectional curvatures are non-positive, then $(-1)^m \chi(M) \geq 0$.

PROOF. From the equation (4) and Lemma*, for $x \in M$,

$$\begin{aligned} K_n(x) &= \langle R_n(e_1 \wedge \cdots \wedge e_n), e_1 \wedge \cdots \wedge e_n \rangle \\ &= \frac{2! (n-2)!}{n!} \sum_{P_1 \in \beta_1} \langle R(P_1) \wedge R_{n-2}(P_1^\perp), M_x \rangle \end{aligned}$$

where $\{e_1, \dots, e_n\}$ is any orthonormal basis for M_x . Here, we use the curvature condition (C) over and over again. If $K \equiv K_2 \geq 0$,

$$K_n(x) \geq \frac{2^m}{n!} \sum_{P_1 \in \beta_1} K(P_1) \sum_{P_2 \in \beta_2} K(P_2) \cdots \sum_{P_{m-1} \in \beta_{m-1}} K(P_{m-1}) \cdot K(P_{m-1}^\perp)$$

where $\beta_1 = \{e_{i_1} \wedge e_{i_2} | 1 \leq i_1 < i_2 \leq n\}$, $\dots$, $\beta_{m-1} = \{e_{i_{n-3}} \wedge e_{i_{n-2}} | 1 \leq i_{n-3} < i_{n-2} \leq n\}$, $P_{m-1}^\perp = e_{i_{n-1}} \wedge e_{i_n}$ and $(i_1, \dots, i_n)$ is an even permutation of $(1, \dots, n)$. If $K \leq 0$,

$$(-1)^m K_n(x) \geq (-1)^m \frac{2^m}{n!} \sum_{P_1 \in \beta_1} K(P_1) \cdots \sum_{P_{m-1} \in \beta_{m-1}} K(P_{m-1}) \cdot K(P_{m-1}^\perp).$$

Comparison with (1) completes the proof.

q.e.d.

For some even p dividing n , we consider the following *curvature condition* (C*): When the p -th sectional curvatures are always non-negative (or, non-positive), there exists the following property

$$\begin{aligned} K_p(P) \cdot K_{(k-1)p}(P^\perp) &\leq \langle R_p(P) \wedge R_{(k-1)p}(P^\perp), W \rangle \\ \text{(or, } (-1)^k K_p(P) \cdot K_{(k-1)p}(P^\perp) &\leq (-1)^k \langle R_p(P) \wedge R_{(k-1)p}(P^\perp), W \rangle) \end{aligned}$$

for any positive integer $k (\leq n/p)$ and all kp -plane $W = P \wedge P^\perp \in G_{kp}(M)$ where P is a p -plane and $P^\perp$ is the orthogonal complement of P in W .

We can easily obtain the following

THEOREM 2. *Let M be a compact oriented Riemannian manifold of even dimension n . Suppose that M has the curvature condition (C*). If, for some even p dividing n , the p -th sectional curvatures are all non-negative, then $\chi(M) \geq 0$, and if the p -th sectional curvatures are non-positive, then $(-1)^{n/p} \chi(M) \geq 0$.*

REMARK 3. *It is clear from [4], that the curvature condition (C*) is satisfied when the p -th sectional curvatures are constant on $G_p(M)$. At this juncture an equality is preserved.*

Theorem 2 contains the result of J. A. Thorpe (Theorem B in [4]).

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