Proceedings of the Edinburgh Mathematical Society (2005) **48**, 479–484 ©
DOI:10.1017/S001309150400032X Printed in the United Kingdom

INVARIANT SUBSPACES AND HANKEL-TYPE OPERATORS ON A BERGMAN SPACE

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(Received 29 April 2004)

Abstract Let $L^2 = L^2(D, rdrd\theta/\pi)$ be the Lebesgue space on the open unit disc D and let $L_a^2 = L^2 \cap \mathrm{Hol}(D)$ be a Bergman space on D. In this paper, we are interested in a closed subspace $\mathcal{M}$ of L^2 which is invariant under the multiplication by the coordinate function z, and a Hankel-type operator from L_a^2 to $\mathcal{M}^\perp$. In particular, we study an invariant subspace $\mathcal{M}$ such that there does not exist a finite-rank Hankel-type operator except a zero operator.

Keywords: Bergman space; invariant subspace; Hankel-type operator

2000 *Mathematics subject classification:* Primary 47B35; 47A15

1. Introduction

Let D be the open unit disc in $\mathbb{C}$ and $\mathrm{Hol}(D)$ be the set of all holomorphic functions on D. Let $\mathrm{d}\mu = rdrd\theta/\pi$ and $L^2 = L^2(D, \mathrm{d}\mu)$ the Lebesgue space. The Bergman space L_a^2 on D is defined by $L_a^2 = L^2 \cap \mathrm{Hol}(D)$. Then L_a^2 is the closed subspace of L^2. When $\mathcal{M}$ is a closed subspace of L^2 and $z\mathcal{M} \subseteq \mathcal{M}$, $\mathcal{M}$ is called an invariant subspace. For φ in $L^\infty = L^\infty(D, \mathrm{d}\mu)$, a Hankel-type operator is defined by

$$H_\varphi^{\mathcal{M}} f = (I - P^{\mathcal{M}})(\varphi f) \quad (f \in L_a^2),$$

where $P^{\mathcal{M}}$ is the orthogonal projection from L^2 onto $\mathcal{M}$. When $\mathcal{M} = L_a^2$, $H_\varphi^{\mathcal{M}}$ is called a big Hankel operator and when $\mathcal{M} = (\bar{z}\overline{L_a^2})^\perp$, $H_\varphi^{\mathcal{M}}$ is called a small Hankel operator. When $L_a^2 \subseteq \mathcal{M} \subseteq (\bar{z}\overline{L_a^2})^\perp$, $H_\varphi^{\mathcal{M}}$ is called an intermediate Hankel operator.

It is easy to see that there does not exist a finite-rank big Hankel operator except a zero one (see [**3**, **6**]). On the other hand, there exist a lot of finite-rank non-zero small Hankel operators (see [**6**]). In fact, it is easy to see the results. Strouse [**7**] described completely all finite-rank intermediate Hankel operators for some invariant subspace. In the previous paper [**6**], we began to study finite-rank intermediate Hankel operators for arbitrary invariant subspace. In [**6**, Theorem 3.2], we gave three necessary and sufficient conditions for $\mathcal{M}$ such that there does not exist a finite-rank intermediate Hankel operator except a zero one. In this paper, without the hypothesis on an invariant subspace $\mathcal{M}$, we give a new necessary and sufficient condition for $\mathcal{M}$ which have a finite-rank Hankel-type operator except a zero one.

For an invariant subspace $\mathcal{M}$ in L^2, $\ker H_\varphi^{\mathcal{M}}$ denotes the kernel of $H_\varphi^{\mathcal{M}}$ and then $\ker H_\varphi^{\mathcal{M}} = \{f \in L_a^2; \varphi f \in \mathcal{M}\}$. Hence $\ker H_\varphi^{\mathcal{M}}$ is also an invariant subspace in L_a^2. Thus each invariant subspace $\mathcal{M}$ in L^2 is related to an invariant subspace in L_a^2 by a Hankel-type operator. In this paper, the following property of invariant subspaces in L^2 is important.

Definition 1.1. Let $\mathcal{M}$ be an invariant subspace of L^2. $\mathcal{M}$ is called weakly divisible if whenever $f \in \mathcal{M}$ and $|f(z)| \leqslant \gamma|z-a|$ for some $a \in D$ and some $\gamma \geqslant 0$ then $f(z) = (z-a)g(z)$ and g is a function in $\mathcal{M}$.

In § 2, we generalize a theorem of Axler and Bourdon [**1**], which will be used later on. In § 3, we show that there does not exist a finite-rank Hankel-type operator $H_\varphi^{\mathcal{M}}$ except a zero one if and only if $\mathcal{M}$ is weakly divisible. In § 4, we give several examples of weakly divisible invariant subspaces.

In this paper $[S]_*$ denotes the weak* closed linear span of a subset S in L^∞ and $[S]_2$ denotes the closed linear span of a subset S in L^2.

2. An invariant subspace and the index

In this section, for a given invariant subspace $\mathcal{M}$ we are interested in two invariant subspaces $\mathcal{M}'$ and $\mathcal{M}''$ such that $\mathcal{M}' \subseteq \mathcal{M} \subseteq \mathcal{M}''$, $\dim \mathcal{M} \ominus \mathcal{M}' < \infty$ and $\dim \mathcal{M}'' \ominus \mathcal{M} < \infty$. Under some conditions on $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{M}''$, we describe $\mathcal{M}'$ and $\mathcal{M}''$ using $\mathcal{M}$. Corollary 2.4 will be used in §§ 3 and 4. Corollary 2.4 (i) is known from [**1**].

When $\mathcal{M}$ is an invariant subspace of L^2, for $a \in \mathbb{C}$ put $\operatorname{ind}_a \mathcal{M} = \dim\{\mathcal{M} \ominus (z-a)\mathcal{M}\}$. $\operatorname{ind}_a \mathcal{M}$ is called the index of $\mathcal{M}$ at a. It is known (cf. [**1**]) that for each n $(0 \leqslant n \leqslant \infty)$ and for any a $(\in D)$ there exists an invariant subspace $\mathcal{M}$ with $\operatorname{ind}_a \mathcal{M} = n$.

Theorem 2.1. *Let $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be invariant subspaces of L^2 and $\mathcal{M}_1 \subseteq \mathcal{M}_2$.*

(i) $\operatorname{ind}_a \mathcal{M} = 0$ *for any $a \notin D$.*

(ii) *If $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 < \infty$, then there exists a polynomial b such that $b\mathcal{M}_2 \subseteq \mathcal{M}_1$, $Z(b) \subset D$ and the degree of $b \leqslant \dim \mathcal{M}_2 \ominus \mathcal{M}_1$ and*

$$\sum(\operatorname{ind}_a \mathcal{M}_2; a \in Z(b)) \geqslant \dim \mathcal{M}_2 \ominus \mathcal{M}_1.$$

Proof. (i) If $|a| > 1$, then $(z-a)^{-1} \in H^\infty$ and $\mathcal{M} = (z-a)\mathcal{M}$. Hence $\operatorname{ind}_a \mathcal{M} = 0$. If $|a| = 1$, then $(z-a)\mathcal{M} = (z-a)\{z - a(1+\varepsilon)\}^{-1}\mathcal{M}$. For any $f \in \mathcal{M}$, it is easy to see that

$$\int_D \left| \frac{z-a}{z-a(1+\varepsilon)} f - f \right|^2 \mathrm{d}\mu \to 0 \quad (\varepsilon \to 0)$$

by Lebesgue's convergence theorem. This implies that $(z-a)\mathcal{M}$ is dense in $\mathcal{M}$ and so $\operatorname{ind}_a \mathcal{M} = 0$ for $|a| = 1$.

(ii) Put $\mathcal{N} = \mathcal{M}_2 \ominus \mathcal{M}_1$ and $\mathcal{S}_z = PM_z|\mathcal{N}$, where M_z is a multiplication operator on L^2 by the coordinate function z and P is the orthogonal projection from L^2 to $\mathcal{N}$. If $n = \dim \mathcal{N} < \infty$, then there exists a polynomial b of degree n such that $\mathcal{S}_b = b(\mathcal{S}_z) = 0$ and so $b\mathcal{M}_2 \subseteq \mathcal{M}_1$. By (i), we may assume that $Z(b) \subset D$. We will prove that $\sum(\operatorname{ind}_a \mathcal{M}_2; a \in Z(b)) \geqslant n$. We can write that $b = a_0 \prod_{j=1}^{n}(z - a_j)$ and so $Z(b) = \{a_1, a_2, \ldots, a_n\}$, where $a_0 \in \mathbb{C}$. If $\sum(\operatorname{ind}_a \mathcal{M}_2; a \in Z(b)) \leqslant n-1$, then we may assume $\operatorname{ind}_{a_1} \mathcal{M}_2 = 0$. Since $[(z-a_1)\mathcal{M}_2]_2 = \mathcal{M}_2$,

$$\prod_{j=2}^{n}(z - a_j)\mathcal{M}_2 \subseteq \mathcal{M}_1 \subset \mathcal{M}_2.$$

Then it is easy to see that $\dim \mathcal{M}_2 \ominus [\prod_{j=2}^{n}(z-a_j)\mathcal{M}_2]_2 \leqslant n-1$ because $\operatorname{ind}_{a_j} \mathcal{M}_2 \leqslant 1$ for $2 \leqslant j \leqslant n$. This contradicts that $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 = n$. □

Corollary 2.2. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be invariant subspaces of L^2 and $\mathcal{M}_1 \subseteq \mathcal{M}_2$. If $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 = 1$, then $(z-a)\mathcal{M}_2 \subseteq \mathcal{M}_1 \subsetneq \mathcal{M}_2$ for some $a \in D$ and $\operatorname{ind}_a \mathcal{M}_2 \geqslant 1$. If $\operatorname{ind}_a \mathcal{M}_1 = 1$ or $\operatorname{ind}_a \mathcal{M}_2 = 1$, then $\mathcal{M}_1 = [(z-a)\mathcal{M}_2]_2$.*

Proof. By Theorem 2.1, $(z-a)\mathcal{M}_2 \subseteq \mathcal{M}_1$ for some $a \in D$ and so $\operatorname{ind}_a \mathcal{M}_2 \geqslant 1$. Since $(z-a)\mathcal{M}_1 \subseteq (z-a)\mathcal{M}_2 \subseteq \mathcal{M}_1 \subsetneq \mathcal{M}_2, \mathcal{M}_1 = [(z-a)\mathcal{M}_2]_2$ if $\operatorname{ind}_a \mathcal{M}_1 = 1$ or $\operatorname{ind}_a \mathcal{M}_2 = 1$. □

Corollary 2.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be invariant subspaces such that $\mathcal{M}_1 \subsetneq \mathcal{M}_2$ and $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 = n < \infty$. Suppose that $(z-a)\mathcal{M}_j$ is closed for any a in D when $j = 1, 2$. If $\operatorname{ind}_a \mathcal{M}_1 = 1$ for any a in D or $\operatorname{ind}_a \mathcal{M}_2 = 1$ for any a in D, then $\mathcal{M}_1 = b\mathcal{M}_2$ and $\mathcal{M}_2 = \langle f_1/b, \ldots, f_n/b\rangle \oplus \mathcal{M}_1$, where $b = \prod_{j=1}^{n}(z-a_j), \{a_j\} \subset D$ and $\{f_j\} \subset \mathcal{M}_1$.*

Proof. By Theorem 2.1 there exists a polynomial b such that $b\mathcal{M}_2 \subseteq \mathcal{M}_1$ and $Z(b) \subset D$ and the degree of $b \leqslant n$. Hence $b = \prod_{j=1}^{\ell}(z-a_j)$ and $\{a_j\} \subset D$ and $\ell \leqslant n$. When $\operatorname{ind}_a \mathcal{M}_2 = 1$ for any a in D, $\dim \mathcal{M}_2 \ominus b\mathcal{M}_2 = \ell$ because $(z-a_j)\mathcal{M}_2$ is closed for $1 \leqslant j \leqslant \ell$ and so $\ell = n$. Hence $\mathcal{M}_1 = b\mathcal{M}_2$. When $\operatorname{ind}_a \mathcal{M}_1 = 1$ for any a in D, $\dim \mathcal{M}_1 \ominus b\mathcal{M}_1 = \ell$ by the same reason. Since $b\mathcal{M}_1 \subseteq b\mathcal{M}_2 \subseteq \mathcal{M}_1$ and $\dim b\mathcal{M}_2 \ominus b\mathcal{M}_1 = n$, $\ell = n$ and so $\mathcal{M}_1 = b\mathcal{M}_2$. Put $\mathcal{M}_2 = \langle \varphi_1, \ldots, \varphi_n\rangle \oplus \mathcal{M}_1$, where $\{\varphi_j\}$ are orthogonal to $\mathcal{M}_1$. What was just proved above, $b\mathcal{M}_2 = \mathcal{M}_1$ and so $b\mathcal{M}_2 = \langle b\varphi_1, \ldots, b\varphi_n\rangle \oplus b\mathcal{M}_1 = \mathcal{M}_1$. Put $f_j = b\varphi_j$ for $j = 1, \ldots, n$, then $\{f_j\}$ are in $\mathcal{M}_1$ and $\mathcal{M}_2 = \langle f_1/b, \ldots, f_n/b\rangle \oplus \mathcal{M}_1$. □

Corollary 2.4. *Let $\mathcal{M}$ be an invariant subspace of L^2.*

(i) *If $\dim L_a^2 \ominus \mathcal{M} = n < \infty$ and $n \neq 0$, then $\mathcal{M} = bL_a^2$, where $b = \prod_{j=1}^{n}(z-a_j)$ and $\{a_j\} \subset D$.*

(ii) *If $\dim \mathcal{M} \ominus L_a^2 = n < \infty$, then $\mathcal{M} = L_a^2$.*

Proof. It is known that $\operatorname{ind}_a L_a^2 = 1$ and $(z-a)L_a^2$ is closed for each $a \in D$. Hence we can apply Corollary 2.3 to $\mathcal{M}_1 = L_a^2$ or $\mathcal{M}_2 = L_a^2$. If $\mathcal{M}_1 = \mathcal{M}$ and $\mathcal{M}_2 = L_a^2$, then (i) follows. If $\mathcal{M}_1 = L_a^2$ and $\mathcal{M}_2 = \mathcal{M}$, then $\mathcal{M} = \langle f_1/b, \ldots, f_n/b\rangle \oplus L_a^2$, where $b = \prod_{j=1}^{n}(z-a_j)$, $\{a_j\} \subset D$ and $\{f_j\} \subset L_a^2$. For each $1 \leqslant \ell \leqslant n$, $f_\ell/b \in L^2$ and so

$f_\ell(a_j) = 0$ for $1 \leqslant j \leqslant n$. Then f_ℓ/b belongs to L^2_a and so $f_\ell/b = 0$ for each ℓ. Thus $\mathcal{M} = L^2_a$ and so (ii) follows. □

3. Finite-rank Hankel-type operators

In this section, we study the relation between finite-rank Hankel-type operators and invariant subspaces.

Theorem 3.1. *Let $\mathcal{M}$ be an invariant subspace of L^2. Then there does not exist a finite-rank Hankel-type operator $H^{\mathcal{M}}_\varphi$ except a zero one if and only if $\mathcal{M}$ is weakly divisible.*

Proof. Suppose $\mathcal{M}$ is weakly divisible. If $H^{\mathcal{M}}_\varphi$ is of finite rank, then $\ker H^{\mathcal{M}}_\varphi$ is an invariant subspace in L^2_a and $\dim L^2_a/\ker H^{\mathcal{M}}_\varphi < \infty$. By (i) of Corollary 2.4, $\ker H^{\mathcal{M}}_\varphi = bL^2_a$ for some polynomial b with $Z(b) \subset D$ and so $b\varphi$ belongs to $\mathcal{M}$. Put $f = b\varphi$, then $|f(z)| \leqslant \gamma|b(z)|$ $(z \in D)$, where $\gamma = \|\varphi\|_\infty$. Suppose $b(z) = a_0 \prod_{j=1}^n (z - a_j)$, where $\{a_j\} \subset D$. For any ℓ with $1 \leqslant \ell \leqslant n$,

$$\left|\frac{f(z)}{z - a_\ell}\right| \leqslant \gamma |a_0| \prod_{j \neq \ell} |z - a_j| \quad (z \in D)$$

and $f(z)/(z - a_\ell)$ belongs to $\mathcal{M}$ because $a_\ell \in D$ and $\mathcal{M}$ is weakly divisible. Thus $\varphi(z) = f(z)/b(z)$ belongs to $\mathcal{M}$. Hence $H^{\mathcal{M}}_\varphi = 0$.

Conversely, if $\mathcal{M}$ is not weakly divisible, then there exists a function f in $\mathcal{M}$ and a point a in D such that $|f(z)| \leqslant \gamma|z - a|$ $(z \in D)$ and $f(z)/(z - a)$ does not belong to $\mathcal{M}$. Put $\varphi = f(z)/(z - a)$, then $\varphi \in L^\infty$ and $H^{\mathcal{M}}_\varphi$ is not zero because $\varphi \notin \mathcal{M}$. On the other hand, $(z - a)\varphi \in \mathcal{M}$ and so the kernel of $H^{\mathcal{M}}_\varphi$ contains $(z - a)L^2_a$. This implies that $H^{\mathcal{M}}_\varphi$ is of rank one because $L^2_a/(z - a)L^2_a = \mathbb{C}$. □

Proposition 3.2. *If there exists a symbol φ such that $r(H^{\mathcal{M}}_\varphi) = n \geqslant 1$, then there exists a symbol φ_j such that $r(H^{\mathcal{M}}_{\varphi_j}) = j$ for any j with $0 \leqslant j \leqslant n - 1$.*

Proof. Suppose $1 \leqslant n = r(H^{\mathcal{M}}_\varphi) < \infty$. Then $\ker H^{\mathcal{M}}_\varphi =$ the kernel of $H^{\mathcal{M}}_\varphi$ is an invariant subspace of L^2_a and $L^2_a/\ker H^{\mathcal{M}}_\varphi$ is of finite dimension n. By Corollary 2.4, $\ker H^{\mathcal{M}}_\varphi = bL^2_a$, where $b = \prod_{\ell=1}^n (z - a_\ell)$ and $(a_\ell) \subset D$. Hence $b\varphi$ belongs to $\mathcal{M}$. Put

$$\varphi_j = \varphi \prod_{\ell=j+1}^{n} (z - a_\ell) \quad \text{for } 1 \leqslant j \leqslant n - 1,$$

then $\varphi_j \notin \mathcal{M}$ for $1 \leqslant j \leqslant n - 1$ and $\varphi_0 = b\varphi$. Since $\ker H^{\mathcal{M}}_{\varphi_j} = b_j L^2_a$ for $1 \leqslant j \leqslant n - 1$, where $b_j = \prod_{\ell=1}^{j} (z - a_\ell)$, $H^{\mathcal{M}}_{\varphi_j}$ is of finite rank j for $0 \leqslant j \leqslant n - 1$. □

Corollary 3.3. *The following two expressions are equivalent for an invariant subspace $\mathcal{M}$.*

(i) *If $r(H^{\mathcal{M}}_\varphi) < \infty$, then $r(H^{\mathcal{M}}_\varphi) = 0$.*

(ii) *If $r(H_\varphi^{\mathcal{M}}) \leqslant 1$, then $r(H_\varphi^{\mathcal{M}}) = 0$.*

Proof. (i) $\Rightarrow$ (ii). This is clear.

(ii) $\Rightarrow$ (i). If (i) is not true, then there exists a symbol φ with $r(H_\varphi^{\mathcal{M}}) = n \geqslant 2$. By Proposition 3.2 there exists a symbol φ_1 such that $r(H_{\varphi_1}^{\mathcal{M}}) = 1$. This contradicts (ii). □

4. Weakly divisible invariant subspaces

For a function f in L_a^2, put $Z(f) = \{a \in D; f(a) = 0\}$ and $Z(G) = \cap\{Z(f); f \in G\}$ for a subset G in L_a^2. For $1 \leqslant p \leqslant \infty$, if E is an open set in D, H_E^p denotes the set of all functions in L^p that are analytic on E. In Corollary 4.2, a weakly divisible invariant subspace $\mathcal{M}$ is described completely when $\mathcal{M}$ is in L_a^2. There exists a non-zero invariant subspace $\mathcal{M}$ in L_a^2 such that $\mathcal{M} \cap L^\infty = \langle 0 \rangle$. For it is known (see [**5**]) that there exists a non-zero function f in L_a^2 such that $Z(f)$ does not satisfy the Blaschke condition.

Theorem 4.1. *Let $\mathcal{M}$ be an invariant subspace of L^2.*

(i) *If $\mathcal{M} \cap L^\infty \subseteq H^\infty$ and $Z(\mathcal{M} \cap L^\infty) = \emptyset$, then $\mathcal{M}$ is weakly divisible.*

(ii) *If $\mathcal{M} \cap L^\infty = H_E^\infty$ for some open set E, then $\mathcal{M}$ is weakly divisible.*

(iii) *If $\mathcal{M} \cap L^\infty = \langle 0 \rangle$, then $\mathcal{M}$ is weakly divisible.*

Proof. (i) If $\{f_n\}$ is a sequence in $\mathcal{M} \cap L^\infty$ which converges pointwise boundedly to f, then $f \in \mathcal{M}$. By the Krein–Schmulian criterion (see [**4**, IV 2.1]), $\mathcal{M} \cap L^\infty$ is weak* closed. Hence, by a well-known theorem of Beurling [**2**] $\mathcal{M} \cap L^\infty = qH^\infty$ for some inner function q. Hence if $f \in \mathcal{M}$ and $|f(z)| \leqslant \gamma|z-a|$ $(z \in D)$ for some $a \in D$, then $f = qh$ for some $h \in H^\infty$. Since $Z(\mathcal{M} \cap L^\infty) = \emptyset$, $|q(z)| > 0$ $(z \in D)$ and so $h(a) = 0$. Hence $f(z)/(z-a) = q(z) \times (h(z)/(z-a)) \in qH^\infty$. Thus $f(z)/(z-a)$ belongs to $\mathcal{M}$.

(ii) If $f \in H_E^\infty$ and $|f(z)| \leqslant \gamma|z-a|$ $(z \in D)$ for some $a \in D$, then $f(z)/(z-a) \in L^\infty$ and $f(z)/(z-a)$ is analytic on E. Hence $f(z)/(z-a)$ belongs to H_E^∞ and so $\mathcal{M}$ is weakly divisible.

(iii) This is clear. □

Corollary 4.2. *Let $\mathcal{M}$ be an invariant subspace of L_a^2. Then $\mathcal{M}$ is weakly divisible if and only if $\mathcal{M} \cap L^\infty = \langle 0 \rangle$ or $Z(\mathcal{M} \cap L^\infty) = \emptyset$.*

Proof. The part of 'if' is a result of (i) and (iii) of Theorem 4.1. Conversely, suppose that $\mathcal{M}$ is weakly divisible. If $\mathcal{M} \cap L^\infty \neq \langle 0 \rangle$, then by a theorem of Beurling there exists an inner function q with $\mathcal{M} \cap L^\infty = qH^\infty$. If $q(a) = 0$ for some $a \in D$, then there exists a finite positive constant γ such that $|q(z)| \leqslant \gamma|z-a|$ $(z \in D)$ and $q/(z-a) \notin \mathcal{M}$. This contradicts the weak divisibility of $\mathcal{M}$ and so $Z(q) = Z(\mathcal{M} \cap L^\infty) = \emptyset$. □

Corollary 4.3. *Let $\mathcal{M}$ be an invariant subspace of L^2.*

(i) *If $\mathcal{M} \subsetneq L_a^2$ and $\dim L_a^2/\mathcal{M} < \infty$, then $\mathcal{M}$ is not weakly divisible.*

(ii) *If $\mathcal{M} \supseteq L_a^2$ and $\dim \mathcal{M}/L_a^2 < \infty$, then $\mathcal{M}$ is weakly divisible.*

Proof. (i) If $\mathcal{M} \subsetneq L_a^2$ and $\dim L_a^2/\mathcal{M} = \ell < \infty$, then by (i) of Corollary 2.4 $\mathcal{M} = bL_a^2$, where $b = \prod_{j=1}^{\ell}(z - a_j)$ and $a_j \in D$ $(1 \leqslant j \leqslant \ell)$. Hence $Z(\mathcal{M} \cap L^\infty) = Z(b) \neq \emptyset$ and so by Corollary 4.2 $\mathcal{M}$ is not weakly divisible.

(ii) By (2) of Corollary 2.4 $\mathcal{M} = L_a^2$ and so $\mathcal{M} \cap L^\infty = H^\infty$. Hence (i) of Theorem 4.1 implies that $\mathcal{M}$ is weakly divisible. □

Corollary 4.4. *If $\mathcal{M} = H_E^2$ for some open set E in D, then $\mathcal{M}$ is weakly divisible.*

Proof. It is a result of (ii) of Theorem 4.1. □

Proposition 4.5. *Suppose that $\mathcal{M}_j$ is a weakly divisible invariant subspace of L^2 for $j = 1, 2, \dots$ and $\mathcal{M}_j \times \mathcal{M}_\ell = \{fg; f \in \mathcal{M}_j \text{ and } g \in \mathcal{M}_\ell\} = \langle 0 \rangle$ if $j \neq \ell$. If $\mathcal{M} = \sum_{j=1}^{\infty} \oplus \mathcal{M}_j$, then $\mathcal{M}$ is a weakly divisible invariant subspace.*

Proof. If $f \in \mathcal{M}$, then $f = \sum_{j=1}^{\infty} f_j$ and $|f(z)| = \sum_{j=1}^{\infty} |f_j(z)|$ $(z \in D)$ by hypothesis. This implies that $\mathcal{M}$ is weakly divisible. □

Corollary 4.6. *Let $1 \leqslant \ell \leqslant \infty$. Suppose D_j is an open set in D with $\mu(\partial D_j) = 0$ for $1 \leqslant j \leqslant \ell$, $D_i \cap D_j = \emptyset (i \neq j)$ and $D = \bigcup_{j=1}^{\ell} D_j$. Then $\mathcal{M} = \sum_{j=1}^{\ell} \oplus L_a^2(D_j)$ is weakly divisible.*

Proof. This is a result of Corollary 4.4 and Proposition 4.5. □

Proposition 4.7. *If $\mathcal{M}$ is a weakly divisible invariant subspace of L^2 and φ is a unimodular function in L^∞, then $\varphi\mathcal{M}$ is a weakly divisible invariant subspace.*

Proof. From the definition of weak divisibility, the proposition follows trivially. □

Corollary 4.8. *If φ is a unimodular function in L^∞, then φL_a^2 is weakly divisible.*

Acknowledgements. This research was partly supported by Grant-in-Aid for Scientific Research, Ministry of Education of Japan.

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