Calculating relaxation time distribution function from power spectrum based on inverse integral transformation method

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A novel method is presented for obtaining the distribution function of relaxation times $G(\tau)$ from power spectrum $1/f^{\alpha}$ ($1 \leq \alpha \leq 2$). It is derived using McWhorter model and its inverse Stieltjes transform. Unlike the pre-assumed conventional $g(\tau)$ distribution, the extracted $G(\tau)$ has a peak whose width increases as the slope of the power spectrum α decreases. The peak position determines the dominant time constant of the system. Our method is unique because the distribution function is directly extracted from the measured power spectrum. We then demonstrate the validity of this method in the analysis of noise in transistor.

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1. Introduction

$S(f) \sim 1/f^{\alpha}$ spectral densities, where f is the frequency and α is power component, are widely observed in nature and are of great interest in many research fields [1-9]. To explain $1/f$ spectrum, McWhorter model has been often used in a low frequency electronic noise in the field of semiconductor research [2,5,6,9]. According to the model, $1/f^{\alpha}$ ($1 \leq \alpha \leq 2$) noise can be obtained by summation of $1/f^2$ noise with the appropriate weights. In other words, the power spectrum of $S(\omega)$, as a function of angular frequency ω, can be expressed as a superposition of an uncoupled Lorentzian kernel in the following Fredholm integral equation of the first kind, namely, the suitable combination of Lorentzian kernel having different time constants τ:

$$S(\omega) = \int_{\tau_1}^{\tau_2} g(\tau) \frac{\tau}{1+\omega^2\tau^2} d\tau \,, \tag{1}$$

where $g(\tau)$ is the distribution function of relaxation times on the linear time scale τ, and τ_1 and τ_2 are the limits of the integration. The relationship between the distribution $g(\tau)$ and power component α has already been studied [4]. It is well known that integrating Eq. (1) with power law distribution $g(\tau) \propto 1/\tau^{(2-\alpha)}$ approximately gives power spectrum $S(\omega) \sim 1/\omega^{\alpha}$. Here, it is assumed that the $g(\tau)$ are expressed as either of relationships given below:

$$g(\tau) = \frac{a}{\tau} \,, \tag{2}$$

$$g(\tau) = a \,, \tag{3}$$

where a is a constant. Integrating Eq. (1) with Eqs. (2) and (3) gives

$$S(\omega) = \frac{a}{\omega}\left[\arctan(\omega\tau_2) - \arctan(\omega\tau_1)\right] \sim \frac{1}{\omega} \tag{4}$$

$$S(\omega) = \frac{a}{2\omega^2} \ln\left[\frac{1+\omega^2\tau_2^2}{1+\omega^2\tau_1^2}\right] \sim \frac{1}{\omega^2} \quad , \tag{5}$$

respectively. Note that the power spectrum $S(\omega)$ could not be always fitted well with an experimentally-obtained power spectrum $1/\omega^{\alpha}$ using the pre-assumed power law distribution $g(\tau)$ [9]. To discuss the distribution $g(\tau)$ in detail, it is absolutely required to define the distribution $g(\tau)$ directly from the experimentally-obtained power spectrum $1/\omega^{\alpha}$. Thus, further investigation is still needed to clarify the relationship between the distribution and power spectrum.

McWhorter model is Fredholm integral equation of the first kind with Lorentzian kernel. Therefore, using an appropriate spectral density, the distribution can be defined by an inverse integral transform. In this study, applying this alternative approach, we demonstrate that we can obtain the generalized distribution function of relaxation times from $1/f^{\alpha}$ spectrum ($1 \leq \alpha \leq 2$) on the basis of an inverse Stieltjes transform.

2. Generalized equation and approximate solutions of relaxation distribution

We have deduced the generalized relaxation time distribution from the viewpoint of the inverse integral transform, as described below. The basic idea of the method applied to solving the integral equation has been shown by Fujita in the field of polymer physics [10].

Owing to the wide time range covered [11], it is even more common and preferred to define the distribution in a logarithm of time instead of taking the linear time scale. Also, the interval of integration $[\tau_1, \tau_2]$ in Eq. (1) is replaced by $[-\infty, \infty]$ leading to

$$S(\omega) = \int_{-\infty}^{\infty} G(\tau) \frac{\tau}{1+\omega^2\tau^2} d\ln\tau . \qquad (6)$$

The connection between both distributions is clearly given by

$$G(\tau) = \tau g(\tau). \qquad (7)$$

The determination of relaxation time distribution $G(\tau)$ can be obtained with inverse Stieltjes transform of power spectrum $S(\omega)$, as detailed in Appendix. From the theory of the inverse transformation [10,12], the second and third order approximation of relaxation time distribution $G(\tau)$ can be given by

$$G_2(\tau) = -\frac{e^2}{2\pi\tau}\left[\frac{dS(\omega)}{d(\ln\omega)} + \frac{1}{2}\frac{d^2S(\omega)}{d(\ln\omega)^2}\right]_{\omega=1/\tau}, \tag{8a}$$

$$G_3(\tau) = -\frac{1}{\tau}\left[\frac{dS(\omega)}{d(\ln\omega)} + \frac{3}{4}\frac{d^2S(\omega)}{d(\ln\omega)^2} + \frac{1}{8}\frac{d^3S(\omega)}{d(\ln\omega)^3}\right]_{\omega=1/\tau}. \tag{8b}$$

The higher order approximate distributions $G_n(\tau)$ can be derived in a similar and straightforward manner (see Appendix).

When the experimentally-obtained power spectrum $S(\omega)$ is smooth enough in the wide frequency range, the distribution function $G(\tau)$ might be obtained from Eqs. (8a), (8b) by using a numerical differentiation (*e.g.*, the Savitzky‑Golay method [13]). Alternatively, the distribution function $G(\tau)$ might be obtained from the power spectrum $S(\omega)$ using numerical inverse integral transform algorithms such as histogram method [14,15], CONTIN [16,17], maximum entropy method [18] and so on. In this study, to avoid the impact of quality of the experimentally-obtained power spectrum [19], we used the following equation for power spectrum $S(\omega)$ on the calculation of the distribution function $G(\tau)$

$$S(\omega) = \frac{c\tau_c}{1 + (\omega\tau_c)^\alpha}, \tag{9}$$

where c is a constant, $\omega = 2\pi f$, $\tau_c = 1/(2\pi f_c)$, f_c is the corner frequency, and α is the slope of the power spectrum. Equation (9) was empirically found to fit experimental data of power spectra well in the case of low-frequency noise in SiN_x insulator-gate GaAs-based etched nanowire field effect transistors (FETs) [9]. The effective time constant τ_c and the slope α were indeed successfully evaluated by using this relatively simple and flexible function. To calculate $G(\tau)$, we used the 6^{th} order approximate distribution (see Appendix, Eq. (A6c)). In general, as the order increases, the higher the order used the residual sum of squares (RSS) decreases asymptotically to a small finite value almost equal to zero. This indicates that the higher the order, the less is the error between the measured data and model. On the other hand, the calculation complexity increases as the order increases. Considering the trade-off between precision and complexity, in this work, we have chosen the 6^{th} order approximation because the RSS value is almost saturated around the 6^{th} order. Any improvement beyond the 6^{th} order is trivial and insignificant. The obtained $G(\tau)$ is shown in **Fig. 1** for several values of noise exponent α and time constant τ_c. We used the constant $c = 1$ in all the calculations. In this plot, the $G(\tau)$ was normalized by using the trapezoidal numerical integration rule,

$$\int_{-\infty}^{\infty} G(\tau) d \ln \tau = 1 . \tag{10}$$

As shown in **Fig. 1(a)**, the distribution width decreases with increasing slope α. For $\alpha = 1$, it reduces to Kirton and Uren trapezium model [5,6], namely the $G(\tau)$ distribution becomes uniform-like distribution. In order to explain $1/f$ noise in semiconductor devices, they used the relationship $g(\tau)d\tau \propto d\tau/\tau = d\ln\tau$. It was therefore assumed that τ values are uniformly distributed along the $\ln\tau$ axis. When α goes to 2, the $G(\tau)$ distribution approaches delta function whose peak position corresponds to relaxation time. For intermediate values of α, the relaxation time constant of the dominant event is readily obtained as it corresponds to the peak position of asymmetrical bell-shape $G(\tau)$ distribution. **Fig. 1(b)** shows calculated $G(\tau)$ distribution with $\alpha = 1.5$ for different values of τ_c which shows that the $G(\tau)$ distribution retains it characteristic asymmetric bell-shape with a peak whose position corresponding to τ_c.

To verify the accuracy of the approximation and obtain back the power spectra $S(\omega)$ from the approximate $G(\tau)$, unnormalized distribution functions are substituted in the integral equation (6) using the trapezoidal numerical integration rule. The results are given in **Fig. 2**.

The open symbols $S(\omega)$ are those obtained from the integral equation (6), and the solid lines represent the original power spectra Eq. (9), as shown in **Figs. 2(a)** and **2(b)**, for the several values of noise exponent α and time constant τ_c, respectively. The power spectra $S(\omega)$ obtained from the approximate distributions $G(\tau)$ are in good agreement with the original power spectra given by Eq. (9). Thus, the approximate distribution function $G(\tau)$ can provide a fairly good approximation to the ideal distribution function $G(\tau)$.

3. Application to low frequency noise in GaAs-based nanowire FETs having a SiN_x gate insulator

In this section, we apply the above inverse integral transformation method to obtain distribution functions of relaxation times $G(\tau)$ from the experimentally-obtained power spectra $S(\omega)$ of the low frequency noise in the drain current from SiN_x insulator-gate GaAs-based etched nanowire FET [9]. **Figure 3** shows a scanning electron microscopy (SEM) image of a fabricated SiN_x-gate GaAs nanowire FET with its schematic illustration. The FET was fabricated using an AlGaAs/GaAs heterostructure having a two-dimensional electron gas (2DEG) on a (001) semi-insulating GaAs substrate. The 2DEG carrier density was 1.1×10^{11} cm^{-2}, and the mobility was 3200 $cm^2V^{-1}s^{-1}$ at 300 K. The nanowire channel was formed on the wafer by electron beam lithography and wet chemical etching. The nanowire length was 1.8 μm, whose direction was $<\bar{1}10>$. The nanowire width was 200 nm. Ni/Ge/Au/Ni/Au ohmic contacts were formed for the source and drain electrodes. A 30-nm-SiN_x insulator was deposited for the gate insulator by electron cyclotron resonance chemical vapor deposition at a substrate temperature of 260ºC. A PtPd wrap gate was formed on the SiN_x layer. The gate length was 200 nm. To measure low-frequency noise in the drain current I_D, a drain voltage V_D of 0.5 V and gate voltage V_G of 0.8 V were applied for all measurements, which were performed in the drain current saturation region. The current was amplified using a FEMTO DLPCA-200 low-noise amplifier. The noise spectrum was obtained using an HP 35670A spectrum analyzer. To minimize the external noise, measurement system was setup in an electromagnetic shielding room. Batteries were used for drain and gate voltage sources. The noise floor of the measurement system was 10^{-20} A^2/Hz, which was sufficiently smaller than the noise in the fabricated devices. All measurements were performed at room temperature. Further details of fabrication process, measurement set-up and basic characteristics of the fabricated device are discussed in ref [9].

Figure 4(a) shows the measured drain current noise spectrum $S_{ID}(f)$ in the fabricated device. The spectrum $S_{ID}(f)$ slope indicated a power law behavior intermediate between those of $1/f^2$ and $1/f$. From the measured drain current spectrum, we then extracted $G(\tau)$ as given above. As shown in **Fig. 4(a)**, the experimental spectrum could be well fitted with Eq. (9). The origin of the $1/f^2$ noise is charging and discharging of the discrete electron traps [5,6,20], and $1/f^\alpha$ ($1 \leq \alpha < 2$) noise is qualitatively understood as the aggregate of such discrete charging and discharging events with various time constants τ. **Figure 4(b)** shows calculated distribution functions of the time constant $G(\tau)$ from the noise power spectrum. In this calculation, we used the 6th order approximation (see Appendix, Eq. (A6c)) together with the fitting parameters. From the experimentally obtained spectrum in **Fig. 4(a)**, the power spectrum obtained has a value of $\alpha = 1.6$. We then calculated $G(\tau)$ distribution shown in **Fig. 4(b)** having a peak position residing at τ = 3.3 ms. When α goes 2, $S(\omega)$ approaches Lorentzian spectrum giving a single-level trap [5,6]. In this case, the $G(\tau)$ distribution approaches delta function whose peak position corresponds to the average of the trapping and detrapping times of electron. It is likely that the intermediate condition between $1/f$ and $1/f^2$ noises represents the aggregate of such discrete trapping and detrapping events with various time constants. It should be mentioned that this observed value of τ is in reasonable agreement with the charge retention time obtained from the transient measurement of the hysteresis in the I_D-V_G characteristics of the SiN_x-gate GaAs nanowire FET [21], therefore, demonstrating the validity of the proposed model. This result also indicated that the origin of the noise was indeed electron traps in SiN_x layer. It should be again emphasized that using this proposed method, we have obtained directly the dominant time constant from the measured power spectrum. We also believe that this new method of obtaining the distribution function of relaxation time from the power spectrum may also be applied in other fields involving similar fluctuations phenomena [7].

Conclusion

In summary, a novel method was presented for obtaining the distribution function of relaxation times $G(\tau)$ from power spectrum $1/f^{\alpha}$ ($1 \leq \alpha \leq 2$). The relaxation time distribution $G(\tau)$ was derived by using McWhorter model and its inverse Stieltjes transform. Unlike the pre-assumed conventional $g(\tau)$ distribution, the extracted $G(\tau)$ from our proposed method has a peak whose width increases as the slope of the power spectrum α decreases. Moreover, the position of the distribution peak determines the dominant time constant of the system on the lifetime scale. Our proposed method is unique in the sense that instead of using a pre-assumed distribution function of relaxation time, the distribution function is directly extracted from the experimentally measured power spectrum. We demonstrated the validity of this method in the analysis of low-frequency noise in FET. We also believe that this new method of obtaining the distribution function of relaxation time from the power spectrum may also be applied in other fields involving similar fluctuations phenomena [7].

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Appendix

Transformation of variables given below

$$x = \omega^2, \quad t = \tau^{-2}, \quad f(x) = S\left(x^{1/2}\right), \quad \phi(t) = \frac{1}{2} G\left(t^{-1/2}\right) t^{-1/2} \tag{A1}$$

reduces Eq. (6) to the form

$$f(x) = \int_0^\infty \frac{\phi(t)}{x+t} dt \,. \tag{A2}$$

Equation (A2) obviously represents that $f(x)$ is indicated in the form of a Stieltjes transform of $\phi(t)$. From the theory of transformation [10,12], if the $\phi(t)t^{1/2}$ is bounded and continuous over the range $0 < t < \infty$, the inverse Stieltjes transformation for an even-ordered $\phi(t)$ is

$$\phi_{2n}(t) = \frac{(-1)^n}{2\pi} \left(\frac{e}{n}\right)^{2n} \left\{ \frac{d^n}{dx^n} \left[x^{2n} \frac{d^n}{dx^n} f(x) \right] \right\}, \tag{A3a}$$

and if the $\phi(t)$ is bounded and continuous over the range $0 < t < \infty$, for an odd-ordered $\phi(t)$,

$$\phi_{2n+1}(t) = \lim_{n\to\infty} \frac{(-1)^n}{(n-1)!(n+1)!} \left\{ \frac{d^{n+1}}{dx^{n+1}} \left[x^{2n+1} \frac{d^n}{dx^n} f(x) \right] \right\}. \tag{A3b}$$

Consequently, the expressions for the $G(\tau)$ are

$$G_{2n}(\tau) = \frac{1}{\pi\tau} \lim_{n\to\infty} (-1)^n \left(\frac{e}{n}\right)^{2n} \left[(D-n+1)\cdots(D-1)D(D+1)\cdots(D+n)S(\omega) \right], \tag{A4a}$$

$$G_{2n+1}(\tau) = \frac{2}{\tau} \lim_{n\to\infty} \frac{(-1)^n}{(n-1)!(n+1)!} \left[(D-n+1)\cdots(D-1)D(D+1)\cdots(D+n+1)S(\omega) \right] , \tag{A4b}$$

respectively for the odd-ordered and even-ordered functions, where the operator D is

$$D = \frac{1}{2} \frac{d}{d(\ln\omega)}. \tag{A5}$$

From Eqs. (A4a) and (A4b), expansions for the approximate distribution $G(\tau)$ are thus obtained in order to give

$$G_4(\tau) = -\frac{e^4}{16\pi\tau} \left[\frac{dS(\omega)}{d(\ln\omega)} + \frac{1}{4}\frac{d^2S(\omega)}{d(\ln\omega)^2} - \frac{1}{4}\frac{d^3S(\omega)}{d(\ln\omega)^3} - \frac{1}{16}\frac{d^4S(\omega)}{d(\ln\omega)^4} \right]_{\omega=1/\tau} , \tag{A6a}$$

$$G_5(\tau) = -\frac{1}{\tau} \left[\frac{dS(\omega)}{d(\ln\omega)} + \frac{5}{12}\frac{d^2S(\omega)}{d(\ln\omega)^2} - \frac{5}{24}\frac{d^3S(\omega)}{d(\ln\omega)^3} - \frac{5}{48}\frac{d^4S(\omega)}{d(\ln\omega)^4} - \frac{1}{96}\frac{d^5S(\omega)}{d(\ln\omega)^5} \right]_{\omega=1/\tau} , \tag{A6b}$$

$$G_6(\tau) = -\frac{2e^6}{243\pi\tau} \left[\frac{dS(\omega)}{d(\ln\omega)} + \frac{1}{6}\frac{d^2S(\omega)}{d(\ln\omega)^2} - \frac{5}{16}\frac{d^3S(\omega)}{d(\ln\omega)^3} - \frac{5}{96}\frac{d^4S(\omega)}{d(\ln\omega)^4} + \frac{1}{64}\frac{d^5S(\omega)}{d(\ln\omega)^5} + \frac{1}{384}\frac{d^6S(\omega)}{d(\ln\omega)^6} \right]_{\omega=1/\tau} . \tag{A6c}$$

The higher order approximate distributions $G(\tau)$ may be derived in a similar and straightforward manner.

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Figure captions

Fig. 1 Normalized distribution function $G(\tau)$ obtained by Eqs. (9), (10) and (A6c) for (a) several values of noise exponent α and (b) time constant τ_c.

Fig. 2 Original power spectrum $S(f)$ given by Eq. (9) (solid line) and the power spectrum obtained by substituting the calculated approximate distribution $G(\tau)$ into the integral equation Eq. (6) (open symbols) the for (a) several values of noise exponent α and (b) time constant τ_c.

Fig. 3 SEM image and schematic illustration of a SiN_x-gate GaAs nanowire FET.

Fig. 4 (a) Measured experimental noise spectrum (black symbol). The red solid curve is the best fit curve (Eq. (9)). (b) Extracted distribution function $G(\tau)$.

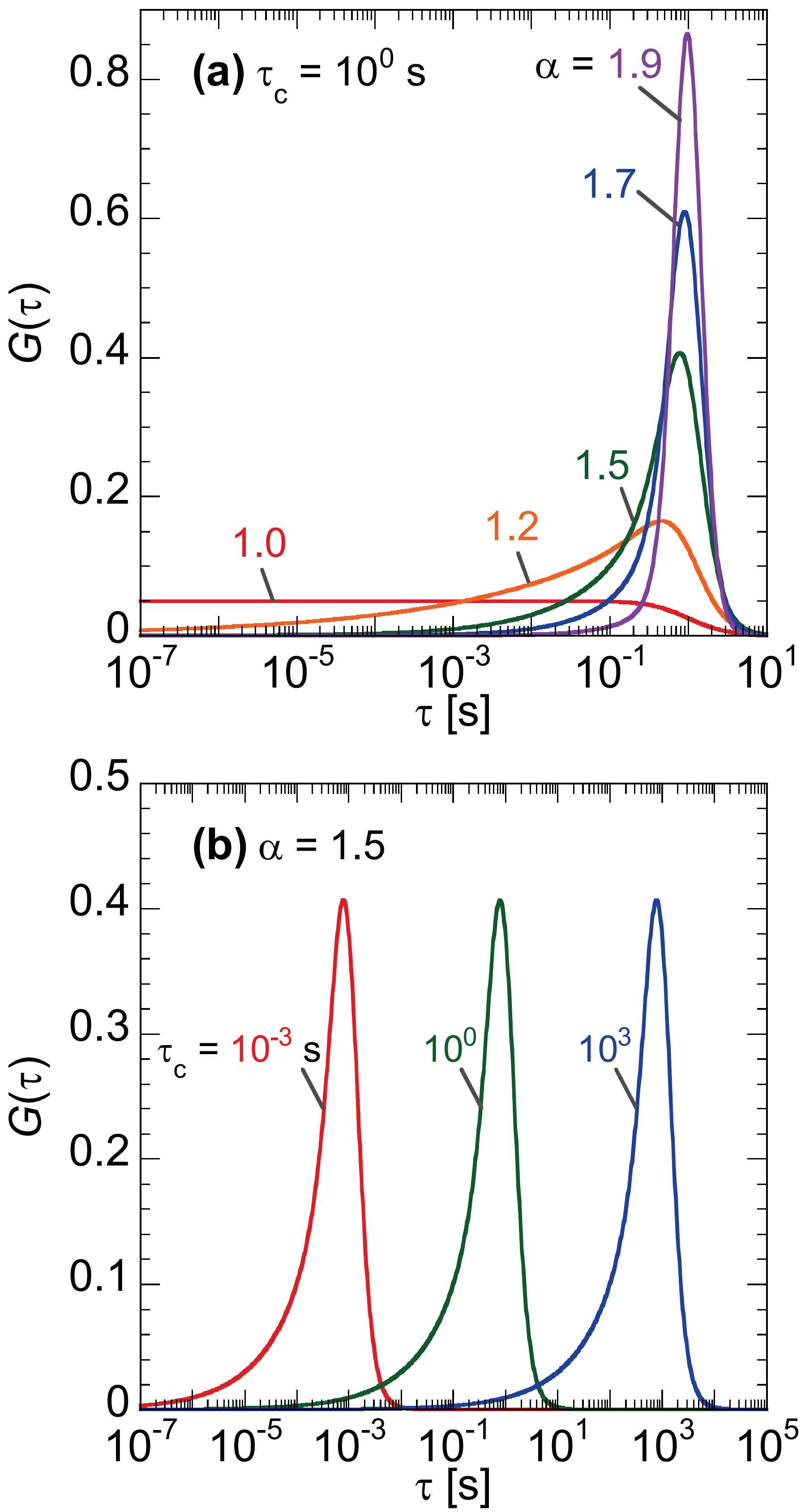

(a) $\tau_c = 10^0$ s
α = 1.9
1.7
1.5
1.2
1.0
$G(\tau)$
τ [s]
(b) α = 1.5
$\tau_c = 10^{-3}$ s
10^0
10^3
$G(\tau)$
τ [s]

Figure 1

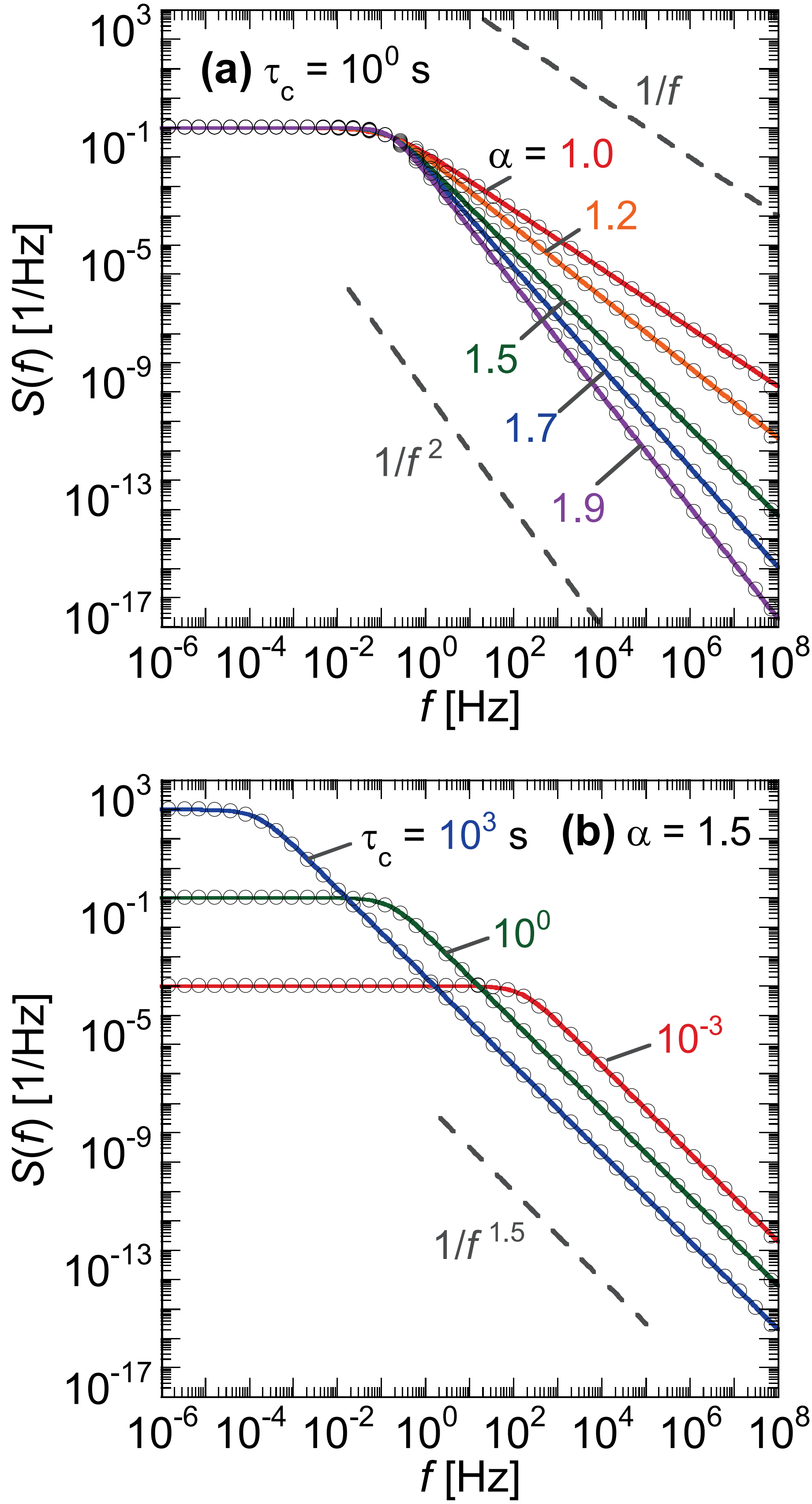

(a) $\tau_c = 10^0$ s
1/f
α = 1.0
1.2
1.5
1.7
1.9
$1/f^2$
(b) α = 1.5
$\tau_c = 10^3$ s
10^0
10^{-3}
$1/f^{1.5}$
S(f) [1/Hz]
f [Hz]

Figure 2

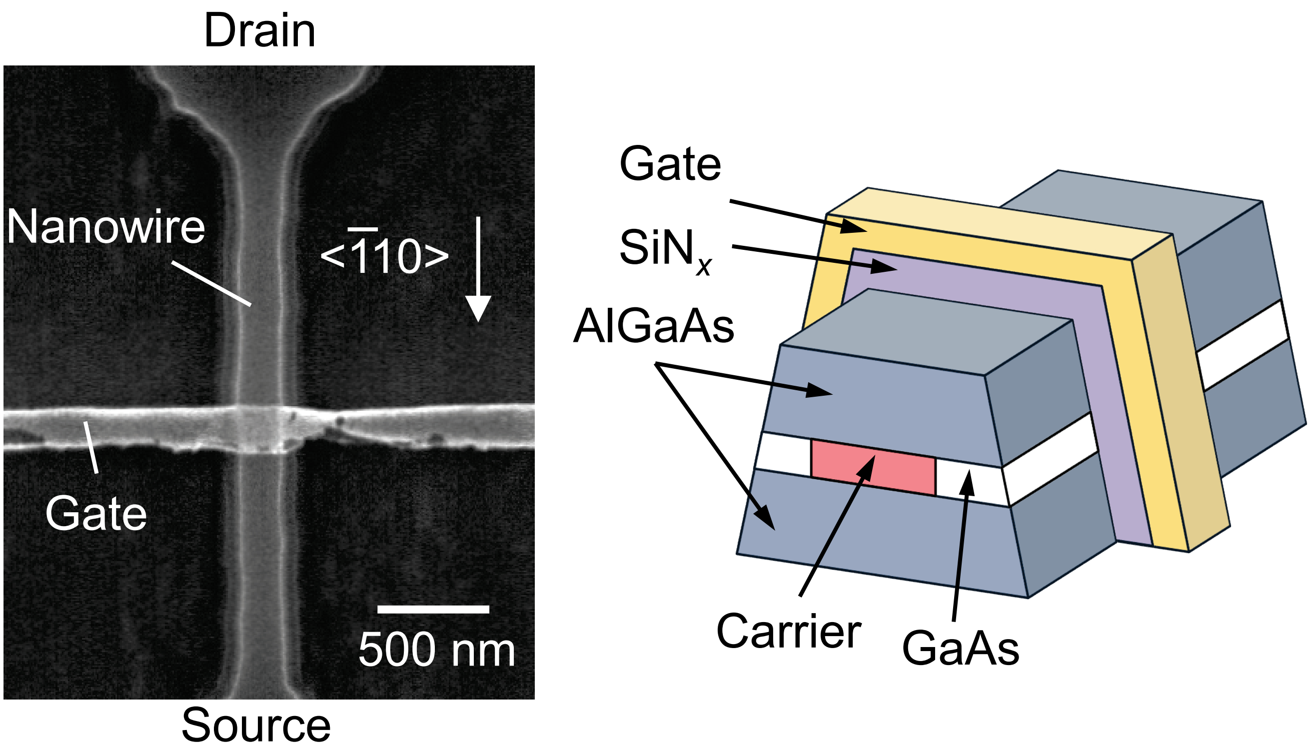
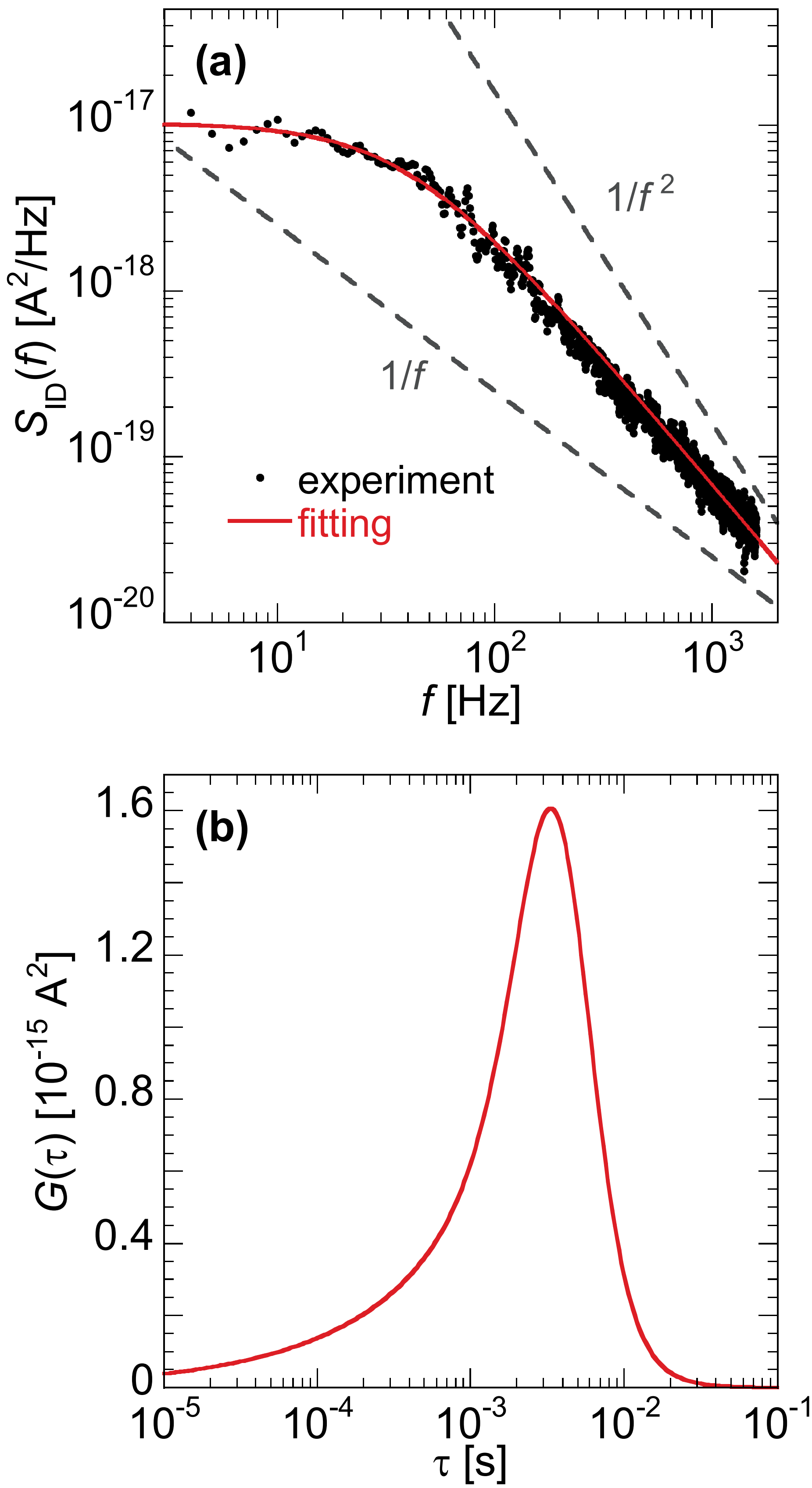

Drain
Nanowire
<$\bar{1}$10>
Gate
500 nm
Source
Gate
SiN_x
AlGaAs
Carrier
GaAs

Figure 3



(a)
10^{-17}
10^{-18}
10^{-19}
10^{-20}
$S_{ID}(f)$ [A^2/Hz]
$1/f^2$
$1/f$
experiment
fitting
10^1
10^2
10^3
f [Hz]
(b)
1.6
1.2
0.8
0.4
0
$G(\tau)$ [10^{-15} A^2]
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
τ [s]

Figure 4