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A GENERAL REGULARITY THEORY FOR WEAK MEAN CURVATURE FLOW

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ABSTRACT. We give a new proof of Brakke's partial regularity theorem up to $C^{1,\varsigma}$ for weak varifold solutions of mean curvature flow by utilizing parabolic monotonicity formula, parabolic Lipschitz approximation and blow-up technique. The new proof extends to a general flow whose velocity is the sum of the mean curvature and any given background flow field in a dimensionally sharp integrability class. It is a natural parabolic generalization of Allard's regularity theorem in the sense that the special time-independent case reduces to Allard's theorem.

1. INTRODUCTION

A family $\{M_t\}_{t \geq 0}$ of k -dimensional surfaces in $\mathbb{R}^n$ is called the mean curvature flow (hereafter abbreviated MCF) if the velocity of M_t is equal to its mean curvature at each point and time. As one of the most fundamental geometric evolution problems, the MCF has been the subject of intensive research since 1980's. The earliest study of MCF goes back to the seminal work of Brakke [7] who used the notion of varifold [1] in geometric measure theory to show the existence of weak solutions for general initial surfaces. More precisely, given any k -dimensional integral varifold V_0 with some mild finiteness assumptions, he showed the existence of a family of varifolds $\{V_t\}_{t \geq 0}$ each of which satisfies the MCF equation in a distributional sense for all $t \geq 0$. For a.e. time, Brakke additionally proved that V_t is integral. Assuming that the density function is equal to 1 almost everywhere, Brakke also claimed that the weak MCF varifold solutions are supported by smooth k -dimensional surfaces almost everywhere. The hypothesis is called the 'unit density hypothesis' and it is a natural assumption since even the time-independent case of Allard's regularity theory has essentially the same hypothesis. Brakke's regularity theory introduced remarkably ingenious tools such as the 'clearing-out lemma' and the 'popping soap film lemma', and it has a significant influence on the analysis of the related singular perturbation problems such as the Allen-Cahn equation [19] and the parabolic Ginzburg-Landau equation [5, 6, 21, 22]. However, the detail of Brakke's regularity theory is technically involved and a clear accessible proof is desired due to its importance.

The aim of the present paper is twofold. The first aim is to give a new and self-contained proof of Brakke's partial regularity theorem up to $C^{1,\varsigma}$. Brakke's proof relies on a long chain of graphical approximations which is complicated and hard to follow (see [7, 6.9, 'Flattening out']). Utilizing the ' L^2 - L^∞ lemma' (which we explain below) we replace this part with Allard-like Lipschitz approximations. The second aim is to generalize the result so that the velocity of motion may have an additional transport term which belongs to a certain

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integrability class. The existence of such flow has been studied by Liu-Sato-Tonegawa [23] and it motivated the authors to investigate the generalization of Brakke's theorem. Also, such extra term may naturally arise when one considers the MCF in Riemannian manifolds via Nash's imbedding theorem. From a view point of regularity theory, the result of the present paper reduces essentially to Allard's regularity theorem [1] in the special case of time-independent case. We note that simple modifications of Brakke's original proof do not seem to yield the theorem of the present paper when one puts the general transport term. Building upon the present work and assuming that the transport term u (see the assumption) is α -Hölder continuous, the second author established $C^{2,\alpha}$ regularity theorem in the forthcoming paper [34]. In particular [34] proves that the unit density Brakke's MCF without transport term (i.e. $u = 0$) is for a.e. time and a.e. everywhere C^∞ , which was originally claimed in [7] (see Section 10.1 for a further comment).

The major difference of our proof compared to [7] is the use of the ' L^2 - L^∞ lemma' of Section 6.2. It shows that the smallness of measurement of 'height' of MCF in the L^2 sense (hereafter called L^2 -height) guarantees that the whole support of MCF lies in a narrow region close to a k -dimensional plane. The proof of this fact utilizes an analogy of the well-known parabolic monotonicity formula due to Huisken [18] (and see [9, 31] and [17] for similar formulas in case of harmonic heat flow and semilinear heat equation, respectively). Note that Brakke's work preceded the discovery of parabolic monotonicity formula and only the 'elliptic' monotonicity formula was available at the time. The L^2 - L^∞ lemma may be considered as a robust version of Brakke's clearing-out lemma in the sense that the former may accommodate general transport terms while the latter apparently has some limitations doing so. With this new input the proof may be outlined as follows. We make a full use of the so-called popping soap film lemma with some modifications from [7, 6.6], which gives a control of the space-time L^2 norm of the mean curvature in terms of the smallness of L^2 -height (corresponding to Section 5). Then the rest of the proof proceeds more or less like Allard's regularity proof with parabolic modifications. Namely we approximate the support of moving varifolds by a Lipschitz graph (with respect to the parabolic metric) utilizing parabolic monotonicity formula. Then through a contradiction argument, we employ the blow-up technique. While the blow-up limit is a harmonic function in Allard's case, it is a solution of the heat equation in the present case. This gives a decay estimate of L^2 -height with respect to a slightly tilted k -dimensional plane in a smaller scale. Iteration argument gives Hölder estimate of the spatial gradient of the graph just like Allard's case. Though the proof of the present paper is lengthy and technically involved, we expect that the researchers familiar with regularity theory would find the details more tractable and natural than Brakke's original one.

There have been a large amount of research on the MCF and we may mention only a small fraction of them related to Brakke's varifold solutions. In particular, numerous important works regarding the behavior of MCF at the first singularity occurrence are not mentioned in this paper. For existence of generalized solutions we mention [3, 8, 13, 14, 24]. In particular Evans-Spruck [14] proved that the almost every level set of viscosity solution of MCF in the context of level set method is MCF in Brakke's sense. These level sets naturally satisfy the aforementioned unit density hypothesis, hence Brakke's theorem applies. White discovered a simple proof of local regularity theorem [39] up to and including the time at which singularities first occur in any classical MCF. He showed the local regularity when the Gaussian density is close to 1. This result is sufficient for many interesting cases (see the introduction of [39]), but nevertheless, does not replace Brakke's theorem in full. The stratification of singularities depending on the structure of tangent cones (or tangent flow

in case of MCF) due to White [35] is also relevant for the setting of this paper. There have been a significant advance of knowledge in the case of mean convex hypersurface, where the salient features of singularities of MCF have been established ([26, 36, 38]). A good reference on the subject is [12], where one can find many useful estimates for Brakke's flow written in smooth setting, and we mention [4, 10, 16, 20, 25, 37] and the references therein on the MCF.

The organization of the paper is as follows. Section 2 lists notations and recalls some well-known results from geometric measure theory. Section 3 contains the assumptions and main result. The content of Section 4, the Lipschitz approximation of varifolds for fixed time, is used in the subsequent Section 5, which deals with the local energy estimate, or tilt-excess estimate in terms of L^2 -height if one looks for an analogy with Allard's regularity theory. Section 6 is independent of the previous two sections, and establishes monotonicity-type formulas and L^2 - L^∞ estimates. Section 7 is independent and gives construction of parabolic Lipschitz graph which approximates moving varifold with good error bounds. Based on the estimates and constructions obtained through Section 5-7, Section 8 shows the decay estimates of L^2 -height via blow-up techniques. Theorem 8.7 is the main local regularity theorem. Section 9 shows that Theorem 8.7 is applicable for a.e. time and concludes the proof of main partial regularity theorem. Section 10 lists a few concluding remarks and Section 11 collects basic results from [1] and [7] for the convenience of the reader.

2. PRELIMINARIES

2.1. Basic notations. Throughout this paper, k and n will be positive integers with $0 < k < n$. By regarding n and k as absolute constants, dependence of constants on k and n may not be stated explicitly. Let $\mathbb{N}$ be the natural number and $\mathbb{R}^+ := \{x \geq 0\}$. For $0 < r < \infty$ and $a \in \mathbb{R}^n$ (or $\mathbb{R}^k$) let

$$B_r(a) := \{x \in \mathbb{R}^n : |x - a| < r\}, \quad B_r^k(a) := \{x \in \mathbb{R}^k : |x - a| < r\}$$

and when $a = 0$ let $B_r := B_r(0)$ and $B_r^k := B_r^k(0)$. We also define $\tau(r)(x) := rx$. For $s \in \mathbb{R}$ define a parabolic cylinder

$$P_r(a, s) := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x - a| < r, |t - s| < r^2\}.$$

Note that it is not an often-used parabolic cylinder with s located at the top end. Let $\bar{A}$ be the closure of $A \subset \mathbb{R}^n$. For $a \in \bar{A}$ let

$$\text{Tan}(A, a)$$

be the closed cone whose intersection with $\{x \in \mathbb{R}^n : |x| = 1\}$ is given by

$$(2.1) \quad \bigcap_{0 < r < \infty} \overline{\{(x - a)/|x - a| : x \in A \cap B_r(a) \setminus \{a\}\}}.$$

We denote by $\mathcal{L}^n$ the Lebesgue measure on $\mathbb{R}^n$ and by $\mathcal{H}^k$ the k -dimensional Hausdorff measure on $\mathbb{R}^n$. The restriction of $\mathcal{H}^k$ to a set A is denoted by $\mathcal{H}^k|_A$. We let $\omega_k := \mathcal{L}^k(B_1^k)$. For an open subset $U \subset \mathbb{R}^n$ let $C_c(U)$ be the set of all compactly supported continuous functions on U and let $C_c(U; \mathbb{R}^n)$ be the set of all compactly supported, continuous vector fields. The upper subscript of $C_c^l(U)$ and $C_c^l(U; \mathbb{R}^n)$ indicates continuous l -th order differentiability. For $g \in C^1(U; \mathbb{R}^n)$, we regard $\nabla g(x)$ as an element of $\text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$. Similarly for $g \in C^2(U)$, we regard the Hessian matrix $\nabla^2 g(x)$ as an element of $\text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$. ∇ always indicates differentiation with respect to the space variables, and not with respect to the time variable.

For any Radon measure μ on $\mathbb{R}^n$ and $\phi \in C_c(\mathbb{R}^n)$ we often write $\mu(\phi)$ for $\int_{\mathbb{R}^n} \phi d\mu$. Let $\text{spt } \mu$ be the support of μ . Thus $x \in \text{spt } \mu$ if $\forall r > 0$, $\mu(B_r(x)) > 0$. Let $\Theta^k(\mu, x)$ be the

k -dimensional density of μ at x , i.e., $\lim_{r \rightarrow 0} \mu(B_r(x)) / (\omega_k r^k)$, when the limit exists. For μ measurable function u , and $1 \leq p \leq \infty$, $u \in L^p(\mu)$ means $(\int |u|^p d\mu)^{1/p} < \infty$.

For $-\infty < t < s < \infty$ and $x, y \in \mathbb{R}^n$, define

$$(2.2) \quad \rho_{(y,s)}(x,t) := \frac{1}{(4\pi(s-t))^{k/2}} \exp\left(-\frac{|x-y|^2}{4(s-t)}\right).$$

$\rho_{(y,s)}$ is the k -dimensional backward heat kernel which is often used throughout this paper.

2.2. The Grassmann manifold and varifolds. Let $\mathbf{G}(n, k)$ be the space of k -dimensional subspaces of $\mathbb{R}^n$ and let $\mathbf{A}(n, k)$ be the space of k -dimensional affine planes of $\mathbb{R}^n$. For $S \in \mathbf{G}(n, k)$, we identify S with the corresponding orthogonal projection of $\mathbb{R}^n$ onto S . Let $S^\perp \in \mathbf{G}(n, n-k)$ be the orthogonal complement of S . For two elements A and B of $\text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$, define a scalar product $A \cdot B := \text{trace}(A^* \circ B)$ where A^* is the transpose of A and $\circ$ indicates the usual composition. The identity of $\text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is denoted by I . Let $a \otimes b \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ be the tensor product of $a, b \in \mathbb{R}^n$. For $A \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ define

$$|A| := \sqrt{A \cdot A}, \quad \|A\| := \sup\{|A(x)| : x \in \mathbb{R}^n, |x| = 1\}.$$

For $A \in \text{Hom}(\mathbb{R}^k, \mathbb{R}^n)$, define J_A as the Jacobian of A , namely,

$$J_A := \sqrt{\det(A^* \circ A)}.$$

For $T \in \mathbf{G}(n, k)$, $a \in \mathbb{R}^n$ and $0 < r < \infty$ we define cylinders

$$C(T, a, r) := \{x \in \mathbb{R}^n : |T(x - a)| < r\}, \quad C(T, r) := C(T, 0, r).$$

We recall some notions related to varifold and refer to [1, 30] for more details. For any open set $U \subset \mathbb{R}^n$, define $G_k(U) := U \times \mathbf{G}(n, k)$. A general k -varifold in U is a Radon measure on $G_k(U)$. Set of all general k -varifolds in U is denoted by $\mathbf{V}_k(U)$. For $V \in \mathbf{V}_k(U)$, let $\|V\|$ be the weight measure of V , namely,

$$\|V\|(\phi) := \int_{G_k(U)} \phi(x) dV(x, S), \quad \forall \phi \in C_c(U).$$

For a proper map $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ define $f_\# V$ as the push-forward of varifold $V \in \mathbf{V}_k(U)$ (see [1] for the definition). Given any $\mathcal{H}^k$ measurable countably k -rectifiable set $M \subset U$ with locally finite $\mathcal{H}^k$ measure, there is a natural k -varifold $|M| \in \mathbf{V}_k(U)$ defined by

$$|M|(\phi) := \int_M \phi(x, \text{Tan}_x M) d\mathcal{H}^k(x), \quad \forall \phi \in C_c(G_k(U)),$$

where $\text{Tan}_x M \in \mathbf{G}(n, k)$ is the approximate tangent space which exists $\mathcal{H}^k$ a.e. on M . In this case, $\| |M| \| = \mathcal{H}^k \llcorner_M$. We say $V \in \mathbf{V}_k(U)$ is integral if

$$V(\phi) = \int_M \phi(x, \text{Tan}_x M) \theta(x) d\mathcal{H}^k(x), \quad \forall \phi \in C_c(G_k(U)),$$

for some $\mathcal{H}^k$ measurable countably k -rectifiable set $M \subset U$ and $\mathcal{H}^k$ a.e. integer-valued locally $\mathcal{H}^k$ integrable function θ defined on M . Note that for such varifold, $\Theta^k(\|V\|, x) = \theta(x) \in \mathbb{N}$, $\mathcal{H}^k$ a.e. on M . Set of all integral k -varifolds in U is denoted by $\mathbf{IV}_k(U)$. We say V is a unit density k -varifold if V is integral and $\theta = 1$ a.e. on M , that is, $V = |M|$. When V is integral, we often write $\int_U (g(x))^\perp d\|V\|(x)$ for $\int_{G_k(U)} S^\perp(g(x)) dV(x, S)$.

2.3. First variation and generalized mean curvature. For $V \in \mathbf{V}_k(U)$ let δV be the first variation of V , namely,

$$\delta V(g) := \int_{G_k(U)} \nabla g(x) \cdot S \, dV(x, S)$$

for $g \in C_c^1(U; \mathbb{R}^n)$. Let $\|\delta V\|$ be the total variation when it exists, and if $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, we have for some $\|V\|$ measurable vector field $h(V, \cdot)$

$$(2.3) \quad \delta V(g) = - \int_U g(x) \cdot h(V, x) \, d\|V\|(x).$$

The vector field $h(V, \cdot)$ is called the generalized mean curvature of V . We say V is stationary if $h(V, \cdot) = 0$, $\|V\|$ a.e. in U , or equivalently, $\delta V(g) = 0$ for all $g \in C_c^1(U; \mathbb{R}^n)$. For any $V \in \mathbf{IV}_k(U)$ with integrable $h(V, \cdot)$, Brakke's perpendicularity theorem of generalized mean curvature [7, Chapter 5] says that we have

$$(2.4) \quad \int_U (g(x))^\perp \cdot h(V, x) \, d\|V\|(x) = \int_U g(x) \cdot h(V, x) \, d\|V\|(x)$$

for all $g \in C_c(U; \mathbb{R}^n)$.

2.4. The right-hand side of MCF equation. For any $V \in \mathbf{V}_k(U)$, $u \in L_{loc}^2(\|V\|)$ and $\phi \in C_c^1(U; \mathbb{R}^+)$, define

$$(2.5) \quad \mathcal{B}(V, u, \phi) := \int_U (-\phi(x)h(V, x) + \nabla \phi(x)) \cdot (h(V, x) + (u(x))^\perp) \, d\|V\|(x)$$

when $V \in \mathbf{IV}_k(U)$, $\|\delta V\|$ is locally finite and absolutely continuous with respect to $\|V\|$, and $h(V, \cdot) \in L_{loc}^2(\|V\|)$. Otherwise we define $\mathcal{B}(V, u, \phi) = -\infty$. Formally, if a family of smooth k -dimensional surfaces $\{M_t\}$ moves by the velocity equal to the mean curvature plus smooth u , then, one can check that $V_t = |M_t|$ satisfies

$$(2.6) \quad \frac{d}{dt} \|V_t\|(\phi) \leq \mathcal{B}(V_t, u, \phi), \quad \forall \phi \in C_c^1(U; \mathbb{R}^+).$$

In fact, (2.6) holds with equality. Conversely, if (2.6) is satisfied, then one can prove that the velocity is equal to the mean curvature plus u . The inequality in (2.6) allows the sudden loss of surface and it is the source of general non-uniqueness of Brakke's formulation.

3. ASSUMPTIONS AND MAIN RESULT

3.1. Assumptions. For an open set $U \subset \mathbb{R}^n$ and $0 < \Lambda \leq \infty$ suppose that we have a family of k -varifolds $\{V_t\}_{0 \leq t < \Lambda}$ and a family of $\mathbb{R}^n$ valued functions $\{u(\cdot, t)\}_{0 \leq t < \Lambda}$ both on U satisfying the followings.

(A1) For a.e. $t \in [0, \Lambda)$, V_t is a unit density k -varifold.

(A2) There exists $1 \leq E_1 < \infty$ such that

$$(3.1) \quad \|V_t\|(B_r(x)) \leq \omega_k r^k E_1, \quad \forall B_r(x) \subset U, \quad \forall t \in [0, \Lambda).$$

(A3) Suppose $2 \leq p < \infty$ and $2 < q < \infty$ satisfy

$$(3.2) \quad \varsigma := 1 - \frac{k}{p} - \frac{2}{q} > 0$$

and assume that u satisfies

$$(3.3) \quad \|u\|_{L^{p,q}(U \times (0,\Lambda))} := \left(\int_0^\Lambda \left(\int_U |u(x,t)|^p d\|V_t\|(x) \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} < \infty.$$

(A4) For all $\phi \in C^1(U \times [0, \Lambda]; \mathbb{R}^+)$ with $\phi(\cdot, t) \in C_c^1(U)$ and $0 \leq t_1 < t_2 < \Lambda$,

$$(3.4) \quad \|V_{t_2}\|(\phi(\cdot, t_2)) - \|V_{t_1}\|(\phi(\cdot, t_1)) \leq \int_{t_1}^{t_2} \mathcal{B}(V_t, u(\cdot, t), \phi(\cdot, t)) dt + \int_{t_1}^{t_2} \int_U \frac{\partial \phi}{\partial t}(\cdot, t) d\|V_t\| dt$$

holds.

As an immediate consequence, for $\phi \in C_c^2(U; \mathbb{R}^+)$, we have from (3.4) and the Cauchy-Schwarz inequality

$$(3.5) \quad \begin{aligned} \|V_{t_2}\|(\phi) - \|V_{t_1}\|(\phi) &\leq \int_{t_1}^{t_2} \int_U -\phi|h|^2 + |\nabla \phi||h| + \phi|h||u| + |\nabla \phi||u| d\|V_t\| dt \\ &\leq \int_{t_1}^{t_2} \int_U -\frac{|h|^2 \phi}{2} + \frac{|\nabla \phi|^2}{\phi} + |u|^2 \phi + |\nabla \phi||u| d\|V_t\| dt \\ &\leq \int_{t_1}^{t_2} \int_U -\frac{|h|^2 \phi}{2} d\|V_t\| dt + c(\|\phi\|_{C^2}, E_1, \|u\|_{L^{p,q}}, \text{spt } \phi). \end{aligned}$$

Thus we have from (3.5) that $h(V_t, \cdot) \in L_{loc}^2(\|V_t\| \times dt)$. In particular

$$(3.6) \quad h(V_t, \cdot) \in L_{loc}^2(\|V_t\|)$$

for a.e. $t \in (0, \Lambda)$, which also follows directly from the definition of $\mathcal{B}(V, u, \phi)$ and (3.4).

3.2. Remarks on the assumptions. Before stating our main theorem, we motivate our assumptions.

The setting that Brakke proved the local regularity theorem corresponds to $u = 0$ case. We note that the weak solution he constructed in [7] satisfies (A2)-(A4) locally in space and time. In this special case, we may relax the assumption (A2) by, for example,

$$\sup_{0 \leq t < \Lambda} \|V_t\|(U) \leq C,$$

or

$$U = B_R, \quad \|V_0\|(U) < \infty \text{ and } \text{spt } \|V_0\| \subset\subset U.$$

The assumption (A2) (restricted for $U' \times (\delta, \Lambda)$ with $\forall U' \subset\subset U$ and $\forall \delta > 0$) follows from the parabolic monotonicity formula of Section 6.1. The inequality (3.4) is the integral form of Brakke's formulation, except that we need time varying test functions. See the discussion in [7, 3.5] for the derivation. The varifolds are also integral for a.e. time if V_0 is assumed to be integral. With parabolic monotonicity formula it is not difficult to prove that there is an initial short time interval during which the varifolds remain unit density if the initial varifold V_0 has (A2) with E_1 close to 1.

As we noted in Section 1, in [14] Evans-Spruck proved that almost all level set of MCF viscosity solution is a unit density MCF in the sense of Brakke. Thus almost all level set satisfies (A1)-(A4) with $u = 0$.

Ilmanen [19] proved that the singular perturbation limit of the Allen-Cahn equation is a rectifiable Brakke's MCF under mild conditions on initial data. In addition the second author [33] proved that the limit varifolds are integral modulo division by a surface energy constant for a.e. time. Thus the limit varifolds arising from the Allen-Cahn equation, unless there are portions where higher (≥ 2) multiplicities occur, satisfy (A1)-(A4). We mention that a simple proof of [19, 33] was obtained by Sato [29] utilizing the result of Röger-Schätzle [27] (see also [28]) for $k = n - 1$ and $n = 2, 3$.

As for $u \neq 0$ case, we note first that (3.2) is dimensionally a sharp condition. A simple dimension analysis of natural re-scaling $\tilde{x} = \lambda^{-1}x$, $\tilde{t} = \lambda^{-2}t$ and $\tilde{u} = \lambda u$ for $\lambda > 0$ shows

$$\|\tilde{u}\|_{L^{p,q}} = \lambda^{1-\frac{k}{p}-\frac{2}{q}} \|u\|_{L^{p,q}}.$$

It is a rule of thumb that (3.2) is essential to successfully obtain a local regularity theorem. We also note that if $V_t = V_0$ for all $t > 0$, then q may be regarded as $+\infty$ and (3.2) requires $p > k$. Time-independence means the velocity is 0. Then we have $0 = h(V, \cdot) + u^\perp$ (see Section 10) and (3.3) gives $h(V, \cdot) \in L^p(\|V\|)$ with $p > k$. It is the condition for the application of Allard's regularity theorem.

The existence of flow, where the velocity has additional transport term u , has been investigated in [23] for $k = n - 1$, $n = 2, 3$. There, given an arbitrary u in $L^p_{\text{loc}}([0, \infty); W^{1,p}(\mathbb{T}^n))$ with $p > \frac{n+2}{2}$, where $\mathbb{T}^n$ is the n -dimensional torus and $W^{1,p}(\mathbb{T}^n) = \{u \in L^p(\mathbb{T}^n) : \nabla u \in L^p(\mathbb{T}^n)\}$, and C^1 initial $\text{spt } \|V_0\|$, it was proved that there exists a family $\{V_t\}_{t \geq 0}$ of $(n-1)$ -varifolds satisfying (A2)-(A4) and a.e. time integrality. $u(\cdot, t)$ in this case is defined as a trace on the support of $\|V_t\|$. The fulfillment of condition (3.3) is not stated explicitly there, but can be verified using [23, Theorem 2.1]. The unit density hypothesis (A1) is satisfied for at least short time [23, Theorem 2.2(c)]. Thus we can apply the present regularity theory to this initial short time. Unlike $u = 0$ case, we are not aware of any short time existence result for classical solutions under low regularity assumptions on u . The extension of [23] for $n > 3$ and $k = n - 1$ which covers the whole range of p and q satisfying (3.2) is currently under investigation with similar approach [32]. We are unaware of any general method in case $k < n - 1$, most relevant being the approximation scheme via the parabolic Ginzburg-Landau equation [6] where $u = 0$, $k = n - 2$ and $n \geq 3$. Note that [6] proved that the limit varifold V_t corresponding to vorticity concentration is rectifiable k -varifold moving by MCF in Brakke's formulation. On the other hand, one apparently does not exclude the possibility of non-integral varifold due to the non-trivial diffuse part of the limit measure.

3.3. Main result.

Definition 3.1. A point $x \in U \cap \text{spt } \|V_t\|$ is said to be a $C^{1,\varsigma}$ regular point if there exists some open neighborhood O in $\mathbb{R}^{n+1}$ containing (x, t) such that $O \cap \cup_{0 < s < \Lambda} (\text{spt } \|V_s\| \times \{s\})$ is an embedded $(k+1)$ -dimensional manifold represented as the image of $f : O' := B_R^k \times (t - R^2, t + R^2) \rightarrow O \subset \mathbb{R}^{n+1}$ with

$$\sup_{(y_j, s_j) \in O', j=1,2} \frac{\|\nabla f(y_1, s_1) - \nabla f(y_2, s_2)\|}{\max\{|y_1 - y_2|^\varsigma, |s_1 - s_2|^{\varsigma/2}\}} < \infty, \quad \sup_{(y, s_j) \in O', j=1,2} \frac{|f(y, s_1) - f(y, s_2)|}{|s_1 - s_2|^{(1+\varsigma)/2}} < \infty.$$

Here ∇ is the differentiation with respect to y -variables.

Here is the main partial regularity theorem.

Theorem 3.2. Under the assumptions (A1)-(A4), for a.e. $t \in (0, \Lambda)$, there exists a (possibly empty) closed set $G_t \subset \text{spt } \|V_t\|$ with $\mathcal{H}^k(G_t) = 0$ such that $\text{spt } \|V_t\| \setminus G_t$ is a set of $C^{1,\varsigma}$ regular points.

Theorem 3.2 is the consequence of Theorem 8.7 and Theorem 9.1 which are the main local regularity theorems. For more detailed local estimates and assumptions, see the statement of Theorem 8.7.

4. LIPSCHITZ APPROXIMATION FOR FIXED TIME

In this section, the main result is Proposition 4.6 which gives a graphical Lipschitz approximation of varifold, and which is one of the essential ingredients for Theorem 5.7 of the next section. The results of this section do not involve time variable. The content of this section corresponds to [7, 5.3, 5.4]. To avoid unnecessary technical complications, we avoid general multiple-sheet situations discussed in [7] which are not needed for the purpose of this paper.

Lemma 4.1. *Corresponding to each $0 < \lambda < 1$ there exists $\gamma > 0$ with the following property. For $V \in \mathbf{IV}_k(B_R)$ assume*

$$(4.1) \quad \Theta^k(\|V\|, 0) = 1,$$

$$(4.2) \quad r\|\delta V\|(B_r) \leq \gamma\|V\|(B_r), \quad \forall r \in (0, R].$$

Then we have

$$(4.3) \quad \|V\|(B_r) \geq \lambda\omega_k r^k, \quad \forall r \in (0, R].$$

Remark 4.2. *If the condition (4.2) is replaced by $\|\delta V\|(B_r) \leq \gamma\|V\|(B_r)$, then the well-known monotonicity formula [1, 5.1] says $r^{-k}\|V\|(B_r) \exp(\gamma r)$ is monotone increasing. Allard essentially posed this stronger condition to define a set of points in $\text{spt } \|V\|$ to be included in some Lipschitz graph (see [1, 8.12]). By using the condition (4.2), Brakke obtained an improved error estimate for Lipschitz approximation as in Proposition 4.6, in the sense that the coverage of $\text{spt } \|V\|$ by Lipschitz graph is better than that of [1, 8.12], and which is one of the key points of the proof. See Remark 4.9 for further comment.*

Proof. Without loss of generality we may assume $R = 1$. Assume that the conclusion were false. Then for each $m \in \mathbb{N}$ there exist $V_m \in \mathbf{IV}_k(B_1)$ such that (4.1) and (4.2) hold for $V = V_m$ and $\gamma = 1/m$, but (4.3) does not hold. Let R_m be the supremum of $\{0\} \cup \{r > 0 : \|V_m\|(B_s) \geq \lambda\omega_k s^k \ \forall s \in (0, r]\}$. By (4.1), we have $R_m > 0$. Since we are assuming the negation of (4.3), we have $\|V_m\|(B_{R_m}) = \lambda\omega_k R_m^k$. Let $\tilde{V}_m = (\tau(R_m^{-1}))_{\#} V_m$. Then we have by (4.2) and by the definition of R_m

$$(4.4) \quad \|\delta \tilde{V}_m\|(B_1) = \frac{1}{R_m^{k-1}} \|\delta V_m\|(B_{R_m}) \leq \frac{1}{m R_m^k} \|V_m\|(B_{R_m}) = \frac{\omega_k \lambda}{m},$$

$$(4.5) \quad \|\tilde{V}_m\|(B_1) = \frac{1}{R_m^k} \|V_m\|(B_{R_m}) = \omega_k \lambda,$$

$$(4.6) \quad \|\tilde{V}_m\|(B_s) = \frac{1}{R_m^k} \|V_m\|(B_{sR_m}) \geq \lambda\omega_k s^k, \quad \forall s \in (0, 1].$$

By (4.4) and (4.5) and the compactness theorem of integral varifolds ([1, 6.4]), there exists a subsequence $\{\tilde{V}_{m_j}\}$ and $V \in \mathbf{IV}_k(B_1)$ such that $\tilde{V}_{m_j} \rightarrow V$ in the sense of varifolds. By (4.4), V is stationary, and by (4.6), $0 \in \text{spt } \|V\|$. From (4.5) and (4.6), we also have

$$(4.7) \quad \|V\|(B_1) = \lambda\omega_k.$$

By the upper-semicontinuity of density function and the integrality of V , we have $\Theta^k(\|V\|, 0) \geq 1$. By the monotonicity formula of stationary varifold, we have $\|V\|(B_1) \geq \omega_k$, which contradicts with (4.7) since $\lambda < 1$. This proves (4.3). $\square$

Lemma 4.3. *Corresponding to each $0 < \lambda < 1$ and $1 \leq E_1 < \infty$, there exists $\gamma > 0$ with the following property. For $V \in \mathbf{IV}_k(B_R)$ assume (4.1), (4.2),*

$$(4.8) \quad \|V\|(B_R) \leq \omega_k R^k E_1,$$

$$(4.9) \quad \int_{G_k(B_R)} \|S - T\| dV(x, S) \leq \gamma \|V\|(B_R)$$

for some $T \in \mathbf{G}(n, k)$. Then for any $b \in T$ with $|b| \leq R/2$,

$$(4.10) \quad \|V\|(B_r(b)) \geq \lambda \omega_k r^k, \quad \forall r \in [R/10, R/2].$$

Proof. Assume $R = 1$ without loss of generality. Assume that the conclusion were false. Then for each $m \in \mathbb{N}$ we have a sequence $V_m \in \mathbf{IV}_k(B_1)$, $b_m \in T$ with $|b_m| \leq 1/2$ and R_m with $1/10 \leq R_m \leq 1/2$ such that (4.1), (4.2), (4.8) and (4.9) hold for $\gamma = 1/m$ and $V = V_m$ but (4.10) fails, that is,

$$(4.11) \quad \|V_m\|(B_{R_m}(b_m)) < \lambda \omega_k R_m^k.$$

Note that we need to have varying T_m , but after orthogonal rotations, we may assume that $T_m = T$. Due to (4.8) and (4.2) we have a subsequential limit $V \in \mathbf{IV}_k(B_1)$ which is stationary, and we may assume that $b_m \rightarrow b_* \in T$ with $|b_*| \leq 1/2$ and $R_m \rightarrow R_*$ with $1/10 \leq R_* \leq 1/2$. By (4.11) we also have

$$(4.12) \quad \|V\|(B_{R_*}(b_*)) \leq \lambda \omega_k R_*^k.$$

By (4.9), we also have

$$(4.13) \quad \int_{G_k(B_1)} \|S - T\| dV(x, S) = 0.$$

Since V is stationary and integral, one can show with (4.13) that V is invariant in T -direction in B_1 . Lemma 4.1 shows that $0 \in \text{spt } \|V\|$, thus $b_* \in \text{spt } \|V\|$. In particular we have $\Theta^k(\|V\|, b_*) \geq 1$. By the monotonicity formula we have $\|V\|(B_{R_*}(b_*)) \geq \omega_k R_*^k$. This contradicts with (4.12) and we may conclude the proof. $\square$

Lemma 4.4. *Corresponding to $1/2 < \lambda < 1$, $1 \leq l < \infty$ and $1 \leq E_1 < \infty$, there exists $\gamma > 0$ with the following property. For $T \in \mathbf{G}(n, k)$ and $V \in \mathbf{IV}_k(C(T, 3R))$, and two distinct points $y_1, y_2 \in C(T, R)$, assume*

$$(4.14) \quad |y_1 - y_2| \leq l |T^\perp(y_1 - y_2)|,$$

$$(4.15) \quad \Theta^k(\|V\|, y_i) = 1 \text{ for } i = 1, 2,$$

$$(4.16) \quad \int_{G_k(B_r(y_i))} \|S - T\| dV(x, S) \leq \gamma \|V\|(B_r(y_i)), \quad \forall r \in (0, 2R], i = 1, 2,$$

$$(4.17) \quad r \|\delta V\|(B_r(y_i)) \leq \gamma \|V\|(B_r(y_i)), \quad \forall r \in (0, 2R], i = 1, 2,$$

$$(4.18) \quad \|V\|(B_r(y_i)) \leq \omega_k r^k E_1, \quad \forall r \in (0, 2R], i = 1, 2.$$

For $i = 1, 2$ set $\tilde{y}_i := T^\perp(y_i) \in \mathbb{R}^n$. Then we have

$$(4.19) \quad \|V\|(B_R(\tilde{y}_1) \cup B_R(\tilde{y}_2)) \geq 2\lambda \omega_k R^k.$$

Remark 4.5. *The above Lemma resembles [1, 6.2] which is one of the essential ingredients for graphical Lipschitz approximation needed for Allard's regularity theory. Note that the difference of above claim is that $\tilde{y}_1$ and $\tilde{y}_2$ are projected points to $T^\perp$ and the two balls centered at these points are inside of the cylinder $C(T, R)$. Roughly speaking, the Lemma claims that if there are two good points placed in vertical positions, there must be almost two parallel sheets inside of $C(T, R)$.*

Proof. Without loss of generality, we may assume $R = 1$. Suppose that the claim were false. Then we would have a sequence of $V_m \in \mathbf{IV}_k(C(T, 3))$ and $y_{1,m} \neq y_{2,m}$ in $C(T, 1)$ such that (4.14)-(4.18) are satisfied with $V = V_m$, $y_1 = y_{1,m}$, $y_2 = y_{2,m}$ and $\gamma = 1/m$ while (4.19) does not hold, i.e.,

$$(4.20) \quad \|V_m\|(B_1(\tilde{y}_{1,m}) \cup B_1(\tilde{y}_{2,m})) < 2\lambda\omega_k,$$

where $\tilde{y}_{i,m} := T^\perp(y_{i,m})$ for $i = 1, 2$. Here we should have varying T_m which, after rotating the coordinates, can be fixed as T . For $m \in \mathbb{N}$, $0 < s \leq 1$ and $i = 1, 2$ set

$$(4.21) \quad A_{i,m}^s := \{x : \text{dist}(y_{i,m} - sT(y_{i,m}), x) < s\}, \quad A_m^s := A_{1,m}^s \cup A_{2,m}^s.$$

Using $y_{i,m} \in C(T, 1)$, one can check that $A_m^s \subset C(T, 1)$. One can also see that $A_m^1 = B_1(\tilde{y}_{1,m}) \cup B_1(\tilde{y}_{2,m})$. We note that

$$(4.22) \quad \frac{1}{\omega_k s^k} \|V_m\|(A_{i,m}^s) \geq \frac{\lambda + 1}{2} (> \lambda) \quad \forall s \in (0, 1)$$

for all sufficiently large m and $i = 1, 2$. To see this, for $s \in (0, 1)$ we consider $V_{m,s} = (\tau(s^{-1}))_\# V_m$ and translate $s^{-1}y_{i,m}$ to the origin. Such change of coordinates transforms $A_{i,m}^s$ to $B_1(-T(y_{i,m}))$. Then by Lemma 4.3 with $b = -T(y_{i,m})$, $R = 2$, λ there replaced by $(\lambda + 1)/2$ and (4.15)-(4.18), we obtain that $\|V_{m,s}\|(B_1(-T(y_{i,m}))) \geq \frac{\lambda+1}{2}\omega_k$ for all large m . This shows (4.22) after changing back to the original coordinates. Next, let R_m be the supremum of

$$(4.23) \quad \{0\} \cup \{r > 0 : \|V_m\|(A_m^s) \geq 2\lambda\omega_k s^k \quad \forall s \in (0, r]\}.$$

By (4.22), for all sufficiently large m and sufficiently small $s > 0$, we have

$$(4.24) \quad \frac{1}{\omega_k s^k} \|V_m\|(A_m^s) = \sum_{i=1,2} \frac{1}{\omega_k s^k} \|V_m\|(A_{i,m}^s) \geq (\lambda + 1) > 2\lambda.$$

Thus (4.24) shows that $R_m > 0$. On the other hand, since $A_m^1 = B_1(\tilde{y}_{1,m}) \cup B_1(\tilde{y}_{2,m})$, (4.20) shows $R_m < 1$. In particular, by the definition of R_m , we have

$$(4.25) \quad \|V_m\|(A_m^{R_m}) = 2\lambda\omega_k R_m^k$$

for all large m . Set $\hat{V}_m := (\tau(R_m^{-1}))_\# V_m$ and $\hat{y}_{i,m} := R_m^{-1}y_{i,m}$. The change of variables $\tau(R_m^{-1})$ transforms $A_{i,m}^{R_m}$ to $B_1(\hat{y}_{i,m} - T(y_{i,m}))$, which we denote by $\hat{B}_{i,m}$. Suppose that $\hat{B}_{1,m_j} \cap \hat{B}_{2,m_j} = \emptyset$ for some subsequence $\{m_j\}_{j=1}^\infty$ (subsequently omitting the sub-index). By Lemma 4.3 with the origin replaced by $\hat{y}_{i,m}$, b replaced by $\hat{y}_{i,m} - T(y_{i,m})$ and $R = 2$, for all sufficiently large m , we may conclude that $\|\hat{V}_m\|(\hat{B}_{i,m}) \geq (\lambda + 1)\omega_k/2$. Here we have used (4.15)-(4.18) with $\gamma = 1/m$. After changing back to the original coordinates, we have $\|V_m\|(A_{i,m}^{R_m}) \geq (\lambda + 1)\omega_k R_m^k/2$. Since we are now assuming $A_{1,m}^{R_m} \cap A_{2,m}^{R_m} = \emptyset$, this contradicts with (4.25). Suppose that $\hat{B}_{1,m} \cap \hat{B}_{2,m} \neq \emptyset$ for all large m . By shifting $x \rightarrow x - \hat{y}_{1,m}$ and re-defining $\hat{B}_{i,m}$ as $\hat{B}_{i,m} - \hat{y}_{1,m}$ for $i = 1, 2$, we may assume that $\hat{B}_{1,m} \cup \hat{B}_{2,m} \subset B_4$. We have $\hat{B}_{1,m} = B_1(-T(y_{1,m}))$ and $\hat{B}_{2,m} = B_1(\hat{y}_{2,m} - \hat{y}_{1,m} - T(y_{2,m}))$. We also shift $\hat{V}_m$ by $x \rightarrow x - \hat{y}_{1,m}$. Since $\hat{y}_{2,m} - \hat{y}_{1,m} \in B_4$ and $|T(y_{i,m})| \leq 1$ for $i = 1, 2$, there exists a

subsequence (denoted by the same index) such that $\hat{y}_{2,m} - \hat{y}_{1,m} \rightarrow \hat{y}_*$, $T(y_{i,m}) \rightarrow b_i \in T$ for $i = 1, 2$, and $\hat{V}_m \rightarrow V$. We also have $\hat{B}_{1,m} \cup \hat{B}_{2,m} \rightarrow B_1(-b_1) \cup B_1(\hat{y}_* - b_2)$ in Hausdorff distance.

Suppose $T^\perp(\hat{y}_*) = 0$. Then $\lim_{m \rightarrow \infty} |T^\perp(\hat{y}_{2,m} - \hat{y}_{1,m})| = 0$, and we would have by (4.14) $\lim_{m \rightarrow \infty} |\hat{y}_{1,m} - \hat{y}_{2,m}| = 0$. This implies $\lim_{m \rightarrow \infty} |T(y_{1,m} - y_{2,m})| = 0$, thus we conclude that $b_1 = b_2$, $\hat{y}_* = 0$ and $B_1(-b_1) \cup B_1(\hat{y}_* - b_2) = B_1(-b_1)$. By (4.15)-(4.18), Lemma 4.1 shows $0 \in \text{spt } \|V\|$. Also V is stationary integral varifold with the same condition as (4.13), thus V is invariant in T -direction on $B_1(-b_1)$ as in the proof of Lemma 4.3. By the definition of R_m as in (4.23), we have

$$(4.26) \quad \|V\|(\{x : \text{dist}(-sb_1, x) < s\}) \geq 2\lambda\omega_k s^k$$

for all $0 < s \leq 1$ and equality holds for $s = 1$. Letting $s \rightarrow 0$, due to the invariance of $\|V\|$ in T -direction and $b_1 \in T$, this implies that $\Theta^k(\|V\|, 0) \geq 2\lambda$. Also note that $\Theta^k(\|V\|, x) \in \mathbb{N}$ is constant in T . Since it is an integer and $\lambda > 1/2$, we have $\Theta^k(\|V\|, 0) \geq 2$. Then this implies that $\Theta^k(\|V\|, -b_1) \geq 2$ because of $-b_1 \in T$. By the monotonicity formula, then, we have $\|V\|(B_1(-b_1)) \geq 2\omega_k$, which contradicts with the equality of (4.26) when $s = 1$.

Next suppose $T^\perp(\hat{y}_*) \neq 0$. By the same line of proof using Lemma 4.1, we have $\Theta^k(\|V\|, 0) \geq 1$ and $\Theta^k(\|V\|, \hat{y}_*) \geq 1$. Thus, the distinct affine planes containing $-b_1$ and $\hat{y}_* - b_2$ respectively have positive integer multiplicities for V . This shows $\|V\|(B_1(-b_1) \cup B_1(\hat{y}_* - b_2)) \geq 2\omega_k$. By the definition of R_m , we on the other hand have the same quantity $= 2\lambda\omega_k$, a contradiction. This concludes the proof. $\square$

Proposition 4.6. *Corresponding to $1 \leq E_1 < \infty$ and $0 < \nu < 1$, there exist $0 < \alpha_1 < 1$, $0 < \beta_1 < 1$ and $1 < P_1 < \infty$ with the following property. For $V \in \mathbf{IV}_k(C(T, 6R))$ with finite mass and which is of unit density, let M be a $\mathcal{H}^k$ measurable countably k -rectifiable set such that $V = |M|$. We identify T with $\mathbb{R}^k \times \{0\}$. Define*

$$(4.27) \quad \alpha^2 := \frac{1}{R^{k-2}} \int_{C(T, 6R)} |h(V, x)|^2 d\|V\|(x),$$

$$(4.28) \quad \beta^2 := \frac{1}{R^k} \int_{G_k(C(T, 6R))} \|S - T\|^2 dV(x, S).$$

Suppose that

$$(4.29) \quad \|V\|(B_r(x)) \leq \omega_k r^k E_1, \quad \forall r \in (0, 4R], \quad \forall x \in M \cap C(T, 2R),$$

$$(4.30) \quad \frac{1}{\omega_k (2R)^k} \|V\|(C(T, 2R)) \leq 2 - \nu$$

and

$$(4.31) \quad \frac{1}{\omega_k R^k} \|V\|(C(T, R)) \geq \nu.$$

Then one of the following (1) or (2) holds.

(1) We have $\alpha \geq \alpha_1$ or $\beta \geq \beta_1$.

(2) There exist Lipschitz maps $f : T \rightarrow \mathbb{R}^{n-k}$ and $F : T \rightarrow \mathbb{R}^n$ such that

$$(4.32) \quad F(x) = (x, f(x)), \quad \forall x \in T,$$

$$(4.33) \quad |f(x_1) - f(x_2)| \leq |x_1 - x_2|, \quad \forall x_1, \forall x_2 \in T.$$

Set $X := M \cap \text{image } F \cap C(T, R)$ and $Y := T(X)$. Then

$$(4.34) \quad \frac{1}{R^k} (\|V\|(C(T, R) \setminus X) + \mathcal{L}^k(B_R^k \setminus Y)) \leq \begin{cases} P_1(\alpha^{\frac{2k}{k-2}} + \beta^2) & \text{if } k > 2, \\ P_1\beta^2 & \text{if } k \leq 2. \end{cases}$$

Remark 4.7. If $h(V, \cdot)$ does not exist or $h(V, \cdot) \notin L^2(\|V\|)$, then we define $\alpha = \infty$ and we have trivially (1). Later we apply the result to V_t for a.e. time, where $h(V_t, \cdot) \in L^2(\|V_t\|)$.

Remark 4.8. For $k > 2$, if (1) holds, we may re-define P_1 sufficiently large depending on α_1 and β_1 so that (4.34) holds trivially. Thus, for $k > 2$, we may consider that the case (1) is a subset of (2), with $F(x) = (x, 0)$, for example. For $k \leq 2$, if $\beta \geq \beta_1$, then we may also choose P_1 sufficiently large similarly. Thus we may drop ‘or $\beta \geq \beta_1$ ’ from (1). We still find it more convenient to write out (1) and (2) as above.

Remark 4.9. The fact that the power $2k/(k-2)$ of α is strictly larger than 2 is of essential importance in proving Theorem 5.7, and this is the reason that we work with $r\|\delta V\|(B_r) \leq \gamma\|V\|(B_r)$ instead of $\|\delta V\|(B_r) \leq \gamma\|V\|(B_r)$, which is done in Allard’s theory.

Proof. We may assume $R = 1$ without loss of generality, and we may assume that $\alpha < \infty$ by Remark 4.7. Let $\gamma > 0$ be the smallest constant obtained in Lemma 4.1-4.4 corresponding to

$$(4.35) \quad \lambda = 1 - \frac{\nu}{2^{3k}},$$

$l = \sqrt{2}$, and E_1 as in the assumption. Let $M_g \subset M \cap C(T, 2)$ be the set of points y such that the following (4.36)-(4.38) all hold,

$$(4.36) \quad \Theta^k(\|V\|, y) = 1,$$

$$(4.37) \quad r\|\delta V\|(B_r(y)) \leq \gamma\|V\|(B_r(y)), \quad \forall r \in (0, 4],$$

$$(4.38) \quad \int_{G_k(B_r(y))} \|S - T\|^2 dV(y, S) \leq \gamma^2\|V\|(B_r(y)), \quad \forall r \in (0, 4].$$

By Schwarz’ inequality, (4.38) implies

$$(4.39) \quad \int_{G_k(B_r(y))} \|S - T\| dV(y, S) \leq \gamma\|V\|(B_r(y)), \quad \forall r \in (0, 4].$$

Take any $x \in T(M_g) \subset B_2^k$. By definition there exists at least one $y \in M_g$ such that $T(y) = x$. Suppose that there are two such y , i.e., there are two distinct $y_1, y_2 \in M_g$ such that $T(y_1) = T(y_2) = x$. Then y_1, y_2 satisfy all the assumptions of Lemma 4.4, namely, we have $|y_1 - y_2| = |T^\perp(y_1 - y_2)|$, thus (4.14) is satisfied with $l = \sqrt{2}$, and (4.15)-(4.18) are satisfied with $R = 2$ due to (4.29), (4.36), (4.37) and (4.39). Since $B_2(\tilde{y}_1) \cup B_2(\tilde{y}_2) \subset C(T, 2)$, (4.19) and (4.35) shows

$$(4.40) \quad \|V\|(C(T, 2)) \geq 2^{k+1}\lambda\omega_k = 2^{k+1}\{1 - (\nu/2^{3k})\}\omega_k.$$

This contradicts with (4.30). Since there is only one such $y \in M_g$ with $T(y) = x$, we may define a map $f : T(M_g) \rightarrow \mathbb{R}^{n-k}$ so that $f(x) = T^\perp(y)$. Next, for any $y_1 \neq y_2$ in M_g , if $|y_1 - y_2| \leq \sqrt{2}|T^\perp(y_1 - y_2)|$, similar reasoning using Lemma 4.4 leads to a contradiction with (4.30). Thus we have $|y_1 - y_2| > \sqrt{2}|T^\perp(y_1 - y_2)|$. This shows that $|f(x_1) - f(x_2)| < |x_1 - x_2|$ for all $x_1, x_2 \in T(M_g)$. By Kirszbraun’s theorem [15, 2.10.43], we have an extension $\tilde{f} : T \rightarrow \mathbb{R}^{n-k}$ of f with the equal Lipschitz constant. Rewrite this $\tilde{f}$ as f . We then define $F : T \rightarrow \mathbb{R}^n$ by $F(x) := (x, f(x))$.

For any $y \in (M \cap C(T, 2)) \setminus M_g$, at least one of (4.36)-(4.38) fails. The set of points for which (4.36) fails is null with respect to $\mathcal{H}^k$, thus it is null with respect to $\|V\|$. Let X'_1 be the set of points for which (4.37) fails, and let X'_2 be the set of points for which (4.38) fails. We first estimate $\|V\|(X'_1)$. For $y \in X'_1$, either (A) there exists a sequence $R_m \rightarrow 0$, $0 < R_m \leq 4$, such that (4.37) fails for $r = R_m$, or (B) we have $R := \inf\{r \in (0, 4] : r\|V\|(B_r(y)) > \gamma\|V\|(B_r(y))\} > 0$. In either case of (A) or (B), there exists some $0 < R(y) < 4$ such that both

$$(4.41) \quad \omega_k R(y)^k / 2 \leq \|V\|(B_{R(y)}(y)),$$

$$(4.42) \quad \gamma\|V\|(B_{R(y)}(y)) < R(y)\|\delta V\|(B_{R(y)}(y))$$

hold. The reason is that, if (A) holds, we use (4.36) to have (4.41) and choose small enough R_m to have (4.42). If (B) holds, by Lemma 4.1 we have $\|V\|(B_{R(y)}(y)) \geq \{1 - (\nu/2^{3k})\}\omega_k R^k$. Thus choosing $R(y)$ to be slightly larger than R if necessary, we have both (4.41) and (4.42) satisfied. By Hölder's inequality applied to (4.42), we have

$$(4.43) \quad \gamma^2\|V\|(B_{R(y)}(y)) < R(y)^2 \int_{B_{R(y)}(y)} |h(V, \cdot)|^2 d\|V\|.$$

Using (4.41) and (4.43), we obtain

$$(4.44) \quad (\gamma^2/2)\omega_k R(y)^{k-2} < \int_{B_{R(y)}(y)} |h(V, \cdot)|^2 d\|V\|.$$

Consider the case $k \leq 2$. Since $y \in C(T, 2)$ and $R(y) < 4$, we have $B_{R(y)}(y) \subset C(T, 6)$ and we have from (4.27) and (4.44)

$$(4.45) \quad \alpha^2 > (\gamma^2/2)\omega_k R(y)^{k-2} > (\gamma^2/2^{5-2k})\omega_k.$$

If $\alpha^2 < (\gamma^2/2^{5-2k})\omega_k$ and $X'_1 \neq \emptyset$, we would obtain a contradiction. We set α_1 so that $\alpha_1^2 < (\gamma^2/2^{5-2k})\omega_k$. Then, either we have $\alpha \geq \alpha_1$, or else $\alpha < \alpha_1$ and $X'_1 = \emptyset$. Thus for $k \leq 2$ and for the former case, we may end the proof, with case (1). For the rest of the proof, for $k \leq 2$, assume that $\alpha < \alpha_1$. Next consider $k > 2$. From (4.45), we have

$$(4.46) \quad R(y)^2 < (2\alpha^2\gamma^{-2}\omega_k^{-1})^{\frac{2}{k-2}}.$$

Substituting (4.46) into (4.43) gives

$$(4.47) \quad \|V\|(B_{R(y)}(y)) < \gamma^{-2}(2\alpha^2\gamma^{-2}\omega_k^{-1})^{\frac{2}{k-2}} \int_{B_{R(y)}(y)} |h(V, \cdot)|^2 d\|V\|.$$

From (4.47), we may choose $R'(y) < R(y)$ such that

$$(4.48) \quad \|V\|(\overline{B_{R'(y)}(y)}) < \gamma^{-2}(2\alpha^2\gamma^{-2}\omega_k^{-1})^{\frac{2}{k-2}} \int_{\overline{B_{R'(y)}(y)}} |h(V, \cdot)|^2 d\|V\|.$$

Thus the Besicovitch covering theorem with (4.48) implies that

$$(4.49) \quad \|V\|(X'_1) \leq \mathbf{B}(n)\gamma^{-2}(2\alpha^2\gamma^{-2}\omega_k^{-1})^{\frac{2}{k-2}} \int_{C(T,6)} |h(V, \cdot)|^2 d\|V\| = \mathbf{B}(n)\gamma^{\frac{-2k}{k-2}}\omega_k^{\frac{-2}{k-2}}\alpha^{\frac{2k}{k-2}}.$$

Here $\mathbf{B}(n)$ is the Besicovitch constant which depends only on n . For the estimate of $\|V\|(X'_2)$ (both $k \leq 2$ and $k > 2$), the Besicovitch covering theorem and (4.28) imply that

$$(4.50) \quad \|V\|(X'_2) \leq \mathbf{B}(n)\gamma^{-2}\beta^2.$$

Combining (4.49) and (4.50), we obtain

$$(4.51) \quad \|V\|(C(T, 2) \setminus M_g) \leq \begin{cases} \mathbf{B}(n)(\gamma^{\frac{-2k}{k-2}} \omega_k^{\frac{-2}{k-2}} \alpha^{\frac{2k}{k-2}} + \gamma^{-2} \beta^2), & k > 2, \\ \mathbf{B}(n) \gamma^{-2} \beta^2, & k \leq 2. \end{cases}$$

Next we estimate $\mathcal{H}^k(B_1^k \setminus T(M_g))$. The set $B_1^k \setminus T(M_g)$ may be regarded as a ‘hole of $\|V\|$ ’ and we need to estimate the size in terms of α and β . First we claim that $T(M_g) \cap B_1^k \neq \emptyset$. Otherwise, $C(T, 1) \cap M_g = \emptyset$. From (4.31), we have

$$(4.52) \quad \nu \omega_k \leq \|V\|(C(T, 1)) \leq \|V\|(C(T, 2) \setminus M_g).$$

By (4.51), if we restrict α and β small depending only on n, k, E_1 and ν , we would have a contradiction to (4.52). Thus we either have $\alpha \geq \alpha_1$ or $\beta \geq \beta_1$, where α_1 and β_1 depend only on n, k, E_1 and ν , or we have $T(M_g) \cap B_1^k \neq \emptyset$. The former case corresponds to the case (1) of the conclusion. We assume for the rest of the proof that $\alpha < \alpha_1$ and $\beta < \beta_1$. Thus we have some $x \in M_g \cap C(T, 1)$. By Lemma 4.3 with the origin there replaced by x , b there replaced by $T^\perp(x)$ and $R = 4$, we conclude that

$$(4.53) \quad \begin{aligned} (1 - \frac{\nu}{2^{3k}}) 2^k \omega_k &\leq \|V\|(B_2(T^\perp(x))) \leq \|V\|(C(T, 2)) \\ &= \|V\|(C(T, 2) \cap M_g) + \|V\|(C(T, 2) \setminus M_g). \end{aligned}$$

On the other hand, since M_g is a Lipschitz graph,

$$(4.54) \quad \begin{aligned} \|V\|(C(T, 2) \cap M_g) &= \int_{T(M_g)} J_{\nabla F} d\mathcal{H}^k \\ &\leq \int_{T(M_g)} (1 + c(n, k) \|\text{image } \nabla F - T\|^2) d\mathcal{H}^k \\ &\leq \mathcal{H}^k(T(M_g)) + c(n, k) \int_{G_k(C(T, 2))} \|S - T\|^2 dV(x, S) \\ &= 2^k \omega_k - \mathcal{H}^k(B_2^k \setminus T(M_g)) + c(n, k) \beta^2. \end{aligned}$$

Combining (4.53) and (4.54), we obtain

$$(4.55) \quad \mathcal{H}^k(B_2^k \setminus T(M_g)) \leq \frac{\nu}{2^{2k}} \omega_k + c(n, k) \beta^2 + \|V\|(C(T, 2) \setminus M_g).$$

The estimates (4.51) and (4.55) show that

$$(4.56) \quad \mathcal{H}^k(B_2^k \setminus T(M_g)) < \frac{1}{4^k} \omega_k$$

holds by further restricting α_1 and β_1 if necessary depending on $1 - \nu$ and k . We set $Q_1 = B_1^k \setminus T(M_g)$ and $Q_2 = B_2^k \setminus T(M_g)$. For $\mathcal{H}^k$ almost all point $x \in Q_1$, due to (4.56), there exists $0 < r(x) < 1$ such that

$$(4.57) \quad \mathcal{H}^k(Q_2 \cap B_{r(x)}^k(x)) = \frac{1}{4^k} \omega_k r(x)^k.$$

By the Besicovitch covering theorem, there exists a set of mutually disjoint balls $\{B_{r(x_i)}(x_i)\}_{i \in I}$ (I : a finite set or $\mathbb{N}$) such that $x_i \in Q_1$ and, denoting $U = \cup_{i \in I} B_{r(x_i)}(x_i)$,

$$(4.58) \quad \mathcal{H}^k(Q_1) \leq \mathbf{B}(k) \mathcal{H}^k(U).$$

We also estimate just like (4.54) that

$$(4.59) \quad \begin{aligned} \|V\|(T^{-1}(U)) &\leq \mathcal{H}^k(T(M_g) \cap U) + c(n, k) \beta^2 + \|V\|(C(T, 2) \setminus M_g) \\ &= \mathcal{H}^k(U) - \mathcal{H}^k(Q_2 \cap U) + c(n, k) \beta^2 + \|V\|(C(T, 2) \setminus M_g). \end{aligned}$$

To obtain a lower bound for (4.59), note that (4.57) guarantees that there exists $y_i \in M_g \cap C(T, 2)$ for each $i \in I$ such that $T(y_i) \in B_{r(x_i)/2}^k(x_i)$. Then Lemma 4.3 with the origin there replaced by y_i , b there replaced by $x_i + T^\perp(y_i)$ (so that $y_i - x_i - T^\perp(y_i) \in T$) and $R = 2r(x_i)$, we conclude that

$$(4.60) \quad \left(1 - \frac{\nu}{2^{3k}}\right) \omega_k r(x_i)^k \leq \|V\|(B_{r(x_i)}(x_i + T^\perp(y_i))) \leq \|V\|(T^{-1}(B_{r(x_i)}^k(x_i))).$$

Then summing over $i \in I$, and (4.59) and (4.60) show that

$$(4.61) \quad \mathcal{H}^k(Q_2 \cap U) \leq \frac{\nu}{2^{3k}} \mathcal{H}^k(U) + c(n, k) \beta^2 + \|V\|(C(T, 2) \setminus M_g).$$

By (4.57), we have

$$(4.62) \quad \frac{1}{4^k} \mathcal{H}^k(U) = \mathcal{H}^k(Q_2 \cap U),$$

and combining (4.61) and (4.62), we obtain

$$(4.63) \quad \left(\frac{1}{4^k} - \frac{\nu}{2^{3k}}\right) \mathcal{H}^k(U) \leq c(n, k) \beta^2 + \|V\|(C(T, 2) \setminus M_g).$$

Thus (4.58) and (4.63) give

$$(4.64) \quad \mathcal{H}^k(Q_1) \leq \mathbf{B}(k) \left(\frac{1}{4^k} - \frac{\nu}{2^{3k}}\right)^{-1} (c(n, k) \beta^2 + \|V\|(C(T, 2) \setminus M_g)).$$

Finally, setting $X = M \cap \text{image } F \cap C(T, 1)$ and $Y = T(X)$, we note that $C(T, 1) \cap M_g \subset X$, $\|V\|(C(T, 1) \setminus X) \leq \|V\|(C(T, 2) \setminus M_g)$, $B_1^k \setminus Y \subset B_1^k \setminus T(M_g) = Q_1$. Thus (4.51) and (4.64) give the desired estimate (4.34) for a suitable choice of P_1 . $\square$

5. ENERGY ESTIMATES

The aim of this section is to prove Theorem 5.7, which claims that the deviation of k -dimensional area of moving varifold from that of flat k -dimensional plane can be estimated in terms of L^2 -height. In Allard's regularity theory, the analogy of this part may be the estimate of tilt-excess in terms of L^2 -height. However, compared to Allard's theory, the proof is more involved. All we have control over is the rate of change of mass in time. The idea is that when the deviation is large, the mean curvature is relatively large. This helps push down the mass more quickly, and the mass becomes very close to that of flat plane in finite time. The set of estimates in this section is a crucial foundation for parabolic Lipschitz approximation and blow-up technique. The content corresponds to [7, 6.6 'Popping soap films'] even though the details have been largely changed and various alterations are made. Some of the results used are relegated to Appendix for convenience of the readers.

Definition 5.1. Fix $\phi \in C^\infty([0, \infty))$ such that $0 \leq \phi \leq 1$,

$$(5.1) \quad \phi(x) \begin{cases} = 1 & \text{for } 0 \leq x \leq (2/3)^{1/k}, \\ > 0 & \text{for } 0 \leq x < (5/6)^{1/k}, \\ = 0 & \text{for } x \geq (5/6)^{1/k}. \end{cases}$$

For $0 < R < \infty$, $x \in \mathbb{R}^n$ and $T \in \mathbf{G}(n, k)$ define

$$(5.2) \quad \phi_{T,R}(x) := \phi(R^{-1}|T(x)|), \quad \phi_T(x) := \phi_{T,1}(x) = \phi(|T(x)|).$$

Set

$$(5.3) \quad \mathbf{c} := \int_T \phi_T^2(x) d\mathcal{H}^k(x).$$

In the next Proposition, we use

$$(5.4) \quad \frac{2}{3}\omega_k < \mathbf{c} < \frac{5}{6}\omega_k$$

which follows easily from (5.1).

Proposition 5.2. *Corresponding to $1 \leq E_1 < \infty$ and $0 < \nu < 1$ there exist $0 < \alpha_2 < 1$, $0 < \mu_1 < 1$ and $1 < P_2 < \infty$ with the following property. For $T \in \mathbf{G}(n, k)$ and unit density varifold $V \in \mathbf{IV}_k(C(T, 1))$ with finite mass, define*

$$(5.5) \quad \alpha^2 := \int_{C(T, 1)} |h(V, x)|^2 \phi_T^2(x) d\|V\|(x),$$

$$(5.6) \quad \mu^2 := \int_{C(T, 1)} |T^\perp(x)|^2 d\|V\|(x).$$

Suppose $\text{spt } \|V\|$ is bounded and

$$(5.7) \quad \|V\|(B_r(x)) \leq \omega_k r^k E_1, \quad \forall B_r(x) \subset C(T, 1).$$

(A) Suppose

$$(5.8) \quad |\|V\|(\phi_T^2) - \mathbf{c}| \leq \frac{\mathbf{c}}{8}, \quad \alpha \leq \alpha_2 \quad \text{and} \quad \mu \leq \mu_1.$$

Then we have

$$(5.9) \quad |\|V\|(\phi_T^2) - \mathbf{c}| \leq \begin{cases} P_2(\alpha^{\frac{2k}{k-2}} + \alpha^{\frac{3}{2}}\mu^{\frac{1}{2}} + \mu^2) & \text{if } k > 2, \\ P_2(\alpha^{\frac{3}{2}}\mu^{\frac{1}{2}} + \mu^2) & \text{if } k \leq 2. \end{cases}$$

(B) Suppose

$$(5.10) \quad \frac{\mathbf{c}}{8} < |\|V\|(\phi_T^2) - \mathbf{c}| \leq (1 - \nu)\mathbf{c} \quad \text{and} \quad \mu \leq \mu_1.$$

Then we have $\alpha \geq \alpha_2$.

Proof. First consider case (A). Define

$$(5.11) \quad \beta^2 := \int_{G_k(C(T, 1))} \|S - T\|^2 \phi_T^2 dV(\cdot, S).$$

By Theorem 11.3 with $R_1 = 1/12$ and $R_2 = 1$, we have

$$(5.12) \quad 12^k \|V\|(\phi_{T, 1/12}^2) - \|V\|(\phi_T^2) \geq -12^k(k\beta^2 \log 12 + \alpha\beta + \beta^2).$$

Since $\phi_{T, 1/12} \leq 1$ and by (5.8), (5.12) and (5.4), we have

$$(5.13) \quad 12^k \|V\|(C(T, 1/12)) \geq \frac{7}{8}\mathbf{c} - c(k)(\alpha\beta + \beta^2) > \frac{7}{12}\omega_k - c(k)(\alpha\beta + \beta^2).$$

By restricting α and β depending only on k , (5.13) guarantees that

$$(5.14) \quad \|V\|(C(T, 1/12)) \geq \frac{1}{2}\omega_k \frac{1}{12^k}.$$

Similarly for $R_1 = (3/2)^{1/k}/6$ and $R_2 = 1$, Theorem 11.3 gives

$$(5.15) \quad R_1^{-k} \|V\|(\phi_{T, R_1}^2) - \|V\|(\phi_T^2) \leq R_1^{-k}(k\beta^2 \log R_1^{-1} + \alpha\beta + \beta^2).$$

By (5.1), we have $\{\phi_{T, R_1} = 1\} \supset C(T, 1/6)$. Thus (5.15), (5.8) and (5.4) give

$$(5.16) \quad R_1^{-k} \|V\|(C(T, 1/6)) \leq c(k)(\beta^2 + \alpha\beta) + \frac{9}{8}\mathbf{c} \leq c(k)(\beta^2 + \alpha\beta) + \frac{15}{16}\omega_k.$$

With the definition of R_1 and (5.16), we have

$$(5.17) \quad \|V\|(C(T, 1/6)) \leq c(k)(\beta^2 + \alpha\beta) + \frac{45}{32} \frac{1}{6^k} \omega_k.$$

Since $45/32 < 3/2$, by restricting α and β depending only on k , we have from (5.17)

$$(5.18) \quad \|V\|(C(T, 1/6)) \leq \frac{3}{2} \omega_k \frac{1}{6^k}.$$

We now use Proposition 4.6 with $R = 1/12$ and $\nu = 1/2$. Due to (5.7), (5.14) and (5.18), the assumptions (4.29), (4.30) and (4.31) are satisfied. The smallness of α and β in (4.27) and (4.28) can be guaranteed by assuming α and β in (5.5) and (5.11) are small due to $C(T, 1/2) \subset \{\phi_T = 1\}$. Thus there exists a Lipschitz function $F : T \rightarrow \mathbb{R}^n$ and $\tilde{P}_1$ (which is a suitable multiple of P_1 by a constant depending only on k) such that

$$(5.19) \quad \|V\|(C(T, 1/12) \setminus X) + \mathcal{H}^k(B_{1/12}^k \setminus Y) \leq \begin{cases} \tilde{P}_1(\alpha^{\frac{2k}{k-2}} + \beta^2) & \text{if } k > 2, \\ \tilde{P}_1\beta^2 & \text{if } k \leq 2. \end{cases}$$

Here $X = C(T, 1/12) \cap M \cap \text{image } F$ and $Y = T(X)$. By using this approximation, we estimate as

$$(5.20) \quad \int_{C(T, 1/12)} \phi_{T, 1/12}^2 d\|V\| = \int_{C(T, 1/12) \setminus X} \phi_{T, 1/12}^2 d\|V\| + \int_Y \phi_{T, 1/12}^2 J_{\nabla F} d\mathcal{H}^k.$$

The first term on the right-hand side of (5.20) is bounded by (5.19). The error of replacing Y by $B_{1/12}^k$ for the second term is also bounded similarly. Just like estimating (4.54), we may obtain

$$(5.21) \quad \left| \int_{C(T, 1/12)} \phi_{T, 1/12}^2 d\|V\| - \int_T \phi_{T, 1/12}^2 d\mathcal{H}^k \right| \leq \begin{cases} \tilde{P}_1(\alpha^{\frac{2k}{k-2}} + \beta^2) & \text{if } k > 2, \\ \tilde{P}_1\beta^2 & \text{if } k \leq 2. \end{cases}$$

where $\tilde{P}_1$ may differ from before only by a multiple of constant depending only on n and k . We now apply Theorem 11.3 again. Observing that $12^k \int_T \phi_{T, 1/12}^2 d\mathcal{H}^k = \mathbf{c}$ and by (5.21), yet with another $\tilde{P}_1$, we obtain

$$(5.22) \quad \left| \|V\|(\phi_T^2(x)) - \mathbf{c} \right| \leq \begin{cases} \tilde{P}_1(\alpha^{\frac{2k}{k-2}} + \beta^2 + \alpha\beta) & \text{if } k > 2, \\ \tilde{P}_1(\beta^2 + \alpha\beta) & \text{if } k \leq 2. \end{cases}$$

Finally, with Lemma 11.2, we have

$$(5.23) \quad \beta^2 \leq 4\alpha\mu + c(\phi)\mu^2 \text{ and } \beta \leq 2\alpha^{\frac{1}{2}}\mu^{\frac{1}{2}} + c(\phi)\mu$$

where $c(\phi) \geq 1$ depends only on $\sup |\phi'|$ which we may consider to be constant. Thus with suitable restrictions on α and μ , β is considered small. By Young's inequality, we also have $\alpha\mu \leq c(\alpha^{3/2}\mu^{1/2} + \mu^2)$ and this combined with (5.22) and (5.23) proves (5.9) with suitable choices of P_2 , α_2 and μ_1 .

For case (B), assume the claim were false. Then for each $m \in \mathbb{N}$ there exists $V_m = |M_m|$ satisfying (5.10) with V there replaced by V_m , while

$$(5.24) \quad \mu_m^2 := \int_{C(T, 1)} |T^\perp(x)|^2 d\|V_m\| < \frac{1}{m}, \text{ and } \alpha_m^2 := \int_{C(T, 1)} |h(V_m, \cdot)|^2 \phi_T^2 d\|V_m\| < \frac{1}{m}.$$

By Lemma 11.2, we also have the corresponding $\beta_m \rightarrow 0$. By (5.7), there exists a convergent subsequence (denoted by the same index) $\{V_m\}$ and its limit V on $C(T, 1)$. By (5.24), on $\{\phi_T > 0\}$, V is stationary and integral. Since the corresponding β and μ for V vanishes, $V|_{\{\phi_T > 0\}} = i|T \cap C(T, 1)|$ for some integer i due to the constancy theorem ([1, 4.6(3)]). On

the other hand, V satisfies (5.10) (with possible $\leq$ sign on the left-hand side). But there is no such integer satisfying $\mathbf{c}/8 \leq |\mathbf{ic} - \mathbf{c}| \leq (1 - \nu)\mathbf{c}$. Hence we obtain a contradiction and we may conclude the proof by choosing smaller μ_1 and α_2 from (A) and (B). $\square$

Corollary 5.3. *Let α_2 , μ_1 and P_2 be the same constants as in Proposition 5.2. Set $\mu_2 = \min\{\mu_1, (\mathbf{c}/32P_2)^{1/2}\}$. For V and T as in Proposition 5.2 define α and μ as in (5.5) and (5.6). Define*

$$(5.25) \quad \hat{E} := \|V\|(\phi_T^2) - \mathbf{c}.$$

Assume (5.7),

$$(5.26) \quad \mu \leq \mu_2$$

and

$$(5.27) \quad 2P_2\mu^2 \leq |\hat{E}| \leq (1 - \nu)\mathbf{c}.$$

Then we have

$$(5.28) \quad \alpha^2 \geq \begin{cases} \min \left\{ \alpha_2^2, (4P_2)^{-\frac{k-2}{k}} |\hat{E}|^{\frac{k-2}{k}}, (4P_2)^{-\frac{4}{3}} \mu^{-\frac{2}{3}} |\hat{E}|^{\frac{4}{3}} \right\} & \text{if } k > 2, \\ \min \left\{ \alpha_2^2, (2P_2)^{-\frac{4}{3}} \mu^{-\frac{2}{3}} |\hat{E}|^{\frac{4}{3}} \right\} & \text{if } k \leq 2. \end{cases}$$

Remark 5.4. *Note that the mean curvature square term α^2 gives a lower bound for the rate of mass decrease, and Corollary 5.3 relates that to the mass itself. This allows one to introduce a differential inequality for the mass, see Lemma 5.5.*

Proof. We only prove the case $k > 2$ since $k \leq 2$ can be handled similarly. If $\alpha \geq \alpha_2$, then we obtain (5.28) trivially. Thus assume $\alpha < \alpha_2$. The case (B) of Proposition 5.2 do not occur under this assumption. The assumptions for Proposition 5.2, (5.7) and (5.8), are all satisfied due to (5.25), (5.26) and (5.27). Then we have

$$(5.29) \quad |\hat{E}| \leq P_2(\alpha^{\frac{2k}{k-2}} + \alpha^{\frac{3}{2}} \mu^{\frac{1}{2}} + \mu^2).$$

By (5.27) and (5.29) we conclude that

$$(5.30) \quad |\hat{E}| \leq 2P_2(\alpha^{\frac{2k}{k-2}} + \alpha^{\frac{3}{2}} \mu^{\frac{1}{2}}).$$

The inequality (5.30) implies that we have either $|\hat{E}| \leq 4P_2\alpha^{2k/(k-2)}$ or $|\hat{E}| \leq 4P_2\alpha^{3/2}\mu^{1/2}$. Thus we have

$$(5.31) \quad (4P_2)^{-\frac{k-2}{k}} |\hat{E}|^{\frac{k-2}{k}} \leq \alpha^2 \text{ or } (4P_2)^{-\frac{4}{3}} \mu^{-\frac{2}{3}} |\hat{E}|^{\frac{4}{3}} \leq \alpha^2.$$

Thus either $\alpha^2 \geq \alpha_2^2$ or (5.31) holds. This shows (5.28). $\square$

Lemma 5.5. *Corresponding to $0 < P < \infty$, $0 < K_1 < \infty$ there exists $0 < \Lambda < \infty$ with the following property. Given μ_* with $0 < \mu_* < 1$, define $f_{\mu_*} : \mathbb{R} \rightarrow \mathbb{R}^+$ by*

$$(5.32) \quad f_{\mu_*}(t) = \begin{cases} P \min\{1, |t|^{\frac{k-2}{k}}, \mu_*^{-\frac{2}{3}} |t|^{\frac{4}{3}}\} & \text{for } k > 2, \\ P \min\{1, \mu_*^{-\frac{2}{3}} |t|^{\frac{4}{3}}\} & \text{for } k \leq 2. \end{cases}$$

Suppose $\Phi : [0, \Lambda] \rightarrow \mathbb{R}$ satisfies

$$(5.33) \quad \Phi(t_2) - \Phi(t_1) \leq - \int_{t_1}^{t_2} f_{\mu_*}(\Phi(s)) ds, \quad 0 \leq \forall t_1 < \forall t_2 \leq \Lambda.$$

(1) Suppose $\Phi(0) \leq \omega_k$. Then we have

$$(5.34) \quad \Phi(\Lambda) \leq K_1 \mu_*^2.$$

(2) Suppose $\Phi(0) < -K_1\mu_*^2$. Then we have

$$(5.35) \quad \Phi(\Lambda) < -\omega_k.$$

Remark 5.6. The point of Lemma 5.5 is that Λ does not depend on μ_* .

Proof. Consider the case $k > 2$ and (1). We set

$$(5.36) \quad \Lambda = \frac{\omega_k - 1}{P} + \frac{k}{2P} + \frac{3}{PK_1^{\frac{1}{3}}}.$$

Since Φ is monotone decreasing, we may assume $\Phi > 0$ on $[0, \Lambda]$ for the conclusion. We have from (5.32)

$$(5.37) \quad f_{\mu_*}(\Phi(s)) = \begin{cases} P & \text{for } 1 \leq \Phi(s), \\ P\Phi(s)^{\frac{k-2}{k}} & \text{for } \mu_*^{\frac{2k}{k+6}} \leq \Phi(s) \leq 1, \\ P\mu_*^{-\frac{2}{3}}\Phi(s)^{\frac{4}{3}} & \text{for } 0 < \Phi(s) \leq \mu_*^{\frac{2k}{k+6}}. \end{cases}$$

Suppose that $1 \leq \Phi \leq \omega_k$ on $[0, \tilde{t}]$. Then (5.33) and (5.37) show that $\tilde{t} \leq (\omega_k - 1)/P$. Suppose that $1 \geq \Phi \geq \mu_*^{\frac{2k}{k+6}}$ on $[\tilde{t}, t_*]$. From (5.33) and (5.37), we have $\Phi' \leq -P\Phi^{(k-2)/k}$ a.e. on $[\tilde{t}, t_*]$. Integrating over the interval gives

$$(5.38) \quad \Phi(t_*)^{\frac{2}{k}} \leq \Phi(\tilde{t})^{\frac{2}{k}} - \frac{2P}{k}(t_* - \tilde{t}).$$

Since $\Phi(\tilde{t}) \leq 1$ and $\Phi > 0$, (5.38) shows that $t_* - \tilde{t} < k/(2P)$. Thus there exists $\hat{t} \in [0, k/(2P) + (\omega_k - 1)/P]$ such that $\Phi(\hat{t}) < \mu_*^{\frac{2k}{k+6}}$. From (5.33) and (5.37), we have $\Phi' \leq -P\mu_*^{-2/3}\Phi^{4/3}$ a.e. on $[\hat{t}, \Lambda]$. Integrating over the interval gives

$$(5.39) \quad \Phi(\Lambda)^{-\frac{1}{3}} \geq \Phi(\hat{t})^{-\frac{1}{3}} + \frac{1}{3}P\mu_*^{-\frac{2}{3}}(\Lambda - \hat{t}).$$

Thus we obtain from (5.39) that $\Phi(\Lambda) \leq 27\mu_*^2P^{-3}(\Lambda - \hat{t})^{-3}$. By (5.36), this shows (5.34) for $k > 2$. For (2), define $\tilde{\Phi}(t) = -\Phi(\Lambda - t)$ for $t \in [0, \Lambda]$. Then one can check that $\tilde{\Phi}$ satisfies (5.33). By the first part of the proof, if $\tilde{\Phi}(0) \leq \omega_k$, then $\tilde{\Phi}(\Lambda) \leq K_1\mu_*^2$. This shows (5.35). The proof for $k \leq 2$ is similar and is omitted. $\square$

Theorem 5.7. Corresponding to $1 \leq E_1 < \infty$ and $0 < \nu < 1$ there exist $0 < \varepsilon_1 < 1$, $0 < \Lambda < \infty$ and $0 < K < \infty$ with the following property. For $U = C(T, 1)$ suppose $\{V_t\}_{0 \leq t \leq 2\Lambda+3}$ and $\{u(\cdot, t)\}_{0 \leq t \leq 2\Lambda+3}$ satisfy (A1)-(A4). Assume

$$(5.40) \quad 0 < \exists C < \infty : \text{spt } \|V_t\| \subset C(T, 1) \cap \{x : |T^\perp(x)| < C\}, \quad \forall t \in (0, 2\Lambda + 3],$$

$$(5.41) \quad \mu_*^2 := \sup_{0 \leq t \leq 2\Lambda+3} \int_{C(T, 1)} |T^\perp(x)|^2 d\|V_t\|(x) \leq \varepsilon_1^2,$$

$$(5.42) \quad 0 \leq \exists \hat{t}_1 \leq 1 : \|V_{\hat{t}_1}\|(\phi_T^2) \leq \mathbf{c}(2 - \nu),$$

$$(5.43) \quad 2\Lambda + 2 \leq \exists \hat{t}_2 \leq 2\Lambda + 3 : \|V_{\hat{t}_2}\|(\phi_T^2) \geq \mathbf{c}\nu,$$

$$(5.44) \quad C(u) := \int_0^{2\Lambda+3} \int_{C(T, 1)} 2|u|^2 \phi_T^2 d\|V_t\| dt \leq \varepsilon_1^2.$$

Then, we have

$$(5.45) \quad \sup_{1+\Lambda \leq t \leq 2+\Lambda} \left| \|V_t\|(\phi_T^2) - \mathbf{c} \right| \leq K(\mu_*^2 + C(u))$$

and

$$(5.46) \quad \int_{1+\Lambda}^{2+\Lambda} \int_{C(T,1)} |h(V_t, \cdot)|^2 \phi_T^2 d\|V_t\| dt \leq 12K(\mu_*^2 + C(u)).$$

Remark 5.8. *Imprecise but helpful picture to keep in mind is to regard the moving varifold $\|V_t\|$ as a hypersurface graph (restricting to $n = k + 1$ case) $y = f(x, t)$, with some extra pieces around. Assume $C(u) = 0$, since it is a perturbative term, and think ϕ_T as a characteristic function of $C(T, 1)$. Assumption (5.41) says that spatial L^2 -norm $\int_{B_1} |f(\cdot, t)|^2 dx$ is kept small uniformly for the whole time interval. (5.42) says that near the initial time, $\int_{B_1} \sqrt{1 + |\nabla f|^2} dx - \omega_k$ plus area of some other pieces is strictly less than ω_k , preventing the situation where two flat sheets remaining still. (5.43) says that near the final time, there is some fixed amount of mass inside the cylinder. This prevents the complete loss of varifold in the intermediate time. The conclusion (5.45) gives the energy bound in the following sense. By Taylor expansion, we may postulate that*

$$\int_{B_1} \sqrt{1 + |\nabla f|^2} dx - \omega_k \approx \int_{B_1} \frac{|\nabla f|^2}{2} dx.$$

Thus, roughly speaking, (5.45) gives a uniform in time Dirichlet energy bound of the graph in terms of L^2 -norm of f . Since the mean curvature is analogous to the Laplacian of f , (5.46) gives a second derivative L^2 -norm (space-time) bound in terms of the L^2 -norm of f . In the next section we prove that μ_*^2 can be estimated in terms of the space-time L^2 -norm of f , $\int \int |f|^2 dx dt$.

Proof. We prove the case $k > 2$. The proof for $k \leq 2$ is similar. We set

$$(5.47) \quad K_2 := 80 \sup(5|\nabla \phi_T|^4 \phi_T^{-2} + |\nabla |\nabla \phi_T||^2),$$

which is a constant depending only on k . By Proposition 5.2 and Corollary 5.3, corresponding to E_1 of assumption (A2) and ν there replaced by $\nu/2$, we choose and fix α_2 , μ_2 and P_2 . We then set

$$(5.48) \quad P := \frac{1}{4 \cdot 2^{4/3}} \min\{\alpha_2^2, (4P_2)^{-\frac{k-2}{k}}, (4P_2)^{-\frac{4}{3}}\}.$$

With this P and

$$(5.49) \quad K_1 := 2P_2,$$

we use Lemma 5.5 to fix Λ . We then choose

$$(5.50) \quad \varepsilon_1^2 := \min \left\{ K_2^{-1} (3 + 2\Lambda)^{-1} \frac{\nu \mathbf{c}}{4}, \frac{\nu \mathbf{c}}{4}, \mu_2^2 \right\}$$

and

$$(5.51) \quad K := \max\{2, 4P_2, 2K_2(2\Lambda + 3)\}.$$

Overall, note that all those constants are fixed depending only on n , k , E_1 and ν . We prove the claim with the constants defined above.

Define

$$(5.52) \quad E(t) := \|V_t\|(\phi_T^2) - \mathbf{c} - \int_0^t \int_{C(T,1)} 2|u|^2 \phi_T^2 d\|V_s\| ds - K_2 \mu_*^2 t.$$

We first prove that

$$(5.53) \quad E(t_2) - E(t_1) \leq -\frac{1}{4} \int_{t_1}^{t_2} \int_{C(T,1)} |h(V_t, \cdot)|^2 \phi_T^2 d\|V_t\| dt, \quad 0 \leq \forall t_1 < \forall t_2 \leq 2\Lambda + 3.$$

The assumption (5.40) and (A2) show that $\|V_t\|(\text{spt } \phi_T)$ is uniformly bounded in t . By (A1), (A3) and (3.6), for a.e. t , we have $V_t \in \mathbf{IV}_k(U)$, $h(V_t, \cdot) \in L^2(\|V_t\|)$ and $u(\cdot, t) \in L^2(\|V_t\|)$. At such time t , by (2.5) and the perpendicularity of mean curvature (2.4), we have (omitting t dependence for simplicity)

$$(5.54) \quad \mathcal{B}(V, u, \phi_T^2) \leq \int_U -|h|^2 \phi_T^2 + \phi_T^2 |h| |u| + |u^\perp \cdot \nabla \phi_T^2| + (\nabla \phi_T^2)^\perp \cdot h d\|V\|.$$

Since ϕ_T depends only on $|T(x)|$, $\nabla \phi_T \in T$. Thus the last term of (5.54) is

$$(5.55) \quad \begin{aligned} \int_{G_k(U)} 2\phi_T S^\perp(\nabla \phi_T) \cdot h dV &= \int_{G_k(U)} 2\phi_T(T - S)(\nabla \phi_T) \cdot h dV \\ &\leq \frac{1}{4} \int_U |h|^2 \phi_T^2 d\|V\| + 4 \int_{G_k(U)} |\nabla \phi_T|^2 \|S - T\|^2 dV. \end{aligned}$$

Similarly,

$$(5.56) \quad \begin{aligned} \int_U |u^\perp \cdot \nabla \phi_T^2| d\|V\| &= \int_{G_k(U)} 2\phi_T |(T - S)(u) \cdot \nabla \phi_T| dV \\ &\leq \int_U \phi_T^2 |u|^2 d\|V\| + \int_{G_k(U)} |\nabla \phi_T|^2 \|S - T\|^2 dV. \end{aligned}$$

To estimate the last term of (5.55) and (5.56) we slightly modify the proof of Lemma 11.2 and obtain

$$(5.57) \quad \begin{aligned} \int_{G_k(U)} |\nabla \phi_T|^2 \|S - T\|^2 dV &\leq 4 \left(\int_U \phi_T^2 |h|^2 d\|V\| \int_U |T^\perp(x)|^2 |\nabla \phi_T|^4 \phi_T^{-2} d\|V\| \right)^{\frac{1}{2}} \\ &\quad + 16 \int_U |T^\perp(x)|^2 |\nabla \phi_T|^2 d\|V\|. \end{aligned}$$

Combining (5.54)-(5.57), we obtain

$$(5.58) \quad \mathcal{B}(V, u, \phi_T^2) \leq \int_U -\frac{|h|^2 \phi_T^2}{4} + 2|u|^2 \phi_T^2 + 80|T^\perp(x)|^2 (5|\nabla \phi_T|^4 \phi_T^{-2} + |\nabla \phi_T|^2) d\|V\|.$$

Since V_t satisfies (A4), integrating (5.58) over $[t_1, t_2]$ and using (5.41) and (5.47), we obtain (5.53). Next, by (5.42) and (5.53), we have for any $t \in [\hat{t}_1, 2\Lambda + 3]$

$$(5.59) \quad E(t) \leq E(\hat{t}_1) \leq \mathbf{c}(1 - \nu).$$

Due to (5.41), (5.44) and (5.50), (5.59) shows

$$(5.60) \quad \|V_t\|(\phi_T^2) - \mathbf{c} \leq \mathbf{c}(1 - \nu) + 2 \times \frac{\nu \mathbf{c}}{4} = \mathbf{c}(1 - \nu/2), \quad \forall t \in [\hat{t}_1, 2\Lambda + 3].$$

Similarly, due to (5.43) and (5.53), we have

$$(5.61) \quad E(t) \geq E(\hat{t}_2) \geq \mathbf{c}(\nu - 1) - 2 \times \frac{\nu \mathbf{c}}{4} = \mathbf{c}(-1 + \nu/2)$$

and

$$(5.62) \quad \|V_t\|(\phi_T^2) - \mathbf{c} \geq \mathbf{c}(-1 + \nu/2)$$

for all $t \in [0, \hat{t}_2]$. To obtain a contradiction assume that there exists $t_* \in [\Lambda + 1, \Lambda + 2]$ with

$$(5.63) \quad \|V_{t_*}\|(\phi_T^2) - \mathbf{c} > K(\mu_*^2 + C(u)).$$

By (5.52), (5.51) and (5.53), (5.63) implies that for all $t \in [0, t_*]$,

$$(5.64) \quad \begin{aligned} \|V_t\|(\phi_T^2) - \mathbf{c} &\geq E(t) \geq E(t_*) > K(\mu_*^2 + C(u)) - C(u) - K_2\mu_*^2(\Lambda + 3) \\ &\geq \frac{K}{2}\mu_*^2 \geq 2P_2\mu_*^2. \end{aligned}$$

By (5.60) and (5.64), we may apply Corollary 5.3 for a.e. $t \in [\hat{t}_1, t_*]$ to obtain

$$(5.65) \quad \int_{C(T,1)} |h(V_t, \cdot)|^2 \phi_T^2 d\|V_t\| \geq 4P \min\{1, E(t)^{\frac{k-2}{k}}, \mu_*^{-\frac{2}{3}} E(t)^{\frac{4}{3}}\},$$

where we also used (5.48), (5.41) and $\|V_t\|(\phi_T^2) - \mathbf{c} \geq E(t)$. We then use Lemma 5.5. With $\Phi(t) = E(t + \hat{t}_1)$, we have $\Phi(0) \leq \mathbf{c}(1 - \nu) < \omega_k$ and (5.33) is satisfied on $[0, \Lambda]$ due to (5.53) and (5.65). We thus conclude that $E(\Lambda + \hat{t}_1) \leq 2P_2\mu_*^2$. On the other hand, from (5.64) and $\Lambda + \hat{t}_1 \in [0, t_*]$, we have $E(\Lambda + \hat{t}_1) > 2P_2\mu_*^2$, which is a contradiction. Similarly, for a contradiction, assume that there exists $t_* \in [\Lambda + 1, \Lambda + 2]$ with

$$(5.66) \quad \|V_{t_*}\|(\phi_T^2) - \mathbf{c} \leq -K(\mu_*^2 + C(u)).$$

Similar computations to (5.64) using (5.52), (5.53), (5.51) and (5.66) show

$$(5.67) \quad \begin{aligned} \|V_t\|(\phi_T^2) - \mathbf{c} &\leq E(t) + C(u) + K_2\mu_*^2(2\Lambda + 3) \leq E(t_*) + C(u) + K_2\mu_*^2(2\Lambda + 3) \\ &\leq \|V_{t_*}\|(\phi_T^2) - \mathbf{c} + C(u) + K_2\mu_*^2(2\Lambda + 3) \leq -\frac{K}{2}(\mu_*^2 + C(u)) \leq -2P_2\mu_*^2 \end{aligned}$$

for $t \in [t_*, 2\Lambda + 3]$. Thus, by (5.62), (5.67) and (5.48), Corollary 5.3 shows

$$(5.68) \quad \int_{C(T,1)} |h(V_t, \cdot)|^2 \phi_T^2 d\|V_t\| \geq 4 \cdot 2^{4/3} P \min\{1, \|V_t\|(\phi_T^2) - \mathbf{c}|^{\frac{k-2}{k}}, \mu_*^{-\frac{2}{3}} \|V_t\|(\phi_T^2) - \mathbf{c}|^{\frac{4}{3}}\}$$

for a.e. $t \in [t_*, \hat{t}_2]$. Since $\|V_t\|(\phi_T^2) - \mathbf{c} \geq K(\mu_*^2 + C(u))/2$ as in (5.67), we have by (5.51)

$$(5.69) \quad |E(t)| \leq \|V_t\|(\phi_T^2) - \mathbf{c} + C(u) + K_2\mu_*^2(2\Lambda + 3) \leq 2\|V_t\|(\phi_T^2) - \mathbf{c}.$$

Combining (5.68) and (5.69), we obtain

$$(5.70) \quad \int_{C(T,1)} |h(V_t, \cdot)|^2 \phi_T^2 d\|V_t\| \geq 4P \min\{1, |E(t)|^{\frac{k-2}{k}}, \mu_*^{-\frac{2}{3}} |E(t)|^{\frac{4}{3}}\}.$$

Then applying Lemma 5.5, we obtain $E(t_* + \Lambda) < -\omega_k$. Since $t_* + \Lambda \in [0, 2 + 2\Lambda]$, this is a contradiction to (5.61). Lastly one observes that (5.53), (5.52) and (5.45) show (5.46). $\square$

For parabolic Lipschitz approximation, we also need the following estimate.

Corollary 5.9. *Under the same assumptions of Theorem 5.7, there exists a constant $\tilde{K}$ depending only on E_1 and ν such that*

$$(5.71) \quad \int_{1+\Lambda}^{2+\Lambda} |2^k \|V_t\|(\phi_{T,1/2}^2) - \mathbf{c}| dt \leq \tilde{K}(\mu_*^2 + C(u)).$$

Proof. By Lemma 11.2, (5.41) and (5.46), we have

$$(5.72) \quad \int_{1+\Lambda}^{2+\Lambda} \int_{G_k(C(T,1))} \|S - T\|^2 \phi_T^2 dV_t(\cdot, S) dt \leq 4(12K(\mu_*^2 + C(u)))^{1/2} \mu_* + c(k) \mu_*^2.$$

Then the difference of $\|V\|(\phi_T^2)$ and $2^k \|V\|(\phi_{T,1/2}^2)$ can be estimated using Theorem 11.3, (5.72) and (5.46). This gives (5.71). $\square$

6. PARABOLIC MONOTONICITY FORMULA AND L^2 - L^∞ ESTIMATE

In this section we prove two important tools which are used in the subsequent sections. They are both derived by modifying the monotonicity formula due to Huisken [18]. Note that Brakke did not utilize these powerful tools since it was not known at the time. He instead used a certain vanishing theorem [7, 6.3. ‘Clearing out’] to obtain a uniform L^∞ estimate of the height of varifold.

6.1. Local monotonicity formula. In this subsection, let $\eta \in C_c^\infty(B_{14/15})$ be a non-negative radially symmetric function such that $\eta \equiv 1$ on $B_{13/15}$, $|\nabla \eta| \leq 30$ and $\|\nabla^2 \eta\| \leq c(n)$. Then define $\hat{\rho}_{(y,s)}(x, t) = \eta(x) \rho_{(y,s)}(x, t)$, where $\rho_{(y,s)}(x, t)$ is defined as in (2.2).

Lemma 6.1. *There exists $c_1 = c_1(n, k)$ with the following property. For $-\infty < t < s < \infty$, $x \in B_1$, $y \in B_{4/5}$ and $S \in \mathbf{G}(n, k)$, we have*

$$(6.1) \quad \left| \frac{\partial \hat{\rho}_{(y,s)}(x, t)}{\partial t} + S \cdot \nabla^2 \hat{\rho}_{(y,s)}(x, t) + \frac{|S^\perp(\nabla \hat{\rho}_{(y,s)}(x, t))|^2}{\hat{\rho}_{(y,s)}(x, t)} \right| \leq c_1.$$

Proof. Here we write ρ for $\rho_{(y,s)}(x, t)$ and $\hat{\rho}$ for $\hat{\rho}_{(y,s)}(x, t)$. First assume that $x \in B_{13/15}$ so that the derivatives of η are 0 and $\hat{\rho} = \rho$ near x . We have

$$\begin{aligned} \frac{\partial \hat{\rho}}{\partial t} &= \left(\frac{k}{2(s-t)} - \frac{|x-y|^2}{4(s-t)^2} \right) \hat{\rho}, \quad \nabla \hat{\rho} = -\frac{x-y}{2(s-t)} \hat{\rho}, \\ \nabla^2 \hat{\rho} &= -\frac{\hat{\rho}}{2(s-t)} I + \frac{\hat{\rho}}{4(s-t)^2} (x-y) \otimes (x-y). \end{aligned}$$

Since $S \cdot I = k$, $S \cdot (x-y) \otimes (x-y) = |S(x-y)|^2$ and $|S^\perp(x-y)|^2 + |S(x-y)|^2 = |x-y|^2$,

$$\frac{\partial \hat{\rho}}{\partial t} + S \cdot \nabla^2 \hat{\rho} + \frac{|S^\perp(\nabla \hat{\rho})|^2}{\hat{\rho}} = 0.$$

When $x \in B_1 \setminus B_{13/15}$, $|x-y| > \frac{1}{15}$. The similar computation in this case produces extra terms which involves $\nabla \eta$ and $\nabla^2 \eta$. Since they are multiplied by $\exp\left(-\frac{|x-y|^2}{4(s-t)}\right) < \exp\left(-\frac{1}{900(s-t)}\right)$, there exists some constant $c_1 = c_1(n, k)$ satisfying the desired estimate. $\square$

Proposition 6.2. *For $U = B_R$, $0 < \Lambda \leq \infty$ and $1 \leq E_1 < \infty$ suppose that $\{V_t\}_{0 \leq t < \Lambda}$ and $\{u(\cdot, t)\}_{0 \leq t < \Lambda}$ satisfy (A1)-(A4). Define $\hat{\rho}_{(y,s)}(x, t) = \eta(x/R) \rho_{(y,s)}(x, t)$. For $0 < t_1 < t_2 < \Lambda$, $t_2 < s$ and $y \in B_{4R/5}$, we have*

$$(6.2) \quad \begin{aligned} & \int_{B_R} \hat{\rho}_{(y,s)}(\cdot, t_2) d\|V_{t_2}\| - \int_{B_R} \hat{\rho}_{(y,s)}(\cdot, t_1) d\|V_{t_1}\| \\ & \leq c_2 \|u\|_{L^{p,q}(B_R \times (t_1, t_2))}^2 E_1^{1-\frac{2}{p}} (t_2 - t_1)^\varsigma + c_1 \omega_k E_1 R^{-2} (t_2 - t_1). \end{aligned}$$

Here $c_2 = c_2(k, p, q)$ and $\varsigma = 1 - k/p - 2/q$.

Proof. After a change of coordinates we may assume $R = 1$ without loss of generality. We compute for a.e. $t \in (t_1, t_2)$ (when V_t is integral and $h(V_t, \cdot) \in L^2(\|V_t\|)$ by (A1) and (3.6)) and $t < s$ using the definition (2.5) as

$$(6.3) \quad \begin{aligned} \mathcal{B}(V_t, u(\cdot, t), \hat{\rho}(\cdot, t)) &= \int_{B_1} (-h\hat{\rho} + \nabla \hat{\rho}) \cdot (h + u^\perp) d\|V_t\| \\ &= \int_{B_1} -|h|^2 \hat{\rho} + 2\nabla \hat{\rho} \cdot h + u^\perp \cdot (-h\hat{\rho} + \nabla \hat{\rho}) - \nabla \hat{\rho} \cdot h d\|V_t\|. \end{aligned}$$

We then complete the square by adding and subtracting $|(\nabla \hat{\rho})^\perp|^2/\hat{\rho}$. Due to the perpendicularity of mean curvature (2.4), we have for $\|V_t\|$ a.e. $h \cdot \nabla \hat{\rho} = h \cdot (\nabla \hat{\rho})^\perp$. We then continue the computation of (6.3) as

$$(6.4) \quad = \int_{B_1} -\hat{\rho} \left| h - \frac{(\nabla \hat{\rho})^\perp}{\hat{\rho}} \right|^2 + \frac{|(\nabla \hat{\rho})^\perp|^2}{\hat{\rho}} + u \cdot (-h\hat{\rho} + (\nabla \hat{\rho})^\perp) + S \cdot \nabla^2 \hat{\rho} d\|V_t\|.$$

By the Cauchy-Schwarz inequality applied to the third term of (6.4), we obtain

$$(6.5) \quad \leq \int_{B_1} \frac{|(\nabla \hat{\rho})^\perp|^2}{\hat{\rho}} + |u|^2 \hat{\rho} + S \cdot \nabla^2 \hat{\rho} d\|V_t\|.$$

Substituting (6.5) into (3.4) and using (3.1) and (6.1), we obtain

$$(6.6) \quad \int_{B_1} \hat{\rho}(\cdot, t) d\|V_t\| \Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \left(\int_{B_1} |u|^2 \hat{\rho} d\|V_t\| + c_1 \omega_k E_1 \right) dt.$$

For the first term of (6.6), by the Hölder inequality, we have

$$(6.7) \quad \begin{aligned} \int_{t_1}^{t_2} \int_{B_1} |u|^2 \hat{\rho} d\|V_t\| dt &\leq \int_{t_1}^{t_2} \left(\int_{B_1} |u|^p \hat{\rho} d\|V_t\| \right)^{\frac{2}{p}} \left(\int_{B_1} \hat{\rho} d\|V_t\| \right)^{1-\frac{2}{p}} dt \\ &\leq E_1^{1-\frac{2}{p}} \int_{t_1}^{t_2} \frac{1}{(4\pi(s-t))^{\frac{k}{p}}} \left(\int_{B_1} |u|^p d\|V_t\| \right)^{\frac{2}{p}} dt. \end{aligned}$$

We used (3.1) in the last line of (6.7). Again using the Hölder inequality, (6.7) may be estimated as

$$(6.8) \quad \begin{aligned} &\leq \left(\int_{t_1}^{t_2} \left(\int_{B_1} |u|^p d\|V_t\| \right)^{\frac{q}{p}} dt \right)^{\frac{2}{q}} \left(\int_{t_1}^{t_2} (4\pi(s-t))^{-\frac{kq}{p(q-2)}} dt \right)^{\frac{q-2}{q}} E_1^{1-\frac{2}{p}} \\ &\leq \|u\|_{L^{p,q}(B_1 \times (t_1, t_2))}^2 (4\pi)^{-\frac{k}{p}} E_1^{1-\frac{2}{p}} \left(- \left(1 - \frac{kq}{p(q-2)} \right)^{-1} (s-t)^{1-\frac{kq}{p(q-2)}} \Big|_{t=t_1}^{t_2} \right)^{\frac{q-2}{q}}. \end{aligned}$$

We have $kq < p(q-2)$ due to (3.2). Since $(s-t_1)^{1-\frac{kq}{p(q-2)}} - (s-t_2)^{1-\frac{kq}{p(q-2)}} \leq (t_2-t_1)^{1-\frac{kq}{p(q-2)}}$, by setting $c_2 = c_2(k, p, q)$, we obtain (6.2). $\square$

Corollary 6.3. *Corresponding to $1 \leq E_1 < \infty$, p and q , there exist $0 < c_3 < 1$ and $0 < c_4 < 1$ with the following property. Suppose $\{V_t\}_{0 \leq t \leq \Lambda}$ and $\{u(\cdot, t)\}_{0 \leq t \leq \Lambda}$ satisfy (A1)-(A4) and assume $B_R(a) \times (t, t + c_3 R^2) \subset\subset U \times (0, \Lambda)$. In addition assume*

$$(6.9) \quad R^\zeta \|u\|_{L^{p,q}(B_R(a) \times (t, t + c_3 R^2))} \leq 1, \quad \|V_t\|(B_{14R/15}(a)) \leq c_4 R^k.$$

Then we have $\|V_{t+c_3 R^2}\|(B_{4R/5}(a)) = 0$.

Proof. After a change of variables, we may assume $R = 1$, $t = 0$ and $a = 0$. First By Proposition 6.2 with $t_2 = c_3$ to be chosen, $s = t_2 + \epsilon$ and $t_1 = 0$, we have for any $y \in B_{4/5}$

$$(6.10) \quad \int_{B_1} \hat{\rho}_{(y, t_2+\epsilon)}(\cdot, t) d\|V_t\| \Big|_{t=0}^{t_2} \leq c_2 \|u\|_{L^{p,q}(B_1 \times (0, t_2))}^2 E_1^{1-\frac{2}{p}} t_2^\zeta + c_1 \omega_k E_1 t_2.$$

Here $\epsilon > 0$ is arbitrary. Since $\hat{\rho}$ has support in $B_{14/15}$ and $\eta \leq 1$, and by (6.9) and (6.10), we have

$$(6.11) \quad \int_{B_1} \hat{\rho}_{(y, t_2+\epsilon)}(\cdot, t_2) d\|V_{t_2}\| \leq (4\pi t_2)^{-k/2} c_4 + c_2 E_1^{1-\frac{2}{p}} t_2^\zeta + c_1 \omega_k E_1 t_2.$$

Suppose that V_{t_2} is unit density, and assume for a contradiction that $\|V_{t_2}\|(B_{4/5}) > 0$. Then there exists some $y \in B_{4/5}$ such that $\Theta^k(\|V_{t_2}\|, y) = 1$ and one can show that

$$(6.12) \quad \lim_{\epsilon \rightarrow 0} \int_{B_1} \hat{\rho}_{(y, t_2 + \epsilon)}(\cdot, t_2) d\|V_{t_2}\| = 1.$$

Combining (6.11) and (6.12), if we choose $t_2 = c_3$ appropriately small and then choose c_4 small depending only on E_1, p, q , we obtain a contradiction. If V_{t_2} is not unit density, we may find a sequence $\lim_{m \rightarrow \infty} t_m = t_2$ with $t_m < t_2$ such that V_{t_m} is unit density by (A1). Then the previous argument shows that $\|V_{t_m}\|(B_{4/5}) = 0$ for all large m . By (3.4), we may conclude that $\|V_{t_2}\|(B_{4/5}) = 0$. $\square$

6.2. L^2 - L^∞ estimate. Proposition 6.4 gives L^∞ bound of the height of varifold relative to T in terms of space-time L^2 -height plus some perturbative term. This has been already observed by Ecker [12] in a smooth setting for mean curvature flow while we are unaware of other applications in the setting of geometric measure theory. The result bridges space-time L^2 -height smallness and time-uniform L^2 -height smallness, the latter being a hypothesis of Theorem 5.7. Also L^∞ estimate makes a good use in the blow-up analysis of Section 8. In [1], Allard proves a similar estimate for stationary varifolds, but unlike the present setting, it does not play a role in his regularity theory.

Proposition 6.4. *For $U = B_R$, $R^2 \leq \Lambda < \infty$ and $1 \leq E_1 < \infty$ suppose that $\{V_t\}_{0 \leq t < \Lambda}$ and $\{u(\cdot, t)\}_{0 \leq t < \Lambda}$ satisfy (A1)-(A4). Then there exists a constant $c_5 = c_5(n, k)$ with the following property. For $T \in \mathbf{G}(n, k)$ set*

$$(6.13) \quad \mu^2 := \frac{c_5}{R^{k+4}} \int_0^\Lambda \int_{B_R} |T^\perp(x)|^2 d\|V_t\| dt + c_2 \|u\|_{L^{p,q}(B_R \times (0, \Lambda))}^2 E_1^{1-\frac{2}{p}} \Lambda^\zeta (2 + \Lambda/R^2).$$

Here c_2 is as in Proposition 6.2. Then for all $t \in (R^2/5, \Lambda)$ we have

$$(6.14) \quad \text{spt } \|V_t\| \cap B_{4R/5} \subset \{|T^\perp(x)| \leq \mu R\}.$$

Proof. Without loss of generality we may assume $R = 1$ and redefine Λ as Λ/R^2 . In the proof let $\eta \in C^\infty(B_1 \times (0, \Lambda))$ be a non-negative function with $\eta \equiv 1$ on $B_{13/15} \times [2/15, \Lambda)$, $\eta \equiv 0$ on $(B_1 \times (0, \Lambda)) \setminus (B_{14/15} \times [1/15, \Lambda))$, $0 \leq \eta \leq 1$ and $|\nabla \eta|, \|\nabla^2 \eta\|, |\frac{\partial \eta}{\partial t}| \leq c(n)$. For $(y, s) \in B_{4/5} \times (1/5, \Lambda)$, we use $\phi(x, t) = |T^\perp(x)|^2 \rho_{(y,s)}(x, t) \eta(x, t)$ in (3.4), over the time interval $t_1 = 0$ and $0 < t_2 < s < \Lambda$. We then obtain (writing $\rho_{(y,s)}(x, t)$ as ρ)

$$(6.15) \quad \begin{aligned} & \left| \int_{B_1} |T^\perp(x)|^2 \rho \eta d\|V_t\| \right|_{t=t_2} \\ & \leq \int_0^{t_2} \int_{B_1} \{-h\rho\eta |T^\perp(x)|^2 + \nabla(\rho\eta |T^\perp(x)|^2)\} \cdot (h + u^\perp) + |T^\perp(x)|^2 \frac{\partial}{\partial t}(\rho\eta) d\|V_t\| dt \end{aligned}$$

since $\eta = 0$ for $t = 0$. The integrand of the right-hand side of (6.15) can be computed as follows for a.e. t . We use the perpendicularity of the mean curvature vector (2.4).

$$\begin{aligned}
& -|h|^2 \rho \eta |T^\perp(x)|^2 + (\nabla \rho \cdot h) \eta |T^\perp(x)|^2 + \rho \nabla(\eta |T^\perp(x)|^2) \cdot h + (-h \rho \eta |T^\perp(x)|^2 \\
& + \eta |T^\perp(x)|^2 \nabla \rho) \cdot u^\perp + \rho \nabla(\eta |T^\perp(x)|^2) \cdot u^\perp + |T^\perp(x)|^2 \frac{\partial}{\partial t}(\rho \eta) \\
(6.16) \quad & \leq -\rho \left| h - \frac{(\nabla \rho)^\perp}{\rho} \right|^2 \eta |T^\perp(x)|^2 - (\nabla \rho \cdot h) \eta |T^\perp(x)|^2 + \frac{|(\nabla \rho)^\perp|^2}{\rho} \eta |T^\perp(x)|^2 \\
& + \rho \nabla(\eta |T^\perp(x)|^2) \cdot h + \rho \left| h - \frac{(\nabla \rho)^\perp}{\rho} \right|^2 \eta |T^\perp(x)|^2 + \rho \eta |T^\perp(x)|^2 |u|^2 \\
& + \rho \nabla(\eta |T^\perp(x)|^2) \cdot u^\perp + |T^\perp(x)|^2 \frac{\partial}{\partial t}(\rho \eta).
\end{aligned}$$

Thus we obtain from (6.15) and (6.16) that

$$\begin{aligned}
(6.17) \quad & \left| \int_{B_1} |T^\perp(x)|^2 \rho \eta d\|V_t\| \right|_{t=t_2} \leq \int_0^{t_2} \int_{B_1} -(\nabla \rho \cdot h) \eta |T^\perp(x)|^2 + \rho \nabla(\eta |T^\perp(x)|^2) \cdot h \\
& + \frac{|(\nabla \rho)^\perp|^2}{\rho} \eta |T^\perp(x)|^2 + \rho \eta |T^\perp(x)|^2 |u|^2 + \rho \nabla(\eta |T^\perp(x)|^2) \cdot u^\perp + |T^\perp(x)|^2 \frac{\partial}{\partial t}(\rho \eta) d\|V_t\| dt.
\end{aligned}$$

The first two terms on the right-hand side of (6.17) are

$$\begin{aligned}
(6.18) \quad & \int_0^{t_2} \int_{G_k(B_1)} \nabla(\eta |T^\perp(x)|^2 \nabla \rho) \cdot S - \nabla\{\rho \nabla(\eta |T^\perp(x)|^2)\} \cdot S dV_t(x, S) dt \\
& = \int_0^{t_2} \int_{G_k(B_1)} (\nabla^2 \rho \cdot S) \eta |T^\perp(x)|^2 - \rho \nabla^2(\eta |T^\perp(x)|^2) \cdot S dV_t(x, S) dt.
\end{aligned}$$

Now we use the identity $\frac{\partial \rho}{\partial t} + \frac{|(\nabla \rho)^\perp|^2}{\rho} + S \cdot \nabla^2 \rho \equiv 0$ to obtain from (6.17) and (6.18) that

$$\begin{aligned}
(6.19) \quad & \left| \int_{B_1} |T^\perp(x)|^2 \rho \eta d\|V_t\| \right|_{t=t_2} \leq \int_0^{t_2} \int_{G_k(B_1)} \left\{ -\nabla^2(\eta |T^\perp(x)|^2) \cdot S + \eta |T^\perp(x)|^2 |u|^2 \right. \\
& \left. + |T^\perp(x)|^2 \nabla \eta \cdot u^\perp + \eta \nabla |T^\perp(x)|^2 \cdot u^\perp + |T^\perp(x)|^2 \frac{\partial \eta}{\partial t} \right\} \rho dV_t(x, S) dt.
\end{aligned}$$

For the first term involving u in (6.19), we use $|T^\perp(x)| \leq 1$ and the similar computation as in the previous subsection then gives

$$(6.20) \quad \int_0^{t_2} \int_{B_1} \eta |T^\perp(x)|^2 |u|^2 \rho d\|V_t\| dt \leq c_2(k, p, q) \|u\|_{L^{p,q}(B_1 \times (0, \Lambda))}^2 E_1^{1-\frac{2}{p}} \Lambda^\varsigma.$$

For the second term involving u in (6.19), we have (again using $|T^\perp(x)| \leq 1$)

$$(6.21) \quad \int_0^{t_2} \int_{B_1} |T^\perp(x)|^2 \nabla \eta \cdot u^\perp \rho d\|V_t\| dt \leq \int_0^{t_2} \int_{B_1} |T^\perp(x)|^2 \frac{|\nabla \eta|^2}{\eta} \rho + \eta |u|^2 \rho d\|V_t\| dt.$$

We note that $\eta^{-1} |\nabla \eta|^2 \leq c(n) \max \|\nabla^2 \eta\|$ and $\eta^{-1}(x, t) |\nabla \eta(x, t)|^2 \rho_{(y,s)}(x, t)$ is uniformly bounded for $(y, s) \in B_{4/5} \times (1/5, \infty)$ and $(x, t) \in A := (B_{14/15} \times [1/15, \Lambda]) \setminus (B_{13/15} \times [2/15, \Lambda])$ since $|x - y| \geq 1/15$. Outside of A , it vanishes. Thus we have some $c(n, k)$ such that

$\eta^{-1}|\nabla\eta|^2\rho \leq c(n, k)$. The second term in the integral of (6.21) can be estimated as in (6.20). For the third term involving u in (6.19),

$$(6.22) \quad \begin{aligned} & \int_0^{t_2} \int_{B_1} \eta(\nabla|T^\perp(x)|^2 \cdot u^\perp) \rho d\|V_t\| dt \leq \frac{1}{\Lambda} \int_0^{t_2} \int_{B_1} |T^\perp(x)|^2 \eta \rho d\|V_t\| dt \\ & + \Lambda \int_0^{t_2} \int_{B_1} \eta|u|^2 \rho d\|V_t\| dt. \end{aligned}$$

The last term of (6.22) can be estimated as in (6.20). For the remaining two terms of (6.19), we have

$$(6.23) \quad \begin{aligned} & -\nabla^2(\eta|T^\perp(x)|^2) \cdot S + |T^\perp(x)|^2 \frac{\partial\eta}{\partial t} \\ & = -(\nabla^2\eta \cdot S)|T^\perp(x)|^2 - 4(\nabla\eta \otimes T^\perp(x)) \cdot S - 2\eta T^\perp \cdot S + |T^\perp(x)|^2 \frac{\partial\eta}{\partial t}. \end{aligned}$$

Note that $T^\perp \cdot S \geq 0$ in general (see (11.3)), and

$$(6.24) \quad 4|(\nabla\eta \otimes T^\perp(x)) \cdot S| \leq 4|T^\perp(x)| |\nabla\eta| \sqrt{T^\perp \cdot S} \leq 2\eta T^\perp \cdot S + 2 \frac{|\nabla\eta|^2}{\eta} |T^\perp(x)|^2.$$

Since $\frac{|\nabla\eta|^2}{\eta} \leq c(n) \|\nabla^2\eta\|$, (6.24) shows that the right-hand side of (6.23) can be bounded from above by $c(n)|T^\perp(x)|^2$ on A and otherwise (6.23) ≤ 0 . Since $\rho \leq c(k)$ on A , we have

$$(6.25) \quad \begin{aligned} & \int_0^{t_2} \int_{G_k(B_1)} \left\{ -\nabla^2(\eta|T^\perp(x)|^2) \cdot S + |T^\perp(x)|^2 \frac{\partial\eta}{\partial t} \right\} \rho dV_t(x, S) dt \\ & \leq c(n, k) \int_0^\Lambda \int_{B_1} |T^\perp(x)|^2 d\|V_t\| dt. \end{aligned}$$

We obtain from (6.19), (6.20), (6.21), (6.22) and (6.25)

$$(6.26) \quad \begin{aligned} & \left| \int_{B_1} |T^\perp(x)|^2 \eta \rho d\|V_t\| \right|_{t=t_2} \leq c_2 \|u\|_{L^{p,q}}^2 E_1^{1-\frac{2}{p}} \Lambda^\varsigma (2 + \Lambda) \\ & + \frac{1}{\Lambda} \int_0^{t_2} \int_{B_1} |T^\perp(x)|^2 \eta \rho d\|V_t\| dt + c(n, k) \int_0^\Lambda \int_{B_1} |T^\perp(x)|^2 d\|V_t\| dt. \end{aligned}$$

holds for all $t_2 \in (0, s)$. It is easy to check that the differential inequality $F'(t) \leq C + F(t)/\Lambda$ for $t \in (0, s)$ with $F(0) = 0$ means $F'(t) \leq Ce$ for all $t \in (0, s)$ with $s < \Lambda$. Hence from (6.26) we obtain

$$(6.27) \quad \begin{aligned} & \left| \int_{B_1} |T^\perp(x)|^2 \eta \rho d\|V_t\| \right|_{t=t_2} \\ & \leq e \left(c_2 \|u\|_{L^{p,q}}^2 E_1^{1-\frac{2}{p}} \Lambda^\varsigma (2 + \Lambda) + c(n, k) \int_0^\Lambda \int_{B_1} |T^\perp(x)|^2 d\|V_t\| dt \right). \end{aligned}$$

We now set the right-hand side of (6.27) as μ^2 in the statement of the present proposition with an appropriate choice of $c_5 = c_5(n, k)$ and replacing c_2 by ec_2 . Finally assume that we have $\|V_{t_0}\|(B_{4/5} \cap \{|T^\perp(x)| > \mu\}) > 0$ for some $t_0 \in (1/5, \Lambda)$ when V_{t_0} is unit density. Then there exists some $x_0 \in B_{4/5}$ with $|T^\perp(x_0)| > \mu$ such that $\Theta^k(\|V_{t_0}\|, x_0) = 1$ and the approximate tangent space exists. One can then show that $\lim_{\varepsilon \rightarrow 0+} \int_{B_1} |T^\perp(x)|^2 \eta \rho_{(x_0, t_0+\varepsilon)} d\|V_{t_0}\| = |T^\perp(x_0)|^2$. From (6.27), we have $|T^\perp(x_0)| \leq \mu$, which is a contradiction. We may easily

extend the same conclusion for all $t \in (1/5, \Lambda)$ instead of a.e. t by (3.4). Thus we conclude the proof. $\square$

6.3. Time uniform L^2 estimate. The following estimate is used in Section 9. The proof is similar to Proposition 6.4.

Proposition 6.5. *Corresponding to $1 \leq E_1 < \infty$, p, q and $1 \leq \Lambda < \infty$ there exists c_6 with the following property. For $U = B_{LR}$, $2 \leq L < \infty$, suppose that $\{V_t\}_{0 \leq t \leq \Lambda R^2}$ and $\{u(\cdot, t)\}_{0 \leq t \leq \Lambda R^2}$ satisfy (A1)-(A4). Then for $T \in \mathbf{G}(n, k)$ and for all $t \in [0, \Lambda R^2]$,*

$$(6.28) \quad R^{-(k+2)} \int_{B_R} |T^\perp(x)|^2 d\|V_t\| \leq \exp(1/(4\Lambda)) R^{-(k+2)} \int_{B_{LR}} |T^\perp(x)|^2 d\|V_0\| \\ + c_6 \{ (R^{2\varsigma} \|u\|^2 + R^\varsigma \|u\|) L^2 + L^{k+2} \exp(-(L-1)^2/(8\Lambda)) \}.$$

Here $\|u\| = \|u\|_{L^{p,q}(B_{LR} \times [0, \Lambda R^2])}$.

Proof. Without loss of generality, we set $R = 1$ after a change of variables. Let $\eta \in C_c^\infty(B_L)$ be a radially symmetric non-negative function with $\eta \equiv 1$ on B_{L-1} , $0 \leq \eta \leq 1$, and $|\nabla \eta|, \|\nabla^2 \eta\| \leq c(n)$. In (3.4), we use $\phi(x, t) = |T^\perp(x)|^2 \rho_{(0, 2\Lambda)}(x, t) \eta(x)$ over the time interval $[0, t_1]$, $t_1 \leq \Lambda$. The computations are analogous to those in the proof of Proposition 6.4, the only difference this time is that η does not depend on time. Writing $\rho = \rho_{(0, 2\Lambda)}(x, t)$, we have (see (6.19))

$$(6.29) \quad \int_0^{t_1} |T^\perp(x)|^2 \rho \eta d\|V_t\| \Big|_{t=0}^{t_1} \leq \int_0^{t_1} \int \{ -\nabla^2(\eta |T^\perp(x)|^2) \cdot S + \eta |T^\perp(x)|^2 |u|^2 \\ + |T^\perp(x)|^2 \nabla \eta \cdot u^\perp + \eta \nabla |T^\perp(x)|^2 \cdot u^\perp \} \rho dV_t(x, S) dt.$$

The last three terms involving u can be estimated as in (6.7) and (6.8), using also $|T^\perp(x)| \leq L$. We also use (6.23) and (6.24) (without $\partial \eta / \partial t$ term) and dropping negative term to obtain from (6.29)

$$(6.30) \quad \int_0^{t_1} |T^\perp(x)|^2 \rho \eta d\|V_t\| \Big|_{t=0}^{t_1} \leq c(n, k) \int_0^{t_1} \int \|\nabla^2 \eta\| L^2 \rho d\|V_t\| dt \\ + c(p, q, k, E_1, \Lambda) (\|u\| + \|u\|^2) L^2.$$

Since $\|\nabla^2 \eta\|$ is zero on B_{L-1} , and since $2\Lambda \geq 2\Lambda - t_1 > \Lambda$, we have $\rho_{(0, 2\Lambda)}(x, t) \leq (4\pi\Lambda)^{-k/2} \exp(-(L-1)^2/8\Lambda)$ there. From (6.30), one obtains

$$(6.31) \quad \int_0^{t_1} |T^\perp(x)|^2 \rho \eta d\|V_t\| \Big|_{t=0}^{t_1} \leq c(p, q, n, k, E_1, \Lambda) \{ (\|u\| + \|u\|^2) L^2 + L^{k+2} \exp(-(L-1)^2/8\Lambda) \}$$

Since $\rho_{(0, 2\Lambda)}(x, 0) \leq (8\pi\Lambda)^{-k/2}$ for all $x \in B_L$, and $\rho_{(0, 2\Lambda)}(x, t) \geq (8\pi\Lambda)^{-k/2} \exp(-1/(4\Lambda))$ for $|x| \leq 1$ and $t \in [0, \Lambda]$, we obtain (6.28) from (6.31) with a suitable choice of constant. $\square$

7. PARABOLIC LIPSCHITZ APPROXIMATION

A main result of this section is Theorem 7.5 which gives a good Lipschitz graph approximation of moving varifold in space-time. The similar Lipschitz approximation has to be constructed in Allard's regularity theory and one can see a clear parallel in that sense. We note that along with the monotonicity type estimates in the previous section, Lipschitz approximation is completely absent in Brakke's original proof. Instead Brakke constructed

approximate graphs by substituting test functions weighted by heat kernel in the flow inequality (see [7, 6.8, 6.9]).

Proposition 7.1. *Given $0 < l < \infty$, there exist constants $c_7 = c_7(l)$, $c_8 = c_8(n, k)$ and $c_9 = c_9(k, p, q)$ with the following property. For $T \in \mathbf{G}(n, k)$, $U = C(T, 1)$, $\Lambda = 1$ and $1 \leq E_1 < \infty$ suppose that $\{V_t\}_{0 \leq t \leq 1}$ and $\{u(\cdot, t)\}_{0 \leq t \leq 1}$ satisfy (A1)-(A4). For $\max\{2, l\} < R < \infty$, $0 < \gamma < 1$ and $(y_1, s_1), (y_2, s_2) \in C(T, 1) \times (0, 1)$, we assume the following.*

(7.1) *The approximate tangent space to V_{s_i} exists at y_i and $\Theta^k(\|V_{s_i}\|, y_i) = 1$, $i = 1, 2$.*

$$(7.2) \quad \sigma := |T^\perp(y_1 - y_2)| \geq l^{-1} \max\{|T(y_1 - y_2)|, |s_1 - s_2|^{\frac{1}{2}}\}.$$

Set $\bar{y} := \frac{y_1 + y_2}{2}$, $\bar{s} := \frac{s_1 + s_2}{2}$ and $R(l, \sigma) := (R^{3/2} + \frac{\sqrt{l^2 + 1}}{2})\sigma$. Then assume further that

$$(7.3) \quad P_{R(l, \sigma)}(\bar{y}, \bar{s}) \subset C(T, 1) \times (0, 1),$$

and for each $i = 1, 2$ we have

$$(7.4) \quad \int_{P_{R^{3/2}\sigma}(y_i, s_i)} \|S - T\|^2 dV_t(\cdot, S) dt \leq \gamma(R^{3/2}\sigma)^{k+2},$$

and R and l satisfy

$$(7.5) \quad R^2 + \frac{l^2}{2} < \frac{(R^{3/2} - 1)^2}{2k}.$$

Under the above assumptions, we have

$$(7.6) \quad \begin{aligned} 2 \leq & \frac{\exp(c_7/\sqrt{R})}{(1 - \frac{l^2}{2R^2})^{k/2}} \int_{B_{R(l, \sigma)}(\bar{y})} \rho_{(\bar{y}, \bar{s})}(\cdot, t) d\|V_t\| \Big|_{t=\bar{s}-R^2\sigma^2} + c_8 E_1 (\exp(-\sqrt{R}) + \sigma^\varsigma R^2) \\ & + c_8 \gamma R^{3(k+2)/2} + c_9 \|u\|_{L^{p, q}(C(T, 1) \times (0, 1))}^2 E_1^{1-\frac{2}{p}} R^{2\varsigma} (\sigma^{2\varsigma} + \sigma^\varsigma). \end{aligned}$$

Remark 7.2. *Proposition 7.1 is analogous to [1, 6.1]. To describe the idea, assume $n = k + 1$ case. Consider T as a horizontal plane. Then σ is the vertical distance of two points, which is typically very small. Later R will be chosen very large so that the coefficient of the first term of the right-hand side of (7.6) is very close to 1. In Lemma 7.3 we prove that there can be only one horizontal sheet in terms of density ratio via monotonicity formula if the first variation is small. Thus we conclude that there cannot be two very close points of density one positioned in a vertical manner. Rough idea of the proof is that we ‘cut’ moving varifold by a horizontal plane to separate these two points, and derive a monotonicity type formula for each piece. The cutting naturally produces extra error terms which are assumed to be small (see (7.4)).*

Proof. Let $\xi_1, \xi_2 \in C^\infty(\mathbb{R})$ be non-negative functions such that

$$(7.7) \quad \begin{aligned} \xi_1(s) &= \begin{cases} 1 & \text{for } s \leq \sigma/4, \\ 0 & \text{for } s \geq \sigma/2, \end{cases} & \xi_2(s) &= \begin{cases} 0 & \text{for } s \leq \sigma/2, \\ 1 & \text{for } s \geq 3\sigma/4, \end{cases} \\ 0 \leq \xi_i &\leq 1, |\xi'_i| \leq 8/\sigma \text{ and } |\xi''_i| \leq 128/\sigma^2 \text{ for } i = 1, 2. \end{aligned}$$

Let $\eta_1, \eta_2 \in C_c^\infty(\mathbb{R}^n)$ be non-negative functions such that for each $i = 1, 2$,

$$(7.8) \quad \begin{aligned} \eta_i(x) &= \begin{cases} 1 & \text{on } B_{(R^{3/2}-1)\sigma}(y_i), \\ 0 & \text{on } \mathbb{R}^n \setminus B_{R^{3/2}\sigma}(y_i), \end{cases} \\ 0 \leq \eta_i &\leq 1, |\nabla \eta_i| \leq 2/\sigma \text{ and } \|\nabla^2 \eta_i\| \leq c(n)/\sigma^2. \end{aligned}$$

Due to (7.2), for each $i = 1, 2$, $|\bar{y} - y_i| = |y_1 - y_2|/2 \leq \sigma\sqrt{l^2 + 1}/2$, and (7.3) shows

$$(7.9) \quad \text{spt } \eta_i \subset B_{R^{3/2}\sigma}(y_i) \subset B_{R^{3/2}\sigma + |\bar{y} - y_i|}(\bar{y}) \subset B_{R(l, \sigma)}(\bar{y}) \subset C(T, 1).$$

For each $i = 1, 2$ and arbitrarily small $\varepsilon > 0$ we use (3.4) with

$$\phi(x, t) = \rho_{(y_i, s_i + \varepsilon)}(x, t) \eta_i(x) \xi_i(|T^\perp(x - y_1)|),$$

$t_1 = \bar{s} - R^2\sigma^2$ and $t_2 = s_i$. By (7.2) and $l < R$, we have $t_2 - t_1 \geq R^2\sigma^2 - |s_1 - s_2|/2 \geq R^2\sigma^2 - l^2\sigma^2/2 > 0$. In the following, we denote $\rho_{(y_i, s_i + \varepsilon)}(x, t)$ (and even after setting $\varepsilon = 0$) by ρ_i . The same computations as those leading to (6.19) give (we omit writing variables with no fear of confusion)

$$(7.10) \quad \int_{t=\bar{s}-R^2\sigma^2}^{s_i} \rho_i \eta_i \xi_i d\|V_t\| \leq \int_{\bar{s}-R^2\sigma^2}^{s_i} dt \int \{-\nabla^2(\eta_i \xi_i) \cdot S + \eta_i \xi_i |u|^2 + \nabla(\eta_i \xi_i) \cdot u^\perp\} \rho_i dV_t(\cdot, S).$$

Due to (7.7) and (7.8), $\eta_i(x) \xi_i(|T^\perp(x - y_1)|)$ is identically 1 in a small neighborhood of $x = y_i$ for each $i = 1, 2$. Hence using (7.1) we have

$$\lim_{\varepsilon \rightarrow 0+} \int \rho_{(y_i, s_i + \varepsilon)} \eta_i \xi_i d\|V_{s_i}\| = \Theta^k(\|V_{s_i}\|, y_i) = 1.$$

Thus by letting $\varepsilon \rightarrow 0$, (7.10) is

$$(7.11) \quad 1 \leq \int_{t=\bar{s}-R^2\sigma^2}^{s_i} \rho_i \eta_i \xi_i d\|V_t\| + \int_{\bar{s}-R^2\sigma^2}^{s_i} dt \int \{-\nabla^2(\eta_i \xi_i) \cdot S + \eta_i \xi_i |u|^2 + \nabla(\eta_i \xi_i) \cdot u^\perp\} \rho_i dV_t(\cdot, S) =: I_1 + I_2 + I_3 + I_4$$

where $\rho_i = \rho_{(y_i, s_i)}(x, t)$. We next estimate the right-hand side of (7.11). We start with I_2 .

Estimate for I_2 .

For the integrand of I_2 we have (writing $v = T^\perp(x - y_1)/|T^\perp(x - y_1)|$)

$$(7.12) \quad \begin{aligned} |-\nabla^2(\eta_i \xi_i) \cdot S| &\leq \xi_i |\nabla^2 \eta_i \cdot S| + 2|(\nabla \eta_i \otimes \nabla \xi_i) \cdot S| + \eta_i |\nabla^2 \xi_i \cdot S| \\ &\leq \xi_i k \|\nabla^2 \eta_i\| + 2|\xi'_i \nabla \eta_i \cdot S(v)| + \eta_i \left| S \cdot \left\{ \left(\xi''_i - \frac{\xi'_i}{|T^\perp(x - y_1)|} \right) v \otimes v + \frac{\xi'_i}{|T^\perp(x - y_1)|} T^\perp \right\} \right|. \end{aligned}$$

Note that (see Lemma 11.1) we have

$$(7.13) \quad \begin{aligned} |S(T^\perp(v))| &\leq \|S - T\| |v|, \\ S \cdot (T^\perp(v) \otimes T^\perp(v)) &= S(T^\perp(v)) \cdot S(T^\perp(v)) \leq \|S - T\|^2 |v|^2, \\ S \cdot T^\perp &\leq k \|S - T\|^2. \end{aligned}$$

Thus with (7.7), (7.8), (7.13), and the fact that $|T^\perp(x - y_1)| \geq \sigma/4$ on $\text{spt } |\nabla \xi_i|$, (7.12) is estimated as

$$(7.14) \quad |-\nabla^2(\eta_i \xi_i) \cdot S| \leq c(n, k) \sigma^{-2} \chi_{\text{spt } |\nabla \eta_i| \cap \text{spt } \xi_i} + c(k) \sigma^{-2} \|S - T\|^2 \chi_{\text{spt } \eta_i \cap \text{spt } |\nabla \xi_i|}$$

with a suitable choice of constants $c(n, k)$ and $c(k)$. We next estimate ρ_i on $\text{spt } |\nabla \eta_i| \cap \text{spt } \xi_i$ and $\text{spt } \eta_i \cap \text{spt } |\nabla \xi_i|$ respectively.

On $\text{spt } |\nabla \eta_i| \cap \text{spt } \xi_i$, since $\text{spt } |\nabla \eta_i| \cap B_{(R^{3/2}-1)\sigma}(y_i) = \emptyset$ by (7.8),

$$(7.15) \quad \rho_i \leq \frac{\exp\left(-\frac{(R^{3/2}-1)^2\sigma^2}{4(s_i-t)}\right)}{(4\pi(s_i-t))^{k/2}}$$

which takes the maximum value at $s_i - t = (R^{3/2} - 1)^2 \sigma^2 / (2k)$ and monotone increasing until that point. Note that $s_i - t$ varies over the interval $(0, s_i - \bar{s} + R^2 \sigma^2) \subset (0, (R^2 + \frac{l^2}{2}) \sigma^2)$. Thus as long as (7.5) is satisfied, the maximum value of (7.15) is estimated as

$$(7.16) \quad \rho_i \leq \frac{\exp\left(-\frac{(R^{3/2}-1)^2}{4(R^2+l^2/2)}\right)}{(4\pi)^{k/2}(R^2+l^2/2)^{k/2}\sigma^k} \leq \frac{\exp\left(-\frac{(R^{3/2}-1)^2}{6R^2}\right)}{(4\pi)^{k/2}R^k\sigma^k} \leq \frac{c(k)\exp(-R^{3/4})}{\sigma^k}$$

on $\text{spt}|\nabla\eta_i| \cap \text{spt}\xi_i$. The last two inequalities follow from $R > \max\{2, l\}$ and due to the exponential decay for large R .

For $x \in \text{spt}\eta_i \cap \text{spt}|\nabla\xi_i|$, we have $|x - y_i| \geq \sigma/4$, and one can check that the maximum value can be estimated as

$$(7.17) \quad \rho_i \leq \frac{\exp\left(-\frac{\sigma^2}{64(s_i-t)}\right)}{(4\pi(s_i-t))^{k/2}} \leq \frac{(32k)^{k/2}\exp(-k/2)}{(4\pi)^{k/2}\sigma^k}.$$

Hence we may estimate I_2 by combining (7.14), (7.16) and (7.17) as (with yet another suitable choice of constants)

$$(7.18) \quad \begin{aligned} I_2 &\leq \frac{c(n, k)\exp(-R^{3/4})}{\sigma^{k+2}} \int_{\bar{s}-R^2\sigma^2}^{s_i} \|V_t\|(B_{R^{3/2}\sigma}(y_i)) dt \\ &\quad + \frac{c(k)}{\sigma^{k+2}} \int_{\bar{s}-R^2\sigma^2}^{s_i} \int_{B_{R^{3/2}\sigma}(y_i)} \|S - T\|^2 dV(\cdot, S) dt. \end{aligned}$$

By (3.1), $|s_i - \bar{s} + R^2\sigma^2| \leq (R^2 + l^2/2)\sigma^2$, $B_{R^{3/2}\sigma}(y_i) \times (\bar{s} - R^2\sigma^2, s_i) \subset P_{R^{3/2}\sigma}(y_i, s_i)$ and (7.4), with a suitable choice of constants we have from (7.18)

$$(7.19) \quad I_2 \leq c(n, k)E_1 \exp(-\sqrt{R}) + c(k)\gamma R^{3(k+2)/2}.$$

Estimate for I_3 .

I_3 can be estimated as in (6.7) and (6.8), so we have

$$(7.20) \quad I_3 \leq c_2(k, p, q) \|u\|_{L^{p,q}(C(T,1) \times (0,1))}^2 E_1^{1-\frac{2}{p}} (3R^2\sigma^2/2)^\varsigma.$$

Estimate for I_4 .

We have

$$(7.21) \quad |\rho_i \nabla(\eta_i \xi_i) \cdot u^\perp| \leq \frac{\sigma^{-\varsigma}}{2} |u|^2 \rho_i \chi_{\text{spt}\eta_i} + \frac{\sigma^\varsigma}{2} |\nabla(\eta_i \xi_i)|^2 \rho_i.$$

The first term of the right-hand side of (7.21) can be handled just like I_3 (with multiplication of $\sigma^{-\varsigma}$). The second term is

$$(7.22) \quad \frac{\sigma^\varsigma}{2} |\nabla(\eta_i \xi_i)|^2 \rho_i \leq \frac{\sigma^\varsigma}{2} (|\nabla\eta_i| + |\nabla\xi_i|)^2 \rho_i \chi_{\text{spt}\eta_i} \leq \frac{\sigma^\varsigma}{2} \left(\frac{10}{\sigma}\right)^2 \rho_i \chi_{\text{spt}\eta_i}$$

by (7.8) and (7.7). Thus (7.21) and (7.22) combined with (3.1) and $|s_i - \bar{s} + R^2\sigma^2| \leq (R^2 + l^2/2)\sigma^2$ show

$$(7.23) \quad I_4 \leq c_2 \|u\|_{L^{p,q}(C(T,1) \times (0,1))}^2 E_1^{1-\frac{2}{p}} (3R^2/2)^\varsigma \sigma^\varsigma + 75\sigma^\varsigma R^2 E_1.$$

Estimate for I_1 .

It is important to note that

$$(7.24) \quad \sum_{i=1}^2 \int \rho_i \eta_i \xi_i d\|V_t\| \Big|_{t=\bar{s}-R^2\sigma^2} \leq \int_{B_{R(l,\sigma)}(\bar{y})} \xi_1 \rho_1 + \xi_2 \rho_2 d\|V_t\| \Big|_{t=\bar{s}-R^2\sigma^2}$$

by (7.9) and that the supports of ξ_1 and ξ_2 are disjoint. We next compute the ratio of ρ_i and $\rho_{(\bar{y}, \bar{s})}$. With $|s_i - \bar{s}| \leq l^2 \sigma^2 / 2$

$$(7.25) \quad \frac{\rho_{(y_i, s_i)}(x, \bar{s} - R^2 \sigma^2)}{\rho_{(\bar{y}, \bar{s})}(x, \bar{s} - R^2 \sigma^2)} = \frac{R^k \sigma^k \exp\left(\frac{|\bar{y}-x|^2}{4R^2 \sigma^2} - \frac{|y_i-x|^2}{4(s_i-\bar{s}+R^2 \sigma^2)}\right)}{(s_i - \bar{s} + R^2 \sigma^2)^{k/2}} \leq \frac{\exp\left(\frac{|\bar{y}-x|^2}{4R^2 \sigma^2} - \frac{|y_i-x|^2}{4(R^2+l^2/2)\sigma^2}\right)}{\left(1 - \frac{l^2}{2R^2}\right)^{k/2}}.$$

One checks that when $|\bar{y} - x| \leq |\bar{y} - y_i| (\leq \sigma \sqrt{l^2 + 1}/2)$, the argument of exponential function in (7.25) is

$$(7.26) \quad \frac{\left(R^2 + \frac{l^2}{2}\right) |\bar{y} - x|^2 - R^2 |y_i - x|^2}{4R^2 \left(R^2 + \frac{l^2}{2}\right) \sigma^2} \leq \frac{|\bar{y} - x|^2}{4R^2 \sigma^2} \leq \frac{l^2 + 1}{16R^2}.$$

When $|\bar{y} - x| > |\bar{y} - y_i|$, we have $|y_i - x| \geq |\bar{y} - x| - |\bar{y} - y_i| > 0$ and $-|y_i - x|^2 \leq -|\bar{y} - x|^2 + 2|\bar{y} - x||\bar{y} - y_i|$. Thus

$$(7.27) \quad \begin{aligned} & \frac{\left(R^2 + \frac{l^2}{2}\right) |\bar{y} - x|^2 - R^2 |y_i - x|^2}{4R^2 \left(R^2 + \frac{l^2}{2}\right) \sigma^2} \leq \frac{\frac{l^2}{2} |\bar{y} - x|^2 + 2R^2 |\bar{y} - x| |\bar{y} - y_i|}{4R^4 \sigma^2} \\ & \leq \frac{\frac{l^2}{2} (R^{3/2} + \sqrt{l^2 + 1}/2)^2 + 2R^2 (R^{3/2} + \sqrt{l^2 + 1}/2)(\sqrt{l^2 + 1}/2)}{4R^4} \leq c(l)/\sqrt{R} \end{aligned}$$

since $|\bar{y} - x| \leq (R^{3/2} + \sqrt{l^2 + 1}/2)\sigma$ and by comparing the exponents of R . Thus, combining (7.25), (7.26) and (7.27), we obtain

$$(7.28) \quad \rho_{(y_i, s_i)}(x, \bar{s} - R^2 \sigma^2) \leq \frac{\exp(c(l)/\sqrt{R})}{\left(1 - \frac{l^2}{2R^2}\right)^{k/2}} \rho_{(\bar{y}, \bar{s})}(x, \bar{s} - R^2 \sigma^2).$$

Combining (7.24) and (7.28), and the fact that ξ_1 and ξ_2 have disjoint supports, we obtain

$$(7.29) \quad \sum_{i=1}^2 \int \rho_i \eta_i \xi_i d\|V_t\| \Big|_{t=\bar{s}-R^2 \sigma^2} \leq \frac{\exp(c(l)/\sqrt{R})}{\left(1 - \frac{l^2}{2R^2}\right)^{k/2}} \int_{B_{R(l, \sigma)}(\bar{y})} \rho_{(\bar{y}, \bar{s})}(\cdot, t) d\|V_t\| \Big|_{t=\bar{s}-R^2 \sigma^2}.$$

Final step.

Finally summing (7.11) for $i = 1, 2$, and by the estimates (7.19), (7.20), (7.23) and (7.29), we obtain (7.6) with a suitable choice of constants. $\square$

In the next Lemma 7.3 and Theorem 7.5 we use

$$\hat{\rho}_{(y, s)}(x, t) = \eta(7x/5) \rho_{(y, s)}(x, t),$$

where η is defined in Section 6.1. In the statement of Proposition 6.2, this corresponds to choosing $R = 5/7$. For ϕ_T as in (5.2), we have

$$(7.30) \quad \text{spt } \hat{\rho}_{(y, s)}(\cdot, t) \subset B_{2/3} \subset \{\phi_T = 1\}, \quad \hat{\rho}_{(y, s)}(\cdot, t) = \rho_{(y, s)}(\cdot, t) \text{ on } B_{13/21}.$$

Lemma 7.3. *Corresponding to $1 \leq E_1 < \infty$, p and q as in (3.2) there exist $0 < r_1 < 1$ and $0 < \varepsilon_2 < 1$ with the following property. For $U = C(T, 1)$, $\Lambda = 1$ and E_1 , suppose $\{V_t\}_{0 \leq t \leq 1}$ and $\{u(\cdot, t)\}_{0 \leq t \leq 1}$ satisfy (A1)-(A4). Let ϕ_T and $\mathbf{c}$ be as in (5.2).*

Assume

$$(7.31) \quad \int_{C(T, 1) \times (0, 1)} |h(V_t, \cdot)|^2 \phi_T^2 d\|V_t\| dt \leq \varepsilon_2 r_1^2 / 4,$$

$$(7.32) \quad \|\|V_t\|(\phi_T^2) - \mathbf{c}\| \leq \varepsilon_2, \quad \forall t \in (0, 1),$$

$$(7.33) \quad \text{spt } \|V_t\| \cap C(T, 1) \subset \{|T^\perp(x)| \leq \varepsilon_2\}, \quad \forall t \in (0, 1),$$

$$(7.34) \quad \|u\|_{L^{p,q}(C(T,1) \times (0,1))} \leq 1.$$

Then for any $y \in C(T, 1/2) \cap \{|T^\perp(x)| \leq \varepsilon_2\}$, $t \in (r_1^2, 1)$ and $0 < r < r_1/\sqrt{2}$, we have

$$(7.35) \quad \int_{C(T,1)} \hat{\rho}_{(y,t+r^2)}(\cdot, t) d\|V_t\| \leq \frac{26}{25}.$$

Remark 7.4. The hypothesis of Lemma 7.3 (7.31)-(7.34) can be satisfied if the L^2 -height and the perturbative term are small due to L^2 - L^∞ estimate and energy estimate. The conclusion is that the weighted density ratio can be estimated from above by a number close to 1 (7.35) for all small radii.

Proof. We fix $0 < r_1 < 1$ so that

$$(7.36) \quad c_2 E_1^{1-\frac{2}{p}} (r_1^2)^\varsigma + c_1 E_1 \omega_k r_1^2 (7/5)^2 < \frac{1}{100},$$

$$(7.37) \quad \sup_{0 < r \leq r_1} \sup_{y \in B_{1/2} \cap T} \left| \int_T \hat{\rho}_{(y,r^2)}(x, 0) d\mathcal{H}^k(x) - 1 \right| < \frac{1}{100}.$$

With regard to (7.37), we may choose such r_1 since $\hat{\rho}_{(y,r^2)}(\cdot, 0) = \rho_{(y,r^2)}(\cdot, 0)$ on $B_{13/21}$ due to (7.30) and $\hat{\rho}_{(y,r^2)}(\cdot, 0)$ approaches the delta mass on T at y as $r \rightarrow 0$. The choice of r_1 depends on E_1 , p and q . By (6.2), for all $y \in B_{4/7}$, $0 < t_1 < t < 1$ and $t - t_1 < r_1^2$ we have by (7.34) and (7.36)

$$(7.38) \quad \int_{B_1} \hat{\rho}_{(y,t+r^2)}(\cdot, t) d\|V_t\| - \int_{B_1} \hat{\rho}_{(y,t+r^2)}(\cdot, t_1) d\|V_{t_1}\| < \frac{1}{100}.$$

We next claim that there exists a positive constant $0 < \varepsilon_2 < 1$ depending on r_1 and E_1 with the following properties. Assume that V is an integral varifold with

$$(7.39) \quad \|V\|(B_1) \leq E_1,$$

$$(7.40) \quad \int_{C(T,1)} |h(V, \cdot)|^2 \phi_T^2 d\|V\| \leq \varepsilon_2,$$

$$(7.41) \quad \text{spt } \|V\| \cap C(T, 1) \subset \{|T^\perp(x)| \leq \varepsilon_2\},$$

$$(7.42) \quad |\|V\|(\phi_T^2) - \mathbf{c}| \leq \varepsilon_2.$$

Then for all $y \in C(T, 1/2) \cap \{|T^\perp(x)| \leq \varepsilon_2\}$ and $r_1/\sqrt{2} \leq r \leq r_1$, we have

$$(7.43) \quad \left| \int_{B_1} \hat{\rho}_{(y,r^2)}(\cdot, 0) d\|V\| - 1 \right| < \frac{1}{50}.$$

If false, then for each $m \in \mathbb{N}$ there exist an integral varifold V_m , $y_m \in C(T, 1/2) \cap \{|T^\perp(x)| \leq 1/m\}$ and $r_1/\sqrt{2} \leq r_m \leq r_1$ satisfying (7.39)-(7.42) for $\varepsilon_2 = 1/m$ but not satisfying (7.43). Then by (7.39) we would have a converging subsequence with limit V . Due to (7.40) and the compactness theorem of integral varifold, V is a stationary integral varifold on $\{\phi_T > 0\}$. Due to (7.41), $\text{spt } \|V\| \subset T$. By the constancy theorem of integral varifold

and (7.42), we may conclude that $\|V\| = \mathcal{H}^k \llcorner_T$ on $\{\phi_T > 0\}$. We may also assume that $y_m \rightarrow y \in \overline{C(T, 1/2)} \cap T$ and $r_m \rightarrow r \in [r_1/\sqrt{2}, r_1]$ and we have

$$\left| \int_{B_1} \hat{\rho}_{(y, r^2)}(x, 0) d\|V\|(x) - 1 \right| \geq \frac{1}{50}.$$

This contradicts with (7.37) since $\text{spt } \hat{\rho}_{(y, r^2)}(\cdot, 0) \subset \{\phi_T > 0\}$ by (7.30) and $\|V\| = \mathcal{H}^k \llcorner_T$.

Now given $y \in C(T, 1/2) \cap \{|T^\perp(x)| \leq \varepsilon_2\}$, $t \in (r_1^2, 1)$ and $0 < r < r_1/\sqrt{2}$, consider the interval $[t+r^2-r_1^2, t+r^2-r_1^2/2]$ which has the length $r_1^2/2$. Then by (7.31), there exists some t_1 in this interval such that (7.40) holds for V_{t_1} . The conditions (7.39), (7.41) and (7.42) are satisfied for V_{t_1} by the assumptions of the present lemma. Since $r_1^2/2 \leq t+r^2-t_1 \leq r_1^2$, we have from (7.43)

$$(7.44) \quad \left| \int_{B_1} \hat{\rho}_{(y, t+r^2)}(\cdot, t_1) d\|V_{t_1}\| - 1 \right| < \frac{1}{50}.$$

Since $0 \leq t - t_1 \leq r_1^2$, combining (7.44) and (7.38), we obtain the desired result. $\square$

Theorem 7.5. *Corresponding to $1 \leq E_1 < \infty$, p and q as in (3.2) there exists $0 < \varepsilon_3 < 1$ and c_{10} with the following property. For $U = C(T, 1)$, $\Lambda = 1$ and $1 \leq E_1 < \infty$, suppose $\{V_t\}_{0 \leq t \leq 1}$ and $\{u(\cdot, t)\}_{0 \leq t \leq 1}$ satisfy (A1)-(A4). Let ϕ_T , $\phi_{T, 1/2}$ and $\mathbf{c}$ be as in (5.2) and let r_1 be as in Lemma 7.3. Write $|M_t| := V_t$ for a.e. t and identify T with $\mathbb{R}^k \times \{0\}$. Suppose that we have*

$$(7.45) \quad \int_{C(T, 1) \times (0, 1)} |h(V_t, \cdot)|^2 \phi_T^2 d\|V_t\| dt \leq \varepsilon_3 r_1^2/4,$$

$$(7.46) \quad \|\|V_t\|(\phi_T^2) - \mathbf{c}\| \leq \varepsilon_3, \quad \forall t \in (0, 1),$$

$$(7.47) \quad \text{spt } \|V_t\| \cap C(T, 1) \subset \{|T^\perp(x)| \leq \varepsilon_3\}, \quad \forall t \in (0, 1),$$

$$(7.48) \quad \|u\|_{L^{p, q}(C(T, 1) \times (0, 1))} \leq 1.$$

Set

$$(7.49) \quad \beta^2 := \int_{G_k(C(T, 1) \times (0, 1))} \|S - T\|^2 \phi_T^2 dV_t(\cdot, S) dt$$

and

$$(7.50) \quad \kappa^2 := \left| \int_0^1 \|V_t\|(\phi_{T, 1/2}^2) dt - \frac{\mathbf{c}}{2^k} \right|.$$

Then there exist maps $f : B_{1/3}^k \times (1/4, 3/4) \rightarrow \mathbb{R}^{n-k}$ and $F : B_{1/3}^k \times (1/4, 3/4) \rightarrow \mathbb{R}^n \times (0, 1)$ such that for all $(x, s), (y, t) \in B_{1/3}^k \times (1/4, 3/4)$,

$$(7.51) \quad \begin{aligned} F(x, s) &= (x, f(x, s), s), \\ |f(x, s) - f(y, t)| &\leq c(n, k) \max\{|x - y|, |s - t|^{1/2}\}, \quad |f(x, s)| \leq \varepsilon_3. \end{aligned}$$

Define $X = \{\cup_{t \in (1/4, 3/4)} (M_t \cap C(T, 1/3)) \times \{t\}\} \cap \text{image } F$ and $Y = (T \times \text{Id}_{\mathbb{R}})(X)$. Then

$$(7.52) \quad \begin{aligned} &(\|V_t\| \times dt)(\{C(T, 1/3) \times (1/4, 3/4)\} \setminus X) \\ &+ \mathcal{L}^{k+1}(\{B_{1/3}^k \times (1/4, 3/4)\} \setminus Y) \leq \kappa^2 + c_{10} \beta^2. \end{aligned}$$

Remark 7.6. Due to (A1), note that V_t can be represented as $|M_t|$ for some $\mathcal{H}^k$ measurable countably k -rectifiable set M_t for a.e. t . We need to be careful that we possibly have $\mathcal{H}^k(\text{spt } \|V_t\| \setminus M_t) > 0$ in general. Once the partial regularity of $\text{spt } \|V_t\|$ is proved, on the other hand, we may identify M_t with $\text{spt } \|V_t\|$.

Proof. We use Proposition 7.1 with $l = 1$ and R, γ, σ chosen as follows. We choose and fix R sufficiently large depending only on n, k, E_1 so that (7.5) is satisfied and

$$(7.53) \quad 1 < \frac{\exp(c_7/\sqrt{R})}{(1 - \frac{1}{2R^2})^{k/2}} < 1 + \frac{1}{100}, \quad c_8 E_1 \exp(-\sqrt{R}) < \frac{1}{100}.$$

After fixing such R , we choose and fix $\gamma > 0$ sufficiently small depending only on n, k, E_1 so that

$$(7.54) \quad c_8 \gamma R^{3(k+2)/2} < \frac{1}{100}.$$

We choose and fix σ sufficiently small depending only on n, k, E_1, p, q so that

$$(7.55) \quad c_9 E_1^{1-\frac{2}{p}} R^{2\varsigma} (\sigma^{2\varsigma} + \sigma^\varsigma) < \frac{1}{100}, \quad c_8 E_1 R^2 \sigma^\varsigma < \frac{1}{100},$$

$$(7.56) \quad (R^{3/2} + \sqrt{2}/2)\sigma \leq r_1, \quad R^2 \sigma^2 \leq r_1^2/2.$$

We set

$$(7.57) \quad \varepsilon_3 = \min\{\varepsilon_2, \sigma/2\}.$$

We then define

$$(7.58) \quad A = \{(x, t) : x \in C(T, 1/2) \cap M_t, |T^\perp(x)| \leq \varepsilon_3, t \in (1/4, 3/4), \\ \sup_{0 < r < r_1} \frac{1}{r^{k+2}} \int_{P_r(x, t)} \|S - T\|^2 dV_s(\cdot, S) ds \leq \gamma\}.$$

We apply Lemma 7.1 and show that A must be contained in a Lipschitz graph. To do so, assume that we have distinct (y_1, s_1) and (y_2, s_2) in A with $|T^\perp(y_1 - y_2)| \geq \max\{|T(y_1 - y_2)|, |s_1 - s_2|^{1/2}\}$. Set $\bar{\sigma} = |T^\perp(y_1 - y_2)|$. By (7.57) and (7.58), $\bar{\sigma} \leq |T^\perp(y_1)| + |T^\perp(y_2)| \leq \sigma$. Since $R^{3/2}\bar{\sigma} \leq R^{3/2}\sigma < r_1$ by (7.56), we have (7.4) satisfied by (7.58) for $i = 1, 2$. We also have

$$(7.59) \quad P_{(R^{3/2} + \frac{\sqrt{2}}{2})\bar{\sigma}}(\bar{y}, \bar{s}) \subset C(T, 13/21) \times (0, 1)$$

due to (7.56) and restricting r_1 if necessary (depending on absolute constant). Thus we have (7.1)-(7.5) satisfied for (y_1, s_1) and (y_2, s_2) . Then we conclude that (7.6) holds. By (7.53)-(7.55) and (7.48) we have

$$(7.60) \quad 2 \leq (1 + \frac{1}{100}) \int_{B_{(R^{3/2} + \sqrt{2}/2)\bar{\sigma}}(\bar{y})} \rho_{(\bar{y}, \bar{s})}(\cdot, t) d\|V_t\| \Big|_{t=\bar{s}-R^2\bar{\sigma}^2} + \frac{4}{100}.$$

The inclusion (7.59) implies that we may replace ρ in (7.60) by $\hat{\rho}$ since $\eta = 1$ on $B_{13/21}$. Note that we have $R^2\bar{\sigma}^2 \leq R^2\sigma^2 \leq r_1^2/2$ by (7.56). Due to (7.45)-(7.48) and (7.57), conditions (7.31)-(7.34) are satisfied. Thus Lemma 7.3 gives

$$2 \leq (1 + \frac{1}{100}) \frac{26}{25} + \frac{4}{100}$$

which is a contradiction. Thus for any two distinct points (y_1, s_1) and (y_2, s_2) in A , we have

$$(7.61) \quad |T^\perp(y_1 - y_2)| \leq \max\{|T(y_1 - y_2)|, |s_1 - s_2|^{1/2}\}.$$

We next consider the projection of A on T . Define

$$(7.62) \quad A' = \{(x, s) \in B_{1/2}^k \times (1/4, 3/4) : x = T(y) \text{ for some } (y, s) \in A\}.$$

The inequality (7.61) shows that $(T^{-1}(x), s) \cap A$ consists of a single element $\{y\}$ for each $(x, s) \in A'$, thus we may define the maps $\tilde{f}(x, s) = T^\perp(y)$ and $\tilde{F}(x, s) = (x, \tilde{f}(x, s), s)$. We also have from (7.61) and (7.58) that

$$|\tilde{f}(x, s) - \tilde{f}(y, t)| \leq \max\{|x - y|, |s - t|^{1/2}\} \quad \text{for } (x, s), (y, t) \in A',$$

$$\sup_{(x, s) \in A'} |\tilde{f}(x, s)| \leq \varepsilon_3.$$

By the standard Lipschitz extension lemma applied with the natural parabolic metric, we may extend $\tilde{f}$ and $\tilde{F}$ to be defined on $T \times \mathbb{R}$. We denote them by f and F , respectively, and they satisfy

$$(7.63) \quad |f(x, s) - f(y, t)| \leq c(n, k) \max\{|x - y|, |s - t|^{1/2}\} \quad \text{for } (x, s), (y, t) \in T \times \mathbb{R},$$

$$(7.64) \quad \sup_{(x, s) \in T \times \mathbb{R}} |f(x, s)| \leq \varepsilon_3.$$

This proves the claim of (7.51) by restricting to $C(T, 1/3)$ from $C(T, 1/2)$. We next estimate the measures of

$$(7.65) \quad \begin{aligned} B &:= \cup_{t \in (1/4, 3/4)} \{(M_t \cap C(T, 1/2)) \times \{t\}\} \setminus A, \\ B' &:= (T \times \text{Id}_{\mathbb{R}})(B) \subset T \times \mathbb{R}, \\ B'' &:= B_{1/3}^k \times (1/4, 3/4) \setminus A'. \end{aligned}$$

B is the subset of $\{M_t\}_{0 \leq t \leq 1}$ which may not be covered by the image of F , B' is the projection to $T \times \mathbb{R}$, and B'' is the subset of $T \times \mathbb{R}$ such that $T^{-1}(x)$ may not intersect with M_t .

For each $(x, s) \in B$, there exists some $0 < r(x, s) < r_1$ such that

$$(7.66) \quad \int_{\overline{P_{r(x, s)}(x, s)}} \|S - T\|^2 dV_t(\cdot, S) dt \geq \gamma(r(x, s))^{k+2}$$

by the definition of A , (7.58). Thus $\{\overline{P_{r(x, s)}(x, s)}\}_{(x, s) \in B}$ is a covering of B . By [15, 2.8.14], which is a generalized version of the Besicovitch covering theorem, there exists a finite number of subfamilies $\mathcal{B}_1, \dots, \mathcal{B}_{\mathbf{B}(n)}$ each of which consists of mutually disjoint parabolic cylinders and $B \subset \cup_{i=1}^{\mathbf{B}(n)} \cup_{\mathcal{B}_i} \overline{P_{r(x, s)}(x, s)}$. Then using (3.1), (7.66) and the disjointness of $\mathcal{B}_i$ we have

$$(7.67) \quad \begin{aligned} (\|V_t\| \times dt)(B) &\leq \sum_{i=1}^{\mathbf{B}(n)} \sum_{\mathcal{B}_i} (\|V_t\| \times dt)(\overline{P_{r(x, s)}(x, s)}) \\ &\leq \sum_{i=1}^{\mathbf{B}(n)} \sum_{\mathcal{B}_i} 2E_1 \omega_k(r(x, s))^{k+2} \leq \sum_{i=1}^{\mathbf{B}(n)} \sum_{\mathcal{B}_i} 2E_1 \omega_k \gamma^{-1} \int_{\overline{P_{r(x, s)}(x, s)}} \|S - T\|^2 dV_t(\cdot, S) dt \\ &\leq 2\mathbf{B}(n) E_1 \omega_k \gamma^{-1} \int_{C(T, 13/24) \times (0, 1)} \|S - T\|^2 dV_t(\cdot, S) dt. \end{aligned}$$

Note that $C(T, 13/24) \subset \{\phi_T = 1\}$ due to (5.1) and (5.2).

For B' , note that $T(P_r(x, s)) \subset T \times \mathbb{R}$ is a $(k+1)$ -dimensional parabolic cylinder with $\mathcal{L}^{k+1}(T(P_r(x, s))) = 2\omega_k r^{k+2}$. Since $\bigcup_{i=1}^{\mathbf{B}(n)} \bigcup_{B_i} T(P_{r(x,s)}(x, s))$ covers B' , we have

$$(7.68) \quad \mathcal{L}^{k+1}(B') \leq \sum_{i=1}^{\mathbf{B}(n)} \sum_{B_i} \mathcal{L}^{k+1}(T(P_{r(x,s)}(x, s))) \leq \sum_{i=1}^{\mathbf{B}(n)} \sum_{B_i} 2\omega_k (r(x, s))^{k+2}$$

and the rest is estimated as in (7.67), but with the estimate not involving E_1 .

We next estimate B'' . Since f is Lipschitz in the space variables, it is differentiable a.e. and the area formula shows (write $C_* := C(T, 1/2) \times (1/4, 3/4)$ here for simplicity)

$$(7.69) \quad \begin{aligned} \int_{C_*} \phi_{T,1/2}^2 d\|V_t\| dt &= \int_A \phi_{T,1/2}^2 d\|V_t\| dt + \int_B \phi_{T,1/2}^2 d\|V_t\| dt \\ &= \int_{A'} \phi_{T,1/2}^2(x) J_{\nabla(x, f(x, t))} d\mathcal{H}^k(x) dt + \int_B \phi_{T,1/2}^2 d\|V_t\| dt \\ &= \int_{A'} \phi_{T,1/2}^2 (J_{\nabla(x, f(x, t))} - 1) d\mathcal{H}^k dt + \int_{C_* \cap T} \phi_{T,1/2}^2 d\mathcal{H}^k dt \\ &\quad - \int_{(C_* \cap T) \setminus A'} \phi_{T,1/2}^2 d\mathcal{H}^k dt + \int_B \phi_{T,1/2}^2 d\|V_t\| dt. \end{aligned}$$

Since $\phi_{T,1/2} = 1$ on $C(T, 1/3) \times (1/4, 3/4)$ and $B'' \subset (C_* \cap T) \setminus A'$, (7.69) shows

$$(7.70) \quad \begin{aligned} \mathcal{L}^{k+1}(B'') &\leq \int_{(C_* \cap T) \setminus A'} \phi_{T,1/2}^2 d\mathcal{H}^k dt \leq \left| \int_{C_*} \phi_{T,1/2}^2 d\|V_t\| dt - \int_{C_* \cap T} \phi_{T,1/2}^2 d\mathcal{H}^k dt \right| \\ &\quad + \int_{A'} \phi_{T,1/2}^2 (J_{\nabla(x, f(x, s))} - 1) d\mathcal{H}^k dt + \int_B \phi_{T,1/2}^2 d\|V_t\| dt. \end{aligned}$$

Note that for f with its Lipschitz constant bounded by $c(n, k)$, we have $J_{\nabla(x, f(x, s))} - 1 \leq c(n, k) \|(\text{image } \nabla(x, f(x, s)) - T)\|^2$, where $\text{image } \nabla(x, f(x, s))$ coincides with the approximate tangent space of M_t a.e. on A' . By (7.50), (7.67) and (7.70), we have

$$(7.71) \quad \mathcal{L}^{k+1}(B'') \leq 2\mathbf{B}(n)\omega_k E_1 \gamma^{-1} \beta^2 + c(n, k) \beta^2 + \kappa^2.$$

Since $\bigcup_{t \in (1/4, 3/4)} (M_t \times \{t\}) \cap \{(C(T, 1/3) \times (1/4, 3/4)) \setminus X \subset B$ and $\{B_{1/3}^k \times (1/4, 3/4)\} \setminus Y \subset (B' \cup B'')$, we proved (7.52) by (7.67), (7.68) and (7.71). $\square$

8. HÖLDER ESTIMATE FOR THE GRADIENT

In this section we prove that if the L^2 -height of varifolds is small, then near the center of the domain, the properly scaled L^2 -height with respect to a slightly tilted plane shows at least a fixed amount of decay. By iteration, this proves that the spatial gradient is Hölder continuous. The method of proof, the blow-up technique, or more technically called as the tilt-excess decay estimate, goes back to De Giorgi [11] and (see also [2]) and it has been used for wide-ranging nonlinear elliptic and parabolic problems. Since the result of the present paper is a parabolic analogue to Allard's regularity theory [1] (see Section 10.2), the proof naturally shares the similarity. On the other hand, there are a few subtle and interesting differences from elliptic case. The first point is that L^∞ estimate is rather essential to show that the blow-up limit satisfies the heat equation. Note that the test function that we can use in the weak inequality (3.4) has to be non-negative. Thus, we need to know the height of varifolds in L^∞ norm, instead of, say, L^2 norm. The second point is that we need to capitalize on the monotone decreasing property (with respect to time variable) of L^2 norm of blow-up sequence to show the strong convergence of space-time L^2 norm. Recall that in

elliptic case, Rellich's compactness theorem shows the strong L^2 convergence immediately. Here, since we do not have any control of time-derivatives of blow-up sequence, we need to use some special feature of parabolic problem.

Proposition 8.1. *Corresponding to $1 \leq E_1 < \infty$, $0 < \nu < 1$, p, q as in (3.2) there exist $0 < \varepsilon_4 < 1$, $2 < \Lambda_* < \infty$, $0 < \theta_* < 1/4$, $1 < c_{11} < \infty$ with the following property. For $W \in \mathbf{G}(n, k)$, $0 < R < \infty$, $U = C(W, 2R)$ and $(0, \Lambda)$ replaced by $(-\Lambda_* R^2, \Lambda_* R^2)$, suppose $\{V_t\}_{-\Lambda_* R^2 \leq t \leq \Lambda_* R^2}$ and $\{u(\cdot, t)\}_{-\Lambda_* R^2 \leq t \leq \Lambda_* R^2}$ satisfy (A1)-(A4). For $W \in \mathbf{G}(n, k)$ let $\phi_{W,R}$ and $\mathbf{c}$ be as in (5.2) and (5.3). Suppose*

$$(8.1) \quad T \in \mathbf{G}(n, k) \text{ satisfies } \|T - W\| < \varepsilon_4,$$

$$(8.2) \quad A \in \mathbf{A}(n, k) \text{ is parallel to } T,$$

$$(8.3) \quad \mu := \left(R^{-(k+4)} \int_{-\Lambda_* R^2}^{\Lambda_* R^2} \int_{C(W, 2R)} \text{dist}(x, A)^2 d\|V_t\| dt \right)^{\frac{1}{2}} < \varepsilon_4,$$

$$(8.4) \quad \|u\| := \|u\|_{L^{p,q}(C(W, 2R) \times (-\Lambda_* R^2, \Lambda_* R^2))},$$

$$(8.5) \quad (-\Lambda_* + 1)R^2 \leq \exists t_1 \leq (-\Lambda_* + 2)R^2 : R^{-k} \|V_{t_1}\|(\phi_{W,R}^2) < (2 - \nu)\mathbf{c},$$

$$(8.6) \quad (\Lambda_* - 2)R^2 \leq \exists t_2 \leq (\Lambda_* - 1)R^2 : R^{-k} \|V_{t_2}\|(\phi_{W,R}^2) > \nu\mathbf{c}.$$

Then there are $\tilde{T} \in \mathbf{G}(n, k)$ and $\tilde{A} \in \mathbf{A}(n, k)$ such that

$$(8.7) \quad \tilde{A} \text{ is parallel to } \tilde{T},$$

$$(8.8) \quad \|T - \tilde{T}\| \leq c_{11}\mu,$$

$$(8.9) \quad \left((\theta_* R)^{-(k+4)} \int_{-\theta_*^2 \Lambda_* R^2}^{\theta_*^2 \Lambda_* R^2} \int_{C(W, 2\theta_* R)} \text{dist}(x, \tilde{A})^2 d\|V_t\| dt \right)^{\frac{1}{2}} \leq \theta_*^\varsigma \max\{\mu, R^\varsigma c_{11} \|u\|\}.$$

Moreover, if $R^\varsigma \|u\| < \varepsilon_4$, we have

$$(8.10) \quad (-\Lambda_* + 1)(\theta_* R)^2 \leq \exists \tilde{t}_1 \leq (-\Lambda_* + 2)(\theta_* R)^2 : (\theta_* R)^{-k} \|V_{\tilde{t}_1}\|(\phi_{W, \theta_* R}^2) < (2 - \nu)\mathbf{c},$$

$$(8.11) \quad (\Lambda_* - 2)(\theta_* R)^2 \leq \exists \tilde{t}_2 \leq (\Lambda_* - 1)(\theta_* R)^2 : (\theta_* R)^{-k} \|V_{\tilde{t}_2}\|(\phi_{W, \theta_* R}^2) > \nu\mathbf{c}.$$

Proof. We may assume that $R = 1$ after a suitable change of variables. We prove (8.7)-(8.9) by contradiction. We will show that (8.10) and (8.11) follow from these at the end. By Theorem 5.7, corresponding to E_1 and ν there replaced by $\nu/2$, we fix $\Lambda, K, \varepsilon_1$ first. We set $\Lambda_* := \Lambda + 5/2$. We will fix $0 < \theta_* < 1/4$ later depending only on E_1, ν, p and q . If the claim were false, then for each $m \in \mathbb{N}$ there exist $\{V_t^{(m)}\}_{-\Lambda_* \leq t \leq \Lambda_*}$, $\{u^{(m)}(\cdot, t)\}_{-\Lambda_* \leq t \leq \Lambda_*}$ satisfying (A1)-(A4) on $C(W^{(m)}, 2) \times [-\Lambda_*, \Lambda_*]$ for $W^{(m)} \in \mathbf{G}(n, k)$ such that, after suitable translations and rotations, we have $T = \mathbb{R}^k \times \{0\}$ and

$$(8.12) \quad \|T - W^{(m)}\| \leq \frac{1}{m},$$

$$(8.13) \quad \mu^{(m)} := \left(\int_{-\Lambda_*}^{\Lambda_*} \int_{C(W^{(m)}, 2)} |T^\perp(x)|^2 d\|V_t^{(m)}\| dt \right)^{\frac{1}{2}} \leq \frac{1}{m},$$

and (8.5) and (8.6) satisfied for $W = W^{(m)}$, $t_1 = t_1^{(m)}$ and $t_2 = t_2^{(m)}$, respectively, but for any $\tilde{T} \in \mathbf{G}(n, k)$ with $\|T - \tilde{T}\| \leq m\mu^{(m)}$ and $\tilde{A} \in \mathbf{A}(n, k)$ which is parallel to $\tilde{T}$,

$$(8.14) \quad \left(\theta_*^{-(k+4)} \int_{-\theta_*^2 \Lambda_*}^{\theta_*^2 \Lambda_*} \int_{C(W^{(m)}, 2\theta_*)} \text{dist}(x, \tilde{A})^2 d\|V_t^{(m)}\| dt \right)^{\frac{1}{2}} > \theta_*^\varsigma \max\{\mu^{(m)}, m\|u^{(m)}\|\}.$$

By taking $\tilde{A} = \tilde{T} = T$ in (8.14), we obtain

$$(8.15) \quad \theta_*^\varsigma m\|u^{(m)}\| < \theta_*^{-(k+4)/2} \mu^{(m)}.$$

Thus (8.15) shows that

$$(8.16) \quad \lim_{m \rightarrow \infty} (\mu^{(m)})^{-1} \|u^{(m)}\| = 0.$$

Lemma 8.2. *For any given $0 < \gamma < 1$, there exists some m_0 such that*

$$(8.17) \quad \text{spt } \|V_t^{(m)}\| \cap \{|T^\perp(x)| > \gamma\} \cap C(W^{(m)}, 3/2) = \emptyset, \quad \forall t \in [-\Lambda_* + 1, \Lambda_*]$$

for all $m \geq m_0$.

Proof of Lemma 8.2. Suppose that (8.17) did not hold for some $\tilde{t}$ such that $V_{\tilde{t}}^{(m)}$ is of unit density. Then there exists some $\tilde{x} \in \{|T^\perp(x)| > \gamma\} \cap C(W^{(m)}, 3/2)$ with $\Theta^k(\|V_{\tilde{t}}^{(m)}\|, \tilde{x}) = 1$ such that the approximate tangent space to $V_{\tilde{t}}^{(m)}$ at $\tilde{x}$ exists. Then by Proposition 6.2 with $U = B_{\gamma/2}(\tilde{x})$, $R = \gamma/2$, $t_2 = \tilde{t}$, $t_1 = \tilde{t} - t$ with $0 < t < 1$, $y = \tilde{x}$ and $s = \tilde{t} + \epsilon$, we have

$$(8.18) \quad \int_{B_{\gamma/2}(\tilde{x})} \hat{\rho}_{(\tilde{x}, \tilde{t}+\epsilon)}(\cdot, \tilde{t}) d\|V_{\tilde{t}}^{(m)}\| \leq \int_{B_{\gamma/2}(\tilde{x})} \hat{\rho}_{(\tilde{x}, \tilde{t}+\epsilon)}(\cdot, \tilde{t} - t) d\|V_{\tilde{t}-t}^{(m)}\| + c_2 \|u^{(m)}\|^2 E_1^{1-\frac{2}{p}} t^\varsigma + 4\gamma^{-2} c_1 \omega_k E_1 t.$$

Set $\epsilon \rightarrow 0+$ and fix $0 < t' < 1$ such that the sum of the second and third term of the right-hand side of (8.18) is less than $1/2$ when $t \leq t'$. We then obtain

$$(8.19) \quad \frac{1}{2} \leq \int_{B_{\gamma/2}(\tilde{x})} \hat{\rho}_{(\tilde{x}, \tilde{t})}(\cdot, \tilde{t} - t) d\|V_{\tilde{t}-t}^{(m)}\|$$

for all $t \in [t'/2, t']$. Since $\hat{\rho}_{(\tilde{x}, \tilde{t})}(y, \tilde{t} - t)$ takes maximum value $(2\pi t')^{-k/2}$ when $y = \tilde{x}$ and $t = t'/2$ in this interval, from (8.19) we obtain

$$(8.20) \quad \frac{1}{2} \leq (2\pi t')^{-k/2} \|V_{\tilde{t}-t}^{(m)}\|(B_{\gamma/2}(\tilde{x}))$$

for all $t \in [t'/2, t']$. Since $B_{\gamma/2}(\tilde{x}) \subset \{|T^\perp(x)| > \gamma/2\} \cap C(W^{(m)}, 2)$, by (8.20), we have

$$(8.21) \quad (\mu^{(m)})^2 \geq \int_{\tilde{t}-t'}^{\tilde{t}-t'/2} \int_{B_{\gamma/2}(\tilde{x})} |T^\perp(x)|^2 d\|V_t^{(m)}\| dt \geq \frac{t'}{2} \left(\frac{\gamma}{2}\right)^2 \frac{(2\pi t')^{k/2}}{2}.$$

Since $\mu^{(m)} \leq 1/m$, (8.21) cannot be true for sufficiently large m . This shows (8.17) holds true for a.e. $t \in [-\Lambda_* + 1, \Lambda_*]$. By using (3.4), we conclude that (8.17) is satisfied for all t . This ends the proof of Lemma 8.2. $\square$

Next we use Proposition 6.4 for $U = B_2$ (or $R = 2$ there) and $(0, \Lambda)$ there replaced by $(-\Lambda_*, \Lambda_*)$, to obtain

$$(8.22) \quad \text{spt } \|V_t^{(m)}\| \cap B_{8/5} \subset \{|T^\perp(x)| \leq 2\tilde{\mu}^{(m)}\}, \quad \forall t \in (-\Lambda_* + 1, \Lambda_*)$$

for

$$(8.23) \quad \tilde{\mu}^{(m)} := \left(\frac{c_5}{2^{k+4}} \int_{-\Lambda_*}^{\Lambda_*} \int_{B_2} |T^\perp(x)|^2 d\|V_t^{(m)}\| dt + c_2 \|u^{(m)}\|^2 E_1^{1-\frac{2}{p}} (2\Lambda_*)^\varsigma (2 + \Lambda_*/2) \right)^{\frac{1}{2}} \\ \leq c_{12}(n, k, E_1) \mu^{(m)}$$

for all sufficiently large m by (8.16) for a suitable c_{12} . In the following, for $t \in (-\Lambda_* + 1, \Lambda_*)$, we set $\|V_t^{(m)}\| = 0$ on $C(T, 1) \cap \{|T^\perp(x)| \geq 1\}$ for all sufficiently large m . Due to Lemma 8.2 and (8.22), this modified $V_t^{(m)}$ on $C(T, 1)$ still satisfies (A1)-(A4). We do this modification for notational simplicity. Otherwise, one can restrict the domain of integration from $C(T, 1)$ to $C(T, 1) \cap \{|T^\perp(x)| < 1\}$ in the following computations. With this convention, we have

$$(8.24) \quad \mu_*^{(m)} := \sup_{-\Lambda_*+1 \leq t \leq \Lambda_*} \left(\int_{C(T,1)} |T^\perp(x)|^2 d\|V_t^{(m)}\| \right)^{\frac{1}{2}} \leq (E_1 \omega_k (8/5)^k)^{1/2} 2\tilde{\mu}^{(m)} \\ \leq c_{13}(E_1, n, k) \mu^{(m)}$$

for a suitable c_{13} by (8.22) and (8.23). We next check that the assumptions for Theorem 5.7 are all satisfied on $(-\Lambda_* + 1, \Lambda_* - 1)$. By the definition of Λ_* , we have $(\Lambda_* - 1) - (-\Lambda_* + 1) = 2\Lambda_* - 2 = 2\Lambda + 3$. (5.42) and (5.43) are satisfied for all sufficiently large m due to (8.5), (8.6) and (8.12). (5.40) follows from (8.22). Thus for all sufficiently large m depending on ε_1 so that (5.41) and (5.44) are satisfied, we conclude from (5.45) and (5.46) as well as Corollary 5.9 that

$$(8.25) \quad \sup_{-1/2 \leq t \leq 1/2} \|\|V_t^{(m)}\|(\phi_T^2) - \mathbf{c}\| \leq K((\mu_*^{(m)})^2 + C(u^{(m)})),$$

$$(8.26) \quad \int_{-1/2}^{1/2} \int_{C(T,1)} |h(V_t^{(m)}, \cdot)|^2 \phi_T^2 d\|V_t^{(m)}\| dt \leq 12K((\mu_*^{(m)})^2 + C(u^{(m)})),$$

$$(8.27) \quad \int_{-1/2}^{1/2} |2^k \|V_t^{(m)}\|(\phi_{T,1/2}^2) - \mathbf{c}| dt \leq \tilde{K}((\mu_*^{(m)})^2 + C(u^{(m)})).$$

The right-hand sides of (8.25)-(8.27) are all bounded by $c_{14}(E_1, \nu, n, k)(\mu^{(m)})^2$ due to (8.24), the definition of $C(u^{(m)})$ and (8.16) for all sufficiently large m . By Proposition 11.2, (8.24) and (8.26), we also have

$$(8.28) \quad \int_{-1/2}^{1/2} \int_{G_k(C(T,1))} \|S - T\|^2 \phi_T^2 dV_t^{(m)}(\cdot, S) dt \leq c_{14}(\mu^{(m)})^2.$$

We next use Theorem 7.5. Let $r_1 = r_1(E_1, p, q)$ be as in Lemma 7.3 and $\varepsilon_3 = \varepsilon_3(E_1, p, q)$ be as in Theorem 7.5. For all sufficiently large m depending on r_1 and ε_3 , we may satisfy the assumptions (7.45)-(7.48). Since β^2 and κ^2 in Theorem 7.5 are bounded by $c_{14}(\mu^{(m)})^2$, we have a corresponding Lipschitz functions $f^{(m)}$ and $F^{(m)}$ defined on $B_{1/3}^k \times (-1/4, 1/4)$ with the corresponding estimates (7.51) and (7.52). We also define $X^{(m)}$ and $Y^{(m)}$ to be the sets X and Y respectively corresponding to $V_t^{(m)}$. We now set

$$(8.29) \quad \tilde{f}^{(m)} := f^{(m)} / \mu^{(m)}.$$

In the following we write $\Omega := C(T, 1/3) \times (-1/4, 1/4)$ and $\Omega' := B_{1/3}^k \times (-1/4, 1/4)$.

Lemma 8.3. *There exists constant c_{15} which depends only on E_1, ν, p, q such that for all sufficiently large m ,*

$$(8.30) \quad \text{spt } \|V_t^{(m)}\| \cap C(T, 1/3) \subset \{|T^\perp(x)| \leq c_{15}\mu^{(m)}\}, \quad \forall t \in (-1/4, 1/4),$$

$$(8.31) \quad \sup_{\Omega'} |\tilde{f}^{(m)}| + \int_{\Omega'} |\nabla \tilde{f}^{(m)}|^2 d\mathcal{H}^{k+1} \leq c_{15},$$

where ∇ denotes the spatial gradient.

Proof of Lemma 8.3. The estimate (8.30) and the supremum estimate of (8.31) follow immediately from (8.22)-(8.24), (7.51) and (8.29). We may proceed just like [1, p.483] for the second term. We split the domain of integration into $Y^{(m)}$ and the complement, and using that the spatial Lipschitz constants of $F^{(m)}$ are $\leq c(n, k)$,

$$(8.32) \quad \begin{aligned} & \int_{Y^{(m)}} |\nabla f^{(m)}|^2 d\mathcal{H}^{k+1} + \int_{\Omega' \setminus Y^{(m)}} |\nabla f^{(m)}|^2 d\mathcal{H}^{k+1} \\ & \leq c(n, k) \int_{Y^{(m)}} |\nabla F^{(m)}|^2 \|\text{image } \nabla F^{(m)} - T\|^2 J_{\nabla F^{(m)}} d\mathcal{H}^{k+1} + c(n, k) \mathcal{L}^{k+1}(\Omega' \setminus Y^{(m)}) \\ & \leq c(n, k) \int_{G_k(C(T, 1/3)) \times (-1/4, 1/4)} \|S - T\|^2 dV_t^{(m)}(\cdot, S) dt + c(n, k) \mathcal{L}^{k+1}(\Omega' \setminus Y^{(m)}) \\ & \leq c(n, k) c_{14}(\mu^{(m)})^2 \end{aligned}$$

where we used (8.28) and (7.52) at the end of (8.32). This shows (8.31). $\square$

Lemma 8.4. *There exist a subsequence $\{\tilde{f}^{(m_j)}\}_{j=1}^\infty$ and $\tilde{f} \in C^\infty(\Omega')$ such that $\lim_{j \rightarrow \infty} \|\tilde{f}^{(m_j)} - \tilde{f}\|_{L^2(\Omega')} = 0$ and $\tilde{f}$ satisfies $\frac{\partial \tilde{f}}{\partial t} - \Delta \tilde{f} = 0$ on Ω' .*

Proof of Lemma 8.4. Let $l \in \{1, \dots, n-k\}$ be fixed. We show that l -th components $\tilde{f}_l^{(m)}$, $\tilde{f}_l$ of $\tilde{f}^{(m)}$, $\tilde{f} \in \mathbb{R}^{n-k}$ respectively satisfy the desired properties. Due to (8.31) we can extract a subsequence (denoted by the same index) so that

$$(8.33) \quad \tilde{f}_l^{(m)} \rightharpoonup \tilde{f}_l \text{ weakly in } L^2(\Omega'), \quad \nabla \tilde{f}_l^{(m)} \rightharpoonup \nabla \tilde{f}_l \text{ weakly in } L^2(\Omega').$$

Due to (8.28) and (7.52), we may further assume that for a.e. $s \in (-1/4, 1/4)$

$$(8.34) \quad \lim_{m \rightarrow \infty} \int_{C(T, 1/3)} \|S - T\|^2 dV_s^{(m)}(\cdot, S) = 0,$$

$$(8.35) \quad \lim_{m \rightarrow \infty} \|V_s^{(m)}\|(C(T, 1/3) \setminus X^{(m)})_{|t=s} = 0, \quad \lim_{m \rightarrow \infty} \mathcal{L}^k(B_{1/3}^k \setminus Y^{(m)})_{|t=s} = 0.$$

Again by (8.31), the Rellich compactness theorem and diagonal argument, we may assume that

$$(8.36) \quad \{\tilde{f}_l^{(m)}(\cdot, s_j)\}_{m \in \mathbb{N}} \text{ is a Cauchy sequence in } L^2(B_{1/3}^k)$$

for a countable dense set $\{s_i\}_{i=1}^\infty$ in $(-1/4, 1/4)$. We may choose these s_i satisfying (8.34) and (8.35) as well. We fix such a subsequence in the following. For arbitrary non-negative $\tilde{\phi} \in C_c^\infty(\Omega')$ we set $\phi^{(m)}(x, t) := (x_l + c_{15}\mu^{(m)})\tilde{\phi}(T(x), t)$ for $x \in \mathbb{R}^n$ and $t \in (-1/4, 1/4)$. By definition $\phi^{(m)}$ is non-negative on $\{x : |x_l| \leq c_{15}\mu^{(m)}\}$. Due to (8.30), we may use $\phi^{(m)}$ as a test function in (3.4) by a slight modification away from T . In the following similar computations using (3.4) we implicitly assume that we do this modification of test functions which does not affect the computations. Now we have

$$(8.37) \quad 0 \leq \int_{\Omega} (-h(V_t^{(m)}, \cdot)\phi^{(m)} + \nabla \phi^{(m)} \cdot (h(V_t^{(m)}, \cdot) + (u^{(m)})^\perp) + \frac{\partial \phi^{(m)}}{\partial t} d\|V_t^{(m)}\|) dt.$$

By the Cauchy-Schwarz inequality and by dropping a negative term, we obtain from (8.37)

$$(8.38) \quad 0 \leq \int_{\Omega} |u^{(m)}|^2 \phi^{(m)} + |u^{(m)}| |\nabla \phi^{(m)}| + \frac{\partial \phi^{(m)}}{\partial t} + (x_l + c_{15} \mu^{(m)}) \nabla \tilde{\phi} \cdot h(V_t^{(m)}, \cdot) \\ + \tilde{\phi} h(V_t^{(m)}, \cdot)_l d\|V_t^{(m)}\| dt =: I_1^{(m)} + I_2^{(m)} + I_3^{(m)} + I_4^{(m)} + I_5^{(m)}$$

where $h(V_t^{(m)}, x)_l$ is the $(k+l)$ -th component of $h(V_t^{(m)}, x) \in \mathbb{R}^n$. In the following we estimate $\lim_{m \rightarrow \infty} (\mu^{(m)})^{-1} I_j^{(m)}$ for each $j = 1, \dots, 5$.

Estimate of $I_1^{(m)}$.

Since $|\phi^{(m)}| \leq 2c_{15} \mu^{(m)} \sup |\tilde{\phi}|$, by the Hölder inequality and (8.16),

$$(8.39) \quad \lim_{m \rightarrow \infty} |(\mu^{(m)})^{-1} I_1^{(m)}| \leq c_{15} c(\tilde{\phi}, p, q) \lim_{m \rightarrow \infty} \|u^{(m)}\|^2 = 0.$$

Estimate of $I_2^{(m)}$.

By the Hölder inequality and (8.16),

$$(8.40) \quad \lim_{m \rightarrow \infty} |(\mu^{(m)})^{-1} I_2^{(m)}| \leq \lim_{m \rightarrow \infty} c(p, q) \|u^{(m)}\| (\sup |\tilde{\phi}| (\mu^{(m)})^{-1} + c_{15} \sup |\nabla \tilde{\phi}|) \\ \leq \lim_{m \rightarrow \infty} o(1) c(\tilde{\phi}, p, q) (1 + c_{15} \mu^{(m)}) = 0.$$

Estimate of $I_3^{(m)}$.

We estimate as

$$(8.41) \quad \int_{\Omega} \frac{\partial \phi^{(m)}}{\partial t} d\|V_t^{(m)}\| dt \\ = \int_{\Omega \setminus X^{(m)}} \frac{\partial \phi^{(m)}}{\partial t} d\|V_t^{(m)}\| dt + \int_{Y^{(m)}} (f_l^{(m)} + c_{15} \mu^{(m)}) \frac{\partial \tilde{\phi}}{\partial t} (J_{\nabla F^{(m)}} - 1) d\mathcal{H}^{k+1} \\ - \int_{\Omega' \setminus Y^{(m)}} (f_l^{(m)} + c_{15} \mu^{(m)}) \frac{\partial \tilde{\phi}}{\partial t} d\mathcal{H}^{k+1} + \int_{\Omega'} (f_l^{(m)} + c_{15} \mu^{(m)}) \frac{\partial \tilde{\phi}}{\partial t} d\mathcal{H}^{k+1}.$$

Using (8.31), (7.52) and (8.28), we can show that the first three terms of (8.41) are of order $(\mu^{(m)})^3$. Thus, using the weak convergence (8.33), we obtain

$$(8.42) \quad \lim_{m \rightarrow \infty} (\mu^{(m)})^{-1} I_3^{(m)} = \int_{\Omega'} (\tilde{f}_l + c_{15}) \frac{\partial \tilde{\phi}}{\partial t} d\mathcal{H}^{k+1}.$$

Estimate of $I_4^{(m)}$.

By (8.31) and (8.26), we have

$$|I_4^{(m)}| \leq 2c_{15} \mu^{(m)} \left(\int_{\Omega} |\nabla \tilde{\phi}|^2 d\|V_t^{(m)}\| dt \right)^{1/2} \left(\int_{\Omega} |h(V_t^{(m)}, \cdot)|^2 d\|V_t^{(m)}\| dt \right)^{1/2} \\ \leq 2c_{15} c(E_1, \tilde{\phi}) c_{14}^{1/2} (\mu^{(m)})^2$$

so that

$$(8.43) \quad \lim_{m \rightarrow \infty} |(\mu^{(m)})^{-1} I_4^{(m)}| = 0.$$

Estimate of $I_5^{(m)}$.

Let $\mathbf{e}_l$ be the unit vector with 1 in the $l + k$ -th component. We may write

$$\begin{aligned}
 & \int_{\Omega} \tilde{\phi} h(V_t^{(m)}, \cdot)_l d\|V_t^{(m)}\| dt = - \int_{\Omega} (\nabla \tilde{\phi} \otimes \mathbf{e}_l) \cdot S dV_t^{(m)} dt \\
 & = - \int_{\Omega \setminus X^{(m)}} (\nabla \tilde{\phi} \otimes \mathbf{e}_l) \cdot S dV_t^{(m)} dt \\
 (8.44) \quad & - \int_{Y^{(m)}} \text{image } \nabla F^{(m)} \cdot (\nabla \tilde{\phi} \otimes \mathbf{e}_l) J_{\nabla F^{(m)}} - \nabla f_l^{(m)} \cdot \nabla \tilde{\phi} d\mathcal{L}^{k+1} \\
 & + \int_{\Omega' \setminus Y^{(m)}} \nabla f_l^{(m)} \cdot \nabla \tilde{\phi} d\mathcal{L}^{k+1} - \int_{\Omega'} \nabla f_l^{(m)} \cdot \nabla \tilde{\phi} d\mathcal{L}^{k+1} := I_{5,1}^{(m)} + I_{5,2}^{(m)} + I_{5,3}^{(m)} + I_{5,4}^{(m)}.
 \end{aligned}$$

By (7.52) $|I_{5,1}^{(m)}|$ is $O((\mu^{(m)})^2)$ and by (8.31) and (7.52), the same for $|I_{5,3}^{(m)}|$. By [1, 8.14], (7.51) and (8.31),

$$(8.45) \quad |I_{5,2}^{(m)}| \leq c(n, k, |\nabla \tilde{\phi}|) \int_{Y^{(m)}} |\nabla f_l^{(m)}|^2 d\mathcal{L}^{k+1} \leq O((\mu^{(m)})^2).$$

By (8.33), it follows from above estimates that we have

$$(8.46) \quad \lim_{m \rightarrow \infty} (\mu^{(m)})^{-1} I_5^{(m)} = \lim_{m \rightarrow \infty} (\mu^{(m)})^{-1} I_{5,4}^{(m)} = - \int_{\Omega'} \nabla \tilde{f}_l \cdot \nabla \tilde{\phi} d\mathcal{H}^{k+1}.$$

Summary.

Due to (8.38)-(8.46), we obtain

$$(8.47) \quad 0 \leq \int_{\Omega'} (\tilde{f}_l + c_{15}) \frac{\partial \tilde{\phi}}{\partial t} - \nabla \tilde{f}_l \cdot \nabla \tilde{\phi} d\mathcal{H}^{k+1}.$$

We may carry out the similar argument with $\phi = (c_{15}\mu^{(m)} - x_l)\tilde{\phi}$, and obtain

$$(8.48) \quad 0 \leq \int_{\Omega'} (c_{15} - \tilde{f}_l) \frac{\partial \tilde{\phi}}{\partial t} + \nabla \tilde{f}_l \cdot \nabla \tilde{\phi} d\mathcal{H}^{k+1}.$$

Combining (8.47) and (8.48) and since $\int_{\Omega'} \frac{\partial \tilde{\phi}}{\partial t} d\mathcal{H}^{k+1} = 0$, we conclude that

$$(8.49) \quad 0 = \int_{\Omega'} \tilde{f}_l \frac{\partial \tilde{\phi}}{\partial t} - \nabla \tilde{f}_l \cdot \nabla \tilde{\phi} d\mathcal{H}^{k+1}$$

for all non-negative $\tilde{\phi} \in C_c^\infty(\Omega')$. This also shows that (8.49) is satisfied for all (not necessarily non-negative) $\tilde{\phi} \in C_c^\infty(\Omega')$, and $\tilde{f}_l$ satisfies the heat equation in a weak sense. The standard regularity theory shows that $\tilde{f}_l$ is $C^\infty(\Omega')$ satisfying the heat equation in the classical sense.

Next we show that for a.e. $s \in (-1/4, 1/4)$, the weak limit is unique and

$$(8.50) \quad \tilde{f}_l^{(m)}(\cdot, s) \rightharpoonup \tilde{f}_l(\cdot, s) \text{ weakly in } L^2(B_{1/3}^k).$$

Take any s with the properties (8.34) and (8.35). Since $|\tilde{f}_l^{(m)}(\cdot, s)| \leq c_{15}$, it is bounded in particular in $L^2(B_{1/3}^k)$. Let $g \in L^2(B_{1/3}^k)$ be any weak limit of a subsequence $\{\tilde{f}_l^{(m_j)}(\cdot, s)\}_{j=1}^\infty$. Similar calculations as in (8.41) using (8.34) and (8.35) show that for any $\tilde{\phi} \in C^\infty(B_{1/3}^k)$

$$(8.51) \quad \lim_{j \rightarrow \infty} \int_{C(T, 1/3)} (\mu^{(m_j)})^{-1} (x_l + c_{15}\mu^{(m_j)}) \tilde{\phi} d\|V_s^{m_j}\| = \int_{B_{1/3}^k} (g + c_{15}) \tilde{\phi} d\mathcal{H}^k.$$

We now proceed just as the first part of the proof. The differences this time are that the domain of integration is changed to $C(T, 1/3) \times (-1/4, s)$ and that we take the subsequence $\{\tilde{f}_l^{(m_j)}\}_{j=1}^\infty$. Then using (8.51), we obtain for any non-negative $\tilde{\phi} \in C_c^\infty(\Omega')$

$$(8.52) \quad \int_{B_{1/3}^k} (g + c_{15}) \tilde{\phi}(\cdot, s) d\mathcal{H}^k \leq \int_{\Omega' \cap \{t \leq s\}} (\tilde{f}_l + c_{15}) \frac{\partial \tilde{\phi}}{\partial t} - \nabla \tilde{f}_l \cdot \nabla \tilde{\phi} d\mathcal{H}^{k+1} = \int_{\Omega' \cap \{t=s\}} (\tilde{f}_l + c_{15}) \tilde{\phi} d\mathcal{H}^k,$$

the last equality follows from $\tilde{f}_l$ being the classical solution for the heat equation. Similarly, we have

$$(8.53) \quad \int_{B_{1/3}^k} (c_{15} - g) \tilde{\phi}(\cdot, s) d\mathcal{H}^k \leq \int_{\Omega' \cap \{t \leq s\}} (c_{15} - \tilde{f}_l) \frac{\partial \tilde{\phi}}{\partial t} + \nabla \tilde{f}_l \cdot \nabla \tilde{\phi} d\mathcal{H}^{k+1} = \int_{\Omega' \cap \{t=s\}} (c_{15} - \tilde{f}_l) \tilde{\phi} d\mathcal{H}^k.$$

Thus (8.52) and (8.53) show $\int_{B_{1/3}^k} (g - \tilde{f}_l(\cdot, s)) \tilde{\phi}(\cdot, s) d\mathcal{H}^k = 0$ for all non-negative function $\tilde{\phi} \in C_c^\infty(\Omega')$, which shows that $g(\cdot) = \tilde{f}_l(\cdot, s)$ a.e. on $B_{1/3}^k$. Since the limit is determined independent of the choice of subsequence, the whole sequence $\{\tilde{f}_l^{(m)}(\cdot, s)\}_{m=1}^\infty$ must converge to $\tilde{f}_l(\cdot, s)$ weakly in $L^2(B_{1/3}^k)$, proving (8.50). The same argument shows that the L^2 Cauchy sequence in (8.36) also converges to $\tilde{f}_l(\cdot, s_j)$. The lower semicontinuity under weak convergence shows that

$$(8.54) \quad \|\tilde{f}_l(\cdot, s)\|_{L^2(B_{1/3}^k)} \leq \liminf_{m \rightarrow \infty} \|\tilde{f}_l^{(m)}(\cdot, s)\|_{L^2(B_{1/3}^k)}$$

for a.e. $s \in (-1/4, 1/4)$. We next show that for any $-1/4 < s_j < s < 1/4$ with s_j satisfying (8.36), s satisfying (8.34) and (8.35), and for any non-negative $\tilde{\phi} \in C_c^\infty(B_{1/3}^k)$,

$$(8.55) \quad \limsup_{m \rightarrow \infty} \|\tilde{\phi} \tilde{f}_l^{(m)}(\cdot, s)\|_{L^2(B_{1/3}^k)}^2 \leq \|\tilde{\phi} \tilde{f}_l(\cdot, s_j)\|_{L^2(B_{1/3}^k)}^2 + c(\tilde{\phi})(s - s_j)^{1/2}.$$

We use $(x_l)^2 \tilde{\phi}$ as a test function in (3.4) with time interval $[s_j, s]$ and divide both sides by $(\mu^{(m)})^2$. By the Cauchy-Schwarz inequality and abbreviating the notations, substitution of $(x_l)^2 \tilde{\phi}$ into (3.4) gives

$$(8.56) \quad \begin{aligned} & \int_{C(T, 1/3)} (x_l)^2 \tilde{\phi} d\|V_s^{(m)}\| - \int_{C(T, 1/3)} (x_l)^2 \tilde{\phi} d\|V_{s_j}^{(m)}\| \\ & \leq \int_{C(T, 1/3) \times (s_j, s)} (-h(V_t^{(m)}, \cdot) (x_l)^2 \tilde{\phi} + \nabla((x_l)^2 \tilde{\phi})) \cdot (h(V_t^{(m)}, \cdot) + (u^{(m)})^\perp) d\|V_t^{(m)}\| dt \\ & \leq \int_{C(T, 1/3) \times (s_j, s)} \nabla((x_l)^2 \tilde{\phi}) \cdot (h(V_t^{(m)}, \cdot) + (u^{(m)})^\perp) + |u^{(m)}|^2 (x_l)^2 \tilde{\phi} d\|V_t^{(m)}\| dt \\ & \leq c(\tilde{\phi}) \left\{ \left(\int (x_l)^2 \right)^{1/2} + \left(\int (x_l)^4 \right)^{1/2} \right\} \left\{ \left(\int |h|^2 \right)^{1/2} + \left(\int |u^{(m)}|^2 \right)^{1/2} \right\} + \int (x_l)^2 |u^{(m)}|^2. \end{aligned}$$

Due to (8.22) and (8.26), we can estimate the right-hand side of (8.56) by $c(\mu^{(m)})^2((s_j - s)^{1/2} + \mu^{(m)})$. On the other hand one can estimate the left-hand side of (8.56) (divided by $(\mu^{(m)})^2$) just like (8.41) using (8.33)-(8.36). It is important to note that we have the strong convergence in $L^2(B_{1/3}^k)$ at $t = s_j$. This shows (8.55). Since $\tilde{f}_l$ is smooth and $\{s_j\}_{j=1}^\infty$ is dense, (8.31), (8.54) and (8.55) show that for a.e. $s \in (-1/4, 1/4)$, $\tilde{f}_l^{(m)}(\cdot, s)$ converges strongly in $L^2(B_{1/3}^k)$ to $\tilde{f}_l(\cdot, s)$ as $m \rightarrow \infty$. Here we also note that due to (8.31), there

is no concentration of L^2 -norm of $\tilde{f}_l^{(m)}(\cdot, s)$ near $\partial B_{1/3}^k$. Since $\|\tilde{f}_l^{(m)}(\cdot, s) - \tilde{f}_l(\cdot, s)\|_{L^2(B_{1/3}^k)}$ is bounded uniformly in $s \in (-1/4, 1/4)$, the dominated convergence theorem shows that $\|\tilde{f}_l^{(m)} - \tilde{f}_l\|_{L^2(\Omega')}$ converges to 0. This ends the proof of Lemma 8.4. $\square$

Next we define $T^{(m)} \in \mathbf{G}(n, k)$ as the image of the map $x \mapsto \mu^{(m)}x \cdot \nabla \tilde{f}(0, 0)$, which is the tangent space of the graph $(x, \mu^{(m)}\tilde{f}(x, 0))$ at $x = 0$. Also define $A^{(m)} \in \mathbf{A}(n, k)$ by $A^{(m)} = T^{(m)} + (0, \mu^{(m)}\tilde{f}(0, 0))$.

Lemma 8.5. *There exists a constant c_{16} which depends only on c_{15} and Λ_* with the following property. For $0 < \theta \leq (16\Lambda_*)^{-1/2}$ we have $\|T - T^{(m)}\| \leq c_{16}\mu^{(m)}$ and*

$$(8.57) \quad \limsup_{m \rightarrow \infty} (\mu^{(m)})^{-2} \int_{C(T, 2\theta) \times (-\theta^2\Lambda_*, \theta^2\Lambda_*)} \text{dist}(A^{(m)}, x)^2 d\|V_t^{(m)}\| dt \leq c_{16}\theta^{k+6}.$$

Proof of Lemma 8.5. We first note that any interior partial derivatives of $\tilde{f}$ may be estimated depending only on the order of differentiations and c_{15} due to the fact that $\tilde{f}$ is a solution of the heat equation and by the standard linear regularity theory. In particular, by the second order Taylor expansion at the origin, we have (with $|f(x, t) - f(x, 0)| \leq c|t| \leq c\theta^2$)

$$(8.58) \quad \int_{B_{2\theta}^k \times (-\theta^2\Lambda_*, \theta^2\Lambda_*)} |\tilde{f}(x, t) - \tilde{f}(0, 0) - x \cdot \nabla \tilde{f}(0, 0)|^2 d\mathcal{H}^{k+1} \leq c_{17}\theta^{k+6}$$

for any $0 < \theta \leq (16\Lambda_*)^{-1/2}$ and for some c_{17} depending only on c_{15} and Λ_* . Next, by the Lipschitz approximation of $V_t^{(m)}$ and $\text{dist}(A^{(m)}, x) \leq c(c_{15})\mu^{(m)}$ on the support of $\|V_t^{(m)}\|$, we may prove (just like (8.41)) that

$$(8.59) \quad \begin{aligned} & (\mu^{(m)})^{-2} \int_{C(T, 2\theta) \times (-\theta^2\Lambda_*, \theta^2\Lambda_*)} \text{dist}(A^{(m)}, x)^2 d\|V_t^{(m)}\| dt \\ &= o(1) + (\mu^{(m)})^{-2} \int_{B_{2\theta}^k \times (-\theta^2\Lambda_*, \theta^2\Lambda_*)} |(T^{(m)})^\perp((x, f^{(m)}(x, t)) - (0, \tilde{f}(0, 0)\mu^{(m)}))|^2 d\mathcal{H}^{k+1}. \end{aligned}$$

Since $(x, x \cdot \nabla \tilde{f}(0, 0)\mu^{(m)}) \in T^{(m)}$,

$$(8.60) \quad \begin{aligned} & |(T^{(m)})^\perp((x, f^{(m)}(x, t)) - (0, \tilde{f}(0, 0)\mu^{(m)}))| \\ &= |(T^{(m)})^\perp((x, f^{(m)}(x, t)) - (0, \tilde{f}(0, 0)\mu^{(m)}) - (x, x \cdot \nabla \tilde{f}(0, 0)\mu^{(m)}))| \\ &\leq |f^{(m)}(x, t) - \tilde{f}(x, t)\mu^{(m)}| + |\tilde{f}(x, t) - \tilde{f}(0, 0) - x \cdot \nabla \tilde{f}(0, 0)|\mu^{(m)} \end{aligned}$$

where we added and subtracted $(x, \tilde{f}(x, t)\mu^{(m)})$ and used the triangle inequality in the second line. Now we substitute (8.60) into (8.59) and use (8.58). Since we have the strong $L^2(\Omega')$ convergence (Proposition 8.4) we obtain (8.57) for a suitable c_{16} . $\square$

Fix θ_* so that

$$(8.61) \quad 0 < \theta_* \leq (16\Lambda_*)^{-1/2}, \quad 3c_{16}\theta_*^2 \leq \theta_*^{2\varsigma}.$$

For $1 < \gamma < 2$ we have (8.57) for $\theta = \gamma\theta_*$ and hence for all large m ,

$$(8.62) \quad \begin{aligned} & \int_{C(T, 2\gamma\theta_*) \times (-\theta_*^2\Lambda_*, \theta_*^2\Lambda_*)} \text{dist}(A^{(m)}, x)^2 d\|V_t^{(m)}\| dt \leq (c_{16}(\gamma\theta_*)^{k+6} + o(1))(\mu^{(m)})^2 \\ & < \frac{1}{2}\gamma^{k+6}\theta_*^{2\varsigma+k+4}(\mu^{(m)})^2 \end{aligned}$$

by (8.61), with $\|T^{(m)} - T\| \leq c_{16}\mu^{(m)}$ and $A^{(m)}$ is parallel to $T^{(m)}$. For all sufficiently large m , due to Lemma 8.2, $C(W^{(m)}, 2\theta_*) \cap \text{spt}\|V_t^{(m)}\| \subset C(T, 2\gamma\theta_*)$. Thus we have from (8.62)

$$(8.63) \quad \limsup_{m \rightarrow \infty} \theta_*^{-(k+4)} (\mu^{(m)})^{-2} \int_{C(W^{(m)}, 2\theta_*) \times (-\theta_*^2 \Lambda_*, \theta_*^2 \Lambda_*)} \text{dist}(A^{(m)}, x)^2 d\|V_t^{(m)}\| dt \\ \leq \frac{1}{2} \gamma^{k+6} \theta_*^{2\varsigma}.$$

By letting $\gamma \rightarrow 1$, (8.63) contradicts with (8.14), and the first part of the Proposition 8.1 is proved.

For (8.10) and (8.11), just as we obtained (8.27) by Corollary 5.9, we may obtain

$$(8.64) \quad \int_{-1/2}^{1/2} |\theta_*^{-k} \|V_t\|(\phi_{T, \theta_*}^2) - \mathbf{c}| dt \leq \tilde{K}(\mu^2 + C(u)).$$

Since $\|W - T\| \leq \varepsilon_4$, $\|u\| < \varepsilon_4$ and $\mu < \varepsilon_4$, by further restricting ε_4 so that the right-hand side of (8.64) is sufficiently small, and choosing generic $\tilde{t}_1$ and $\tilde{t}_2$ satisfying the same inequality as (8.64) (with W and a different constant), we may conclude that such t can be chosen. This concludes the proof of Proposition 8.1 $\square$

Corollary 8.6. (cf. [1, 8.17]) *Corresponding to E_1 , ν , p , q there exist $0 < \varepsilon_5 < 1$ and $1 < c_{18} < \infty$ with the following property. Under the assumptions of Proposition 8.1 where ε_4 is replaced by ε_5 , and with $R^\varsigma \|u\| < \varepsilon_5$,*

- (1) *there exists a unique element $a \in \text{spt}\|V_0\| \cap \{x : W(x) = 0\}$,*
- (2) *there exists $T_\infty \in \mathbf{G}(n, k)$ such that*

$$(8.65) \quad \text{Tan}(\text{spt}\|V_0\|, a) \subset T_\infty$$

and

$$(8.66) \quad \|T_\infty - T\| \leq c_{18} \max\{\mu, c_{11} R^\varsigma \|u\|\},$$

- (3) *whenever $0 < s < R$, there are $T_s \in \mathbf{G}(n, k)$ and $A_s \in \mathbf{A}(n, k)$ such that A_s is parallel to T_s and*

$$(8.67) \quad \|T_s - T_\infty\| + \left(s^{-k-4} \int_{-s^2 \Lambda_*}^{s^2 \Lambda_*} \int_{C(W, 2s)} \text{dist}(x, A_s)^2 d\|V_t\| dt \right)^{\frac{1}{2}} \\ \leq c_{18} (s/R)^\varsigma \max\{\mu, c_{11} R^\varsigma \|u\|\}.$$

Proof. We may assume $R = 1$ after a change of variables. We choose $0 < \varepsilon_5 < 1$ and $1 < c_{18} < \infty$ so that

$$(8.68) \quad c_{11} \varepsilon_5 < \varepsilon_4,$$

$$(8.69) \quad \varepsilon_5 + \sum_{j=1}^{\infty} (c_{11})^2 \theta_*^{(j-1)\varsigma} \varepsilon_5 < \varepsilon_4,$$

$$(8.70) \quad 2c_{11} \sum_{j=0}^{\infty} \theta_*^{(j-1)\varsigma} \leq c_{18},$$

$$(8.71) \quad 2\theta_*^{-(k+4)/2-\varsigma} \leq c_{18}.$$

Set $T_0 = T$ and $A_0 = A$. We inductively prove that for $j = 1, 2, \dots$, there are $T_j \in \mathbf{G}(n, k)$ and $A_j \in \mathbf{A}(n, k)$ such that A_j is parallel to T_j and the following hold:

$$(8.72) \quad \|T_j - T_{j-1}\| \leq c_{11}\theta_*^{(j-1)\varsigma} \max\{\mu, c_{11}\|u\|\},$$

$$(8.73) \quad \mu_j := \left(\theta_*^{-j(k+4)} \int_{-\theta_*^{2j}\Lambda_*}^{\theta_*^{2j}\Lambda_*} \int_{C(W, 2\theta_*^j)} \text{dist}(x, A_j)^2 d\|V_t\| dt \right)^{\frac{1}{2}} \leq \theta_*^{j\varsigma} \max\{\mu, c_{11}\|u\|\},$$

$$(8.74) \quad (-\Lambda_* + 1)\theta_*^{2j} \leq \exists t_{1,j} \leq (-\Lambda_* + 2)\theta_*^{2j} : \theta_*^{-kj}\|V_{t_{1,j}}\|(\phi_{W, \theta_*^j}^2) < (2 - \nu)\mathbf{c},$$

$$(8.75) \quad (\Lambda_* - 2)\theta_*^{2j} \leq \exists t_{2,j} \leq (\Lambda_* - 1)\theta_*^{2j} : \theta_*^{-kj}\|V_{t_{2,j}}\|(\phi_{W, \theta_*^j}^2) > \nu\mathbf{c}.$$

Since $\varepsilon_5 < \varepsilon_4$, Proposition 8.1 provides the proof for the validity of $j = 1$ case. Under the inductive assumption up to $j = l$, we have

$$(8.76) \quad \mu_l \leq \theta_*^{l\varsigma} \max\{\mu, c_{11}\|u\|\} < \varepsilon_4$$

by (8.3), $\|u\| < \varepsilon_5$ and (8.68),

$$(8.77) \quad \begin{aligned} \|T_l - W\| &\leq \|T_0 - W\| + \sum_{j=1}^l \|T_j - T_{j-1}\| \leq \|T - W\| \\ &+ \sum_{j=1}^l c_{11}\theta_*^{(j-1)\varsigma} \max\{\mu, c_{11}\|u\|\} \leq \varepsilon_5 + \sum_{j=1}^l (c_{11})^2 \varepsilon_5 \theta_*^{(j-1)\varsigma} < \varepsilon_4 \end{aligned}$$

by (8.1), (8.3), $\|u\| < \varepsilon_5$, (8.72) and (8.69). For $R = \theta_*^l$, the assumptions (8.1)-(8.6) are now satisfied due to (8.76), (8.77), (8.74), (8.75) as well as $\theta_*^{l\varsigma}\|u\| < \varepsilon_4$. Thus we have $T_{l+1} \in \mathbf{G}(n, k)$ and $A_{l+1} \in \mathbf{A}(n, k)$ such that A_{l+1} is parallel to T_{l+1} ,

$$(8.78) \quad \|T_{l+1} - T_l\| \leq c_{11}\mu_l \leq c_{11}\theta_*^{l\varsigma} \max\{\mu, c_{11}\|u\|\}$$

by (8.8) and (8.73),

$$(8.79) \quad \begin{aligned} \mu_{l+1} &:= \left(\theta_*^{-(l+1)(k+4)} \int_{-\theta_*^{2(l+1)}\Lambda_*}^{\theta_*^{2(l+1)}\Lambda_*} \int_{C(W, 2\theta_*^{l+1})} \text{dist}(x, A_{l+1})^2 d\|V_t\| dt \right)^{\frac{1}{2}} \\ &\leq \theta_*^{l\varsigma} \max\{\mu_l, c_{11}\theta_*^{l\varsigma}\|u\|\} \leq \theta_*^{(l+1)\varsigma} \max\{\mu, c_{11}\|u\|\} \end{aligned}$$

by (8.9) and (8.73). (8.74) and (8.75) for $j = l+1$ are also satisfied due to (8.10) and (8.11). Hence the next inductive assumptions (8.72)-(8.75) are satisfied. It is now clear from (8.72) that there exists $T_\infty = \lim_{j \rightarrow \infty} T_j \in \mathbf{G}(n, k)$ satisfying (8.66) due to (8.72) and (8.70). To prove (1), first assume that $\text{spt } \|V_0\| \cap \{x : W(x) = 0\} = \emptyset$. By definition, there exists some $\gamma > 0$ such that $\|V_0\|C(W, \gamma) = 0$. By the unit density assumption (A1) and Proposition 6.2, we may then prove that there exists $0 < \gamma' < \gamma$ such that $\|V_t\|(C(W, \gamma')) = 0$ for $0 \leq t \leq \gamma'$. But this contradicts with (8.75) for all sufficiently large j . Thus we prove $\text{spt } \|V_0\| \cap \{x : W(x) = 0\} \neq \emptyset$. To prove the uniqueness, we observe that

$$(8.80) \quad \text{spt } \|V_t\| \cap \{x : \text{dist}(x, A_j) > o(1)\theta_*^j\} \cap C(W, 3\theta_*^j/2) = \emptyset, \quad \forall t \in [(-\Lambda_* + 1)\theta_*^{2j}, \Lambda_*\theta_*^{2j}]$$

where $o(1)$ here means $0 < o(1) \rightarrow 0$ as $j \rightarrow \infty$. The proof is almost identical to that of Lemma 8.2 after a change of variables and with some notational modifications. Let $a_j := A_j \cap \{x : W(x) = 0\}$. One can prove $\lim_{j \rightarrow \infty} a_j = a := \text{spt } \|V_0\| \cap \{x : W(x) = 0\}$, by means of (8.80), which proves (1). Also (8.80) shows that $\text{Tan}(\text{spt } \|V_0\|, a) \subset T_\infty$ (recall

(2.1)), proving (2). Finally for $0 < s < 1$, choose j such that $\theta_*^{j+1} \leq s < \theta_*^j$, and let $T_s = T_j$, $A_s = A_j$. Then we have

$$\begin{aligned}
 \|T_s - T_\infty\| &\leq c_{11} \max\{\mu, c_{11}\|u\|\} \sum_{l=0}^{\infty} \theta_*^{(j+l)\varsigma} \\
 (8.81) \quad &= c_{11} \sum_{l=0}^{\infty} \theta_*^{(l-1)\varsigma} \max\{\mu, c_{11}\|u\|\} \theta_*^{(j+1)\varsigma} \leq c_{18} \max\{\mu, c_{11}\|u\|\} s^\varsigma / 2
 \end{aligned}$$

by (8.70) and

$$\begin{aligned}
 (8.82) \quad &\left(s^{-(k+4)} \int_{-s^2\Lambda_*}^{s^2\Lambda_*} \int_{C(W, 2s)} \text{dist}(x, A_s)^2 d\|V_t\| dt \right)^{\frac{1}{2}} \leq \theta_*^{-(k+4)/2} \mu_j \\
 &\leq \theta_*^{-(k+4)/2} \max\{\mu, c_{11}\|u\|\} \theta_*^{j\varsigma} \leq \theta_*^{-(k+4)/2-\varsigma} \max\{\mu, c_{11}\|u\|\} s^\varsigma \\
 &\leq c_{18} \max\{\mu, c_{11}\|u\|\} s^\varsigma / 2
 \end{aligned}$$

by (8.71). Combining (8.81) and (8.82), we obtain (8.67). $\square$

Theorem 8.7. *Corresponding to $1 \leq E_1 < \infty$, $0 < \nu < 1$, p and q as in (3.2), there exist $0 < \varepsilon_6 < 1$, $0 < \sigma \leq 1/2$, $2 < \Lambda_3 < \infty$ and $1 < c_{19} < \infty$ with the following property. For $T \in \mathbf{G}(n, k)$, $0 < R < \infty$, $U = C(T, 3R)$ and $(0, \Lambda)$ replaced by $(-\Lambda_3 R^2, \Lambda_3 R^2)$, suppose $\{V_t\}_{-\Lambda_3 R^2 \leq t \leq \Lambda_3 R^2}$ and $\{u(\cdot, t)\}_{-\Lambda_3 R^2 \leq t \leq \Lambda_3 R^2}$ satisfy (A1)-(A4). Suppose*

$$(8.83) \quad \mu := \left(R^{-(k+4)} \int_{-\Lambda_3 R^2}^{\Lambda_3 R^2} \int_{C(T, 3R)} |T^\perp(x)|^2 d\|V_t\| dt \right)^{\frac{1}{2}} < \varepsilon_6,$$

$$(8.84) \quad R^\varsigma \|u\| := R^\varsigma \|u\|_{L^{p,q}(C(T, 3R) \times (-\Lambda_3 R^2, \Lambda_3 R^2))} < \varepsilon_6,$$

$$(8.85) \quad (-\Lambda_3 + 3/2)R^2 \leq \exists t_1 \leq (-\Lambda_3 + 2)R^2 \quad : \quad R^{-k} \|V_{t_1}\|(\phi_{T,R}^2) < (2 - \nu)\mathbf{c},$$

$$(8.86) \quad (\Lambda_3 - 2)R^2 \leq \exists t_2 \leq (\Lambda_3 - 3/2)R^2 \quad : \quad R^{-k} \|V_{t_2}\|(\phi_{T,R}^2) > \nu\mathbf{c},$$

where $\phi_{T,R}$ and $\mathbf{c}$ are defined as in Definition 5.1. Denote $\tilde{D} := (T \cap B_{\sigma R}) \times (-R^2/4, R^2/4)$. Then there are $f : \tilde{D} \rightarrow T^\perp$ and $F : \tilde{D} \rightarrow \mathbb{R}^n$ such that $T(F(y, t)) = y$ and $T^\perp(F(y, t)) = f(y, t)$ for all $(y, t) \in \tilde{D}$,

$$(8.87) \quad \text{spt } \|V_t\| \cap C(T, \sigma R) = \text{image } F(\cdot, t) \quad \forall t \in (-R^2/4, R^2/4),$$

$$(8.88) \quad f(x, t) \text{ is differentiable with respect to } x \text{ at every point of } \tilde{D},$$

$$(8.89) \quad R^{-1} |f(y, s)| + \|\nabla f(y, s)\| \leq c_{19} \max\{\mu, R^\varsigma \|u\|\}, \quad \forall (y, s) \in \tilde{D},$$

$$\begin{aligned}
 (8.90) \quad \|\nabla f(y_1, s_1) - \nabla f(y_2, s_2)\| &\leq c_{19} \max\{\mu, R^\varsigma \|u\|\} \left(R^{-1} \max\{|y_1 - y_2|, |s_1 - s_2|^{1/2}\} \right)^\varsigma, \\
 &\quad \forall (y_1, s_1), \forall (y_2, s_2) \in \tilde{D},
 \end{aligned}$$

$$\begin{aligned}
 (8.91) \quad R^{-1} |f(y, s_1) - f(y, s_2)| &\leq c_{19} \max\{\mu, R^\varsigma \|u\|\} (R^{-1} |s_1 - s_2|^{1/2})^{1+\varsigma}, \\
 &\quad \forall (y, s_1), \forall (y, s_2) \in \tilde{D}.
 \end{aligned}$$

Proof. Let Λ_* , θ_* , c_{18} , ε_5 be fixed by Proposition 8.1 and Corollary 8.6 corresponding to E_1 and ν there replaced by $\nu/2$. We set $\Lambda_3 = \Lambda_* + 1/4$. Without loss of generality we may assume $R = 1$. Set $\sigma := \min\{1/2, \mathbf{c}\nu(2^{k+1}E_1 \sup |\nabla(\phi_T^2)|)^{-1}\}$. Then $\sigma \leq 1/2$ and we have for any $y \in B_\sigma^k$

$$(8.92) \quad \int_{C(T,2)} \phi_T^2(\cdot - y) d\|V_{t_1}\| \leq \int_{C(T,2)} \phi_T^2 d\|V_{t_1}\| + |y|\|V_{t_1}\|(C(T, 3/2)) \sup |\nabla(\phi_T^2)| \\ \leq (2 - \nu)\mathbf{c} + |y|\|V_{t_1}\|(B_2) \sup |\nabla(\phi_T^2)| \leq (2 - \nu/2)\mathbf{c}$$

where we used (8.85), $C(T, 3/2) \cap \text{spt } \|V_{t_1}\| \subset B_2$ for sufficiently small ε_6 which follows from Proposition 6.4, (A2) and the definition of σ . A similar bound for t_2 may be obtained. Note that for any $s \in (-1/4, 1/4)$, $t_1 \in (-\Lambda_* + s + 1, -\Lambda_* + s + 2)$ and $t_2 \in (\Lambda_* + s - 2, \Lambda_* + s - 1)$ due to the intervals in (8.85) and (8.86). Thus all the assumptions for Corollary 8.6 are satisfied for domain centered at (y, s) , i.e., for $C(T, y, 2) \times (-\Lambda_* + s, \Lambda_* + s)$. We may conclude that there exists a unique element which we define to be $F(y, s) \in \text{spt } \|V_s\| \cap \{x : T(x) = y\}$ for each $(y, s) \in \tilde{D}$. Define $f(y, s) := T^\perp(F(y, s))$. It is also clear that $F(y, s)$ is differentiable with respect to space variables and that $\text{image } \nabla F(y, s) = T_\infty(y, s)$, where T_∞ is as in Corollary 8.6 applied to the flow after composing it with the translation taking (y, s) to the origin. Then (8.89) follows from (8.66) and Proposition 6.4 for a suitable c_{19} . For $(y_1, s_1), (y_2, s_2) \in \tilde{D}$, set $d := \max\{|y_1 - y_2|, |s_1 - s_2|^{1/2}\}$ and $w := 2d$. We have $T_w(y_1, s_1) \in \mathbf{G}(n, k)$ and $A_w(y_1, s_1) \in \mathbf{A}(n, k)$ which is parallel to $T_w(y_1, s_1)$ satisfying

$$(8.93) \quad \|T_w(y_1, s_1) - T_\infty(y_1, s_1)\| + \left(w^{-(k+4)} \int_{s_1 - w^2\Lambda_*}^{s_1 + w^2\Lambda_*} \int_{C(T, y_1, 2w)} \text{dist}(x, A_w(y_1, s_1))^2 d\|V_t\| dt \right)^{\frac{1}{2}} \\ \leq c_{18}w^\varsigma \max\{\mu, c_{11}\|u\|\}.$$

By the definition of w and d , one can check that

$$(8.94) \quad C(T, y_2, 2d) \times (s_2 - d^2\Lambda_*, s_2 + d^2\Lambda_*) \subset C(T, y_1, 2w) \times (s_1 - w^2\Lambda_*, s_1 + w^2\Lambda_*).$$

By (8.94),

$$(8.95) \quad \left(d^{-(k+4)} \int_{s_2 - d^2\Lambda_*}^{s_2 + d^2\Lambda_*} \int_{C(T, y_2, 2d)} \text{dist}(x, A_w(y_1, s_1))^2 d\|V_t\| dt \right)^{\frac{1}{2}} \\ \leq 2^{(k+4)/2} \left(w^{-(k+4)} \int_{s_1 - w^2\Lambda_*}^{s_1 + w^2\Lambda_*} \int_{C(T, y_1, 2w)} \text{dist}(x, A_w(y_1, s_1))^2 d\|V_t\| dt \right)^{\frac{1}{2}}.$$

Since we may choose ε_6 small so that $\|T_w(y_1, s_1) - T\| \leq \varepsilon_5$, and conditions (8.5) and (8.6) corresponding to the domain $C(T, y_2, 2d) \times (s_2 - d^2\Lambda_*, s_2 + d^2\Lambda_*)$ are satisfied, we may apply Corollary 8.6 to this domain by restricting ε_6 small. Thus we obtain with (8.93) and (8.94) that

$$(8.96) \quad \|T_w(y_1, s_1) - T_\infty(y_2, s_2)\| \leq c_{18} \max\{2^{(k+4)/2} c_{18} w^\varsigma \max\{\mu, c_{11}\|u\|\}, c_{11} d^\varsigma \|u\|\}.$$

(8.93) and (8.96) shows with an appropriate c_{19} that

$$(8.97) \quad \|T_\infty(y_1, s_1) - T_\infty(y_2, s_2)\| \leq c_{19} d^\varsigma \max\{\mu, \|u\|\}.$$

Since $\text{image } \nabla F(y, s) = T_\infty(y, s)$, (8.97) shows (8.90) via [1, 8.9(5)] with a new suitable c_{19} . For (8.91), we observe that (8.93) combined with Proposition 6.4 gives the estimate with a suitable c_{19} . $\square$

Remark 8.8. Note that Theorem 8.7 shows that $\text{spt } \|V_t\| \cap C(T, \sigma R)$ for $t \in (-R^2/4, R^2/4)$ is a set of $C^{1,\varsigma}$ regular points.

9. PARTIAL REGULARITY

In this section, we prove the main partial regularity result. The main idea of proof comes from [7, 6.12], though we greatly simplify the overall computations. First we give a more convenient form of regularity criterion which is suitable for establishing partial regularity.

Proposition 9.1. Corresponding to $1 \leq E_1 < \infty$, $0 < \nu < 1$, p, q with (3.2) there exists $2 < L < \infty$, $3 < \Lambda_4 < \infty$, $0 < \varepsilon_7 < 1$ with the following property. Assume $\{V_t\}_{0 < t < \Lambda}$ and $\{u(\cdot, t)\}_{0 < t < \Lambda}$ satisfy (A1)-(A4). For $(a, s) \in U \times (0, \Lambda)$, assume that for some $R > 0$, $B_{RL}(a) \times (s - R^2\Lambda_4, s + R^2\Lambda_4) \subset U \times (0, \Lambda)$ and for some $T \in \mathbf{G}(n, k)$,

$$(9.1) \quad \left(R^{-(k+2)} \int_{B_{RL}(a)} |T^\perp(x-a)|^2 d\|V_{s-R^2\Lambda_4}\| \right)^{\frac{1}{2}} < \varepsilon_7,$$

$$(9.2) \quad R^\varsigma \|u\|_{L^{p,q}(B_{RL}(a) \times (s-R^2\Lambda_4, s+R^2\Lambda_4))} < \varepsilon_7,$$

$$(9.3) \quad s - R^2(\Lambda_4 - 5/2) \leq \exists t_1 \leq s - R^2(\Lambda_4 - 3) : R^{-k} \|V_{t_1}\|(\phi_R^2(\cdot - a)) < (2 - \nu)\mathbf{c},$$

$$(9.4) \quad s + R^2(\Lambda_4 - 3) \leq \exists t_2 \leq s + R^2(\Lambda_4 - 5/2) : R^{-k} \|V_{t_2}\|(\phi_R^2(\cdot - a)) > \nu\mathbf{c}.$$

Here, $\phi_R(x) := \phi(|x|/R)$ with ϕ defined in (5.1) and $\mathbf{c}$ defined in (5.3). Then (a, s) is a $C^{1,\varsigma}$ regular point.

Proof. We check that Theorem 8.7 is applicable for the conclusion. We may set $R = 1$, $a = 0$ and $s = 0$ after a change of variables. Let Λ_3 and ε_6 be constants corresponding to E_1 and ν there replaced by $\nu/2$. Set $\Lambda_4 = \Lambda_3 + 1$. We use Proposition 6.5 with $\Lambda = 2\Lambda_4$ and obtain c_6 . Fix a large $L > 2$ so that we have

$$(9.5) \quad c_6 L^{k+2} \exp(-(L/4 - 1)^2/(16\Lambda_4)) < \varepsilon_6^2/(8\Lambda_4).$$

Then restrict $\varepsilon_7 \leq \varepsilon_6$ so that

$$(9.6) \quad 4^{k+2} c_6 (4^{2\varsigma} \varepsilon_7^2 + 4^\varsigma \varepsilon_7) (L/4)^2 < \varepsilon_6^2/(8\Lambda_4).$$

By (6.28) with $R = 4$ and L there replaced by $L/4$ with L given in the assumption, and (9.5), (9.6), (9.2), we obtain

$$(9.7) \quad \int_{B_4} |T^\perp(x)|^2 d\|V_t\| \leq \exp(1/(8\Lambda_4)) \int_{B_L} |T^\perp(x)|^2 d\|V_{-\Lambda_4}\| + \varepsilon_6^2/(4\Lambda_4)$$

for all $t \in [-\Lambda_4, \Lambda_4]$. By integraring (9.7) and by (9.1), we have

$$(9.8) \quad \int_{-\Lambda_4}^{\Lambda_4} \int_{B_4} |T^\perp(x)|^2 d\|V_t\| dt \leq 2\Lambda_4 \exp(1/(8\Lambda_4)) \varepsilon_7^2 + \varepsilon_6^2/2.$$

By (9.8) and Proposition 6.4, by further restricting ε_7 if necessary, we have

$$(9.9) \quad \text{spt } \|V_t\| \cap B_{16/5} \subset \{|T^\perp(x)| \leq 1/5\}$$

for $t \in [-\Lambda_4 + 1, \Lambda_4 - 1] = [-\Lambda_3, \Lambda_3]$. By re-defining $V_t = 0$ on $C(T, 3) \setminus B_{16/5}$ for $t \in [-\Lambda_3, \Lambda_3]$, one can check that the newly defined $\{V_t\}$ satisfies (3.4) on $C(T, 3) \times [-\Lambda_3, \Lambda_3]$. By restricting ε_7 further, we may guarantee that (8.83) is satisfied by (9.8). For conditions (8.85) and (8.86), note that $\int |\phi_T^2 - \phi_1^2| d\|V_t\|$ can be made arbitrarily small by Proposition 6.4 and choosing small ε_7 , i.e., by making $\text{dist}(T, \text{spt } \|V_t\|)$ sufficiently small. Thus, having assumed

(9.3) and (9.4), we can guarantee that (8.85) and (8.86) hold for $\nu/2$ instead of ν . We now have all the assumptions for Theorem 8.7, and hence (a, s) is a $C^{1,\varsigma}$ regular point. $\square$

Lemma 9.2. *Under the assumptions (A1)-(A4), there exists a co-countable set $G \subset (0, \Lambda)$ such that $\|V_t\|(\phi)$ is continuous at $t \in G$ for any $\phi \in C_c^2(U; \mathbb{R}^+)$.*

Proof. Fix $\phi \in C_c^2(U; \mathbb{R}^+)$. Computing as in (3.5), we have for any $0 < t_1 < t_2 < \Lambda$

$$\begin{aligned} \|V_{t_2}\|(\phi) - \|V_{t_1}\|(\phi) &\leq \int_{t_1}^{t_2} \left(\int_U \frac{|\nabla \phi|^2}{\phi} + |u|^2 \phi + |\nabla \phi| |u| d\|V_t\| \right) dt \\ (9.10) \quad &=: \int_{t_1}^{t_2} \Phi(t) dt, \end{aligned}$$

where Φ is integrable due to (A2) and (A4). (9.10) shows $\|V_t\|(\phi) - \int_0^t \Phi(s) ds$ is monotone decreasing, and there are at most countably many discontinuities for such function. Since $\int_0^t \Phi(s) ds$ is continuous, such discontinuities are caused only by $\|V_t\|(\phi)$. Next, choose a countable set of functions $\{\phi_j\}_{j \in \mathbb{N}} \subset C_c^2(U; \mathbb{R}^+)$ such that it is dense with respect to C^0 topology in $C_c^2(U; \mathbb{R}^+)$. We may choose such $\{\phi_j\}_{j \in \mathbb{N}}$ so that, for any $\phi \in C_c^2(U; \mathbb{R}^+)$, there exist some $\tilde{U} \subset\subset U$ with $\text{spt } \phi \subset \tilde{U}$ and a subsequence $\{\phi_{j_k}\}_{k \in \mathbb{N}} \subset C_c^2(\tilde{U}; \mathbb{R}^+)$ such that $\lim_{k \rightarrow \infty} \sup |\phi_{j_k} - \phi| = 0$. Define

$$G := (0, \Lambda) \setminus \bigcup_{j \in \mathbb{N}} \{t \in (0, \Lambda) : \|V_s\|(\phi_j) \text{ is discontinuous at } s = t\}.$$

Then G is co-countable. Take any $\phi \in C_c^2(U; \mathbb{R}^+)$ and $\hat{t} \in G$, and let $\tilde{U}$ and $\{\phi_{j_k}\}_{k \in \mathbb{N}}$ be as described above. Then we have for t' close to $\hat{t}$

$$\begin{aligned} (9.11) \quad &|||V_{t'}\|(\phi) - \|V_{\hat{t}}\|(\phi)| \leq |||V_{t'}\|(\phi) - \|V_{t'}\|(\phi_{j_k})| + |||V_{t'}\|(\phi_{j_k}) - \|V_{\hat{t}}\|(\phi_{j_k})| \\ &+ |||V_{\hat{t}}\|(\phi_{j_k}) - \|V_{\hat{t}}\|(\phi)| \\ &\leq 2 \sup |\phi - \phi_{j_k}| \sup_{s \in (0, \Lambda)} \|V_s\|(\tilde{U}) + |||V_{t'}\|(\phi_{j_k}) - \|V_{\hat{t}}\|(\phi_{j_k})|. \end{aligned}$$

Given any $\varepsilon > 0$, we fix a large k so that the first term of (9.11) is less than $\varepsilon/2$, and then use the continuity of $\|V_s\|(\phi_{j_k})$ at $s = \hat{t}$ so that the second term is less than $\varepsilon/2$ when $|t' - \hat{t}|$ is small enough. This proves that $\|V_s\|(\phi)$ is continuous at $s = \hat{t}$ for any $\phi \in C_c^2(U; \mathbb{R}^+)$ and we may conclude the proof. $\square$

We focus our attention at time $t' \in (0, \Lambda)$ when $V_{t'}$ is unit density and $t' \in G$. By (A1) and Lemma 9.2, such time is a.e. on $(0, \Lambda)$. Assume $V_{t'} = |M|$. Without loss of generality, we may assume $M \subset \text{spt } \|V_{t'}\|$. By the standard measure-theoretic argument ([15, 2.10.19]), for $\mathcal{H}^k$ a.e. $x \in \text{spt } \|V_{t'}\| \setminus M$, we have $\Theta^k(\|V_{t'}\|, x) = 0$. For $\mathcal{H}^k$ a.e. $x \in M$, $\Theta^k(\|V_{t'}\|, x) = 1$ and a unique approximate tangent space $\text{Tan}_x M$ exists.

Definition 9.3. *Set*

$$(9.12) \quad A_1 := \text{spt } \|V_{t'}\| \setminus M, \quad A_2 := M \setminus \{C^{1,\varsigma} \text{ regular points}\}$$

We aim to prove that $\mathcal{H}^k(A_1) = \mathcal{H}^k(A_2) = 0$. By definition, $A_1 \cup A_2$ is closed.

Lemma 9.4. *We have $\mathcal{H}^k(A_1) = 0$.*

Proof. It suffices to prove $\mathcal{H}^k(A_1 \cap B_r(a)) = 0$ for arbitrary $B_{3r}(a) \subset\subset U$. After a change of variables, let $a = 0$. Assume for a contradiction that $\mathcal{H}^k(A_1 \cap B_r) > 0$. Let c_3 and c_4 be constants given in Corollary 6.3. Fix a sufficiently small $0 < R_0 < r/2$ so that

$$(9.13) \quad R_0^\varsigma \|u\|_{L^{p,q}(U \times (0, \Lambda))} \leq 1 \quad \text{and} \quad B_{R_0}(x) \times (t' - c_3 R_0^2, t') \subset\subset U \times (0, \Lambda)$$

for all $x \in B_r$. Then by Corollary 6.3 with (9.13), for any $x \in A_1 \cap B_r$ and $0 < R \leq R_0$, we have

$$(9.14) \quad \|V_{t'-c_3R^2}\|(B_{14R/15}(x)) > c_4R^k,$$

since we would have $\|V_{t'}\|(B_{4R/5}(x)) = 0$ otherwise, contradicting $x \in \text{spt } \|V_{t'}\|$. Next let

$$(9.15) \quad A_{1,m} := \{x \in A_1 \cap B_r : \|V_{t'}\|(B_R(x)) \leq c_4R^k/2, \forall R \in (0, r/m]\}.$$

Since $\Theta^k(\|V_{t'}\|, x) = 0$ for $\mathcal{H}^k$ a.e. $x \in A_1 \cap B_r$, we have $\mathcal{H}^k(A_1 \cap B_r) = \mathcal{H}^k(\cup_{m=1}^\infty A_{1,m})$. Since $A_{1,m}$ is an increasing sequence and since $\mathcal{H}^k(A_1 \cap B_r) > 0$, there exists some m_0 such that $\mathcal{H}^k(A_{1,m_0}) > 0$. By the definition of the Hausdorff measure, there exists some $\delta_0 > 0$ such that

$$(9.16) \quad \mathbf{b} := \mathcal{H}_{\delta_0}^k(A_{1,m_0}) := \inf \left\{ \sum_{j=1}^\infty \omega_k \left(\frac{\text{diam } G_j}{2} \right)^k : A_{1,m_0} \subset \cup_{j=1}^\infty G_j, \text{diam } G_j < 2\delta_0 \right\} > 0,$$

where inf is taken over all such covering of A_{1,m_0} . Note that $\mathbf{b} < \infty$ even if $\mathcal{H}^k(A_{1,m_0}) = \infty$. In the following we fix such $R_0, m_0, \delta_0, A_{1,m_0}$ and $\mathbf{b}$.

For any R with $0 < R < \min\{R_0, r/m_0, \delta_0\}$, consider the covering $\{\overline{B_R(b)}\}_{b \in A_{1,m_0}}$ of A_{1,m_0} . By the Besicovitch covering theorem, there exists a family of collections $\mathcal{B}_1, \dots, \mathcal{B}_{\mathbf{B}(n)}$ each of which consists of mutually disjoint balls and which satisfies $A_{1,m_0} \subset \cup_{j=1}^{\mathbf{B}(n)} \cup_{\overline{B_R(b)} \in \mathcal{B}_j} \overline{B_R(b)}$. Due to (9.16) and $R < \delta_0$, we have for this covering

$$(9.17) \quad \mathbf{b} \leq \omega_k \sum_{j=1}^{\mathbf{B}(n)} (\# \text{ of elements of } \mathcal{B}_j) R^k$$

By (9.17), there exists at least one of the collections, say, $\mathcal{B}_{j_0}$, consisting of closures of mutually disjoint $B_R(b_1), \dots, B_R(b_N)$, where N is the number of elements of $\mathcal{B}_{j_0}$, and such that

$$(9.18) \quad \mathbf{b} \leq \omega_k \mathbf{B}(n) N R^k$$

holds. Since $R < r/m_0$ and $b_j \in A_{1,m_0}$, (9.15) shows

$$(9.19) \quad \sum_{j=1}^N \|V_{t'}\|(B_R(b_j)) = \|V_{t'}\|(\cup_{j=1}^N B_R(b_j)) \leq N c_4 R^k / 2.$$

On the other hand, by (9.14) and $R < R_0$, we have

$$(9.20) \quad \sum_{j=1}^N \|V_{t'-c_3R^2}\|(B_{14R/15}(b_j)) = \|V_{t'-c_3R^2}\|(\cup_{j=1}^N B_{14R/15}(b_j)) > N c_4 R^k.$$

Hence (9.18)-(9.20) show

$$(9.21) \quad \|V_{t'}\|(\cup_{j=1}^N B_R(b_j)) - \|V_{t'-c_3R^2}\|(\cup_{j=1}^N B_{14R/15}(b_j)) < -\frac{c_4 \mathbf{b}}{2\omega_k \mathbf{B}(n)},$$

where we note that the right-hand side is a negative constant independent of R . Let $\phi \in C_c^\infty(B_1)$ be a radially symmetric function such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B_{14/15}$ and

$|\nabla\phi| \leq 30$. Then define $\phi_0(x) := \phi(x/(2r))$ and for $j = 1, \dots, N$, $\phi_j(x) := \phi((x - b_j)/R)$. Since $\cup_{j=1}^N B_R(b_j) \subset B_{3r/2}$ and $\{B_R(b_j)\}_{j=1}^N$ are mutually disjoint, we have

$$(9.22) \quad 1 \geq \tilde{\phi} := \phi_0 - \sum_{j=1}^N \phi_j \geq 0.$$

Due to the definition of ϕ_j and by (9.21), we also have

$$(9.23) \quad \|V_{t'}\|(\sum_{j=1}^N \phi_j) - \|V_{t'-c_3R^2}\|(\sum_{j=1}^N \phi_j) < -\frac{c_4\mathbf{b}}{2\omega_k\mathbf{B}(n)}.$$

We have by (9.22) and (9.23)

$$(9.24) \quad \|V_{t'}\|(\phi_0) - \|V_{t'-c_3R^2}\|(\phi_0) < \|V_{t'}\|(\tilde{\phi}) - \|V_{t'-c_3R^2}\|(\tilde{\phi}) - \frac{c_4\mathbf{b}}{2\omega_k\mathbf{B}(n)}.$$

Since $\tilde{\phi} \in C_c^\infty(U; \mathbb{R}^+)$, by (3.4) and writing $t' - c_3R^2 =: s$ (and omitting $d\|V_t\|dt$),

$$(9.25) \quad \begin{aligned} \|V_{t'}\|(\tilde{\phi}) - \|V_s\|(\tilde{\phi}) &\leq \int_s^{t'} \int (-\tilde{\phi}|h|^2 + |\nabla\tilde{\phi}||h| + \tilde{\phi}|h||u| + |\nabla\tilde{\phi}||u|) \\ &\leq \left\{ \left(\int_s^{t'} \int |\nabla\tilde{\phi}|^2 \right)^{1/2} + \left(\int_s^{t'} \int_{B_{2r}} |u|^2 \right)^{1/2} \right\} \left(\int_s^{t'} \int_{B_{2r}} |h|^2 \right)^{1/2} \\ &\quad + \left(\int_s^{t'} \int |\nabla\tilde{\phi}|^2 \right)^{1/2} \left(\int_s^{t'} \int_{B_{2r}} |u|^2 \right)^{1/2}. \end{aligned}$$

Since $|\nabla\tilde{\phi}| \leq 30/R$, and by (A2) and (A3), we have

$$(9.26) \quad \int_{t'-c_3R^2}^{t'} \int |\nabla\tilde{\phi}|^2 \leq 900\omega_k c_3 (2r)^k E_1, \quad \lim_{R \rightarrow 0} \int_{t'-c_3R^2}^{t'} \int_{B_{2r}} |u|^2 = 0.$$

Combining (9.24)-(9.26), for all small $R > 0$, we obtain

$$(9.27) \quad \|V_{t'}\|(\phi_0) - \|V_{t'-c_3R^2}\|(\phi_0) \leq (900\omega_k c_3 (2r)^k E_1 + 1)^{1/2} \left(\int_{t'-c_3R^2}^{t'} \int_{B_{2r}} |h|^2 \right)^{1/2} - \frac{c_4\mathbf{b}}{4\omega_k\mathbf{B}(n)}.$$

Since $\|V_t\|(\phi_0)$ is continuous at $t = t'$, for all small $R > 0$, we finally obtain

$$(9.28) \quad c_{20} := \left(\frac{c_4\mathbf{b}}{8\omega_k\mathbf{B}(n)} \right)^2 (900\omega_k c_3 (2r)^k E_1 + 1)^{-1} \leq \int_{t'-c_3R^2}^{t'} \int_{B_{2r}} |h|^2.$$

Note that c_{20} is independent of R . Now consider $\hat{\phi}(x) = \phi(x/3r)$ so that $\hat{\phi} \in C_c^\infty(B_{3r})$ and $\hat{\phi} = 1$ on B_{2r} . We have by (3.4), (3.5) and (9.28)

$$(9.29) \quad \|V_{t'}\|(\hat{\phi}) - \|V_{t'-R^2}\|(\hat{\phi}) \leq - \int_{t'-R^2}^{t'} \int \frac{\hat{\phi}|h|^2}{2} + o(1) \leq -\frac{c_{20}}{2} + o(1)$$

for all sufficiently small $R > 0$. But this contradicts the continuity of $\|V_t\|(\hat{\phi})$ as $t \nearrow t'$. Hence we complete the proof. $\square$

Lemma 9.5. *We have $\mathcal{H}^k(A_2) = 0$.*

Proof. Just as in the beginning of proof for Lemma 9.4, for a contradiction, assume $B_{3r} \subset\subset U$ and $\mathcal{H}^k(A_2 \cap B_r) > 0$. Since $A_2 \subset M$, it follows that $\mathcal{H}^k(A_2 \cap B_r) < \infty$. Corresponding to $\nu = 1/2$, we fix L , Λ_4 and ε_7 by Proposition 9.1. For each $m \in \mathbb{N}$ with $(L+1) \leq m$, define

$$(9.30) \quad A_{2,m} := \left\{ x \in A_2 \cap B_r : |R^{-k} \|V_{t'}\|(\phi_R^2(\cdot - x)) - \mathbf{c}| \leq \mathbf{c}/4, \forall R \in (0, r/m] \text{ and } R^{-k-2} \int_{B_{(L+1)R}(x)} |(\text{Tan}_x M)^\perp(y-x)|^2 d\|V_{t'}\|(y) \leq \varepsilon_7^2/2, \forall R \in (0, r/m] \right\},$$

where $\phi_R(x)$ and $\mathbf{c}$ are as in Proposition 9.1. Since there exists an approximate tangent space $\text{Tan}_x M$ for $\mathcal{H}^k$ a.e. $x \in A_2 \cap B_r$, we have $\mathcal{H}^k(A_2 \cap B_r) = \mathcal{H}^k(\cup_{(L+1) \leq m \in \mathbb{N}} A_{2,m})$. Since $\mathcal{H}^k(A_2 \cap B_r) > 0$, for some $(L+1) \leq m_0 \in \mathbb{N}$, we have $\mathcal{H}^k(A_{2,m_0}) > 0$. Similar to (9.16), we may choose $\delta_0 > 0$ such that $\mathcal{H}_{\delta_0}^k(A_{2,m_0}) > 0$. Choose small $R_0 > 0$ so that

$$(9.31) \quad R_0^\zeta \|u\|_{L^{p,q}(U \times (0, \Lambda))} \leq \varepsilon_7, \text{ and } B_{(L+1)R_0}(x) \times (t' - \Lambda_4 R_0^2, t' + \Lambda_4 R_0^2) \subset\subset U \times (0, \Lambda)$$

for all $x \in B_r$. For each $0 < R \leq R_0$, we define

$$(9.32) \quad \begin{aligned} A_{2,m_0}^1(R) &:= \{x \in A_{2,m_0} : (9.1) \text{ fails for any } T \in \mathbf{G}(n, k)\}, \\ A_{2,m_0}^2(R) &:= \{x \in A_{2,m_0} : (9.3) \text{ fails}\}, \quad A_{2,m_0}^3(R) := \{x \in A_{2,m_0} : (9.4) \text{ fails}\} \end{aligned}$$

with $\nu = 1/2$ in (9.3) and (9.4). Since any point of A_{2,m_0} is not $C^{1,\zeta}$ regular, and by (9.31), Proposition 9.1 shows $A_{2,m_0} \subset \cup_{l=1}^3 A_{2,m_0}^l(R)$ for all $0 < R \leq R_0$ (if not, then $x \in A_{2,m_0} \setminus \cup_{l=1}^3 A_{2,m_0}^l(R)$ satisfies (9.1)-(9.4) and Proposition 9.1 applies to (x, t')). Since $\mathcal{H}_{\delta_0}^k$ is sub-additive, we have $\mathcal{H}_{\delta_0}^k(A_{2,m_0}) \leq \sum_{l=1}^3 \mathcal{H}_{\delta_0}^k(A_{2,m_0}^l(R))$ and we have for $0 < R \leq R_0$

$$(9.33) \quad 0 < \mathbf{b} := \mathcal{H}_{\delta_0}^k(A_{2,m_0})/3 \leq \max_{l=1,2,3} \{\mathcal{H}_{\delta_0}^k(A_{2,m_0}^l(R))\}.$$

We prove that we will have a contradiction whichever of three quantities takes the maximum value. We proceed with the assumption that $0 < R \leq \min\{R_0, r/(L+1)m_0, \delta_0/(L+1)\}$.

Case 1. When $\mathbf{b} \leq \mathcal{H}_{\delta_0}^k(A_{2,m_0}^1(R))$.

Consider the covering $\{\overline{B_{(L+1)R}(x)}\}_{x \in A_{2,m_0}^1(R)}$ of $A_{2,m_0}^1(R)$. Just as in Lemma 9.4, there exist mutually disjoint $B_{(L+1)R}(b_1), \dots, B_{(L+1)R}(b_N)$ with $b_1, \dots, b_N \in A_{2,m_0}^1(R)$ and with

$$(9.34) \quad \mathbf{b} \leq \omega_k \mathbf{B}(n)(L+1)^k N R^k.$$

Since $b_j \in A_{2,m_0}$, (9.30) shows

$$(9.35) \quad \sum_{j=1}^N \int_{B_{(L+1)R}(b_j)} |T_j^\perp(x - b_j)|^2 d\|V_{t'}\| \leq \varepsilon_7^2 N R^{k+2}/2,$$

where we denote $T_j := \text{Tan}_{b_j} M$. On the other hand, since $b_j \in A_{2,m_0}^1$, by (9.32), we have

$$(9.36) \quad \sum_{j=1}^N \int_{B_{LR}(b_j)} |T_j^\perp(x - b_j)|^2 d\|V_{t' - \Lambda_4 R^2}\| \geq \varepsilon_7^2 N R^{k+2}.$$

Let $\phi \in C_c^\infty(B_{L+1})$ be a radially symmetric function such that $0 \leq \phi \leq 1$, $\phi = 1$ on B_L and $|\nabla \phi| \leq 2$. For $j = 1, \dots, N$, define

$$(9.37) \quad \phi_j(x) = \phi((x - b_j)/R) \quad \text{and} \quad \xi_j(x) = \{(L+1)R\}^{-2} \phi_j(x) |T_j^\perp(x - b_j)|^2.$$

Note that $\{\xi_j\}_{j=1}^N$ have mutually disjoint supports and that $0 \leq \xi_j \leq 1$. Now combining (9.34)-(9.37), we then obtain

$$(9.38) \quad \|V_{t'}\|(\sum_{j=1}^N \xi_j) - \|V_{t'-\Lambda_4 R^2}\|(\sum_{j=1}^N \xi_j) \leq -\frac{\varepsilon_7^2 \mathbf{b}}{2\omega_k \mathbf{B}(n)(L+1)^{k+2}}.$$

Let $\phi_0 \in C_c^\infty(B_{2r})$ be a radially symmetric function with $0 \leq \phi_0 \leq 1$, $\phi_0 = 1$ on $B_{3r/2}$ and $|\nabla \phi_0| \leq 3/r$. Since $\text{spt } \xi_j \subset B_{3r/2}$, we have

$$(9.39) \quad 1 \geq \tilde{\phi} := \phi_0 - \sum_{j=1}^N \xi_j \geq 0,$$

and by (9.38),

$$(9.40) \quad \|V_{t'}\|(\phi_0) - \|V_{t'-\Lambda_4 R^2}\|(\phi_0) \leq \|V_{t'}\|(\tilde{\phi}) - \|V_{t'-\Lambda_4 R^2}\|(\tilde{\phi}) - \frac{\varepsilon_7^2 \mathbf{b}}{2\omega_k \mathbf{B}(n)(L+1)^{k+2}}.$$

Now the rest of the argument proceeds just like (9.25)-(9.29) since newly defined $\tilde{\phi}$ also satisfies $|\nabla \tilde{\phi}| \leq 3/R$, and we similarly obtain a contradiction to the continuity at $t = t'$. Thus Case 1 cannot occur for sufficiently small $R > 0$.

Case 2. When $\mathbf{b} \leq \mathcal{H}_{\delta_0}^k(A_{2,m_0}^2(R))$.

Consider the covering $\{\overline{B_R(x)}\}_{x \in A_{2,m_0}^2(R)}$ of $A_{2,m_0}^2(R)$. Just as in Lemma 9.4, there exist mutually disjoint $B_R(b_1), \dots, B_R(b_N)$ with $b_1, \dots, b_N \in A_{2,m_0}^2(R)$ with

$$(9.41) \quad \mathbf{b} \leq \omega_k \mathbf{B}(n) N R^k.$$

Since $b_j \in A_{2,m_0}$, (9.30) shows

$$(9.42) \quad R^{-k} \|V_{t'}\|(\phi_R^2(\cdot - b_j)) \leq 5\mathbf{c}/4,$$

while $b_j \in A_{2,m_0}^2(R)$ implies

$$(9.43) \quad R^{-k} \|V_{t'-(\Lambda_4-3)R^2}\|(\phi_R^2(\cdot - b_j)) \geq 3\mathbf{c}/2.$$

Hence (9.41)-(9.43) show

$$(9.44) \quad \|V_{t'}\|(\sum_{j=1}^N \phi_R^2(\cdot - b_j)) - \|V_{t'-(\Lambda_4-3)R^2}\|(\sum_{j=1}^N \phi_R^2(\cdot - b_j)) \leq -\frac{\mathbf{c}\mathbf{b}}{4\omega_k \mathbf{B}(n)}.$$

Define $\phi_0 \in C_c^\infty(B_{2r})$ as in Case 1 and

$$(9.45) \quad 1 \geq \tilde{\phi} := \phi_0 - \sum_{j=1}^N \phi_R^2(\cdot - b_j) \geq 0.$$

Then by (9.44) and (9.45), we have

$$(9.46) \quad \|V_{t'}\|(\phi_0) - \|V_{t'-(\Lambda_4-3)R^2}\|(\phi_0) \leq \|V_{t'}\|(\tilde{\phi}) - \|V_{t'-(\Lambda_4-3)R^2}\|(\tilde{\phi}) - \frac{\mathbf{c}\mathbf{b}}{4\omega_k \mathbf{B}(n)}.$$

Note that $|\nabla \tilde{\phi}| \leq c/R$, and the rest of the argument proceeds just like (9.25)-(9.29), leading to a contradiction for all sufficiently small $R > 0$.

Case 3. When $\mathbf{b} \leq \mathcal{H}_{\delta_0}^k(A_{2,m_0}^3(R))$.

The argument is similar to Case 2. With the same kind of covering balls, we have

$$(9.47) \quad R^{-k} \|V_{t'}\|(\phi_R^2(\cdot - b_j)) \geq 3\mathbf{c}/4$$

in place of (9.42) and

$$(9.48) \quad R^{-k} \|V_{t' + (\Lambda_4 - 3)R^2}\|(\phi_R^2(\cdot - b_j)) \leq \mathbf{c}/2$$

in place of (9.43), leading to

$$(9.49) \quad \|V_{t' + (\Lambda_4 - 3)R^2}\| \left(\sum_{j=1}^N \phi_R^2(\cdot - b_j) \right) - \|V_{t'}\| \left(\sum_{j=1}^N \phi_R^2(\cdot - b_j) \right) \leq -\frac{\mathbf{c}\mathbf{b}}{4\omega_k \mathbf{B}(n)}.$$

Similar argument leads to a contradiction to the continuity of $\|V_t\|(\hat{\phi})$ as $t \searrow t'$.

Thus all three alternatives lead to a contradiction and this shows the claim of Lemma 9.5. $\square$

By Lemma 9.4 and Lemma 9.5, now we conclude the proof of Theorem 3.2.

10. CONCLUDING REMARKS

10.1. C^∞ regularity. In case $u = 0$, Brakke claimed that the support of moving varifold is a.e. time and a.e. everywhere C^∞ . Unfortunately, in examining his proof, there is an essential gap in the argument for going from $C^{1,\varsigma}$ to C^2 estimates. After showing that the spatial gradient is Hölder continuous in the middle of page 202 of [7], Brakke proceeds to prove that the graph can be approximated by some quadratic function, claiming that the graph is $C^{2,1/8}$ in the space variables. For the proof, from (21) of page 203 to (22) of the next page, he substitutes $p + q = R^\varepsilon$ with $\varepsilon = 1/100$, which one can easily see from (22) that it should be $p + q = R^{3/2}$ instead. In the 4th line of (25), Brakke substitutes $p + q = R^\varepsilon$, which instead can only be $R^{3/2}$ from the previous computations. This is a crucial step to show the decay rate is good enough. If $p + q = R^{3/2}$, no improvement is achieved, and his argument has an essential gap in this regard. To remedy the situation, the forthcoming paper [34] gives a proof that, assuming that u is α -Hölder continuous, the support of moving varifold is a.e. $C^{2,\alpha}$ (in parabolic sense) and it satisfies the motion law classically. Thus by the standard parabolic regularity theory, if u is C^∞ , and in particular $u = 0$, the support is a.e. C^∞ , which is Brakke's original claim. The idea of proof is to carry out another blow up argument with second order approximation and is similar to the proof in Section 8 in spirit.

10.2. Time independent case. Suppose that V_t and u do not depend on t while (A1)-(A4) are satisfied. Then (3.4) reduces simply to

$$(10.1) \quad 0 \leq \mathcal{B}(V, u, \phi), \quad \forall \phi \in C_c^1(U; \mathbb{R}^+).$$

We have $V = |M|$ for some countably k -rectifiable set $M \subset U$ by (A1), $h(V) \in L^2(\|V\|)$ by (3.6) and $u \in L^p(\|V\|)$ with $p > k$ by (3.2). We claim that

Lemma 10.1. $h(V) + u^\perp = 0$, $\|V\|$ a.e. on U .

Proof. Let $\hat{x} \in M$ be any Lebesgue point of $h(V)$ and $u^\perp$ with respect to $\mathcal{H}^k$ measure. We may also assume that the approximate tangent space for M exists at $\hat{x}$ and $h(V) = h(V)^\perp$. Set of such points is a full measure set on M . For any $\phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^+)$ and $0 < r < 1$, define $\phi_r(y) := \phi((y - \hat{x})/r)$. For all sufficiently small $r > 0$, $\phi_r \in C_c^1(U; \mathbb{R}^+)$. By (10.1) with this ϕ_r , we have

$$(10.2) \quad 0 \leq \int \{-h\phi_r + r^{-1}\nabla\phi((\cdot - \hat{x})/r)\} \cdot (h + u^\perp) d\|V\|.$$

By change of variables $\tilde{y} = (y - \hat{x})/r$, dividing (10.2) by r^{k-1} , and letting $r \rightarrow 0$, we obtain

$$(10.3) \quad 0 \leq \int_{\text{Tan}_{\hat{x}} M} \nabla \phi(\tilde{y}) d\mathcal{H}^k(\tilde{y}) \cdot (h(V, \hat{x}) + u^\perp(\hat{x})).$$

Note that the vector $\int_{\text{Tan}_{\hat{x}} M} \nabla \phi(\tilde{y}) d\mathcal{H}^k(\tilde{y})$ is perpendicular to $\text{Tan}_{\hat{x}} M$, but otherwise it may be arbitrarily chosen depending on ϕ . Thus for (10.3) to be true, $h(V, \hat{x}) + u^\perp(\hat{x}) = 0$. Thus this holds for $\|V\|$ a.e. $\hat{x}$ on U . $\square$

Thus, in this case, we have $h(V, \cdot) \in L^p(\|V\|)$ with $p > k$. Since V has density equal to 1 a.e., Allard's regularity theorem applies. In this sense, the partial regularity theorem of the present paper is a natural generalization of Allard's regularity theorem.

11. APPENDIX

11.1. Inequalities for $\mathbf{G}(n, k)$. In this subsection we collect some well-known facts which are used in this paper. For completeness we include their proofs.

Lemma 11.1. *For $S, T \in \mathbf{G}(n, k)$ and $v \in \mathbb{R}^n$, we have the following.*

$$(11.1) \quad I \cdot T = k, \quad T^t = T, \quad T \circ T = T, \quad T \circ T^\perp = 0.$$

$$(11.2) \quad 0 \leq k - S \cdot T = S^\perp \cdot T \leq k \|S - T\|^2.$$

$$(11.3) \quad 0 \leq \|S - T\|^2 \leq (S - T) \cdot (S - T) = 2T^\perp \cdot S.$$

$$(11.4) \quad |T(S^\perp(v))| \leq \|T - S\| |v|.$$

$$(11.5) \quad |T(S^\perp(T(v)))| \leq \|T - S\|^2 |v|.$$

Proof. By taking a set of orthonormal vectors $\{v_1, \dots, v_n\}$ such that $v_1, \dots, v_k \in T$, $I \cdot T = \text{tr}(T) = \sum_{i=1}^k v_i \cdot T(v_i) = k$. It is clear that $T^t = T$ if represented for such basis, and thus it is symmetric matrix. $T \circ T = T$ and $T \circ T^\perp = 0$ are clear and this ends the proof of (11.1)

By (11.1), and for the orthonormal vectors $\{v_1, \dots, v_n\}$ as above,

$$\begin{aligned} k - S \cdot T &= I \cdot T - S \cdot T = S^\perp \cdot T = \text{tr}(S^\perp \circ S^\perp \circ T \circ T) = \text{tr}(S^\perp \circ T \circ T \circ S^\perp) \\ &= \text{tr}((T - S) \circ T \circ T \circ (T - S)) = \sum_{i=1}^n v_i^t \cdot (T \circ (T - S) \circ (T - S) \circ T)(v_i) \\ &= \sum_{i=1}^k v_i^t \cdot ((T - S) \circ (T - S))(v_i) \leq k \|S - T\|^2. \end{aligned}$$

It is also clear from the intermediate expression that the above is nonnegative. Thus this proves (11.2).

By choosing a unit vector v such that $\|S - T\| = |(S - T)(v)|$, and noticing

$$|(S - T)(v)|^2 = v^t \cdot ((S - T) \circ (S - T))(v) \leq (S - T) \cdot (S - T),$$

we have $\|S - T\|^2 \leq (S - T) \cdot (S - T)$. Also

$$(S - T) \cdot (S - T) = 2k - 2T \cdot S = 2(I \cdot S - T \cdot S) = 2T^\perp \cdot S$$

proves (11.3).

For (11.4) and (11.5), we have

$$\begin{aligned} |T(S^\perp(v))| &= |T((I - S)(v))| = |T((T - S)(v))| \leq \|T - S\| |v|, \\ |T(S^\perp(T(v)))| &= |T((I - S)(T - S)(v))| = |T((T - S)^2(v))| \leq \|T - S\|^2 |v|. \end{aligned}$$

□

11.2. Additional results. In this subsection we include some results from [7] along with their proofs for the convenience of readers.

Lemma 11.2. (*Tilt of tangent planes* [7, 5.5], [1, 8.13]) *If $U \subset \mathbb{R}^n$ is open, $V \in \mathbf{IV}_k(U)$, $T \in \mathbf{G}(n, k)$, $\phi \in C_c^1(U; \mathbb{R}^+)$,*

$$\begin{aligned} \alpha &= \left(\int_U |h(V, x)|^2 \phi^2(x) d\|V\|(x) \right)^{\frac{1}{2}}, \quad \mu = \left(\int_U |T^\perp(x)|^2 \phi^2(x) d\|V\|(x) \right)^{\frac{1}{2}}, \\ \tilde{\mu} &= \left(\int_U |T^\perp(x)|^2 |\nabla \phi(x)|^2 d\|V\|(x) \right)^{\frac{1}{2}}, \quad \beta = \left(\int_{G_k(U)} \|S - T\|^2 \phi^2(x) dV(x, S) \right)^{\frac{1}{2}}, \end{aligned}$$

then we have

$$(11.6) \quad \beta^2 \leq 4\alpha\mu + 16\tilde{\mu}^2.$$

Proof. Let $g(x) = \phi^2(x)T^\perp(x)$ for $x \in U$. Then for $S \in \mathbf{G}(n, k)$ we have

$$(11.7) \quad \nabla g(x) \cdot S = 2\phi(x)S(T^\perp(x)) \cdot \nabla \phi(x) + \phi^2(x)T^\perp \cdot S.$$

By (11.3) and (11.5), (11.7) implies

$$\begin{aligned} (11.8) \quad \frac{1}{2}\phi^2(x)\|S - T\|^2 &\leq \nabla g(x) \cdot S + 2\phi(x)|S(T^\perp(x)) \cdot \nabla \phi(x)| \\ &\leq \nabla g(x) \cdot S + 2\phi(x)\|S - T\| |T^\perp(x)| |\nabla \phi(x)| \\ &\leq \nabla g(x) \cdot S + \frac{1}{4}\phi^2(x)\|S - T\|^2 + 4|T^\perp(x)|^2 |\nabla \phi(x)|^2. \end{aligned}$$

Since

$$(11.9) \quad \int_{G_k(U)} \nabla g(x) \cdot S dV(x, S) = - \int_U \phi^2(x) h(V, x) \cdot T^\perp(x) d\|V\|(x) \leq \alpha\mu$$

by Schwarz' inequality, (11.8) and (11.9) give (11.6). □

Theorem 11.3. (*Cylindrical growth rates* [7, 6.4]) *Suppose $T \in \mathbf{G}(n, k)$, $0 < R_1 < R_2 < \infty$, $0 \leq \alpha < \infty$, $0 \leq \beta < \infty$, $V \in \mathbf{IV}_k(C(T, R_2))$ is finite and $\text{spt } \|V\|$ is bounded. Suppose $\phi \in C^3(C(T, 1); \mathbb{R}^+)$ depends only on $|T(x)|$ and $\text{spt } \phi \subset C(T, 1)$. Moreover suppose*

$$(11.10) \quad \int_{C(T, R_2)} |h(V, x)|^2 \phi(x/r) d\|V\|(x) \leq \alpha^2 r^k, \quad \forall r \in [R_1, R_2],$$

$$(11.11) \quad \int_{G_k(C(T, R_2))} \|S - T\|^2 \phi(x/r) dV(x, S) \leq \beta^2 r^k, \quad \forall r \in [R_1, R_2].$$

Then we have

$$(11.12) \quad |R_2^{-k} \|V\|(\phi(x/R_2)) - R_1^{-k} \|V\|(\phi(x/R_1))| \leq k\beta^2 \log(R_2/R_1) + \alpha\beta(R_2 - R_1) + \beta^2.$$

Proof. For each fixed $r \in [R_1, R_2]$ we use the vector field $g(x) = r^{-1}\phi(x/r)T(x)$ in the first variation. Since $\text{spt } \|V\|$ is bounded in $C(T, R_2)$, g can be modified suitably so that it has a compact support in $C(T, R_2)$ and so that it does not affect the computations. We have

$$(11.13) \quad \delta V(g) = r^{-1} \int_{G_k(C(T, R_2))} \phi(x/r) T \cdot S + T(x) \otimes \nabla_x \phi(x/r) \cdot S \, dV(x, S).$$

Since ϕ depends only on $|T(x)|$, we can derive that

$$(11.14) \quad \nabla_x \phi(x/r) = -r \frac{\partial \phi(x/r)}{\partial r} \frac{T(x)}{|T(x)|^2}.$$

Using the perpendicularity of mean curvature (2.4), Schwarz' inequality, (11.4), (11.10) and (11.11), we have

$$(11.15) \quad |\delta V(g)| = \left| \int_{G_k(C(T, R_2))} h(V, x) \cdot S^\perp(T(x)/r) \phi(x/r) \, dV(x, S) \right| \leq \alpha \beta r^k.$$

We use (11.13)-(11.15) and (11.18) of Lemma 11.4 to derive

$$(11.16) \quad \left| \frac{d}{dr} \int_{G_k(C(T, R_2))} |S(T(x))|^2 |T(x)|^{-2} \phi(x/r) \, dV(x, S) - \frac{k}{r} \int_{G_k(C(T, R_2))} |S(T(x))|^2 |T(x)|^{-2} \phi(x/r) \, dV(x, S) \right| \leq r^{-1} k \beta^2 r^k + \alpha \beta r^k.$$

Dividing both sides of (11.16) by r^k and integrating this from R_1 to R_2 , we obtain

$$(11.17) \quad \left| r^{-k} \int_{G_k(C(T, R_2))} |S(T(x))|^2 |T(x)|^{-2} \phi(x/r) \, dV(x, S) \right|_{r=R_1}^{R_2} \leq k \beta^2 \log(R_2/R_1) + \alpha \beta (R_2 - R_1).$$

By (11.19) of Lemma 11.4, (11.11) and (11.17), we obtain (11.12). $\square$

Lemma 11.4. For $x \in \mathbb{R}^n$, $S, T \in \mathbf{G}(n, k)$, we have

$$(11.18) \quad |T \cdot S - k|S(T(x))|^2 |T(x)|^{-2}| \leq k \|S - T\|^2,$$

$$(11.19) \quad 0 \leq 1 - |S(T(x))|^2 |T(x)|^{-2} \leq \|S - T\|^2.$$

Proof. We verify (11.18) first. We have

$$(11.20) \quad \begin{aligned} |S(T(x))|^2 &= T(x) \cdot T(S(T(x))) = T(x) \cdot T((S - I)(T(x))) + |T(x)|^2 \\ &= -T(x) \cdot T(S^\perp(T(x))) + |T(x)|^2. \end{aligned}$$

Thus we obtain from (11.20)

$$(11.21) \quad k|S(T(x))|^2 |T(x)|^{-2} - T \cdot S = k - T \cdot S - k \frac{T(x) \cdot T(S^\perp(T(x)))}{|T(x)|^2}.$$

Writing $v = T(x)$, we have

$$(11.22) \quad T(x) \cdot T(S^\perp(T(x))) = v \cdot S^\perp(v) = |S^\perp(v)|^2 \geq 0.$$

Combining (11.2), (11.5), (11.21), and (11.22), we prove (11.18). Observing (11.20) and (11.5), we see that (11.19) holds. $\square$

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