On the GIT moduli of semistable pairs consisting of a plane cubic curve and a line

(平面三次曲線と直線からなる半安定な組の GIT モジュライについて)

Masamichi Kuroda

Department of Mathematics
Hokkaido University, Japan
March 2015

Abstract

We study in some detail the GIT moduli $\overline{P}_{1,3}$ of semistable pairs, each consisting of a plane cubic curve and a line, following the method of the Geometric Invariant Theory. We classify unstable, semistable and stable pairs, respectively, and we give nontrivial identifications of semistable pairs in $\overline{P}_{1,3}$.

This moduli $\overline{P}_{1,3}$ is expected to be close to $\overline{AP}_{1,3}$ which is the first nontrivial case of Alexeev's moduli. For instance, both spaces are of dimension 3. We construct a birational map from $\overline{P}_{1,3}$ to $\overline{AP}_{1,3}$. Each of the base loci of this birational map and its birational inverse are projective lines.

In order to see rough structures of $\overline{P}_{1,3}$ and $\overline{AP}_{1,3}$, we construct natural maps from blowing-ups of $SQ_{1,3} \times \mathbb{P}(V)$ to $\overline{AP}_{1,3}$ and $\overline{P}_{1,3}$, respectively, where $SQ_{1,3}$ is the moduli of Hesse cubics defined by Nakamura and V is the three dimensional vector space, that is, $\mathbb{P}(V)$ is the space of lines. We give a relation between the above birational map and these maps, and hence a relation between known compactifications $SQ_{1,3}$ and $\overline{AP}_{1,3}$.

key words and phrases: Moduli, Stability, Cubic curves.

2010 *Mathematics Subject Classification*: 14H10, 14K10.

Contents

Chapter 1

Introduction

The purpose of this paper is to study in some detail the GIT moduli $\overline{P}_{1,3}$ of semistable pairs (C, L), each with C a plane cubic curve and L a line, following the method of [MFK94]. This is of some interest in connection to the comparison problem of known compactifications $SQ_{1,3}$ of the moduli spaces of level 3 elliptic curves [Na99] and Alexeev's complete moduli $\overline{AP}_{1,3}$ [A02]. Note that dim $SQ_{1,3} = 1$ and dim $\overline{P}_{1,3} =$ dim $\overline{AP}_{1,3} = 3$.

Let k be an algebraic closed field of characteristic $\neq 2, 3$ and let V be the dual space of the space of homogeneous polynomials of degree one on the projective space $\mathbb{P}^2 = \mathrm{Proj}(k[x_0, x_1, x_2])$, that is, $V^\vee = k[x_0, x_1, x_2]_1$. Each $F \in (S^3V)^\vee = S^3V^\vee = k[x_0, x_1, x_2]_3$ (resp. $V^\vee$) corresponds to the cubic curve (resp. line) $V(F)$, where $V(F)$ is the zero set of F in $\mathbb{P}^2$. Then we can regard $\mathbb{P}(S^3V) = \mathrm{Proj}(\mathrm{Sym}(S^3V))$ (resp. $\mathbb{P}(V) = \mathrm{Proj}(\mathrm{Sym}(V))$) as the space of cubic curves (resp. lines). SL(3) acts on $S^3V^\vee$ (resp. $V^\vee$) by $g \cdot F(x) = F(g^{-1} \cdot x)$ for all $F \in S^3V^\vee$ (resp. $V^\vee$). We have the natural morphism $p : \mathrm{SL}(3) \to \mathrm{PGL}(3)$ which is surjective with the finite kernel $I := \{aE \mid a^3 = 1\}$, where E is the identity matrix. For any $g \in \mathrm{PGL}(3)$, let $\tilde{g} \in \mathrm{SL}(3)$ be a matrix such that $p(\tilde{g}) = g$, and we put $g \cdot V(F) := V(\tilde{g} \cdot F)$ for all $F \in S^3V^\vee$ (resp. $V^\vee$). This definition is independent of a choice of $\tilde{g}$, since $\tilde{g}_1\tilde{g}_2^{-1} \in I$ for any two matrices $\tilde{g}_1$ and $\tilde{g}_2$ such that $p(\tilde{g}_1) = p(\tilde{g}_2) = g$. Thus we get an action of PGL(3) on $\mathbb{P}(S^3V) \times \mathbb{P}(V)$:

$$\begin{array}{rcl} \mathrm{PGL}(3) \times \big(\mathbb{P}(S^3V) \times \mathbb{P}(V)\big) & \longrightarrow & \mathbb{P}(S^3V) \times \mathbb{P}(V), \\ (g, (V(F), V(S))) & \longmapsto & (g \cdot V(F), g \cdot V(S)). \end{array}$$

Here we define the GIT moduli $\overline{P}_{1,3}$ of semistable pairs as

$$\overline{P}_{1,3} := (\mathbb{P}(S^3V) \times \mathbb{P}(V))^{ss} /\!/ \mathrm{PGL}(3).$$

By using the numerical criterion due to Hilbert and Mumford, we can classify unstable, semistable and stable pairs completely (see Proposition 3.1.6 and

Table 3.1). Moreover by constructing suitable semistable limits, we give nontrivial identifications of semistable pairs in $\overline{P}_{1,3}$ (see Proposition 3.3.2).

This moduli $\overline{P}_{1,3}$ is expected to be close to $\overline{AP}_{1,3}$ that is the first nontrivial case of Alexeev's moduli $\overline{AP}_{g,d}$ constructed in [A02]. $\overline{AP}_{1,3}$ is the moduli of pairs (C, L), each consisting of a cubic curve C with at worst nodal singularities and a line L which does not pass through singularities of C. Then both spaces are of dimension 3, and a simple computation shows the following (see Theorem 4.1.2 for a more precise statement).

Theorem. *There exists a birational map $f : \overline{P}_{1,3} \to \overline{AP}_{1,3}$. The base locus of f is $\mathbb{P}^1$ which is the set of all semistable pairs (C, L), each consisting of a cuspidal curve C and a line L intersecting with C at smooth points of C. Furthermore the base locus of the birational inverse f^{-1} of f is $\mathbb{P}^1$ which is the set of all abelian pairs (C, L), each consisting of an irreducible cubic curve C and a triple tangent L at a smooth point of C.*

Let $X := SQ_{1,3} \times \mathbb{P}(V)$, where $SQ_{1,3}$ is the moduli of Hesse cubics (see [Na99]). Since X is the space of pairs, each consisting of a Hesse cubic and a line, we have rational maps from X to $\overline{AP}_{1,3}$ and from X to $\overline{P}_{1,3}$. By constructing suitable blowing-ups $\tilde{X}$ and $\hat{X}$ of X along base loci of these rational maps, we can extend these rational maps to natural maps $\varphi : \tilde{X} \to \overline{AP}_{1,3}$ and $\psi : \hat{X} \to \overline{P}_{1,3}$. In fact, $\tilde{X}$ (resp. $\hat{X}$) is a blowing-up of X (resp. $\tilde{X}$) three times. By these maps, we can understand rough structures of $\overline{AP}_{1,3}$ and $\overline{P}_{1,3}$. Moreover by construction, we can obtain a relation between the birational map f and these maps φ and ψ, and hence a relation between known compactifications $SQ_{1,3}$ and $\overline{AP}_{1,3}$ as follows (see Proposition 5.4.5 for a more precise statement).

Proposition. *We have a commutative diagram*

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\psi} & \overline{P}_{1,3} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ \tilde{X} & \xrightarrow{\varphi} & \overline{AP}_{1,3} \\ {\scriptstyle \pi}\downarrow & & \\ X & & \end{array}$$

where π and p are compositions of three blowing-ups, respectively. Then the inverse image of the base locus of f (resp. f^{-1}) by ψ (resp. φ) is the exceptional set (resp. the center) of the composition p of three blowing-ups.

This paper is organized as follows. Chapter 2 is a quick review of some basic results on the Geometric Invariant Theory. In Chapter 3, we study the moduli $\overline{P}_{1,3}$. Especially we classify unstable, semistable and stable pairs, respectively, and we give nontrivial identifications of semistable pairs in $\overline{P}_{1,3}$. In Chapter 4, we compare $\overline{P}_{1,3}$ and $\overline{AP}_{1,3}$ by constructing the birational map $f : \overline{P}_{1,3} \to \overline{AP}_{1,3}$. In Chapter 5, we first give a quick review on the compactification $SQ_{1,3}$ of the moduli space of level 3 elliptic curves. Next we see structures of $\overline{AP}_{1,3}$ and $\overline{P}_{1,3}$ roughly, and give a relation between known compactifications $SQ_{1,3}$ and $\overline{AP}_{1,3}$.

Chapter 2

The Geometric Invariant Theory

In this chapter, we recall briefly some basic results from [MFK94]. We use mainly definitions and properties in [MFK94], [Ne78] and [D03].

2.1 The GIT quotient

An algebraic group over k is a group G together with a structure of algebraic variety over k on G such that the multiplication morphism $\mu : G \times G \to G$ and the inverse morphism $\beta : G \to G$ are morphism of algebraic variety. We define an action of an algebraic group G on a variety X to be a morphism $\sigma : G \times X \to X$ satisfying the usual axioms of an action, that is,

$$\sigma(g, \sigma(g', x)) = \sigma(\mu(g, g'), x), \quad \sigma(e, x) = x,$$

or, in simplified notation,

$$g \cdot (g' \cdot x) = (gg') \cdot x, \quad e \cdot x = x$$

for all $g, g' \in G$, $x \in X$, where e is the identity element of G.

For the construction of a quotient of a variety X by an algebraic group G which acts on X, we desire the quotient to have a natural structure of variety. However, in general the obit space does not have it. Thus we need to modify the definition of the quotient.

If G acts on X, then we obtain an action on the functions on X. For any function $f \in \mathcal{O}_X(U)$ on any open $U \subseteq X$, we define $g \cdot f(x) := f(g^{-1} \cdot x)$ for all $g \in G$. Then we denote by $\mathcal{O}_X(U)^G$ the subring of G-invariant functions:

$$\mathcal{O}_X(U)^G = \{f \in \mathcal{O}_X(U) \mid g \cdot f = f \;\text{ for any }\; g \in G\}.$$

Definition 2.1.1. Let G be an algebraic group acting on a variety X. Then

1. a G-invariant morphism $p : X \to Y$ (that is, $p(g \cdot x) = p(x)$ for all $g \in G$, $x \in X$) is called a categorical quotient, if for any variety Z and any G-invariant morphism $f : X \to Z$ there exists a unique morphism $h : Y \to Z$ such that $h \circ p = f$,

2. a G-invariant morphism $p : X \to Y$ is called a good categorical quotient (or GIT quotient), if p satisfies the following properties:

   (i) For all open $U \subseteq Y$, $p^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(p^{-1}(U))$ is an isomorphism onto the subring $\mathcal{O}_X(p^{-1}(U))^G$ of G-invariant functions.

   (ii) If $W \subseteq X$ is closed and G-invariant (that is, $g \cdot W = W$ for all $g \in G$), then $p(W) \subseteq Y$ is closed.

   (iii) If $V_1, V_2 \subseteq X$ are closed, G-invariant, and $V_1 \cap V_2 = \emptyset$, then $p(V_1) \cap p(V_2) = \emptyset$.

   We denoted by $X//G$ the good categorical quotient (or GIT quotient) of X by G, when it exists.

Proposition 2.1.2. *Let $p : X \to Y$ be a good categorical quotient. Then*

(i) *$p : X \to Y$ is a categorical quotient,*

(ii) *$p(x_1) = p(x_2)$ if and only if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$, where $\overline{G \cdot x_i}$ is the closure of $G \cdot x_i$.*

Proof. (i) For any variety Z and G-invariant morphism $f : X \to Z$, $f^* : \mathcal{O}_Z(Z) \to \mathcal{O}_X(X)$ must be an embedding into $\mathcal{O}_X(X)^G$, since f is G-invariant. By the first condition of the definition of a good categorical quotient, $p^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)^G$ is an isomorphism. Thus we can define a morphism $h^* := (p^*)^{-1} \circ f^* : \mathcal{O}_Z(Z) \to \mathcal{O}_X(X)^G \to \mathcal{O}_Y(Y)$. Then the factoring $p^* \circ h^* = f^*$ is unique, and hence the dual $f = h \circ p$ is a unique factoring of f through p.

(ii) Suppose $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} = \emptyset$. Then since $\overline{G \cdot x_1}$ and $\overline{G \cdot x_2}$ are disjoint closed G-invariant subsets of X, we have $p\left(\overline{G \cdot x_1}\right) \cap p\left(\overline{G \cdot x_2}\right) = \emptyset$ by the third condition of the definition of a good categorical quotient. Hence $p(x_1) \neq p(x_2)$.

Conversely, if $p(x_1) \neq p(x_2)$, then $p^{-1}(p(x_1))$ and $p^{-1}(p(x_2))$ are disjoint closed subsets of X and $p^{-1}(p(x_i)) \supseteq G \cdot x_i$. Hence we obtain $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \subseteq p^{-1}(p(x_1)) \cap p^{-1}(p(x_2)) = \emptyset$. $\square$

Firstly, we construct GIT quotient of affine variety $X = \mathrm{Spec}\ \mathcal{O}_X(X)$ by G. We expect $X//G = \mathrm{Spec}\ \mathcal{O}_X(X)^G$, but for this we require that $\mathcal{O}_X(X)^G$ is finitely generated k-algebra. By the fundamental theorem due to Nagata, this is provided by the group G being reductive (see Theorem 3.4 in [Ne78]).

Definition 2.1.3. 1. A linear algebraic group is a closed subgroup of $\mathrm{GL}(n)$ for some n.

2. A linear algebraic group G is called reductive, if the radical of G (that is, maximal connected solvable normal subgroup) is a torus (that is, isomorphic to $\mathbb{G}_m^r$ for some r).

Example 2.1.4. $\mathrm{PGL}(n)$ is a reductive group. We first show that $\mathrm{PGL}(n)$ is a linear algebraic group. $\mathrm{GL}(n)$ acts on the linear space of $n \times n$ matrices M_n by conjugation. That is, we define $A \cdot M = AMA^{-1}$ for $A \in \mathrm{GL}(n)$, $M \in \mathrm{M}_n$. This defines a morphism $\phi : \mathrm{GL}(n) \to \mathrm{GL}(\mathrm{M}_n) \cong \mathrm{GL}(n^2)$. Then we have $\mathrm{Ker}(\phi) = \{aE \mid a \in k^\times\} \cong \mathbb{G}_m$, where E is the identity matrix. Hence we get an inclusion $\mathrm{PGL}(n) = \mathrm{GL}(n)/\mathbb{G}_m \hookrightarrow \mathrm{GL}(n^2)$.

Next we show that $\mathrm{PGL}(n)$ is reductive. Note that the radical of $\mathrm{GL}(n)$ is the diagonal group $\mathbb{D}_n = \{\mathrm{Diag}(a_1, \ldots, a_n) \mid a_i \in k^\times\}$, and the surjection $\pi : \mathrm{GL}(n) \to \mathrm{PGL}(n)$ has solvable kernel, since $\mathrm{Ker}(\pi) \cong \mathbb{G}_m$. Hence it follows that the radical of $\mathrm{PGL}(n)$ is $\pi(\mathbb{D}_n) \cong \mathbb{G}_m^{n-1}$ (see [B91]).

Theorem 2.1.5. *Let X be an affine variety and G be a reductive group which acts on X. Then $\mathcal{O}_X(X)^G \subseteq \mathcal{O}_X(X)$ is a finitely generated k-algebra and the canonical morphism $p : X \to Y := \mathrm{Spec}\ \mathcal{O}_X(X)^G$ is a good categorical quotient.*

In order to prove this theorem, we need the following geometric property.

Lemma 2.1.6. *Let G be a reductive group acting on an affine variety X. Let Z_1 and Z_2 be two closed G-invariant subsets of X with $Z_1 \cap Z_2 = \emptyset$. Then there exists a G-invariant function $F \in \mathcal{O}_X(X)^G$ such that $F(Z_1) = 1$ and $F(Z_2) = 0$.*

Proof. Since $Z_1 \cap Z_2 = \emptyset$, we have $(1) = I(Z_1 \cap Z_2) = I(Z_1) + I(Z_2)$, where $I(Z_i)$ is the ideal associated with Z_i. Then there exists $f_i \in I(Z_i)$ such that $f_1 + f_2 = 1$. Let $W \subset \mathcal{O}_X(X)$ be the linear subspace spanned by $\{g \cdot f_2 \mid g \in G\}$. We show that W is finite dimensional. We define $F_2 \in \mathcal{O}_{G\times X}(G \times X)$ by $F_2(g, x) = f_2(g \cdot x)$. Since G and X are affine, $\mathcal{O}_{G\times X}(G \times X) = \mathcal{O}_G(G) \otimes \mathcal{O}_X(X)$. Hence we can write F_2 as a finite sum of the form

$$F_2 = \sum_{i=1}^{r} a_i \otimes \varphi_i, \quad a_i \in \mathcal{O}_G(G),\ \varphi_i \in \mathcal{O}_X(X).$$

Then we have

$$g \cdot f_2(x) = f_2(g^{-1} \cdot x) = F_2(g^{-1}, x) = \sum_{i=1}^{r} a_i(g^{-1})\varphi_i(x) \quad \text{for any } g \in G.$$

Thus W is finite dimensional with basis $\{\varphi_1, \dots, \varphi_r\}$.

Let $\varphi : X \to \mathbb{A}^r$, $x \mapsto (\varphi_1(x), \dots, \varphi_r(x))$. Since Z_2 is G-invariant and $f_2 = 0$ on Z_2, we have $g \cdot f_2(z_2) = f_2(g^{-1} \cdot z_2) = 0$ for any $g \in G$, $z_2 \in Z_2$, and hence $\varphi_j = 0$ on Z_2 $(1 \leq j \leq r)$. On the other hand, since Z_1 is G-invariant, $f_1 = 0$ on Z_1 and $f_1 + f_2 = 1$, we have $g \cdot f_2(z_1) = f_2(g^{-1} \cdot z_1) = 1$ for any $g \in G$, $z_1 \in Z_1$, and hence $\varphi_j = 1$ on Z_1 $(1 \leq j \leq r)$. Thus $\varphi(Z_1) = (1, \dots, 1)$ and $\varphi(Z_2) = (0, \dots, 0)$. Since G is reductive, G is geometrically reductive, and hence there exists a nonconstant G-invariant homogeneous polynomial $F' \in k[x_1, \dots, x_r]$ such that $F'(1, \dots, 1) = 1$ (see [Ha75]). Then $F := \varphi^* F'$ has the desired property. $\square$

Proof of Theorem 2.1.5. Let $R := \mathcal{O}_X(X)$. The first assertion follows from Nagata's theorem (see Theorem 3.4 in [Ne78]). We will show that p is a good categorical quotient.

Firstly, we show that p is a G-invariant morphism. For any $g \in G$, the action of g on X is an automorphism of X. Then for all $f \in R^G$, we have $g^* \circ p^*(f) = p^*(f)$, since $p^*(f)$ is G-invariant. Thus we have $p \circ g = p$, that is, p is G-invariant.

Secondly, we show that the first condition of the definition of a good categorical quotient is satisfied. For any open $U \subseteq Y$, there exists $f_i \in R^G$ such that $U = \bigcup_{i=1}^r Y_{f_i}$. For each f_i, we have

$$p^{-1}(Y_{f_i}) = \{x \in X \mid f_i(p(x)) \neq 0\} = \{x \in X \mid p^*(f_i)(x) \neq 0\} = X_{f_i},$$

since p^* is inclusion. Hence we obtain

$$p^{-1}(U) = p^{-1}\left(\bigcup_{i=1}^{r} Y_{f_i}\right) = \bigcup_{i=1}^{r} p^{-1}\left(Y_{f_i}\right) = \bigcup_{i=1}^{r} X_{f_i}.$$

On the other hand, for any $h/f_i^m \in R_{f_i}$, h/f_i^m is contained in $\left(R^G\right)_{f_i} = \mathcal{O}_Y(Y_{f_i})$ if and only if h/f_i^m is G-invariant. Thus $\mathcal{O}_Y(Y_{f_i}) = R^G_{f_i} = \mathcal{O}_X(X_{f_i})^G$. Hence

$$\begin{aligned} \mathcal{O}_Y(U) &= \mathcal{O}_Y\left(\bigcup_{i=1}^{r} Y_{f_i}\right) = \bigcap_{i=1}^{r} \mathcal{O}_Y(Y_{f_i}) = \bigcap_{i=1}^{r} \mathcal{O}_X(X_{f_i})^G \\ &= \mathcal{O}_X\left(\bigcup_{i=1}^{r} X_{f_i}\right)^G = \mathcal{O}_X(p^{-1}(U))^G. \end{aligned}$$

Thirdly, we show that the second condition of the definition of a good categorical quotient is satisfied. Let W be an any closed G-invariant subset of X. Suppose $p(W)$ is not closed, so there exists $y \in \overline{p(W)} \setminus p(W)$. Then the closed subset $Z := p^{-1}(y)$ of X is G-invariant, since p is G-invariant. Clearly $Z \cap W = \emptyset$. By Lemma 2.1.6, there exists $F \in \mathcal{O}_X(X)^G$ such that $F(W) = 1$ and $F(Z) = 0$. Since $p^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)^G$ is an isomorphism, we have $F = p^*(f)$ for some $f \in \mathcal{O}_Y(Y)$. Then we have $f(\overline{p(W)}) = 1$ and $f(y) = f(p(Z)) = 0$, which is a contradiction, since $y \in \overline{p(W)} \setminus p(W)$.

Finally, we show that the third condition of the definition of a good categorical quotient is satisfied. Let $V_1, V_2 \subseteq X$ be closed, G-invariants, and $V_1 \cap V_2 = \emptyset$. By Lemma 2.1.6, there exists $F \in \mathcal{O}_X(X)^G$ such that $F(V_1) = 1$ and $F(V_2) = 0$. Since $p^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)^G$ is an isomorphism, we have $F = p^*(f)$ for some $f \in \mathcal{O}_Y(Y)$. Then we have $f(p(V_1)) = 1$ and $f(p(V_2)) = 0$, and hence $p(V_1) \cap p(V_2) = \emptyset$. $\square$

Next we consider the case where X is a projective variety in $\mathbb{P}^n$. The idea is to cover X by open affine G-invariant sets U_i and then to construct the categorical quotient $X//G$ by gluing together quotients $U_i//G$. For this, we need an extension of the G action on X to a G action on $\mathbb{P}^n$.

Definition 2.1.7. Let G be a linear algebraic group acting on a projective variety X which has an embedding $\varphi : X \hookrightarrow \mathbb{P}^n$.

1. A linearization of the action of G with respect to φ is a linear action of G on $\mathbb{A}^{n+1}$ which induces the given action on X.

2. A linear action of G on X is an action of G together with a linearization of this action.

Unfortunately, for a projective variety X, the GIT quotient $X//G$ does not exist in general. Instead we can take an open G-invariant subset of X which has a good categorical quotient. This open subset is the set of semistable points.

Definition 2.1.8. Let G be a reductive group acting on a projective variety X which has an embedding $\varphi : X \hookrightarrow \mathbb{P}^n$. Let φ_G be a linearization of the action of G with respect to φ. A point $x \in X$ is called

(i) semistable (with respect to φ_G), if there exists some G-invariant homogeneous polynomial f of positive degree such that $f(x) \neq 0$,

(ii) stable (with respect to φ_G), if there exists f as in (i) and additionally $G_x := \{g \in G \mid g \cdot x = x\}$ is finite and all orbits of G in $X_f := \{y \in X \mid f(y) \neq 0\}$ are closed,

(iii) unstable (with respect to φ_G), if it is not semistable.

We write $X^{ss}(\varphi_G)$ (resp. $X^s(\varphi_G)$, $X^{us}(\varphi_G)$) for the set of semistable (resp. stable, unstable) points of X with respect to φ_G, or just X^{ss} (resp. X^s, X^{us}) when the linearization is clear from the context.

Theorem 2.1.9. *Let G be a reductive group acting on a projective variety X which has an embedding $\varphi : X \hookrightarrow \mathbb{P}^n$. Let φ_G be a linearization of the action of G with respect to φ. Let R be the coordinate ring of X. Then there exists a good categorical quotient $p : X^{ss}(\varphi_G) \to X^{ss}(\varphi_G)//G \cong \mathrm{Proj}\ R^G$.*

Proof. By Nagata's theorem, R^G is finite generated k-algebra. Let $f_0, \ldots, f_m$ be generators of R^G. By the definition of semistable points, we have an affine open covering $X^{ss}(\varphi_G) = \bigcup_{i=0}^m X_{f_i}$. Then X_{f_i} is G-invariant, since f_i is G-invariant. Thus for each X_{f_i}, we can take the good geometric quotient

$$\begin{aligned} p_i : X_{f_i} \to X_{f_i}//G &= \mathrm{Spec}\ \mathcal{O}_X(X_{f_i})^G \\ &= \mathrm{Spec}\ (R[1/f_i]_0)^G = \mathrm{Spec}\ R^G[1/f_i]_0 \end{aligned}$$

by Theorem 2.1.5. Take the surjective map

$$k[T_0, \ldots, T_m] \longrightarrow R^G,\ T_i \longmapsto f_i$$

with the kernel $I \subset k[T_0, \ldots, T_m]$. Let Y be the projective variety Proj $R^G \cong V(I) \subset \mathbb{P}^m$. For any $y \in Y \subset \mathbb{P}^m$, there exists $0 \le i \le m$ such that $T_i(y) \neq 0$, so $f_i(y) \neq 0$. Thus we have $y \in Y_{f_i}$, and hence we obtain an affine open covering $Y = \bigcup_{i=0}^m Y_{f_i}$. Then we have

$$Y_{f_i} = \mathrm{Spec}\ \mathcal{O}_Y(Y_{f_i}) = \mathrm{Spec}\ R^G[1/f_i]_0.$$

This gives morphisms $p_i : X_{f_i} \to Y_{f_i}$ $(0 \le i \le m)$. We will show that we can glue these morphisms to get a good categorical quotient

$$p : X \to Y = \mathrm{Proj}\ R^G.$$

Firstly, we show that this gluing of maps is well-defined and p is G-invariant. If p is well-defined then p is G-invariant, since each p_i and X_{f_i} are G-invariant. For any $x \in X_{f_i} \cap X_{f_j} = X_{f_i f_j}$, we need to show $p_i(x) = p_j(x)$. Let $p_{ij} : X_{f_i f_j} \to Y_{f_i f_j}$ be the quotient morphism, and let ι_i, ι_j (resp. κ_i, κ_j) be inclusions from the intersection $Y_{f_i f_j}$ (resp. $X_{f_i f_j}$) to Y_{f_i}, Y_{f_j} (resp. X_{f_i}, X_{f_j}), respectively. For any $x \in X_{f_i f_j}$, $f \in R^G[1/f_i f_j]_0$, we have

$$\begin{aligned} (\kappa_i^* \circ p_i^*)\, f(x) &= \kappa_i^* f(x) = f(\kappa_i(x)) = f(x) \\ &= f(\iota_i(x)) = \iota_i^* f(x) = \left(p_{ij}^* \circ \iota_i^*\right) f(x), \end{aligned}$$

since p_i^*, p_{ij}^*, ι_i, κ_i are inclusion morphisms. Thus we have $\kappa_i^* \circ p_i^* = p_{ij}^* \circ \iota_i^*$, and hence we have $p_i \circ \kappa_i = \iota_i \circ p_{ij}$. Similarly, we have $p_j \circ \kappa_j = \iota_j \circ p_{ij}$. Therefore we get $p_i(x) = p_j(x)$ for all $x \in X_{f_i f_j}$.

Secondly, we show that the first condition of the definition of a good categorical quotient is satisfied. For any open $U \subseteq Y$, we put $U_i := U \cap Y_{f_i}$. Then $\mathcal{O}_Y(U) = \mathcal{O}_Y\left(\bigcup_{i=0}^m U_i\right) = \bigcap_{i=0}^m \mathcal{O}_Y(U_i)$. Since p is surjective, p^* is injective. Thus $p^*\left(\bigcap_{i=0}^m \mathcal{O}_Y(U_i)\right) = \bigcap_{i=0}^m p^*\left(\mathcal{O}_Y(U_i)\right)$. On the other hand, we have $p^{-1}(V) = p_i^{-1}(V)$ for any subset $V \subseteq Y_{f_i}$. In fact, $p^{-1}(V) \supseteq p_i^{-1}(V)$ is clear. Conversely, for any $x \in p^{-1}(V)$, if $x \in X_{f_j} \setminus X_{f_i}$ for some j then

$$f_i(p_j(x)) = p_j^* f_i(x) = f_i(x) = 0,$$

since p_j^* is inclusion. Then we obtain $p(x) = p_j(x) \notin Y_{f_i}$, which is a contradiction, since $p(x) \in V \subseteq Y_{f_i}$. Thus $x \in X_{f_i}$, and hence $p_i(x) = p(x) \in V$. In particular, for each $U_i \subseteq Y_{f_i}$, we obtain an isomorphism

$$p^* : \mathcal{O}_Y(U_i) \to \mathcal{O}_X(p_i^{-1}(U_i))^G = \mathcal{O}_X(p^{-1}(U_i))^G.$$

Therefore the image of $\mathcal{O}_Y(U)$ under p^* is

$$\begin{aligned} p^*\left(\mathcal{O}_Y(U)\right) &= p^*\left(\bigcap_{i=0}^m \mathcal{O}_Y(U_i)\right) = \bigcap_{i=0}^m p^*\left(\mathcal{O}_Y(U_i)\right) = \bigcap_{i=0}^m \mathcal{O}_X(p^{-1}\left(U_i\right))^G \\ &= \mathcal{O}_X\left(\bigcup_{i=0}^m p^{-1}(U_i)\right)^G = \mathcal{O}_X\left(p^{-1}(U)\right)^G. \end{aligned}$$

Since p^* is injective, it follows that $p^* : \mathcal{O}_Y(U) \to \mathcal{O}_X\left(p^{-1}(U)\right)^G$ is an isomorphism.

Thirdly, we show that the second condition of the definition of a good categorical quotient is satisfied. Let W be an any closed G-invariant subset of $X^{ss}(\varphi_G)$. Then $W_i := W \cap X_{f_i}$ is closed G-invariant subset of X_{f_i} clearly. Since p_i's are good categorical quotients, $p_i(W_i) \subseteq Y_{f_i}$ are closed, and hence $p(W) = \bigcup_{i=0}^m p_i(W_i) \subseteq Y$ is closed.

Finally, we show that the third condition of the definition of a good categorical quotient is satisfied. Let V, $W \subseteq X$ be closed, G-invariants, and $V \cap W = \emptyset$. Then $V_i := V \cap X_{f_i}$ and $W_i := W \cap X_{f_i}$ are closed G-invariant subsets of X_{f_i} clearly. Since we know $p^{-1}(Y_{f_i}) = p_i^{-1}(Y_{f_i}) \subseteq X_{f_i}$, we get $p(W) \cap Y_{f_i} = p(W_i) = p_i(W_i)$. Therefore since p_i is a good categorical quotient, we have

$$p(V_i) \cap p(W) = p_i(V_i) \cap p(W) = p_i(V_i) \cap p(W) \cap Y_{f_i} = p_i(V_i) \cap p_i(W_i) = \emptyset,$$

and hence $p(V) \cap p(W) = \bigcup p(V_i) \cap p(W) = \emptyset$. $\square$

2.2 The numerical criterion of stability

Here we give a criterion for stability (so-called the numerical criterion of stability) due to Hilbert and Mumford. We suppose that G is a reductive group acting on a projective variety X embedded in $\mathbb{P}^n$ with a linearization φ_G. The basic idea is to compare the action of G with those of its one parameter subgroups.

Definition 2.2.1. An one parameter subgroup (1-PS) of an algebraic group G is a nontrivial homomorphism of algebraic groups $\lambda : \mathbb{G}_m \to G$.

Let $x^* \in \mathbb{A}^{n+1}$ be a point above $x \in X$. Then we have

$$x \in X^{ss}(\varphi_G) \text{ if and only if } 0 \notin \overline{G \cdot x^*}. \tag{2.1}$$

In fact, if $x \in X^{ss}(\varphi_G)$, then there exists some G-invariant homogeneous polynomial f of positive degree such that $f(x) \neq 0$, and hence $f(x^*) \neq 0$. Then for any $g \in G$, we have $f(g \cdot x^*) = g^{-1} \cdot f(x^*) = f(x^*) \neq 0$, and hence $0 \notin \overline{G \cdot x^*}$. Conversely, if $0 \notin \overline{G \cdot x^*}$, then there exists by Lemma 2.1.6 a G-invariant polynomial f such that $f(0) = 0$ and $f\left(\overline{G \cdot x^*}\right) = 1$. Thus f has constant term 0 and some homogeneous part of f of positive degree must be nonzero at x^*. Hence x is semistable.

Let λ be an 1-PS of G. Then λ induces a linear action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$. Any such action can be diagonalized (see Section 4.6 in [B91]). Hence, there exists a basis $e_0, \ldots, e_n$ of $\mathbb{A}^{n+1}$ such that $\lambda(t) \cdot e_i = t^{m_i} e_i$ for some integer m_i. Here we write $x^* = \sum x_i e_i$. Then we have

$$\lambda(t) \cdot x^* = \sum_{i=0}^{n} t^{m_i} x_i e_i.$$

We define

$$\mu(x, \lambda) = \max\{-m_i \mid x_i \neq 0\}.$$

Note that $\mu(x, \lambda)$ is independent of a choice of the basis of $\mathbb{A}^{n+1}$. In fact, let $e'_0, \ldots, e'_n$ be another basis of $\mathbb{A}^{n+1}$ such that $\lambda(t) \cdot e'_j = t^{m'_j} e'_j$ for some integer m'_j. We write $x^* = \sum x'_j e'_j$, so that $\lambda(t) \cdot x^* = \sum_{i=0}^n t^{m'_j} x'_j e'_i$ and define $\mu'(x, \lambda) = \max\{-m'_j \mid x'_j \neq 0\}$. Then we need to show $\mu(x, \lambda) = \mu'(x, \lambda)$. Let $A = (a_{ij})$ be the $(n+1) \times (n+1)$ matrix which satisfies $(e'_0, \ldots, e'_n) = (e_0, \ldots, e_n)A$. Since $x^* = \sum_{j=0}^n x'_j e'_j = \sum_{i=0}^n \left(\sum_{j=0}^n a_{ij} x'_j\right) e_i$, we have $x_i = \sum_{j=0}^n a_{ij} x'_j$. Since $\lambda(t) \cdot e'_j = \sum_{i=0}^n a_{ij} t^{m_i} e_i$ and $\lambda(t) \cdot e'_j = t^{m'_j} e'_j = \sum_{i=0}^n a_{ij} t^{m'_j} e_i$, we have $a_{ij}\left(t^{m_i} - t^{m'_j}\right) = 0$ for all $0 \leq i, j \leq n$. Suppose $\mu(x, \lambda) = -m_{i_0}$. Then since $x_{i_0} = \sum_{j=0}^n a_{i_0 j} x'_j$ is nonzero, there exists some

$0 \leq j_0 \leq n$ such that $a_{i_0 j_0} x'_{j_0}$ is nonzero. Thus we obtain $\mu(x, \lambda) = -m_{i_0} = -m'_{j_0} \leq \mu'(x, \lambda)$. Similarly we get $\mu'(x, \lambda) \leq \mu(x, \lambda)$. Hence $\mu(x, \lambda) = \mu'(x, \lambda)$.

Suppose all m_i for which $x_i \neq 0$ are strictly positive. Then we have

$$\lim_{t \to 0} \lambda(t) \cdot x^* = \lim_{t \to 0} \sum_{i=0}^{n} t^{m_i} x_i e_i = 0.$$

Thus $0 \in \overline{G \cdot x^*}$, and hence x is unstable by (2.1). Similarly, if all m_i for which $x_i \neq 0$ are strictly negative, then we have $\lim_{t \to 0} \lambda^{-1}(t) \cdot x^* = \lim_{t \to 0} \lambda(t^{-1}) \cdot x^* = 0$, and hence x is unstable. Thus we get

$$x \in X^{ss}(\varphi_G) \Longrightarrow \mu(x, \lambda) \geq 0 \text{ for all 1-PS } \lambda. \tag{2.2}$$

We assume $x \in X^{ss}(\varphi_G)$ and $\mu(x, \lambda) = 0$ for some λ. Let $I := \{i \mid x_i \neq 0,\ m_i > 0\}$. Since $\mu(x, \lambda) = 0$, we have $x_j = 0$ or $m_j = 0$ for any $j \notin I$. Let $y := \sum y_i e_i$, where $y_i = x_i$ if $i \notin I$, and $y_i = 0$ if $i \in I$. Then we have $\lim_{t \to 0} \lambda(t) \cdot x^* = y$, and hence $y \in \overline{\lambda(\mathbb{G}_m) \cdot x^*} \subset \overline{G \cdot x^*}$. If $y \notin G \cdot x^*$, then $G \cdot x^*$ is not closed, and hence x is not stable. Suppose $y \in G \cdot x^*$. By the construction of y, $\lambda(\mathbb{G}_m)$ fixes y. Hence G_y is not finite, and hence y is not stable, and hence x is not stable. Therefore we obtain

$$x \in X^{s}(\varphi_G) \Longrightarrow \mu(x, \lambda) > 0 \text{ for all 1-PS } \lambda. \tag{2.3}$$

Note that the sufficiency of conditions (2.2) and (2.3) are satisfied too.

Theorem 2.2.2. *Let G be a reductive group acting on a projective variety X embedded in $\mathbb{P}^n$ with a linearization φ_G. Then*

$$\begin{aligned} x \in X^{ss}(\varphi_G) &\iff \mu(x, \lambda) \geq 0 \text{ for all 1-PS } \lambda, \\ x \in X^{s}(\varphi_G) &\iff \mu(x, \lambda) > 0 \text{ for all 1-PS } \lambda. \end{aligned}$$

Proof. We have already proved the necessity of the conditions. All we have to do is to show the sufficiency:

$$\begin{aligned} x \in X^{us}(\varphi_G) &\Longrightarrow \mu(x, \lambda) < 0 \text{ for some 1-PS } \lambda, \\ x \in X \setminus X^{s}(\varphi_G) &\Longrightarrow \mu(x, \lambda) \leq 0 \text{ for some 1-PS } \lambda. \end{aligned}$$

Let R be the ring of formal power series $k[[T]]$ and let K be its field of fractions $k((T))$. If $x \in X \setminus X^s(\varphi_G)$, then the morphism $\sigma_{x^*} : G \to \mathbb{A}^{n+1}$, $g \mapsto g \cdot x^*$ is not proper (see Proposition 4.7 in [Ne78]). By the valuative criterion of properness, there exists $g \in G(K) \setminus G(R)$ such that $g \cdot x^* \in \mathbb{A}^{n+1}_R$. Then by Lemme 9.1 in [D03], we can write $g = g_1[\lambda]g_2$, where $g_1, g_2 \in G(R)$, and $[\lambda] \in G(K)$ comes from an 1-PS λ of G in the following sense. We consider λ as a $k(T)$-point of G and identify $k(T)$ with a subfield of $K = k((T))$.

Let $\bar{g}_2$ be the image of g_2 under the residue map $G(R) \to G(k)$ induced by the residue homomorphism $R \to k$, $\sum a_i T^i \mapsto a_0$. Then we have

$$\bar{g}_2^{-1} g_1^{-1} g = \bar{g}_2^{-1} g_1^{-1} \left(g_1[\lambda]g_2\right) = \left(\bar{g}_2^{-1}[\lambda]\bar{g}_2\right) \bar{g}_2^{-1} g_2.$$

Put $\lambda' := \bar{g}_2^{-1}\lambda\bar{g}_2 : \mathbb{G}_m \to G$, $t \mapsto \bar{g}_2^{-1}\lambda(t)\bar{g}_2$. Then we get $[\lambda'] = \bar{g}_2^{-1}[\lambda]\bar{g}_2$ clearly. For the 1-PS λ', there exists a basis $e_0, \ldots, e_n$ of $\mathbb{A}^{n+1}$ such that $\lambda'(t) \cdot e_i = t^{m_i} e_i$ for some integers m_i. This is equivalent to $[\lambda'] \cdot e_i = T^{m_i} e_i$. Here we write $x^* = \sum_{i=0}^n x_i e_i$. Then we have $\mu(x, \lambda') = \max\{-m_i \mid x_i \neq 0\}$ by definition. We show that $\mu(x, \lambda') \leq 0$. For this, we show that $-m_i \leq 0$ if $x_i \neq 0$. For each i, we have

$$\left(\bar{g}_2^{-1} g_1^{-1} g \cdot x^*\right)_i = \left([\lambda'] \cdot \left(\bar{g}_2^{-1} g_2 \cdot x^*\right)\right)_i = T^{m_i} \left(\bar{g}_2^{-1} g_2 \cdot x^*\right)_i.$$

Since $g \cdot x^* \in \mathbb{A}_R^{n+1}$, we obtain

$$\left(\bar{g}_2^{-1} g_2 \cdot x^*\right)_i = T^{-m_i} \left(\bar{g}_2^{-1} g_1^{-1} g \cdot x^*\right)_i \in T^{-m_i} k[[T]]. \tag{2.4}$$

Thus we obtain $-m_i \leq 0$ if $x_i \neq 0$, since $\left(\bar{g}_2^{-1} g_2 \cdot x^*\right)_i \in k[[T]]$ has constant term x_i. Hence we get $\mu(x, \lambda') \leq 0$.

If $x \in X^{us}(\varphi_G)$, then $0 \in \overline{G \cdot x^*}$ by (2.1). By the valuative criterion of properness, there exists $g \in G(K) \setminus G(R)$ such that $g \cdot x^* \in \mathbb{A}_R^{n+1}$ and $g \cdot x^* \equiv 0 \mod (T)$. Then by the same argument, the left-hand side of (2.4) belongs to $T^{-m_i+1} k[[T]]$, and hence we get $-m_i < 0$ if $x_i \neq 0$. Thus $\mu(x, \lambda') < 0$. □

Here we give a observation which will be useful in calculating $\mu(x, \lambda)$.

Lemma 2.2.3. *$\mu(g \cdot x, \lambda) = \mu(x, g^{-1}\lambda g)$ for all $g \in G$.*

Proof. For 1-PS λ, there exists a basis $e_0, \ldots, e_n$ of $\mathbb{A}^{n+1}$ such that $\lambda(t) \cdot e_i = t^{m_i} e_i$ for some integers m_i. Then we have

$$(g^{-1}\lambda(t)g) \cdot (g^{-1} \cdot e_i) = g^{-1} \cdot (\lambda(t) \cdot e_i) = t^{m_i} g^{-1} \cdot e_i.$$

For a basis $g^{-1} \cdot e_0, \ldots, g^{-1} \cdot e_n$ of $\mathbb{A}^{n+1}$, we write $x^* = \sum_{i=0}^n x_i g^{-1} \cdot e_i$. Then we have $g \cdot x^* = \sum_{i=0}^n x_i e_i$ and

$$\lambda(t) \cdot (g \cdot x^*) = \sum_{i=0}^n t^{m_i} x_i e_i, \quad \left(g^{-1}\lambda(t)g\right) \cdot x^* = \sum_{i=0}^n t^{m_i} x_i g^{-1} \cdot e_i.$$

Thus we obtain $\mu(g \cdot x, \lambda) = \mu(x, g^{-1}\lambda g)$ by definition. □

This lemme allows us to replace λ by a convenient conjugate in calculating μ. In particular, for our case $G = \mathrm{PGL}(3)$, any 1-PS λ is conjugate to one of the form

$$\lambda(t) = \mathrm{Diag}(t^{r_0}, t^{r_1}, t^{r_2}), \tag{2.5}$$

with $r_i \in \mathbb{Z}$, $r_0 \geq r_1 \geq r_2$ and $r_0+r_1+r_2 = 0$. Such 1-PS is called normalized.

Chapter 3

The moduli $\overline{P}_{1,3}$

In this chapter, we study the GIT quotient

$$p : \left(\mathbb{P}(S^3V) \times \mathbb{P}(V)\right)^{ss} \longrightarrow \overline{P}_{1,3} = \left(\mathbb{P}(S^3V) \times \mathbb{P}(V)\right)^{ss} //\mathrm{PGL}(3).$$

For this, we first classify unstable pairs.

3.1 Unstable pairs

We give explicitly the unstable pairs of $\mathbb{P}(S^3V) \times \mathbb{P}(V)$ under the action of PGL(3) defined in Chapter 1. By the above argument, to analyze stability we can use normalized one parameter subgroups. Let

$$\begin{array}{rcl} F & = & a_{30}x_1^3 + a_{21}x_1^2x_2 + a_{12}x_1x_2^2 + a_{03}x_2^3 \\ & & +a_{20}x_0x_1^2 + a_{11}x_0x_1x_2 + a_{02}x_0x_2^2 \\ & & +a_{10}x_0^2x_1 + a_{01}x_0^2x_2 \\ & & +a_{00}x_0^3, \\ S & = & b_0x_0 + b_1x_1 + b_2x_2, \end{array}$$

and let $F = F_3 + x_0F_2 + x_0^2F_1 + x_0^3F_0$, $F_k(x) = F_k(x_1, x_2)$, $\deg F_k = k$. Then the image of $z = (V(F), V(S)) \in \mathbb{P}(S^3V) \times \mathbb{P}(V)$ under the Segre embedding $\varphi : \mathbb{P}(S^3V) \times \mathbb{P}(V) \hookrightarrow \mathbb{P}(S^3V \otimes V)$ is $V(H)$, where

$$H(x, y) := F(x)S(y) = \sum a_{ij}b_kx_0^{3-i-j}x_1^ix_2^jy_k.$$

For any normalized 1-PS λ given by the form (2.5), we have

$$\lambda(t) \cdot V(H) = V(H(\lambda(t)^{-1} \cdot x, \lambda(t)^{-1} \cdot y)) = V\left(\sum a_{ij}b_kt^{-R_{ijk}}x_0^{3-i-j}x_1^ix_2^jy_k\right),$$

where $R_{ijk} = (3-i-j)r_0 + ir_1 + jr_2 + r_k$. Thus we get

$$\mu(z, \lambda) = \max\{R_{ijk} \mid a_{ij}b_k \neq 0\}$$

by definition. In what follows, we denote it by $\mu(H, \lambda)$.

Remark 3.1.1. The action of PGL(3) on $\mathbb{P}(S^3V) \times \mathbb{P}(V)$ is not linear, but we see that the definitions of stability (see Def. 2.1.8) are well-defined. For this, we consider the natural morphism $p : \mathrm{SL}(3) \to \mathrm{PGL}(3)$ which is surjective with a finite kernel. Then we have a liner action of SL(3) on $\mathbb{P}(S^3V) \times \mathbb{P}(V)$ which induces the action of PGL(3) on $\mathbb{P}(S^3V) \times \mathbb{P}(V)$. In fact, actions of SL(3) on $S^3V^\vee$ and $V^\vee$ defined in Chapter 1 induce an action of SL(3) on $S^3V^\vee \otimes V^\vee$, hence on $\mathrm{Sym}(S^3V \otimes V)$, hence on $\mathbb{A}^{30} = \mathrm{Spec}\,(\mathrm{Sym}(S^3V \otimes V))$. The action of SL(3) on $\mathbb{A}^{30}$ is a linearization of SL(3) with respect to the Segre embedding $\varphi : \mathbb{P}(S^3V) \times \mathbb{P}(V) \hookrightarrow \mathbb{P}(S^3V \otimes V) = \mathbb{P}^{29}$.

Then since $p : \mathrm{SL}(3) \to \mathrm{PGL}(3)$ is surjective with a finite kernel, and the sets of invariant homogeneous polynomials (resp. the orbits of any point in $\mathbb{P}(S^3V) \times \mathbb{P}(V)$) with respect to these actions are the same, we can regard semistable (stable) points with respect to the action of SL(3) as semistable (stable) points with respect to the action of PGL(3).

To calculate $\mu(H, \lambda)$ we study relations between integer numbers

$$R_{ijk} = 3r_0 + i(r_1 - r_0) + j(r_2 - r_0) + r_k.$$

Since $r_0 \geq r_1 \geq r_2$, we have $R_{ij0} \geq R_{ij1} \geq R_{ij2}$. We fix k and consider relations between R_{ijk}. Since we have $R_{ijk} - R_{i'j'k} = (i-i')(r_1 - r_0) + (j-j')(r_2 - r_0)$ and $r_0 \geq r_1 \geq r_2$, we obtain that $R_{ijk} \geq R_{i'j'k}$ if $i \leq i'$ and $j \leq j'$. This is the vertical relations in the diagram (3.1) below. On the other hand, if $(i-i', j-j') = (1, -1)$, then $R_{ijk} - R_{i'j'k} = r_1 - r_2 \geq 0$. This is the horizontal relations in the diagram (3.1) below. Therefore we obtain the following diagram for fixed k:

$$\begin{array}{ccccccc}
R_{00k} & & & & & & \\
\rotatebox{90}{$\leq$} & & & & & & \\
R_{10k} & \geq & R_{01k} & & & & \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & & & \\
R_{20k} & \geq & R_{11k} & \geq & R_{02k} & & \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
R_{30k} & \geq & R_{21k} & \geq & R_{12k} & \geq & R_{03k}.
\end{array} \tag{3.1}$$

We classify all the geometric pairs (C, L) of H with $\mu(H, \lambda) < 0$ for some normalized 1-PS λ, where $C : F = 0$, $L : S = 0$ and $H = FS$. We know that any unstable pair is equivalent to one of such pairs under the action of PGL(3). In what follows in this section, we assume $\mu(H, \lambda) < 0$ for some normalized λ.

Lemma 3.1.2. $a_{00} = a_{10} = 0$.

Proof. Suppose $a_{00} \neq 0$ or $a_{10} \neq 0$. Then by (3.1), we get $\mu(H, \lambda) \geq R_{10k} \geq R_{102} = 2r_0 + r_1 + r_2 = r_0 \geq 0$, which is absurd. Hence $a_{00} = a_{10} = 0$. $\square$

Lemma 3.1.3. *Let $a_{01} \neq 0$. Then $b_0 = b_1 = a_{20} = 0, b_2 \neq 0$. In this case,*

(i) *L is a triple tangent to C at a smooth point of C, or*

(ii) *$C = L + Q$, $L \not\subset Q_{\mathrm{red}}$.*

Conversely if (C, L) is one of the above two cases, then (C, L) is unstable.

Proof. If $b_0 \neq 0$ or $b_1 \neq 0$, then $\mu(H, \lambda) \geq R_{01k} \geq R_{011} = 2r_0 + r_1 + r_2 = r_0 \geq 0$, which is absurd. Hence $b_0 = b_1 = 0$. If $a_{20} \neq 0$, then by (3.1),

$$\begin{aligned} 0 > \mu(H, \lambda) &= \max\{S_{012}, S_{202}\} = \max\{2r_0 + 2r_2, r_0 + 2r_1 + r_2\} \\ &= \max\{-2r_1, r_1\} \end{aligned}$$

for some nontrivial λ, which is absurd. This shows $a_{20} = 0$. Thus we obtain

$$C : F = x_2(a_{21}x_1^2 + a_{12}x_1x_2 + a_{03}x_3^2 + a_{11}x_0x_1 + a_{02}x_0x_2 + a_{01}x_0^2) + a_{30}x_1^3 = 0$$

and $L : S = x_2 = 0$, where $a_{01} \neq 0$. Assume $a_{30} \neq 0$. Then we get $L \cap C = 3(1 : 0 : 0)$. Since $a_{01} \neq 0$, C is smooth at $P := (1 : 0 : 0)$. Hence L is a triple tangent to C at a smooth point P. Next assume $a_{30} = 0$. Then $F(x) = x_2G(x)$, where $G(x) := a_{21}x_1^2 + a_{12}x_1x_2 + a_{03}x_3^2 + a_{11}x_0x_1 + a_{02}x_0x_2 + a_{01}x_0^2$. Let $Q : G = 0$. Then $C = L + Q$, and $L \not\subset Q$ by $a_{01} \neq 0$.

Conversely in the above two cases, we can choose coordinates such that $L : S = x_2 = 0$ and $C : F = x_2(a_{01}x_0^2 + \cdots) + a_{30}x_1^3 = 0$. In particular, $b_0 = b_1 = 0, b_2 \neq 0$ and $a_{00} = a_{10} = a_{20} = 0$. Then by (3.1), we obtain

$$\begin{aligned} \mu(H, \lambda) &\leq \max\{R_{012}, R_{302}\} = \max\{2r_0 + 2r_2, 3r_1 + r_2\} \\ &= \max\{-2r_1, 2r_1 - r_0\}. \end{aligned}$$

Hence $\mu(H, \lambda) \leq -1 < 0$ for $r = (3, 1, -4)$. $\square$

Lemma 3.1.4. *Assume $a_{01} = 0$ and $F_2 \neq 0$. Then $b_0 = 0$, or $b_0 \neq 0$ and $a_{20} = a_{11} = a_{30} = a_{21} = 0$. In these cases, L passes through a double point of C. Conversely in this case, (C, L) is unstable.*

Proof. If $b_0 = 0$, then $L : S = b_1x_1 + b_2x_2 = 0$ passes through a double point $(1 : 0 : 0)$ of $C : F = F_3 + x_0F_2 = 0$. Suppose $b_0 \neq 0$. Then $a_{20} = a_{11} = a_{30} = a_{21} = 0$. In fact, if $a_{20} \neq 0$, $a_{11} \neq 0$, $a_{30} \neq 0$ or $a_{21} \neq 0$, then by (3.1), we obtain

$$\begin{aligned}\mu(H,\lambda) &\geq \max\{R_{020}, R_{210}\} = \max\{2r_0 + 2r_2, 2r_1 + r_2 + r_0\}\\ &= \max\{-2r_1, r_1\} \geq 0,\end{aligned}$$

which is absurd. Hence $a_{20} = a_{11} = a_{30} = a_{21} = 0$ and $a_{02} \neq 0$. Then $C = 2L' + L''$, $L' : x_2 = 0$ and $L'' : x_0 + a_{12}x_1 + a_{03}x_2 = 0$. Thus L passes through a double point $P := L' \cap L$.

Conversely if L passes through a double point P of C, then we can choose coordinates such that $S = x_2$, $P = (1 : 0 : 0)$ and $F = x_0F_2 + F_3$. In particular, $a_{00} = a_{10} = a_{01} = 0$ and $b_0 = b_1 = 0$. Thus we obtain $\mu(H,\lambda) \leq R_{202} = r_0 + 2r_1 + r_2 = r_1$ by (3.1). Thus $\mu(H,\lambda) \leq -1 < 0$ for $r = (2,-1,-1)$. □

Lemma 3.1.5. *Assume $a_{01} = 0$ and $F_2 = 0$. Then (C, L) is unstable. In this case, C has a triple point, to be more precise, C is a nonreduced 3-lines, or a reduced 3-lines with a triple point. Conversely if C has a triple point, then (C, L) is unstable.*

Proof. By (3.1), we obtain $\mu(H,\lambda) \leq R_{300} = 3r_1 + r_0 = -1 < 0$ for $r = (2,-1,-1)$, and hence (C, L) is unstable. On the other hand, we have $C : F = a_{30}x_1^3 + a_{21}x_1^2x_2 + a_{12}x_1x_2^2 + a_{03}x_2^3 = 0$. Thus C has a triple point $(1 : 0 : 0)$.

Conversely if C has a triple point, then we can choose coordinates such that $C : F_3 = 0$. In particular, $F_0 = F_1 = F_2 = 0$. Then (C, L) is unstable by the above argument. □

In summary, we can prove

Proposition 3.1.6. *The pair (C, L) is unstable if and only if one of the following is true:*

(i) *L is a triple tangent to C,*

(ii) *$L \subset C_{\mathrm{red}}$,*

(iii) *L passes through a double point of C,*

(iv) *C has a triple point,*

(v) *C is nonreduced.*

Proof. All we have to do is to show that if C is nonreduced, then (C, L) is unstable. If C is nonreduced, then C contains a double line or a triple line. If it contains a double line, then L passes through a double point of C, while it contains a triple line, then C has a triple point. Hence if C is nonreduced, then (C, L) is unstable by Lemmas 3.1.4 and 3.1.5. □

Remark 3.1.7. We here introduce the well-known results for stability of plane cubic curves. If a cubic curve $C : F = 0$ is unstable, that is,

$$\mu(F, \lambda) = \max\{R_{ij} := (3 - i - j)r_0 + ir_1 + jr_2 \mid a_{ij} \neq 0\} < 0$$

for some normalized 1-PS λ, then we obtain $a_{00} = a_{10} = a_{01} = a_{20} = a_{11} = 0$ by (3.1) and $R_{11} = r_0 + r_1 + r_2 = 0$. Conversely if $a_{00} = a_{10} = a_{01} = a_{20} = a_{11} = 0$, then $\mu(F, \lambda) \leq \max\{R_{02}, R_{30}\} = \max\{r_0 + 2r_2, 3r_1\} = -1$ for $r = (3, -1, -2)$, and hence C is unstable. If further a_{30} or $a_{02} = 0$, then C is either three lines with a triple point or a line plus a conic tangent to each other. If further $a_{30}a_{02} \neq 0$, then C has a cusp. Conversely in any of these three cases, C is unstable. Therefore C is semistable if and only if C is a 3-gon, a line plus a conic transversal to each other, a nodal curve, or a smooth curve.

Moreover C is stable if and only if C is a smooth curve. In fact, if semistable cubic C has a singular point p, then we may assume that $p = (1 : 0 : 0)$, so that $a_{00} = a_{10} = a_{01} = 0$. Then $\mu(F, \lambda) \leq R_{20} = r_0 + 2r_1 = 0$ for $r = (2, -1, -1)$. Hence C is not stable. Conversely, if C is smooth, then we have a_{00}, a_{10} or $a_{01} \neq 0$. Then $\mu(F, \lambda) \geq \max\{R_{01}, R_{20}\} = \max\{r_0 - r_1, r_1 - r_2\} > 0$ for all nontrivial normalized 1-PS λ. Hence C is stable.

3.2 Semistable and stable pairs

From Proposition 3.1.6, we infer

Proposition 3.2.1. *The pair (C, L) is semistable if and only if any of the following is true:*

(i) *C is reduced and free from triple points. Therefore it is either smooth or a 3-gon or a line plus a conic* possibly tangent to each other*, or a rational curve with a node* or a cusp,

(ii) *L is not contained in C,*

(iii) *L does not pass through any double point of C,*

(iv) *L is not a triple tangent to C.*

Next we classify stable pairs. For it, we classify semistable pairs (C, L) with $\mu(H, \lambda) = 0$ for some nontrivial normalized λ. In this case, (C, L) is semistable but not stable. In what follows in this section, we always assume that (C, L) is semistable.

Definition 3.2.2. We say that L *is* 2*-tangent to* C if L is tangent to an irreducible cubic C, but not triply tangent, at a smooth point of C, and L *is* 3*-tangent to* C if L is triply tangent to C.

Lemma 3.2.3. *Assume* $\mu(H, \lambda) = 0$ *for some nontrivial normalized* λ. *Then* $a_{00} = a_{10} = 0$.

Proof. Suppose $a_{00} \neq 0$ or $a_{10} \neq 0$. Then we obtain $\mu(H, \lambda) \geq R_{102} = r_0 > 0$ for any nontrivial normalized λ, which is absurd. Hence $a_{00} = a_{10} = 0$. □

Lemma 3.2.4. *Assume* $\mu(H, \lambda) = 0$ *for some nontrivial normalized* λ, *and* $a_{01} \neq 0$. *Then* $b_0 = b_1 = 0, b_2 \neq 0$, *and* $a_{20} \neq 0$. *In this case,*

(i) *C is irreducible and L is 2-tangent to C, and if further C is singular, then L does not pass through a double point of C, or*

(ii) *C is an irreducible conic Q plus a line L', and L is tangent to Q and L' is possibly tangent to Q.*

Conversely if (C, L) is one of the above two cases, then (C, L) is semistable but not stable.

Proof. If $b_0 \neq 0$ or $b_1 \neq 0$, then by (3.1), we obtain $\mu(H, \lambda) \geq R_{011} = r_0 > 0$ for any nontrivial normalized λ, which is absurd. Hence $b_0 = b_1 = 0$, $b_2 \neq 0$. If $a_{20} = 0$, then H is unstable by Lemma 3.1.3. Hence $a_{20} \neq 0$. In this case, we have $C : F = x_2(a_{01}x_0^2 + \cdots) + a_{20}x_0x_1^2 + a_{30}x_1^3 = 0$ with $a_{01}a_{20} \neq 0$ and $L : S = x_2 = 0$. Thus L is 2-tangent to C at $P := (1 : 0 : 0)$, which is a smooth point of C by $a_{01} \neq 0$. Since (C, L) is semistable, by Prop. 3.2.1, (i) or (ii) is true.

Conversely, in the first case, we can choose coordinates such that $L : S = x_2 = 0$ and $C : F = a_{01}x_0^2x_2 + A(x)x_2 + a_{20}x_0x_1^2 + a_{30}x_1^3 = 0$ for some quadratic $A(x)$ of degree at most one in x_0. Since L does not pass through any singular points of C, $(1 : 0 : 0)$ is not a singular point of C and hence $a_{01} \neq 0$. Since L is not a triple tangent to C, we have $a_{20} \neq 0$. Therefore $a_{00} = a_{10} = b_0 = b_1 = 0$ and $a_{01}a_{20} \neq 0$. By (3.1), we obtain

$$\mu(H, \lambda) = \max\{R_{012}, R_{202}\} = \max\{-2r_1, r_1\} \geq 0$$

and $\mu(H, \lambda) = 0$ for $r = (1, 0, -1)$. It follows that (C, L) is semistable but not stable.

In the second case, we can choose coordinates such that $L : S = x_2 = 0$, $Q : G = x_0x_2 + x_1^2 = 0$, $L' : S' = x_0 + c_1x_1 + c_2x_2 = 0$ and $P = (1 : 0 : 0)$. In particular, $a_{00} = a_{10} = b_0 = b_1 = 0$ and $a_{01}a_{20} \neq 0$ and hence (C, L) is semistable but not stable by the above argument. Note that L' is tangent to Q if and only if $c_1^2 + 4c_2 = 0$. □

Lemma 3.2.5. *Assume $\mu(H, \lambda) = 0$ for some nontrivial normalized λ, and $a_{01} = 0$. Then $a_{30} = a_{20} = a_{11} = 0$ and $a_{02}a_{21}b_0 \neq 0$. In this case, $C = L'+Q$, L' is a line and Q is an irreducible conic such that $L \not\subset Q$, $L' \neq L$, L' is tangent to Q, L is possibly tangent to Q. Conversely in this case, (C, L) is semistable but not stable.*

Proof. Since (C, L) is semistable, we have $b_0 \neq 0$ by Lemma 3.1.4. If $a_{20} \neq 0$ or $a_{11} \neq 0$, then by (3.1), $\mu(H, \lambda) \geq R_{110} = r_0 > 0$, which is absurd. Hence $a_{20} = a_{11} = 0$. By Lemma 3.1.5, $a_{02} \neq 0$. If $a_{30} \neq 0$, then for some nontrivial normalized λ, we obtain

$$0 = \mu(H, \lambda) = \max\{R_{020}, R_{300}\} = \max\{-2r_1, 2r_1 - r_2\}.$$

Thus $r_1 = r_2 = 0$, and hence this is a contradiction. Hence $a_{30} = 0$. By Lemma 3.1.4, $a_{21} \neq 0$. In this case, we have

$$C : F = x_2(a_{21}x_1^2 + a_{12}x_1x_2 + a_{03}x_2^2 + a_{02}x_0x_2) = 0$$

with $a_{21}a_{02} \neq 0$ and $L : S = x_0 + b_1x_1 + b_2x_2 = 0$. Let $L' : x_2 = 0$ and $Q : a_{21}x_1^2 + x_2(a_{02}x_0 + a_{12}x_1 + a_{03}x_2) = 0$. By $a_{21}a_{02} \neq 0$, Q is an irreducible conic and $L \not\subset Q$. L' is tangent to Q at $(1 : 0 : 0)$.

Conversely in the above case, we can choose coordinates such that $L' : S' = x_2 = 0$, $Q : G = x_2(a_{02}x_0 + a_{12}x_1 + a_{03}x_2) + x_1^2 = 0$, $L : S = x_0 = 0$. Since Q is irreducible, we have $a_{02} \neq 0$. In particular, $a_{00} = a_{10} = a_{01} = a_{20} = a_{11} = a_{30} = 0$ and $a_{21}a_{02}b_0 \neq 0$. Then by (3.1), we obtain

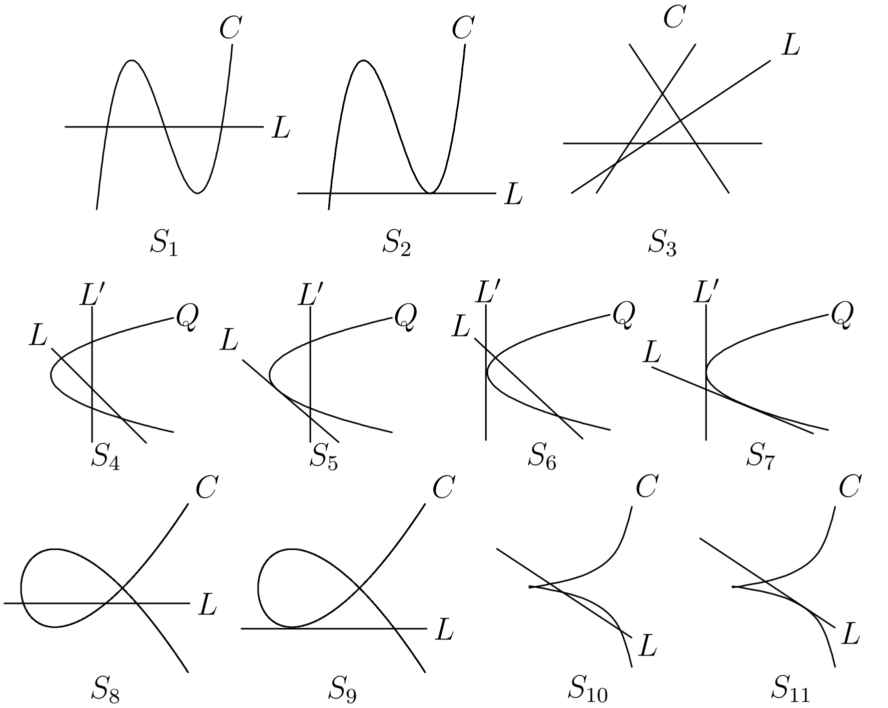
$$\mu(H, \lambda) = \max\{R_{020}, R_{210}\} = \max\{-2r_1, r_1\} \geq 0$$

and $\mu(H, \lambda) = 0$ for $r = (1, 0, -1)$, hence (C, L) is semistable but not stable. Note that L is tangent to Q if and only if $a_{12}^2 = 4a_{03}$. □

By the above Lemmas, we obtain the classification of the semistable and stable pairs (C, L), which is given in Table 3.1. We denote by S_k the locus of semistable pairs in the column k in Table 3.1. Then each semistable pairs in S_k has the following geometric properties.

Table 3.1: The semistable and stable pairs (C, L)

C	C and L	stability	k
smooth	L : transv. to C	stable	1
	L : 2-tangent to C	semistable	2
3-gon	L : transv. to C	stable	3
line L' + conic Q	L, L' transv. to Q	stable	4
	L : tangent to Q	semistable	5
	L' : tangent to Q	semistable	6
	L, L' : tangent to Q	semistable	7
irred. a node	L : transv. to C	stable	8
	L : 2-tangent to C	semistable	9
irred. a cusp	L : transv. to C	stable	10
	L : 2-tangent to C	semistable	11



3.3 Nontrivial identifications in $\overline{P}_{1,3}$

In this section, we give nontrivial identifications of semistable pairs in $\overline{P}_{1,3}$.

Lemma 3.3.1. (1) *The stable loci S_1, S_3, S_4, S_8 and S_{10} are disjoint from each other.*

(2) *We define semistable pairs* $z_i \in S_i$ *(*$i = 3, 5, 6, 7, 11$*) as follows:*

$$\begin{aligned} z_3 &= (V(x_0x_1x_2), V(x_0 + x_1 + x_2)), \\ z_5 &= (V(x_0(x_0x_2 + x_1(x_1 + x_2))), V(x_2)), \\ z_6 &= (V(x_0(x_0x_2 + x_1(x_0 + x_1))), V(x_2)), \\ z_7 &= \left(V(x_0(x_0x_2 + x_1^2)), V(x_2)\right), \\ z_{11} &= \left(V(x_0^2x_2 + x_1^2(x_0 + x_1)), V(x_2)\right). \end{aligned}$$

Then we have $S_3 = \mathrm{PGL}(3) \cdot z_3$*,* $S_5 = \mathrm{PGL}(3) \cdot z_5 + \mathrm{PGL}(3) \cdot z_7$*,* $S_6 = \mathrm{PGL}(3) \cdot z_6 + \mathrm{PGL}(3) \cdot z_7$*,* $S_7 = \mathrm{PGL}(3) \cdot z_7$ *and* $S_{11} = \mathrm{PGL}(3) \cdot z_{11}$*.*

Proof. (1) For any $(C_i, L_i) \in S_i$ ($i = 1, 3, 4, 8, 10$), each C_i has distinct geometric condition each other. Hence the stable loci S_1, S_3, S_4, S_8 and S_{10} are disjoint from each other.

(2) For any semistable pair $(C, L) \in S_3$, there exists $g_1 \in \mathrm{PGL}(3)$ such that $g_1 \cdot C = V(x_0x_1x_2)$, since C is a 3-gon. We write $g_1 \cdot L = V(b_0x_0 + b_1x_1 + b_2x_2)$. Since $g_1 \cdot L$ is transversal to $g_1 \cdot C$, we have $b_0b_1b_2 \neq 0$. Hence $g_2 := \mathrm{Diag}(b_0, b_1, b_2) \in \mathrm{PGL}(3)$ and $g_2g_1 \cdot (C, L) = z_3$.

For any semistable pair $(C, L) = (L' + Q, L) \in S_5$, there exists $g_1 \in \mathrm{PGL}(3)$ such that $g_1 \cdot L' = V(x_0)$ and $g_1 \cdot L = V(x_2)$. We write

$$g_1 \cdot Q = V\left(a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 + a_{10}x_0x_1 + a_{01}x_0x_2 + a_{00}x_0^2\right).$$

Since $g_1 \cdot L \cap g_1 \cdot Q = V(x_2, a_{20}x_1^2 + a_{10}x_0x_1 + a_{00}x_0^2)$ is a double point, we have $a_{10}^2 = 4a_{20}a_{00}$. If $a_{20} = 0$, then $a_{10} = 0$, and hence $g_1 \cdot C$ has a double point $(0 : 1 : 0) \in g_1 \cdot L' \cap g_1 \cdot Q$. Then $g_1 \cdot L$ passes through this point, which is absurd. Thus we get $a_{20} \neq 0$. Then there exists $g_2 \in \mathrm{PGL}(3)$ such that $g_2g_1 \cdot L' = V(x_0)$, $g_2g_1 \cdot L = V(x_2)$ and $g_2g_1 \cdot Q = V\left(x_1^2 + x_2(a_{01}x_0 + a_{11}x_1)\right)$. Since $g_2g_1 \cdot Q$ is irreducible, we have $a_{01} \neq 0$. Therefore we obtain

$$g \cdot C = V(x_0(x_0x_2 + x_1(x_1 + ax_2))) \ \text{ and } \ g \cdot L = V(x_2)$$

for some $g \in \mathrm{PGL}(3)$. If $a = 0$, then $g \cdot (C, L) = z_7$, and if $a \neq 0$, then this pair $g \cdot (C, L)$ is equivalent to z_5 under $\mathrm{PGL}(3)$.

For any semistable pair $(C, L) \in S_6$ (resp. S_7), by a similar argument, we obtain

$$g \cdot C = V(x_0(x_0x_2 + x_1(bx_0 + x_1))) \ \text{ and } \ g \cdot L = V(x_2)$$

(resp. $g \cdot (C, L) = z_7$) for some $g \in \mathrm{PGL}(3)$. If $b = 0$, then $g \cdot (C, L) = z_7$, and if $b \neq 0$, then this pair $g \cdot (C, L)$ is equivalent to z_6 under $\mathrm{PGL}(3)$.

For any semistable pair $(C, L) \in S_{11}$, there exists $g_1 \in \mathrm{PGL}(3)$ such that $g_1 \cdot C = V(x_0^2x_2 + x_1^3)$, since C is a cuspidal curve. We write $g_1 \cdot L =$

$V(b_0x_0 + b_1x_1 + b_2x_2)$. Since $g_1 \cdot L$ does not passes through the cusp point $(0 : 0 : 1)$ of $g_1 \cdot C$, we may assume $b_2 = 1$. Let $g_2 \in \mathrm{PGL}(3)$ which satisfies $g_2^{-1} : (x_0, x_1, x_2) \mapsto (x_0, x_1, -b_0x_0 - b_1x_1 + x_2)$. Then we have $g_2g_1 \cdot L = V(x_2)$ and

$$g_2g_1 \cdot C = V(x_0^2x_2 + x_1^3 - b_1x_0^2x_1 - b_0x_0^3).$$

Since $g_2g_1 \cdot L$ is 2-tangent to $g_2g_1 \cdot C$, we have $x_1^3 - b_1x_0^2x_1 - b_0x_0^3 = (x_1 - ax_0)^2(x_1 - bx_0)$ for some $a \neq b$. Hence $b_0 = -2a^3$, $b_1 = 3a^2$ and $b = -2a$. In particular, $a \neq 0$. Let $g_3 \in \mathrm{PGL}(3)$ which satisfies $g_3^{-1} : (x_0, x_1, x_2) \mapsto (x_0/(3a), x_1 + x_0/3, 9a^2x_2)$ and put $g := g_3g_2g_1 \in \mathrm{PGL}(3)$. Then we have $g \cdot (C, L) = z_{11}$ □

Proposition 3.3.2. *$p(S_3)$, $p(S_5)$, $p(S_6)$, $p(S_7)$ and $p(S_{11})$ are single points in $\overline{P}_{1,3}$ and $p(S_5) = p(S_6) = p(S_7) = p(S_{11})$.*

Proof. By (2) in Lemma 3.3.1, $p(S_3)$, $p(S_7)$ and $p(S_{11})$ are single points.

We prove $p(S_5) = p(S_7)$. For it, by (2) in Lemma 3.3.1, we have to show $p(z_5) = p(z_7)$. It is equivalent to

$$\overline{\mathrm{PGL}(3) \cdot z_5} \cap \overline{\mathrm{PGL}(3) \cdot z_7} \cap (\mathbb{P}(S^3V) \times \mathbb{P}(V))^{ss} \neq \emptyset \tag{3.2}$$

by Proposition 2.1.2. For any $a \in k^\times$, let $g_a \in \mathrm{PGL}(3)$ which satisfies $g_a^{-1} : (x_0, x_1, x_2) \mapsto (x_0/a, x_1, ax_2)$. Then we have

$$g_a \cdot z_5 = (V(x_0(x_0x_2 + x_1(x_1 + ax_2))), V(x_2)) \in (\mathbb{P}(S^3V) \times \mathbb{P}(V))^{ss}.$$

Thus we get $\lim_{a\to 0} g_a \cdot z_5 = z_7$, and hence we obtain (3.2).

Similarly $p(S_6)$ (resp. $p(S_{11})$) is equal to $p(S_7)$. In fact, for any $b \in k^\times$, let g_b, $h_b \in \mathrm{PGL}(3)$ which satisfy $g_b^{-1} : (x_0, x_1, x_2) \mapsto (bx_0, x_1, x_2/b)$, $h_b^{-1} : (x_0, x_1, x_2) \mapsto (x_0, bx_1, b^2x_2)$, respectively. Then we have

$$\begin{aligned} g_b \cdot z_6 &= (V(x_0(x_0x_2 + x_1(bx_0 + x_1))), V(x_2)) \in (\mathbb{P}(S^3V) \times \mathbb{P}(V))^{ss}, \\ h_b \cdot z_{11} &= \big(V(x_0^2x_2 + x_1^2(x_0 + bx_1))), V(x_2)\big) \in (\mathbb{P}(S^3V) \times \mathbb{P}(V))^{ss}. \end{aligned}$$

Then we obtain $\lim_{b\to 0} g_b \cdot z_6 = z_7$ and $\lim_{b\to 0} h_b \cdot z_{11} = z_7$. □

Definition 3.3.3. We define $W_{\mathrm{Cusp}} := p(S_{10}) + p(S_{11}) \subset \overline{P}_{1,3}$.

Lemma 3.3.4. *$W_{\mathrm{Cusp}} \simeq \mathbb{P}^1$ in $\overline{P}_{1,3}$, which is covered with two affine subsets*

$$\begin{aligned} U_1^{\mathrm{Cusp}} &= \{(C, L) \mid C : x_0x_2^2 = x_1^3, L : x_0 = bx_1 + x_2\} \simeq \mathrm{Spec}\ k[b^3], \\ U_2^{\mathrm{Cusp}} &= \{(C, L) \mid C : x_0x_2^2 = x_1^3, L : x_0 = x_1 + cx_2\} \simeq \mathrm{Spec}\ k[c^2], \end{aligned}$$

where b^3 is identified with $1/c^2$.

Proof. For any stable pair $(C, L) \in S_{10}$, there exists $g \in \mathrm{PGL}(3)$ such that

$$g \cdot C : x_0 x_2^2 = x_1^3, \ g \cdot L : x_0 = b_1 x_1 + b_2 x_2$$

with $(b_1, b_2) \neq (0, 0)$. Then the discriminant $D = 4b_1^3 - 27b_2^2$ is nonzero, since $g \cdot L$ is transversal to $g \cdot C$. Since any linear automorphism of $g \cdot C$ keeping the cusp stable is of the form $(x_0, x_1, x_2) \mapsto (x_0, \lambda^2 x_1, \lambda^3 x_2)$, $p(S_{10})$ is the set $\{(b_1, b_2) \mid D \neq 0\}/\sim$, where the equivalence relation $(b_1, b_2) \sim (c_1, c_2)$ is defined by $(c_1, c_2) = (\lambda^2 b_1, \lambda^3 b_2)$ for some $\lambda \in k^\times$. Thus $p(S_{10})$ is the weighted homogeneous space $\mathbb{P}(2, 3)$ minus a single point defined by the discriminant $D = 4b_1^3 - 27b_2^2 = 0$. This point is $p(S_{11})$. In fact, if $4b_1^3 = 27b_2^2$, then $g \cdot L$ is 2-tangent to $g \cdot C$. Thus we obtain $W_{\mathrm{Cusp}} = p(S_{10}) + p(S_{11}) = \mathbb{P}(2, 3) = \mathbb{P}^1$, which is covered with U_1^{Cusp} and U_2^{Cusp} as above. $\square$

Chapter 4

A comparison between $\overline{AP}_{1,3}$ and $\overline{P}_{1,3}$

We define $\overline{AP}_{1,3}$ to be the moduli of pairs, each consisting of a cubic curve C with at worst nodal singularities and a line L which does not pass through singular points of C. It is a proper separated algebraic space over $\mathbb{Z}$. This is the same as Alexeev's complete moduli $\overline{AP}_{1,3}$, because there always exists a semiabelian group scheme acting C in a functorial manner (see [A02]).

In this chapter, we give a birational map from $\overline{P}_{1,3}$ to $\overline{AP}_{1,3}$.

4.1 A birational map from $\overline{P}_{1,3}$ to $\overline{AP}_{1,3}$

Let $W_{\mathrm{T}} \subset \overline{AP}_{1,3}$ be the subset of abelian pairs (C, L) consisting of a cubic curve C and a line L which is 3-tangent to C. We denote by Y (resp. Z) the open dense subset $\overline{P}_{1,3} \setminus W_{\mathrm{Cusp}}$ (resp. $\overline{AP}_{1,3} \setminus W_{\mathrm{T}}$). Then the identity map $f : Y \to Z$, $(C, L) \mapsto (C, L)$ defines a rational map $f : \overline{P}_{1,3} \to \overline{AP}_{1,3}$ and the inverse map $f^{-1} : Z \to Y$ defines the inverse rational map of f. Hence $f : \overline{P}_{1,3} \to \overline{AP}_{1,3}$ is a birational map. The base loci of f, f^{-1} are W_{Cusp}, W_{T}, respectively.

We denote by $G(f)$ the graph of f:

$$G(f) = \mathrm{Im}\,((\mathrm{id}, f) : Y \longrightarrow Y \times Z) \subset \overline{P}_{1,3} \times \overline{AP}_{1,3}.$$

Lemma 4.1.1. *Let $\Gamma := W_{\mathrm{Cusp}} \times W_{\mathrm{T}} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then for any $x \in \Gamma$, there exists a family of semistable pairs $(\mathcal{C}_t, \mathcal{L}_t) \in Y = \overline{P}_{1,3} \setminus W_{\mathrm{Cusp}}$ such that*

$$\lim_{t \to 0} (\mathcal{C}_t, \mathcal{L}_t) = \mathrm{pr}_1(x) \quad \text{and} \quad \lim_{t \to 0} f(\mathcal{C}_t, \mathcal{L}_t) = \mathrm{pr}_2(x),$$

where pr_1 and pr_2 are the canonical projections from Γ to W_{Cusp}, W_{T}, respectively. In particular, we have $\Gamma \subseteq \overline{G(f)} \setminus G(f)$.

Proof. For any $x \in \Gamma$, we have $\mathrm{pr}_1(x) \in W_{\mathrm{Cusp}}$ and $\mathrm{pr}_2(x) \in W_{\mathrm{T}}$. Let $\mathrm{pr}_1(x) = (C, L)$ and $\mathrm{pr}_2(x) = (C', L')$. Since C is a cuspidal curve, L does not pass through the cusp point of C, and L is not 3-tangent to C, we may assume that

$$C : x_0x_2^2 = x_1^3, \quad L : x_0 = b_1x_1 + b_2x_2$$

with $(b_1, b_2) \neq (0, 0)$ clearly. Since C' is a smooth or nodal curve, and L' is a 3-tangent at a smooth point of C', we may assume that

$$C' : x_0x_2^2 = x_1^3 + B_1x_0^2x_1 + B_2x_0^3, \quad L' : x_0 = 0$$

with $(B_1, B_2) \neq (0, 0)$. In fact, assume $L' : x_0 = 0$. Then $C' \cap L'$ is a triple point, hence C' has the form

$$C' : a_{02}x_0x_2^2 + a_{11}x_0x_1x_2 + a_{01}x_0^2x_2 = x_1^3 + a_{20}x_0x_1^2 + a_{10}x_0^2x_1 + a_{00}x_0^3.$$

If $a_{02} = 0$, then C' has a singular point $(0 : 0 : 1)$ and L' passes through this point, which is absurd. Hence $a_{02} \neq 0$. We may assume that $a_{02} = 1$. Thus C' has a Weierstrass form. Since characteristic of k is not equal to 2 or 3, we may assume that C' has an above form.

For $t \neq 0$, we define a pair $(\mathcal{C}_t, \mathcal{L}_t)$ as follows:

$$\begin{aligned} \mathcal{C}_t &: x_0x_2^2 = x_1^3 + B_1t^4x_0^2x_1 + B_2t^6x_0^3, \\ \mathcal{L}_t &: x_0 = b_1x_1 + b_2x_2. \end{aligned}$$

Let $g = \mathrm{Diag}\,(1, t^{-2}, t^{-3}) \in \mathrm{PGL}(3)$. Then $g^{-1} \cdot (x_0, x_1, x_2) = (x_0, t^2x_1, t^3x_2)$, and hence

$$\begin{aligned} g \cdot \mathcal{C}_t &: x_0x_2^2 = x_1^3 + B_1x_0^2x_1 + B_2x_0^3, \\ g \cdot \mathcal{L}_t &: x_0 = b_1t^2x_1 + b_2t^3x_2. \end{aligned}$$

Therefore we have $\lim\limits_{t\to 0} (\mathcal{C}_t, \mathcal{L}_t) = (C, L) = \mathrm{pr}_1(x)$ and

$$\lim_{t\to 0} f(\mathcal{C}_t, \mathcal{L}_t) = \lim_{t\to 0} (\mathcal{C}_t, \mathcal{L}_t) = \lim_{t\to 0} (g \cdot \mathcal{C}_t, g \cdot \mathcal{L}_t) = (C', L') = \mathrm{pr}_2(x).$$

□

Theorem 4.1.2. $f : \overline{P}_{1,3} \to \overline{AP}_{1,3}$ *is a birational map and*

$$W_{\mathrm{Cusp}} \times W_{\mathrm{T}} = \mathrm{pr}_1^{-1}(W_{\mathrm{Cusp}}) \cap \mathrm{pr}_2^{-1}(W_{\mathrm{T}}) = \overline{G(f)} \setminus G(f),$$

where pr_1, pr_2 *are the canonical projections from* $\overline{P}_{1,3} \times \overline{AP}_{1,3}$ *to* $\overline{P}_{1,3}$, $\overline{AP}_{1,3}$, *respectively.*

Proof. Let $\Gamma^\dagger := \mathrm{pr}_1^{-1}(W_{\mathrm{Cusp}}) \cap \mathrm{pr}_2^{-1}(W_{\mathrm{T}})$. Clearly, we have $\Gamma = W_{\mathrm{Cusp}} \times W_{\mathrm{T}} \subset \Gamma^\dagger$. We first show that $\Gamma^\dagger \subset \Gamma$. We denote by ι (resp. $\iota^\dagger$) the canonical inclusion from Γ (resp. $\Gamma^\dagger$) to $\overline{P}_{1,3} \times \overline{AP}_{1,3}$. Since $\mathrm{pr}_1 |_{\Gamma^\dagger} \colon \Gamma^\dagger \to W_{\mathrm{Cusp}}$, $\mathrm{pr}_2 |_{\Gamma^\dagger} \colon \Gamma^\dagger \to W_{\mathrm{T}}$ satisfy a commutative diagram with the given morphisms $W_{\mathrm{Cusp}} \to \mathrm{Spec}\ k$, $W_{\mathrm{T}} \to \mathrm{Spec}\ k$, there exists a unique morphism $\theta : \Gamma^\dagger \to \Gamma$ such that $\mathrm{pr}_i |_{\Gamma} \circ \theta = \mathrm{pr}_i |_{\Gamma^\dagger}$ $(i = 1, 2)$ by the definition of fibered product Γ. Then $\iota^\dagger$ and $\iota \circ \theta$ are morphisms from $\Gamma^\dagger$ to $\overline{P}_{1,3} \times \overline{AP}_{1,3}$ which satisfy $\mathrm{pr}_i \circ \iota^\dagger = \mathrm{pr}_i |_{\Gamma^\dagger}$ and $\mathrm{pr}_i \circ (\iota \circ \theta) = \mathrm{pr}_i |_{\Gamma} \circ \theta = \mathrm{pr}_i |_{\Gamma^\dagger}$ for $i = 1, 2$. By the definition of fibered product $\overline{P}_{1,3} \times \overline{AP}_{1,3}$, such morphism is unique. Hence $\iota^\dagger = \iota \circ \theta$. Therefore $\Gamma^\dagger \subset \Gamma$, since ι and $\iota^\dagger$ are canonical inclusions.

By Lemma 4.1.1, we have $\Gamma \subset \overline{G(f)} \setminus G(f)$. Finally we show that $\overline{G(f)} \setminus G(f) \subset \Gamma^\dagger$. For any $x \in \overline{G(f)} \setminus G(f)$, there exists $x_t \in G(f)$ such that $\lim_{t \to 0} x_t = x$. Suppose $\mathrm{pr}_1(x) \in Y = \overline{P}_{1,3} \setminus W_{\mathrm{Cusp}}$. Then we have

$$Z \ni f(\mathrm{pr}_1(x)) = \lim_{t \to 0} f(\mathrm{pr}_1(x_t)) = \lim_{t \to 0} \mathrm{pr}_2(x_t) = \mathrm{pr}_2(x).$$

Thus $x \in G(f)$, which is absurd. Hence $\mathrm{pr}_1(x) \in W_{\mathrm{Cusp}}$. Since f is birational, we get $\mathrm{pr}_2(x) \in W_{\mathrm{T}}$ similarly. Thus we obtain $\overline{G(f)} \setminus G(f) \subset \mathrm{pr}_1^{-1}(W_{\mathrm{Cusp}}) \cap \mathrm{pr}_2^{-1}(W_{\mathrm{T}}) = \Gamma^\dagger$. $\square$

Chapter 5

Maps from blowing-ups of $SQ_{1,3} \times \mathbb{P}(V)$ to $\overline{AP}_{1,3}$ and $\overline{P}_{1,3}$

In this chapter, in order to see rough structures of $\overline{AP}_{1,3}$ and $\overline{P}_{1,3}$, we first construct natural maps from blowing-ups of $SQ_{1,3} \times \mathbb{P}(V)$ to $\overline{AP}_{1,3}$ and $\overline{P}_{1,3}$, respectively, where $SQ_{1,3}$ is the moduli of Hesse cubics defined in [Na99] and V is the three dimensional vector space. Next we will give a relation between the birational in the previous chapter and these two maps.

We give a quick review on the known compactification $SQ_{1,3}$ of the moduli space of level 3 elliptic curves, before we construct natural maps from blowing-ups of $SQ_{1,3} \times \mathbb{P}(V)$ to $\overline{AP}_{1,3}$ and $\overline{P}_{1,3}$, respectively. For this, we first recall the moduli space $A^{\mathrm{CL}}_{1,3}$ of smooth cubics with the classical level-3 structure and its compactification $SQ^{\mathrm{CL}}_{1,3}$.

5.1 The moduli space of smooth cubics with the classical level-3 structure

Let k be an algebraically closed field of characteristic $\neq 3$ and let $K = (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$, $\{e_1, e_2\}$ a standard basis of K. Let $e_K : K \times K \longrightarrow \mu_3$ be a standard symplectic form of K, where μ_3 is the group of triple roots of unity. That is, e_K is alternating nondegenerate bilinear form such that

$$e_K(e_1, e_2) = e_K(e_2, e_1)^{-1} = \zeta_3, \quad e_K(e_i, e_i) = 1,$$

where ζ_3 is a primitive triple root of unity. Note that for any pair $(ae_1 + be_2, ce_1 + de_2) \in K \times K$, we have

$$e_K(ae_1 + be_2, ce_1 + de_2) = \zeta_3^{ad-bc}.$$

Let C be a smooth cubic with zero O. We denote by $C[3]$ the group of 3-division points $\mathrm{Ker}(3\,\mathrm{id}_C)$. Let $e_C : C[3] \times C[3] \longrightarrow \mu_3$ be the Weil pairing (see III. 8 in [S86]). Since e_C is nondegenerate, e_C is surjective. Hence there exist S, $T \in C[3]$ such that $e_C(S,T) = \zeta_3$. Then $\{S,T\}$ is a basis of $C[3]$. Let $\iota : C[3] \to K$ be an isomorphism defined by $\iota(S) = e_1$, $\iota(T) = e_2$. Then for any pair $(aS + bT, cS + dT) \in C[3] \times C[3]$, we have

$$\begin{aligned} e_C(aS + bT, cS + dT) &= \zeta_3^{ad-bc} = e_K(ae_1 + be_2, ce_1 + de_2) \\ &= e_K \circ (\iota \times \iota)(aS + bT, cS + dT), \end{aligned}$$

and hence $e_C = e_K \circ (\iota \times \iota)$. Thus ι is a symplectic group isomorphism.

Definition 5.1.1. We define $(C, C[3], \iota)$ to be a (planar) cubic with classical level 3-structure if C is a smooth cubic, $C[3]$ is the group of 3-division points, and $\iota : (C[3], e_C) \to (K, e_K)$ is a symplectic group isomorphism.

Example 5.1.2. Let C be a Hesse cubic $C(\mu)$ on $\mathbb{P}^2_k$, that is, C is defined by

$$x_0^3 + x_1^3 + x_2^3 = 3\mu x_0 x_1 x_2,$$

where $\mu \in k$ or $\mu = \infty$ (in which case we define $C(\infty)$ as $x_0x_1x_2 = 0$). We see

(i) $C(\mu)$ is smooth for $\mu \neq \infty$, 1, ζ_3, ζ_3^2,

(ii) $C(\mu)$ is a 3-gon for $\mu = \infty$, 1, ζ_3, ζ_3^2,

(iii) any $C(\mu)$ contains

$$K_0 := \left\{(0 : 1 : -\zeta_3^i), (-\zeta_3^i : 0 : 1), (1 : -\zeta_3^i : 0) \mid i = 0, 1, 2\right\}$$

which is independent of μ,

(iv) for $\mu \neq \infty$, 1, ζ_3, ζ_3^2, K_0 is identified with the group of 3-division points $C(\mu)[3]$ by choosing $(0 : 1 : -1)$ the zero.

We set $S_0 := (0 : 1 : -\zeta_3)$, $T_0 := (1 : -1 : 0) \in K_0$. Then $\{S_0, T_0\}$ is a basis of K_0 and $e_{C(\mu)}(S_0, T_0) = \zeta_3$. Hence we have a symplectic group isomorphism

$$\iota_0 : K_0 \longrightarrow K, \quad \iota_0(S_0) = e_1, \ \iota_0(T_0) = e_2.$$

Thus $(C(\mu), K_0, \iota_0)$, $\mu \neq \infty$, 1, ζ_3, ζ_3^2 is a cubic with classical level 3-structure.

Definition 5.1.3. Two cubics with classical level 3-structure $(C, C[3], \iota)$ and $(C', C'[3], \iota')$ are defined to be isomorphic if there exists an isomorphism $f : C \to C'$ such that $f\,|_{C[3]}: C[3] \to C'[3]$ is a symplectic isomorphism and $\iota' \circ f = \iota$.

Lemma 5.1.4. *For any cubic with classical level* 3*-structure* $(C, C[3], \iota)$*, there exists an unique* $\mu \neq \infty$*,* 1*,* ζ_3*,* ζ_3^2 *such that* $(C, C[3], \iota) \cong (C(\mu), K_0, \iota_0)$*.*

Proof. Firstly, we show the existence of μ. Since C is smooth, there exists an isomorphism $f : C \to C(\mu')$ for some $\mu' \neq \infty$, 1, ζ_3, ζ_3^2 (see Chapter 4, Section 2 in [Hu86]). Let O be the zero of C. We may assume that $f(O) = (0 : 1 : -1)$, if necessary by composing the translation $T_{-f(O)}$. Then since f is an isogeny of degree 1, f is a group isomorphism and the dual isogeny $\hat{f}$ is equal to the inverse isomorphism f^{-1}. Let $S = \iota^{-1}(e_1)$, $T = \iota^{-1}(e_2) \in C[3]$. Since f and $\hat{f} = f^{-1}$ are dual with respect to the Weil pairing, we have

$$e_{C(\mu')}(f(S), f(T)) = e_C(S, \hat{f} \circ f(T)) = e_C(S, T)$$

(see Proposition 8.2 in [S86]). Hence $f\mid_{C[3]}: C[3] \to K_0$ is a symplectic group isomorphism. Then $\iota' := \iota \circ \big(f\mid_{C[3]}\big)^{-1} : K_0 \to K$ is a symplectic isomorphism and we have $f : (C, C[3], \iota) \xrightarrow{\sim} (C(\mu'), K_0, \iota')$. Let $f(S) = aS_0 + bT_0$, $f(T) = cS_0 + dT_0 \in K_0$. Then we have

$$\zeta_3 = e_K(e_1, e_2) = e_C(S, T) = e_{C(\mu')}(f(S), f(T)) = \zeta_3^{ad-bc},$$

and hence $ad - bc \equiv 1 \bmod 3$, so that the pair $(f(S), f(T)) \in K_0 \times K_0$ is contained in $R \cup 2R$, where R consists of

$$\begin{gathered}(S_0, T_0), (S_0, S_0 + T_0), (S_0, 2S_0 + T_0),\\ (T_0, 2S_0), (T_0, 2S_0 + 2T_0), (T_0, 2S_0 + T_0),\\ (S_0 + T_0, 2S_0), (S_0 + T_0, T_0), (S_0 + T_0, S_0 + 2T_0),\\ (S_0 + 2T_0, S_0), (S_0 + 2T_0, 2S_0 + 2T_0), (S_0 + 2T_0, T_0).\end{gathered}$$

For each $(f(S), f(T)) \in R \cup 2R$, there exists $g \in \langle \sigma_1, \sigma_3 \rangle \subset \mathrm{PGL}(3) = \mathrm{Aut}(\mathbb{P}^2)$ such that $g\mid_{C(\mu')}: C(\mu') \longrightarrow C(\mu)$ is a group isomorphism for some $\mu \neq \infty$, 1, ζ_3, ζ_3^2 with $g(f(S)) = S_0$, $g(f(T)) = T_0$, where

$$\sigma_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

In fact, if $(f(S), f(T)) \in R$, then we have the following Table 5.1. We replace each g in Table 5.1 with $\sigma_3^3 g$, if $(f(S), f(T)) \in 2R$.

Since $g\mid_{C(\mu')}$ is an isogeny of degree 1, $g\mid_{K_0}: K_0 \to K_0$ is a symplectic isomorphism by the above argument. By construction, $\iota' = \iota_0 \circ g\mid_{K_0}$, so that we have

$$(C, C[3], \iota) \overset{f}{\underset{\sim}{\longrightarrow}} (C(\mu'), K_0, \iota') \overset{g}{\underset{\sim}{\longrightarrow}} (C(\mu), K_0, \iota_0).$$

Table 5.1: g and μ corresponding to $(f(S), f(T))$

$(f(S), f(T))$	g	μ
(S_0, T_0)	id	μ'
(S_0, S_0+T_0)	σ_3^2	$\zeta_3\mu'$
$(S_0, 2S_0+T_0)$	σ_3^4	$\zeta_3^2\mu'$
$(T_0, 2S_0)$	σ_1	$(\mu'+2)/(\mu'-1)$
$(T_0, 2S_0+2T_0)$	$\sigma_1\sigma_3^2$	$(\zeta_3\mu'+2)/(\zeta_3\mu'-1)$
$(T_0, 2S_0+T_0)$	$\sigma_1\sigma_3^4$	$(\zeta_3^2\mu'+2)/(\zeta_3^2\mu'-1)$
$(S_0+T_0, 2S_0)$	$\sigma_3^2\sigma_1$	$\zeta_3(\mu'+2)/(\mu'-1)$
(S_0+T_0, T_0)	$\sigma_1\sigma_3\sigma_1$	$\zeta_3(\zeta_3\mu'+2)/(\zeta_3\mu'-1)$
(S_0+T_0, S_0+2T_0)	$\sigma_3^5\sigma_1\sigma_3$	$\zeta_3(\zeta_3^2\mu'+2)/(\zeta_3^2\mu'-1)$
(S_0+2T_0, S_0)	$\sigma_3\sigma_1$	$\zeta_3^2(\mu'+2)/(\mu'-1)$
$(S_0+2T_0, 2S_0+2T_0)$	$\sigma_3^5\sigma_1\sigma_3\sigma_1$	$\zeta_3^2(\zeta_3\mu'+2)/(\zeta_3\mu'-1)$
(S_0+2T_0, T_0)	$\sigma_3^4\sigma_1\sigma_3$	$\zeta_3^2(\zeta_3^2\mu'+2)/(\zeta_3^2\mu'-1)$

Secondly, we show the uniqueness of μ. Suppose that $f : (C(\mu), K_0, \iota_0) \to (C(\mu'), K_0, \iota_0)$ is an isomorphism. Then since $\iota_0 = f\ |_{K_0} \circ \iota_0$, we have $f\ |_{K_0} = \mathrm{id}_{K_0}$. In particular, the line $x_0 = 0$ (resp. $x_1 = 0$, $x_2 = 0$) which passes through $(0 : 1 : -\zeta_3^i)$ (resp. $(-\zeta_3^i : 0 : 1)$, $(1 : -\zeta_3^i : 0)$) is fixed by f. Hence $f = \mathrm{Diag}(a_0, a_1, a_2)$ $(a_i \neq 0)$. Moreover since $(0 : 1 : -1)$ and $(-1 : 0 : 1)$ are fixed by f, we have $a_0 = a_1 = a_2$, and hence $f = \mathrm{id}$. Thus we obtain $\mu = \mu'$. □

Proposition 5.1.5. *Let $A_{1,3}^{\mathrm{CL}}$ be the set of isomorphism classes of cubics with classical level 3-structure, and let $SQ_{1,3}^{\mathrm{CL}}$ be the union of $A_{1,3}^{\mathrm{CL}}$ and the 3-gons in Example 5.1.2 (ii). Then we have*

$$\begin{aligned} A_{1,3}^{\mathrm{CL}} &= \{(C(\mu), K_0, \iota_0) \mid \mu \neq \infty, 1, \zeta_3, \zeta_3^2\} \simeq \mathrm{Spec}\ k\left[\mu, \frac{1}{\mu^3-1}\right], \\ SQ_{1,3}^{\mathrm{CL}} &= \{(C(\mu), K_0, \iota_0) \mid \mu \in k \text{ or } \mu = \infty\} \simeq \mathrm{Proj}\ k\left[\mu_0, \mu_1\right], \end{aligned}$$

where $\mu = \mu_1/\mu_0$.

Proof. The assertions follow immediately from Lemma 5.1.4. □

5.2 The moduli space of smooth cubics with a new level-3 structure

If we keep naively using the same definition of the classical level structure in higher dimension, then we will have nonseparated moduli spaces in general.

To construct a separated moduli, we use a lifting of the translation action on C by $C[3]$ to (C, L) as the alternative, where $L := \mathcal{O}_C(1)$ is the hyperplane bundle (see [Na10] in detail). In [Na10], Nakamura considers the moduli spaces over $\mathbb{Z}[\zeta_3, 1/3]$, however in this paper, we consider the moduli space of cubics over an algebraically closed field k of characteristic $\neq 3$.

We set $H := \mathbb{Z}/3\mathbb{Z}$ and $V_H := k[H^\vee] = \bigoplus_{i=0}^{2} k\ v(i)$, where $\{v(i)\}$ is a free k-basis of V_H.

We first construct a $G(3)$-action on $(C(\mu), L(\mu))$ lifting the translation action by K_0 for a smooth Hesse cubic $C(\mu)$ on $\mathbb{P}(V_H) = \mathbb{P}^2_k$ with $L(\mu) = \mathcal{O}_{C(\mu)}(1)$, where K_0 is the group of 3-division points of $C(\mu)$ with zero $O = (0 : 1 : -1)$, we identify V_H with $k[x_0, x_1, x_2]_1$ by the map $v(i) \mapsto x_i$, and $G(3) := \langle \sigma, \tau \rangle \subset \mathrm{SL}(3)$ is the group generated by

$$\sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{bmatrix},\ \tau = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $\sigma\tau = \zeta_3 \tau\sigma$, we have

$$G(3) = \left\{ \zeta_3^\alpha \sigma^a \tau^b \mid \alpha, a, b \in \mathbb{Z}/3\mathbb{Z} \right\}.$$

$G(3)$ acts on $V_H = k[x_0, x_1, x_2]_1$ by

$$\sigma(x_i) = \zeta_3^i x_i, \quad \tau(x_i) = x_{i+1}. \tag{5.1}$$

Then this action induces a group homomorphism $G(3) \to \mathrm{End}(V_H)$. This is called the Schrödinger representation. We denote by U_H this representation. Let $V_H^\vee := \mathrm{Hom}(V_H, k)$ and $\langle\ ,\ \rangle : V_H^\vee \times V_H \to k$ the dual pairing. Using this pairing we have an action of $G(3)$ on $V_H^\vee$ as follows

$$\langle g \cdot F, \gamma \rangle := \langle F, g^{-1}(\gamma) \rangle \quad \text{for any} \quad \gamma \in V_H,$$

where $F \in V_H^\vee$. Hence $G(3)$ acts on $\mathbb{P}^2_k = \mathbb{P}(V_H)$ by $g \cdot V(F) := V(g \cdot F)$. Then we have $g \cdot (x_0 : x_1 : x_2) = (g^{-1}(x_0) : g^{-1}(x_1) : g^{-1}(x_2))$. In particular,

$$\sigma \cdot (x_0 : x_1 : x_2) = (x_0 : \zeta_3^2 x_1 : \zeta_3 x_2), \quad \tau \cdot (x_0 : x_1 : x_2) = (x_2 : x_0 : x_1). \tag{5.2}$$

Then $C(\mu)$ is stable by this action, and hence $G(3)$ acts on $C(\mu)$. For any $P = (a_0 : a_1 : a_2) \in C(\mu)$, we have

$$\sigma \cdot P = P + S, \quad \tau \cdot P = P + T,$$

where $S := \sigma \cdot O = (0 : 1 : -\zeta_3^2)$ and $T := \tau \cdot O = (1 : 0 : -1)$. In fact, if $P = S$ (resp. $2S$), then $\sigma \cdot P = (0 : 1 : -\zeta_3) = 2S$ (resp. $\sigma \cdot P = (0 :$

$1 : -1) = O = 3S$). Suppose $P \neq S,\ 2S$. Then the line $\ell_{P,S}$ (resp. $\ell_{\sigma\cdot P,O}$) passing through P and S (resp. $\sigma \cdot P$ and O) is

$$\ell_{P,S} : \left(\zeta_3^2 a_1 + a_2\right) x_0 - a_0(\zeta_3^2 x_1 + x_2) = 0$$
$$(\text{resp. } \ell_{\sigma\cdot P,O} : \left(\zeta_3^2 a_1 + \zeta_3 a_2\right) x_0 - a_0(x_1 + x_2) = 0).$$

Then $Q := \ell_{P,S} \cap \ell_{\sigma\cdot P,O} = (a_0 : \zeta_3 a_2 : \zeta_3^2 a_1) \in C(\mu)$ and hence we have $P + S = -Q = \sigma \cdot P$. Similarly, we get $\tau \cdot P = P + T$. Therefore for any $g = \zeta_3^\alpha \sigma^a \tau^b \in G(3)$, we have

$$g \cdot P = P + aS + bT.$$

Hence the action of $G(3)$ on $C(\mu)$ is a lifting of the translation action by K_0.

We denote by $p : L(\mu) \to C(\mu)$ the natural projection. Since the action of $G(3)$ on $C(\mu)$ is linear (see Def. 2.1.7), it induces an $L(\mu)$-linear action (see Remark 5.2.1), that is, there is an action of $G(3)$ on $L(\mu)$ such that $p(g \cdot \zeta) = g \cdot p(\zeta)$ for any $\zeta \in L(\mu)$, $g \in G(3)$, and the map $L(\mu)_z \to L(\mu)_{g\cdot z}$, $\zeta \mapsto g \cdot \zeta$ is linear for any $z \in C(\mu)$, $g \in G(3)$, where $L(\mu)_z$ is the fiber $p^{-1}(z)$. Note that this $L(\mu)$-linear action is induced from the Schrödinger representation U_H by the above arguments and this action induces the Schrödinger representation (see Remark 5.2.4), so that we denote by U_H the $L(\mu)$-linear action:

$$U_H : G(3) \times (C(\mu), L(\mu)) \longrightarrow (C(\mu), L(\mu)).$$

By construction, U_H is a $G(3)$-action of weight one, that is, $U_H(a)$ acts as $(\mathrm{id}_{C(\mu)}, a\ \mathrm{id}_{L(\mu)})$ for any a in μ_3 which is the center of $G(3)$.

Remark 5.2.1. An action $\rho : G(3) \to \mathrm{End}(V)$ induces a $\mathbb{H}$-linear action of $G(3)$ on $\mathbb{P}(V)$ as follows, where V is a three dimensional vector space over k and $\mathbb{H} := \mathcal{O}_{\mathbb{P}(V)}(1)$ is the hyperplane bundle. Let $V^\vee := \mathrm{Hom}(V, k)$ and $\langle\ ,\ \rangle : V^\vee \times V \to k$ the dual pairing. We define an action $G(3) \times V^\vee \to V^\vee$, $(g, F) \mapsto \rho(g) \cdot F$,

$$\langle \rho(g) \cdot F, \gamma \rangle := \langle F, \rho\left(g^{-1}\right)(\gamma) \rangle \quad \text{for any } \gamma \in V.$$

This induces an action of $G(3)$ on $(\mathbb{P}(V), \mathbb{T})$, where $\mathbb{T}$ is the tautological line bundle:

$$\mathbb{T} = \left\{(z, v) \in \mathbb{P}(V) \times k^3 \mid v \in \ell_z\right\},$$

where ℓ_z is the line passing through the points above z and the origin of k^3. In fact, for any $(z, v) = (V(F), aF) \in (\mathbb{P}(V), \mathbb{T})$, $F \in V^\vee \setminus \{0\}$, $a \in k$, we define

$$\rho(g) \cdot (z, v) := \left(V\left(\rho(g) \cdot F\right), \rho(g) \cdot (aF)\right).$$

Since the hyperplane bundle $\mathbb{H}$ is the dual of the tautological line bundle $\mathbb{T}$, we have

$$\mathbb{H} = \{(z, f) \in \mathbb{P}(V) \times (k^3)^\vee \mid f \text{ is a linear functional on } \ell_z\},$$

and we can define an action of $G(3)$ on $\mathbb{H}$ as follows:

$$\rho(g) \cdot (z, f) = (\rho(g) \cdot z, \rho(g) \cdot f)\,,$$
$$\rho(g) \cdot f : \ell_{\rho(g)\cdot z} \to k,\ v \mapsto (\rho(g) \cdot f)(v) := f\left(\rho\left(g^{-1}\right) \cdot v\right).$$

This action is a $\mathbb{H}$-linear action of $G(3)$ on $\mathbb{P}(V)$. We denote it by

$$S(\rho(g)) : (\mathbb{P}(V), \mathbb{H}) \longrightarrow (\mathbb{P}(V), \mathbb{H}) \text{ for any } g \in G(3).$$

Moreover if $C \subset \mathbb{P}(V)$ is stable by this action, we have a L-linear action of $G(3)$ on C by the restriction of the above action to (C, L), where $L := \mathcal{O}_C(1)$.

Next let C be any smooth cubic, $L = \mathcal{O}_C(1)$ the line bundle viewed as a scheme over C. By Lemma 5.1.4, there exists a group isomorphism $f : C \to C(\mu)$ such that $f\mid_{C[3]}: C[3] \to K_0$ is a symplectic group isomorphism. Then this isomorphism induces a $G(3)$-action on (C, L) lifting the translation by $C[3]$. We denote it by

$$t : G(3) \times (C, L) \longrightarrow (C, L).$$

In fact, t is defined by

$$\begin{aligned} t : (g, (z, \zeta)) \ \mapsto\ & (t(g) \cdot z, t(g) \cdot \zeta) \\ & := (f^{-1}(U_H(g) \cdot f(z)), F^{-1}(U_H(g) \cdot F(\zeta))), \end{aligned}$$

where we put $F : L = f^*L(\mu) = C \times_{C(\mu)} L(\mu) \overset{\mathrm{pr}_2}{\to} L(\mu)$. Using this $G(3)$-action, we define a new level-3 structure. In a word, new level 3-structure fixes the matrix form of the action of $G(3)$ on the space of sections $V := H^0(C, L)$.

Definition 5.2.2. We define (C, ϕ, t) to be a (planar) cubic with level-$G(3)$ structure (or a level $G(3)$-cubic) if

(i) (C, L) is a (planar) cubic with $L = \mathcal{O}_C(1)$.

(ii) t is a $G(3)$-action of weight one on (C, L), that is, $t(a)$ acts on (C, L) as $(\mathrm{id}_C, a\ \mathrm{id}_L)$ for any $a \in \mu_3$, the center of $G(3)$. Additionally, if C is elliptic, then t induces a translation of C.

(iii) $\phi : C \to \mathbb{P}(V_H)$ is an inclusion, and

$$(\phi, \Phi) : (C, L) \longrightarrow (\mathbb{P}(V_H), \mathbb{H})$$

is a $G(3)$-equivariant morphism by t, where $\Phi : L = \phi^*\mathbb{H} \to \mathbb{H}$ is the natural bundle morphism. That is,

$$(\phi, \Phi)t(g) = S(\rho(\phi, t)(g))(\phi, \Phi) \quad \text{for any } g \in G(3),$$

where $\rho(\phi, t) : G(3) \to \mathrm{End}(V_H)$, $g \mapsto (\phi^*)^{-1} \circ \rho_{t,L}(g) \circ \phi^*$, and $\rho_{t,L} : G(3) \to \mathrm{End}(V)$ is defined by

$$\rho_{t,L}(g)(\theta)(z) = t(g) \cdot \left(\theta\left(t\left(g^{-1}\right) \cdot z\right)\right) \tag{5.3}$$

for any $g \in G(3)$, $\theta \in V$, $z \in C$. Note that $S(\rho(\phi, t)(g))$ is a $\mathbb{H}$-linear action of $g \in G(3)$ on $\mathbb{P}(V)$ induced by $\rho(\phi, t)$ (see Remark 5.2.1).

Remark 5.2.3. $\rho_{t,L} : G(3) \to \mathrm{End}(V)$ is a homomorphism. In fact, for any g_1, $g_2 \in G(3)$, $\theta \in V$, $z \in C$,

$$\begin{aligned}
\left(\rho_{t,L}(g_1) \circ \rho_{t,L}(g_2)(\theta)\right)(z) &= t(g_1) \cdot \left(\rho_{t,L}(g_2)(\theta)\left(t(g_1^{-1}) \cdot z\right)\right) \\
&= t(g_1) \cdot \left(t(g_2) \cdot \left(\theta\left(t(g_2^{-1}) \cdot t(g_1^{-1}) \cdot z\right)\right)\right) \\
&= t(g_1 g_2) \cdot \left(\theta\left(t\left((g_1 g_2)^{-1}\right) \cdot z\right)\right) \\
&= \rho_{t,L}(g_1 g_2)(\theta)(z).
\end{aligned}$$

Hence $\rho_{t,L}(g_1) \circ \rho_{t,L}(g_2) = \rho_{t,L}(g_1 g_2)$.

Remark 5.2.4. Let V be a three dimensional vector space over k. An action $\rho : G(3) \to \mathrm{End}(V)$ induces a $\mathbb{H}$-linear action $S(\rho(g))$ of $g \in G(3)$ on $\mathbb{P}(V)$ (see Remark 5.2.1). For simplicity, we denote by S this action. This action induces a homomorphism $\rho_{S,\mathbb{H}} : G(3) \to \mathrm{End}(V)$ as above (see (5.3) and Remark 5.2.3). Then $\rho_{S,\mathbb{H}} = \rho$. In fact, for any $g \in G(3)$, $\theta \in V$, $z \in \mathbb{P}(V)$,

$$\begin{aligned}
\rho_{S,\mathbb{H}}(g)(\theta)(z) &= S(g) \cdot \left(\theta\left(S\left(g^{-1}\right) \cdot z\right)\right) \\
&= S(g) \cdot \left(\theta\left(\rho\left(g^{-1}\right) \cdot z\right)\right) \\
&= S(g) \cdot \left(\rho\left(g^{-1}\right) \cdot z, \theta\,|_{\ell_{\rho(g^{-1}) \cdot z}}\right) \\
&= \left(z, \rho\,(g) \cdot \left(\theta\,|_{\ell_{\rho(g^{-1}) \cdot z}}\right)\right),
\end{aligned}$$

where $\ell_{\rho(g^{-1}) \cdot z}$ is the line passing through the points above $\rho\left(g^{-1}\right) \cdot z$ and the origin of k^3. Note that $\theta \in V$ is a section of $\mathbb{H}$:

$$\theta : \mathbb{P}(V) \longrightarrow \mathbb{H}, \quad w \longmapsto (w, \theta|_{\ell_w}),$$

and $\theta\,|_{\ell_w}$ is a linear functional on ℓ_w defined by

$$\theta\,|_{\ell_w}\colon \ell_w \longrightarrow k, \quad aF \longmapsto \langle aF, \theta\rangle,$$

where $w = V(F)$, $F \in V^\vee \setminus \{0\}$, $a \in k$, and $\langle\ ,\ \rangle : V^\vee \times V \to k$ is the dual pairing. Recall that $\rho\,(g) \cdot \left(\theta\,|_{\ell_{\rho(g^{-1})\cdot z}}\right) : \ell_z \to k$ is given by

$$\begin{aligned}
\rho\,(g) \cdot \left(\theta\,|_{\ell_{\rho(g^{-1})\cdot z}}\right)(v) &= \theta\,|_{\ell_{\rho(g^{-1})\cdot z}} \left(\rho\left(g^{-1}\right) \cdot v\right) \\
&= \langle \rho\left(g^{-1}\right) \cdot v, \theta\rangle \\
&= \langle v, \rho\,(g)\,(\theta)\rangle = \rho(g)(\theta)\,|_{\ell_z}\,(v)
\end{aligned}$$

(see Remark 5.2.1). Thus we obtain

$$\rho_{S,\mathbb{H}}(g)(\theta)(z) = (z, \rho(g)(\theta)\,|_{\ell_z}) = \rho(g)(\theta)(z),$$

and hence we have $\rho_{S,\mathbb{H}} = \rho$.

Example 5.2.5. Let C be a $G(3)$-invariant cubic on $\mathbb{P}(V_H) = \mathbb{P}^2_k$ under the action induced from the Schrödinger representation U_H (see (5.2)). Then C is a Hesse cubic $C(\mu)$ or one of the following:

$$\begin{aligned}
&C_1(b) : x_0^3 + bx_1^3 + b^2x_2^3 = 0,\\
&C_2(b) : x_1x_2^2 + bx_0x_1^2 + b^2x_0^2x_2 = 0,\\
&C_3(b) : x_1^2x_2 + bx_0x_2^2 + b^2x_0^2x_1 = 0,
\end{aligned}$$

where $b^3 = 1$. Then $(C(\mu), i, U_H)$ with i the inclusion of $C(\mu)$ into $P(V_H)$ is a level $G(3)$-cubic (see Remarks 5.2.1 and 5.2.4). However $(C_j(b), i, U_H)$, $(j, b) \neq (1, 1)$ are not $G(3)$-cubics. Note that $C_1(1)$ is a Hesse cubic $C(\mu)$ with $\mu = 0$. In fact, $(1 : 1 : 1) \in C_1(b)$, $b \neq 1$ (resp. $(1 : 0 : 0) \in C_j(b)$, $j = 2, 3$) is $U_H(\tau)$ (resp. $U_H(\sigma)$)-invariant, and hence τ (resp. σ) is not a translation of $C_1(b)$, $b \neq 1$ (resp. $C_j(b)$, $j = 2, 3$).

Definition 5.2.6. Two level $G(3)$-cubics (C, ϕ, t) and (C', ϕ', t') are defined to be isomorphic if there exists an isomorphism $f : C \longrightarrow C'$ such that

(i) there exists a bundle isomorphism $h : L \to f^*L'$,

(ii) (f, F) is a $G(3)$-isomorphism, that is, $(f, F)t(g) = t'(g)(f, F)$ for any $g \in G(3)$,

where f^*L' is the fibered product

$$\begin{array}{ccc}
f^*L' & \xrightarrow{\mathrm{pr}_2} & L' \\
{\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow \\
C & \xrightarrow{f} & C'
\end{array}$$

and $F := \mathrm{pr}_2 \circ h : L \to f^*L' \to L'$.

Let $V = H^0(C, L)$ and $V' = H^0(C', L')$. If two level $G(3)$-cubics (C, ϕ, t) and (C', ϕ', t') are isomorphic, then by Definition 5.2.6, we have an isomorphism

$$h^{-1} \circ f^* : V' \longrightarrow f^*V' \longrightarrow V,\ \theta' \longmapsto f^*\theta' \longmapsto h^{-1} \circ f^*\theta'.$$

Note that $f^*\theta'$ is a section of f^*L', and hence $h^{-1} \circ f^*\theta'$ is a section of L.

Lemma 5.2.7. *If $f : (C, \phi, t) \xrightarrow{\sim} (C', \phi', t')$, then we have*

$$\rho_{t,L}(g) \circ \left(h^{-1} \circ f^*\right) = \left(h^{-1} \circ f^*\right) \circ \rho_{t',L'}(g) \quad \text{for any} \quad g \in G(3),$$

where $h : L \to f^*L'$ is a bundle isomorphism in Definition 5.2.6, (i).

Proof. For this, we show that

$$f^*\left(\rho_{t',L'}(g)(\theta')\right)(z) = h\left(\rho_{t,L}(g)(h^{-1} \circ f^*\theta')(z)\right) \in f^*L'$$

for any $g \in G(3)$, $\theta' \in V'$, $z \in C$. Since $h : L \to f^*L'$ is a bundle isomorphism, we have $p = \mathrm{pr}_1 \circ h$, where $p : C \to L$ and $\mathrm{pr}_1 : f^*L' \to C$ are natural projections. Hence we have

$$\begin{aligned} \mathrm{pr}_1\left(f^*\left(\rho_{t',L'}(g)(\theta')\right)(z)\right) &= z \\ &= p\left(\rho_{t,L}(g)(h^{-1} \circ f^*\theta')(z)\right) \\ &= \mathrm{pr}_1\left(h\left(\rho_{t,L}(g)(h^{-1} \circ f^*\theta')(z)\right)\right). \end{aligned}$$

By Definition 5.2.6, we have

$$\begin{aligned} \mathrm{pr}_2\left(f^*\left(\rho_{t',L'}(g)(\theta')\right)(z)\right) &= \rho_{t',L'}(g)(\theta')(f(z)) \\ &= t'(g) \cdot \left(\theta'(t'(g^{-1}) \cdot f(z))\right) \\ &= t'(g) \cdot \left(\theta'(f(t(g^{-1}) \cdot z))\right) \\ &= t'(g) \cdot \left(\mathrm{pr}_2(f^*\theta'(t(g^{-1}) \cdot z))\right) \\ &= t'(g) \cdot \left(F \circ h^{-1}(f^*\theta'(t(g^{-1}) \cdot z))\right) \\ &= F\left(t(g) \cdot \left(\left(h^{-1} \circ f^*\theta'\right)(t(g^{-1}) \cdot z)\right)\right) \\ &= F\left(\rho_{t,L}(g)(h^{-1} \circ f^*\theta')(z)\right) \\ &= \mathrm{pr}_2\left(h\left(\rho_{t,L}(g)(h^{-1} \circ f^*\theta')(z)\right)\right). \end{aligned}$$

Thus we obtain $f^*\left(\rho_{t',L'}(g)(\theta')\right)(z) = h\left(\rho_{t,L}(g)(h^{-1} \circ f^*\theta')(z)\right)$. $\square$

Lemma 5.2.8. *Any level $G(3)$-cubic (C, ϕ, t) is isomorphic to a unique Hesse cubic $(C(\mu), i, U_H)$ (see Example 5.2.5).*

Proof. Firstly, we show the existence of μ. Since we have

$$\begin{aligned}\rho(\phi,t)(g_1g_2) &= (\phi^*)^{-1}\rho_{t,L}(g_1g_2)\phi^* \\ &= (\phi^*)^{-1}\rho_{t,L}(g_1)\phi^*(\phi^*)^{-1}\rho_{t,L}(g_2)\phi^* = \rho(\phi,t)(g_1)\rho(\phi,t)(g_2),\end{aligned}$$

V_H is an irreducible $G(3)$-module of weight one through $\rho(\phi,t)$. By Schur's lemma, there exists $A \in \mathrm{GL}(V_H)$ such that

$$U_H(g) = A^{-1}\rho(\phi,t)(g)A = (\phi^*A)^{-1}\rho_{t,L}(g)\phi^*A.$$

Hence it suffices to choose a closed immersion $f : C \to \mathbb{P}(V_H)$ by $f^* = \phi^*A$. Then we have $\rho(f,t)(g) = U_H(g)$ and $(C,\phi,t) \cong (C,f,t)$. Therefore we obtain $f : (C,f,t) \xrightarrow{\sim} (f(C), i, U_H)$ and $f(C) = C(\mu)$ for some μ, because $f(C)$ is a U_H-invariant subscheme of $\mathbb{P}(V_H)$ (see Example 5.2.5).

Secondly, we show the uniqueness of μ. Suppose that $f : (C(\mu), i, U_H) \longrightarrow (C(\mu'), i, U_H)$ is an isomorphism. Then since (f,F) is $G(3)$-isomorphism, by Lemma 5.2.7, we have

$$U_H(g) \circ f^* = f^* \circ U_H(g) \quad \text{for any} \quad g \in G(3).$$

Since the Schrödinger representation U_H is irreducible, f^* is scalar by Schur's lemma. Hence $f = \mathrm{id}_{\mathbb{P}(V_H)}$. Thus we have $C(\mu) = C(\mu')$, so that $\mu = \mu'$. $\square$

Proposition 5.2.9. *Let $SQ_{1,3}$ be the set of isomorphism classes of level $G(3)$-cubics. Then we have*

$$SQ_{1,3} = \{(C(\mu), i, U_H) \mid \mu \in k \textit{ or } \mu = \infty\} \simeq \mathrm{Proj}\ k\,[\mu_0, \mu_1]\,,$$

where $\mu = \mu_1/\mu_0$.

Proof. The assertion follows immediately from Lemma 5.2.8. $\square$

Remark 5.2.10. Any smooth level $G(3)$-structure (C,ϕ,t) induces a classical level 3-structure $(C, C[3], \iota)$ as follows. Since t is a $G(3)$-action of weight one and induces a translation of C, we have $t(g)\cdot P = P + aS + bT$ for any $P \in C$, $g = \zeta_3^{\alpha}\sigma^a\tau^b \in G(3)$, where $S := t(\sigma)\cdot O$, $T := t(\tau)\cdot O$. Then since we have $(n+1)S = nS + S = t(\sigma)\cdot(nS)$ for any $n \in \mathbb{N}$, we get $3S = t(\sigma^3)\cdot O = O$. Similarly, $3T = O$. Thus we obtain

$$C[3] = G(3)\cdot O = \{aS + bT \mid a, b \in \mathbb{Z}/3\mathbb{Z}\}.$$

Let $e_C(S,T) = \zeta_3^{\beta}$, where $e_C : C[3] \times C[3] \to \mu_3$ is the Weil pairing. By the surjectivity of the Weil pairing, we have $\beta = 1$ or 2. We define $\iota : C[3] \to K$

by $\iota(aS+bT) := a\beta e_1 + be_2$, where $\{e_1, e_2\}$ is a standard basis of $K = (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$. Then we have

$$\begin{aligned} e_K \circ (\iota \times \iota)(aS+bT, cS+dT) &= e_K(a\beta e_1 + be_2, c\beta e_1 + de_2) \\ &= \zeta_3^{\beta(ad-bc)} = e_C(aS+bT, cS+dT). \end{aligned}$$

Thus ι is a symplectic group isomorphism and $(C, C[3], \iota)$ is a classical level 3-structure. Therefore we see $SQ_{1,3} \cong SQ_{1,3}^{CL}$. We note that this isomorphism satisfies over $\mathbb{Z}[\zeta_3, 1/3]$ (see [NaT01] for detail).

5.3 A map from a blowing-up of $SQ_{1,3} \times \mathbb{P}(V)$ to $\overline{AP}_{1,3}$

In what follows, k is an algebraic closed field of characteristic $\neq 2$, 3.

Let $X = SQ_{1,3} \times \mathbb{P}(V) = \mathbb{P}^1 \times \mathbb{P}^2$, where $V = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ is the irreducible representation of $G(3)$ of weight one, say, three dimensional space with actions σ and τ. We want to define a map $\varphi : X \to \overline{AP}_{1,3}$,

$$\begin{gathered} \varphi : (\mu_0 : \mu_1) \times (b_0 : b_1 : b_2) \longmapsto (C, L), \qquad (5.4) \\ C : \mu_0(x_0^3 + x_1^3 + x_2^3) = 3\mu_1 x_0 x_1 x_2, \quad L : b_0 x_0 + b_1 x_1 + b_2 x_2 = 0. \end{gathered}$$

However φ is not well-defined when (C, L) is not an abelian pair. If $\mu_1/\mu_0 = \infty$, 1, ζ_3, ζ_3^2, then C is a 3-gon. Only in this case, it is possible that (C, L) is not an abelian pair. Then (C, L) is not an abelian pair if and only if L passes through a singular point of C. Thus φ is not well-defined on subsets of X defined by

$$\begin{aligned} V^{(0)} &:= \{(0:1) \times (b_0 : b_1 : b_2) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid b_0 b_1 b_2 = 0\}, && (5.5) \\ V^{(\beta)} &:= \{(\beta : 1) \times (b_0 : b_1 : b_2) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid \\ &\qquad (b_0 + b_1 + \beta b_2)(b_0 + \zeta_3 b_1 + \beta\zeta_3^2 b_2)(b_0 + \zeta_3^2 b_1 + \beta\zeta_3 b_2) = 0\}, && (5.6) \end{aligned}$$

where $\beta^3 = 1$. We will construct a scheme $\tilde{X}$ such that φ can be extended on it. $\tilde{X}$ will be given by the blowing-up of X three times.

For $a \in \{0,\ 1,\ \zeta_3,\ \zeta_3^2, \infty\}$, we put

$$\mu_0^{(a)} := \begin{cases} \mu_0 & (a = 0), \\ a\mu_1 - \mu_0 & (a^3 = 1), \\ \mu_1 & (a = \infty), \end{cases}$$

$$\mu_1^{(a)} := \begin{cases} \mu_1 & (a = 0), \\ a\mu_1 + 2\mu_0 & (a^3 = 1), \\ \mu_0 & (a = \infty), \end{cases}$$

$$\begin{aligned}
b_i^{(a)} &:= \begin{cases} b_i & (a=0,\infty), \\ b_0+\zeta_3^i b_1 + a\zeta_3^{2i} b_2 & (a^3=1), \end{cases} \qquad (0 \le i \le 2) \\
u^{(a)} &:= \mu_0^{(a)}/\mu_1^{(a)}, \\
s_{i,j}^{(a)} &:= b_i^{(a)}/b_j^{(a)}
\end{aligned}$$

and we denote by $U^{(a,i)}$ the affine open subset of X defined by

$$U^{(a,i)} = \mathrm{Spec}\left(k\left[u^{(a)}, s_{i+1,i}^{(a)}, s_{i+2,i}^{(a)}, \frac{1}{\left(u^{(a)}\right)^3-1}\right]\right). \tag{5.7}$$

Then X has an affine open covering

$$X = \bigcup_{\substack{a=0,\infty \text{ or } a^3=1,\\ 0\le i\le 2}} U^{(a,i)}$$

and φ is well-defined on $U^{\infty} := \bigcup U^{(\infty,i)}$, but φ is not well-defined on only $V\left(u^{(a)}, s_{i+1,i}^{(a)}\right) \cup V\left(u^{(a)}, s_{i+2,i}^{(a)}\right) \subset U^{(a,i)}$ ($a=0$ or $a^3=1$, $0 \le i \le 2$).

Firstly we give $\tilde{X}$ as the blowing-up of X three times:

$$X \xleftarrow{\pi_1} X' \xleftarrow{\pi_2} X'' \xleftarrow{\pi_3} \tilde{X}, \tag{5.8}$$

where π_1 is the blowing-up of X at 12 points $P_k^{(a)} := V\left(\nu^{(a)}, b_i^{(a)}, b_j^{(a)}\right)$ ($a=0$ or $a^3=1$ and $\{i,j,k\}=\{0,1,2\}$). π_2 is the blowing-up of X' along 12 lines $\tilde{L}_i^{(a)}$, where $\tilde{L}_i^{(a)}$ is the strict transform of $L_i^{(a)} := V\left(\nu^{(a)}, b_i^{(a)}\right)$ ($a=0$ or $a^3=1$, $0\le i\le 2$) under the blowing-up π_1. π_3 is the blowing-up of X'' along 12 lines $L''^{(a,i)}$ ($a=0$ or $a^3=1$, $0\le i\le 2$) in the exceptional sets of π_2, where $L''^{(a,i)}$ is defined as follows.

X'' has an affine open covering which consists of $U^{(\infty,i)}$ ($0\le i\le 2$) and 60 affine open subsets $U_j''^{(a,i)}$ ($a=0$ or $a^3=1$, $0\le i\le 2$, $1\le j\le 5$) defined by

$$\begin{aligned}
U_1''^{(a,i)} &:= \mathrm{Spec}\left(k\left[u^{(a)}, \frac{s_{i+1,i}^{(a)}}{u^{(a)}}, \frac{s_{i+2,i}^{(a)}}{u^{(a)}}, \frac{1}{\left(u_1^{(a,i)}\right)^3-1}\right]\right), \\
U_2''^{(a,i)} &:= \mathrm{Spec}\left(k\left[s_{i+2,i}^{(a)}, \frac{u^{(a)}}{s_{i+2,i}^{(a)}}, \frac{s_{i+1,i}^{(a)}}{u^{(a)}}, \frac{1}{\left(u_2^{(a,i)}\right)^3-1}\right]\right), \\
U_3''^{(a,i)} &:= \mathrm{Spec}\left(k\left[s_{i+2,i}^{(a)}, \frac{u^{(a)}}{s_{i+1,i}^{(a)}}, s_{i+1,i+2}^{(a)}, \frac{1}{\left(u_3^{(a,i)}\right)^3-1}\right]\right),
\end{aligned}$$

$$U_4''^{(a,i)} := \mathrm{Spec}\left(k\left[s^{(a)}_{i+1,i}, s^{(a)}_{i+2,i+1}, \frac{u^{(a)}}{s^{(a)}_{i+2,i}}, \frac{1}{\left(u_4^{(a,i)}\right)^3 - 1}\right]\right),$$

$$U_5''^{(a,i)} := \mathrm{Spec}\left(k\left[s^{(a)}_{i+1,i}, \frac{s^{(a)}_{i+2,i}}{u^{(a)}}, \frac{u^{(a)}}{s^{(a)}_{i+1,i}}, \frac{1}{\left(u_5^{(a,i)}\right)^3 - 1}\right]\right),$$

where $u_j^{(a,i)}$ is the function which gives $u^{(a)}$ on $U''^{(a,i)}$. We define

$$\begin{aligned} L_3''^{(a,i)} &:= V\left(u^{(a)}/s^{(a)}_{i+1,i}, s^{(a)}_{i+1,i+2}\right) \subset U_3''^{(a,i)}, \\ L_4''^{(a,i)} &:= V\left(s^{(a)}_{i+2,i+1}, u^{(a)}/s^{(a)}_{i+2,i}\right) \subset U_4''^{(a,i)}. \end{aligned}$$

By gluing these lines, we get 12 projective lines in the exceptional sets of π_2:

$$L''^{(a,i)} := L_3''^{(a,i)} \cup L_4''^{(a,i+2)} \quad (a = 0 \text{ or } a^3 = 1,\ 0 \le i \le 2).$$

Then $\tilde{X}$ has an open covering

$$\tilde{X} = \bigcup_{0 \le i \le 2} U^{(\infty,i)} \cup \bigcup_{\substack{a=0 \text{ or } a^3=1, \\ 0 \le i \le 2,\ 1 \le j \le 7}} U_j^{(a,i)},$$

where each open subsets $U_j^{(a,i)}$ is given by

$$U_1^{(a,i)} := \mathrm{Spec}\left(k\left[u^{(a)}, \frac{s^{(a)}_{i+1,i}}{u^{(a)}}, \frac{s^{(a)}_{i+2,i}}{u^{(a)}}, \frac{1}{\left(u_1^{(a,i)}\right)^3 - 1}\right]\right), \tag{5.9}$$

$$U_2^{(a,i)} := \mathrm{Spec}\left(k\left[s^{(a)}_{i+2,i}, \frac{u^{(a)}}{s^{(a)}_{i+2,i}}, \frac{s^{(a)}_{i+1,i}}{u^{(a)}}, \frac{1}{\left(u_2^{(a,i)}\right)^3 - 1}\right]\right), \tag{5.10}$$

$$U_3^{(a,i)} := \mathrm{Spec}\left(k\left[s^{(a)}_{i+2,i}, \frac{u^{(a)}}{s^{(a)}_{i+1,i}}, \frac{s^{(a)}_{i+1,i}s^{(a)}_{i+1,i+2}}{u^{(a)}}, \frac{1}{\left(u_3^{(a,i)}\right)^3 - 1}\right]\right), \tag{5.11}$$

$$U_4^{(a,i)} := \mathrm{Spec}\left(k\left[s^{(a)}_{i+2,i}, \frac{u^{(a)}}{s^{(a)}_{i+1,i}s^{(a)}_{i+1,i+2}}, s^{(a)}_{i+1,i+2}, \frac{1}{\left(u_4^{(a,i)}\right)^3 - 1}\right]\right), \tag{5.12}$$

$$U_5^{(a,i)} := \operatorname{Spec}\left(k\left[s_{i+1,i}^{(a)}, s_{i+2,i+1}^{(a)}, \frac{u^{(a)}}{s_{i+2,i}^{(a)} s_{i+2,i+1}^{(a)}}, \frac{1}{\left(u_5^{(a,i)}\right)^3 - 1}\right]\right), \quad (5.13)$$

$$U_6^{(a,i)} := \operatorname{Spec}\left(k\left[s_{i+1,i}^{(a)}, \frac{s_{i+2,i}^{(a)} s_{i+2,i+1}^{(a)}}{u^{(a)}}, \frac{u^{(a)}}{s_{i+2,i}^{(a)}}, \frac{1}{\left(u_6^{(a,i)}\right)^3 - 1}\right]\right), \quad (5.14)$$

$$U_7^{(a,i)} := \operatorname{Spec}\left(k\left[s_{i+1,i}^{(a)}, \frac{s_{i+2,i}^{(a)}}{u^{(a)}}, \frac{u^{(a)}}{s_{i+1,i}^{(a)}}, \frac{1}{\left(u_7^{(a,i)}\right)^3 - 1}\right]\right), \quad (5.15)$$

and $u_j^{(a,i)}$ is the function which gives $u^{(a)}$ on $U_j^{(a,i)}$. In what follows, we denote by $v_0^{(a,i,j)}$, $v_1^{(a,i,j)}$, $v_2^{(a,i,j)}$ the above coordinates of $U_j^{(a,i)}$. Then we have

$$u_j^{(a,i)} = \begin{cases} v_0^{(a,i,1)} & (j=1), \\ v_0^{(a,i,j)} v_1^{(a,i,j)} & (j=2,7), \\ v_0^{(a,i,j)} \left(v_1^{(a,i,j)}\right)^2 v_2^{(a,i,j)} & (j=3,6), \\ v_0^{(a,i,j)} v_1^{(a,i,j)} \left(v_2^{(a,i,j)}\right)^2 & (j=4,5). \end{cases} \quad (5.16)$$

Next we define a map φ from $\tilde{X} = U^\infty \cup \bigcup U_j^{(a,i)}$ to $\overline{AP}_{1,3}$. For this, we will define a map φ^∞, $\varphi_j^{(a,i)}$ from U^∞, $U_j^{(a,i)}$ to $\overline{AP}_{1,3}$, respectively, and we will glue these maps. We define φ^∞ by (5.4). Let

$$\left(C_{(a)}, L_{(a)}\right) = \left(V\left(F_{(a)}(x)\right), V\left(S_{(a)}(x)\right)\right), \quad (5.17)$$

$$F_{(a)}(x) := u^{(a)}\left(x_0^3 + x_1^3 + x_2^3\right) - 3x_0x_1x_2, \ S_{(a)}(x) := b_0^{(a)} x_0 + b_1^{(a)} x_1 + b_2^{(a)} x_2$$

and we define matrices $M_{a,i,j}$ as follows. When $i = 2$,

$$M_{a,2,j} = \begin{cases} \operatorname{Diag}\left(1, 1, 1/u^{(a)}\right) & (j=1), \\ \operatorname{Diag}\left(\left(b_1^{(a)} b_2^{(a)} u^{(a)}\right)^{\frac{1}{2}}, b_1^{(a)}, b_2^{(a)}\right) & (j=2,3), \\ \operatorname{Diag}\left(b_0^{(a)}, b_1^{(a)}, b_2^{(a)}\right) & (j=4,5), \\ \operatorname{Diag}\left(b_0^{(a)}, \left(b_0^{(a)} b_2^{(a)} u^{(a)}\right)^{\frac{1}{2}}, b_2^{(a)}\right) & (j=6,7). \end{cases}$$

When $i = 0$ or 1, we define $M_{a,i,j}$ as the matrices given by replacing $b_k^{(a)}$ with $b_{k+i+1}^{(a)}$ in $\tau^{i+1} \cdot M_{a,2,j} \cdot \tau^{2(i+1)}$, where $\tau = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then we define the image of $\varphi_j^{(a,i)}$ as the abelian pair given by

$$M_{a,i,j} \cdot \left(C_{(a)}, L_{(a)}\right) = \left(V\left(F_{(a)}\left(M_{a,i,j}^{-1} \cdot x\right)\right), V\left(S_{(a)}\left(M_{a,i,j}^{-1} \cdot x\right)\right)\right).$$

Specifically, we can write $\varphi_j^{(a,i)}$ as follows.

Definition 5.3.1. For each affine open subsets $U_j^{(a,i)}$ ($a = 0$ or $a^3 = 1$, $0 \leq i \leq 2$, $1 \leq j \leq 7$), we put $J = (a,i,j)$. Then we define $\varphi_j^{(a,i)} : U_j^{(a,i)} \longrightarrow \overline{AP}_{1,3}$ as follows. When $i = 2$,

$$\varphi_1^{(a,2)}\left(v_0^J, v_1^J, v_2^J\right) := \begin{cases} C : y_0^3 + y_1^3 + \left(v_0^J\right)^3 y_2^3 = 3y_0y_1y_2, \\ L : v_1^J y_0 + v_2^J y_1 + y_2 = 0, \end{cases} \tag{5.18}$$

$$\varphi_2^{(a,2)}\left(v_0^J, v_1^J, v_2^J\right) := \begin{cases} C : y_0(y_0^2 - 3y_1y_2) = -\left(v_1^J\right)^{3/2}\left(y_1^3 + \left(v_0^J\right)^3 y_2^3\right), \\ L : \left(v_1^J\right)^{1/2} v_2^J y_0 + y_1 + y_2 = 0, \end{cases} \tag{5.19}$$

$$\varphi_3^{(a,2)}\left(v_0^J, v_1^J, v_2^J\right) := \begin{cases} C : y_0(y_0^2 - 3y_1y_2) = -\left(v_1^J\right)^3\left(v_2^J\right)^{3/2}\left(y_1^3 + \left(v_0^J\right)^3 y_2^3\right), \\ L : \left(v_2^J\right)^{1/2} y_0 + y_1 + y_2 = 0, \end{cases} \tag{5.20}$$

$$\varphi_4^{(a,2)}\left(v_0^J, v_1^J, v_2^J\right) := \begin{cases} C : 3y_0y_1y_2 = v_1^J\left(y_0^3 + \left(v_2^J\right)^3\left(y_1^3 + \left(v_0^J\right)^3 y_2^3\right)\right), \\ L : y_0 + y_1 + y_2 = 0, \end{cases} \tag{5.21}$$

$$\varphi_5^{(a,2)}\left(v_0^J, v_1^J, v_2^J\right) := \begin{cases} C : 3y_0y_1y_2 = v_2^J\left(y_1^3 + \left(v_1^J\right)^3\left(y_0^3 + \left(v_0^J\right)^3 y_2^3\right)\right), \\ L : y_0 + y_1 + y_2 = 0, \end{cases} \tag{5.22}$$

$$\varphi_6^{(a,2)}\left(v_0^J, v_1^J, v_2^J\right) := \begin{cases} C : y_1(y_1^2 - 3y_0y_2) = -\left(v_2^J\right)^3\left(v_1^J\right)^{3/2}\left(y_0^3 + \left(v_0^J\right)^3 y_2^3\right), \\ L : y_0 + \left(v_1^J\right)^{1/2} y_1 + y_2 = 0, \end{cases} \tag{5.23}$$

$$\varphi_7^{(a,2)}\left(v_0^J, v_1^J, v_2^J\right) := \begin{cases} C : y_1(y_1^2 - 3y_0y_2) = -\left(v_2^J\right)^{3/2}\left(y_0^3 + \left(v_0^J\right)^3 y_2^3\right), \\ L : y_0 + \left(v_2^J\right)^{1/2} v_1^J y_1 + y_2 = 0. \end{cases} \tag{5.24}$$

When $i = 0$ or 1, we define the images of $\varphi_j^{(a,i)}$ as abelian pairs given by replacing $\left(y_k, v_\ell^{(a,2,j)}\right)$ with $\left(y_{k+i+1}, v_\ell^{(a,i,j)}\right)$ in (5.18), …, (5.24).

The definitions (5.19), (5.20), (5.23), (5.24) are independent of a choice of a square root of v_1^J or v_2^J. In fact, for (5.19), (5.20) (resp. (5.23), (5.24)), we obtain same abelian pairs in $\overline{AP}_{1,3}$ under the transformation $y_0 \mapsto -y_0$ (resp. $y_1 \mapsto -y_1$).

We can show that (5.18), ..., (5.24) are abelian pairs. Here we only consider in the case (5.21). The other cases can be checked by the same arguments. For (5.21), that is, $J = (a, 2, 4)$, if $v_1^J = 0$, then C is 3-gon $V(y_0y_1y_2)$ and $L = V(y_0 + y_1 + y_2)$ does not pass through the singular points

$$P_0 := (1 : 0 : 0),\ P_1 := (0 : 1 : 0),\ P_2 := (0 : 0 : 1)$$

of C, and hence (C, L) is an abelian pair. If $v_1^J \neq 0$ and $v_2^J = 0$, then we have $C = L' + Q$, $L' = V(y_0)$ and $Q = V\left(v_1^J y_0^2 - 3y_1y_2\right)$. Thus (C, L) is an abelian pair, since L does not pass through the singular points P_1, P_2 of C. If $v_1^J v_2^J \neq 0$ and $v_0^J = 0$, then we have

$$C = V\left(v_1^J\left(y_0^3 + \left(v_2^J\right)^3 y_1^3\right) - 3y_0y_1y_2\right).$$

Since C is a nodal curve with nodal point P_2 and L does not pass through P_2, (C, L) is an abelian pair. If $v_0^J v_1^J v_2^J \neq 0$, then by the transformation $y_0 = v_0^J v_2^J x_0$, $y_1 = v_0^J x_1$, $y_2 = x_2$, we obtain

$$\begin{cases} C : v_0^J v_1^J \left(v_2^J\right)^2 (x_0^3 + x_1^3 + x_2^3) = 3x_0x_1x_2, \\ L : v_2^J v_0^J x_0 + v_0^J x_1 + x_2 = 0. \end{cases}$$

Then we have $\left(v_0^J v_1^J \left(v_2^J\right)^2\right)^3 \neq 1$ by (5.12) and (5.16). Thus C is a nonsingular cubic curve, and hence (C, L) is an abelian pair.

Theorem 5.3.2. *We can glue the maps φ^∞ and $\varphi_j^{(a,i)}$, and we obtain a map*

$$\varphi : \tilde{X} = U^\infty \cup \bigcup_{\substack{a=0 \text{ or } a^3=1, \\ 0\leq i\leq 2,\ 1\leq j\leq 7}} U_j^{(a,i)} \longrightarrow \overline{AP}_{1,3}.$$

This map is an extension of (5.4) and $\varphi^{-1}(W_{\mathrm{T}})$ is the union of the strict transforms of $A_i^{(j)}$ under the composition $\pi := \pi_1 \circ \pi_2 \circ \pi_3 : \tilde{X} \to X$, where

$$A_i^{(j)} := \left\{(\mu_0 : \mu_1) \times (b_0 : b_1 : b_2) \in X \mid b_{i+1} = b_{i+2}\zeta_3^{2j},\ b_{i+2}\mu_1 = b_i\zeta_3^{2j}\mu_0\right\}.$$

Proof. The second assertion is clear by construction. We show the first assertion. Let $\sigma_{(0)} = \mathrm{id}$ and let $\sigma_{(a)} = \sigma_1^3 \varrho_a$, where

$$\sigma_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{bmatrix},\ \varrho_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^2 \end{bmatrix}.$$

Then we have $\sigma_{(a)} \cdot \left(C_{(0)}, L_{(0)}\right) = \left(C_{(a)}, L_{(a)}\right)$. By definition, the image of $\varphi_j^{(a,i)}$ is given by

$$M_{a,i,j} \cdot \left(C_{(a)}, L_{(a)}\right) = \left(M_{a,i,j}\sigma_{(a)}\right) \cdot \left(C_{(0)}, L_{(0)}\right).$$

Thus for $J = (a, i, j)$, we can write formally

$$\begin{gathered}\left(M_{a,i,j}\sigma_{(a)}\right)^{-1} \cdot \varphi_j^{(a,i)}\left(v_0^J, v_1^J, v_2^J\right) = \left(\mathfrak{C}_{(J)}, \mathfrak{L}_{(J)}\right), \\ \mathfrak{C}_{(J)} : \mathfrak{u}^J(x_0^3 + x_1^3 + x_2^3) = 3x_0x_1x_2, \ \mathfrak{L}_{(J)} : \mathfrak{b}_0^J x_0 + \mathfrak{b}_1^J x_1 + \mathfrak{b}_2^J x_2 = 0,\end{gathered}$$

where $\mathfrak{u}^J$, $\mathfrak{b}_k^J$ are functions on $U_j^{(a,i)}$ which give $u^{(0)}$, b_k, respectively. By construction, we have

$$\mathfrak{u}^J = \begin{cases} u_j^{(0,i)} & (a = 0), \\ \frac{1 - u_j^{(a,i)}}{1 + 2u_j^{(a,i)}} a & (a^3 = 1), \end{cases} \tag{5.25}$$

$$\mathfrak{b}_k^J = \begin{cases} b_k^J & (a = 0), \\ \left(b_0^J + \zeta_3^{2k} b_1^J + \zeta_3^k b_2^J\right) a^{k(k+2)} & (a^3 = 1), \end{cases} \tag{5.26}$$

where $u_j^{(a,i)}$ are defined by (5.16) and b_k^J are defined by

$$b_k^J := \begin{cases} 1 & (k = i), \\ v_0^J & (k = i+1 \text{ (resp. } i+2), \ j = 5, 6, 7 \text{ (resp. } 2, 3, 4)), \\ v_0^J v_1^J & (k = i+1 \text{ (resp. } i+2), \ j = 1 \text{ (resp. } 5)), \\ v_0^J v_2^J & (k = i+1 \text{ (resp. } i+2), \ j = 4 \text{ (resp. } 1)), \\ v_0^J v_1^J v_2^J & (k = i+1 \text{ (resp. } i+2), \ j = 2, 3 \text{ (resp. } 6, 7)). \end{cases}$$

For any two maps $\varphi_j^{(a,i)}$, $\varphi_{j'}^{(a',i')}$ on $U_j^{(a,i)}$, $U_{j'}^{(a',i')}$, respectively, we show that $\varphi_j^{(a,i)} = \varphi_{j'}^{(a',i')}$ on $U_j^{(a,i)} \cap U_{j'}^{(a',i')}$. Then for $J = (a, i, j)$, $J' = (a', i', j')$, we can construct functions

$$v_m^J = F_m\left(v_0^{J'}, v_1^{J'}, v_2^{J'}\right) \quad (0 \le m \le 2)$$

by solving equations

$$\mathfrak{u}^J = \mathfrak{u}^{J'}, \quad \left(\mathfrak{b}_0^J : \mathfrak{b}_1^J : \mathfrak{b}_2^J\right) = \left(\mathfrak{b}_0^{J'} : \mathfrak{b}_1^{J'} : \mathfrak{b}_2^{J'}\right) \tag{5.27}$$

for v_0^J, v_1^J, v_2^J. These functions F_m give the gluing $U_j^{(a,i)}$ and $U_{j'}^{(a',i')}$. We will construct F_m explicitly in a nontrivial case (see Example 5.3.3). Then for any $(v_0^{J'}, v_1^{J'}, v_2^{J'}) = (F_0, F_1, F_2) \in U_j^{(a,i)} \cap U_{j'}^{(a',i')}$, we have $\left(\mathfrak{C}_{(J')}, \mathfrak{L}_{(J')}\right) = \left(\mathfrak{C}_{(J)}, \mathfrak{L}_{(J)}\right)$ by (5.27), and hence we obtain

$$\begin{aligned}
\varphi_{j'}^{(a',i')}(v_0^{J'}, v_1^{J'}, v_2^{J'}) &= \left(M_{a',i',j'} \cdot \sigma_{(a')}\right)^{-1} \cdot \varphi_{j'}^{(a',i')}(v_0^{J'}, v_1^{J'}, v_2^{J'}) \\
&= \left(\mathfrak{C}_{(J')}, \mathfrak{L}_{(J')}\right) \\
&= \left(\mathfrak{C}_{(J)}, \mathfrak{L}_{(J)}\right) \\
&= \left(M_{a,i,j} \cdot \sigma_{(a)}\right)^{-1} \cdot \varphi_j^{(a,i)}(F_0, F_1, F_2) \\
&= \varphi_j^{(a,i)}(F_0, F_1, F_2)
\end{aligned}$$

in $\overline{AP}_{1,3}$. Therefore we get $\varphi_j^{(a,i)} = \varphi_{j'}^{(a',i')}$ on $U_j^{(a,i)} \cap U_{j'}^{(a',i')}$.

Next we show that $\varphi_j^{(a,i)} = \varphi^\infty$ on $U_j^{(a,i)} \cap U^\infty$. For it, we need to show that $\varphi_j^{(a,i)} = \varphi^\infty$ on $U_j^{(a,i)} \cap U^{(\infty,k)}$, since $U^\infty = \bigcup U^{(\infty,k)}$. Recall that $U^{(\infty,k)}$ has coordinates $u^{(\infty)}$, $s_{k+1,k}^{(\infty)} (= b_{k+1}/b_k)$, $s_{k+2,k}^{(\infty)} (= b_{k+2}/b_k)$ and

$$\varphi^\infty \left(u^{(\infty)}, s_{k+1,k}^{(\infty)}, s_{k+2,k}^{(\infty)}\right) = \left(C_{(\infty)}, L_{(\infty)}\right),$$

$$C_{(\infty)} : x_0^3 + x_1^3 + x_2^3 = 3u^{(\infty)} x_0 x_1 x_2, \ L_{(\infty)} : x_k + s_{k+1,k}^{(\infty)} x_{k+1} + s_{k+2,k}^{(\infty)} x_{k+2} = 0.$$

Then by solving equations

$$\mathfrak{u}^{(a,i,j)} = 1/u^{(\infty)}, \quad \mathfrak{b}_{k+1}^{(a,i,j)}/\mathfrak{b}_k^{(a,i,j)} = s_{k+1,k}^{(\infty)}, \quad \mathfrak{b}_{k+2}^{(a,i,j)}/\mathfrak{b}_k^{(a,i,j)} = s_{k+2,k}^{(\infty)}$$

for $v_0^{(a,i,j)}$, $v_1^{(a,i,j)}$, $v_2^{(a,i,j)}$, we can construct functions

$$v_m^{(a,i,j)} = G_m \left(u^{(\infty)}, s_{k+1,k}^{(\infty)}, s_{k+2,k}^{(\infty)}\right) \quad (0 \le m \le 2)$$

which glue $U_j^{(a,i)}$ and $U^{(\infty,k)}$ together. Hence we obtain $\varphi_j^{(a,i)} = \varphi^\infty$ on $U_j^{(a,i)} \cap U^{(\infty,k)}$ similarly to above. □

Example 5.3.3. We here consider the case where $J = (1,2,3)$ and $J' = (\zeta_3, 0, 4)$. For simplicity, we write $(v_0, v_1, v_2) = (v_0^J, v_1^J, v_2^J)$, $(v_0', v_1', v_2') = (v_0^{J'}, v_1^{J'}, v_2^{J'})$. Then we have

$$\begin{aligned}
\mathfrak{u}^J &= \frac{1 - v_0 v_1^2 v_2}{1 + 2 v_0 v_1^2 v_2}, \\
\mathfrak{b}_k^J &= v_0 v_1 v_2 + \zeta_3^{2k} v_0 + \zeta_3^k \quad (0 \le k \le 2), \\
\mathfrak{u}^{J'} &= \frac{1 - v_0' v_1' v_2'^2}{1 + 2 v_0' v_1' v_2'^2} \zeta_3, \\
\mathfrak{b}_k^{J'} &= \left(1 + \zeta_3^{2k} v_0' v_2' + \zeta_3^k v_0'\right) \zeta_3^{k(k+2)} \quad (0 \le k \le 2).
\end{aligned}$$

Then by solving equations (5.27), we obtain rational functions $F_k(v_0', v_1', v_2')$ and $G_k(v_0, v_1, v_2)$ such that $v_k = F_k(G_0, G_1, G_2)$, $v_k' = G_k(F_0, F_1, F_2)$. In fact, we can write

$$\begin{aligned}
F_0 &= \frac{\zeta_3 + v_0'v_2' + v_0'}{1 + v_0'v_2'\zeta_3 + v_0'},\\
F_1 &= \frac{(1 + v_0'v_2'\zeta_3 + v_0')(v_0'v_1'{v_2'}^2(\zeta_3 + 2) + 1 - \zeta_3)}{(1 + v_0'v_2' + v_0'\zeta_3)(2v_0'v_1'{v_2'}^2(1 - \zeta_3) + 1 + 2\zeta_3)},\\
F_2 &= \frac{(1 + v_0'v_2' + v_0'\zeta_3)^2(2v_0'v_1'{v_2'}^2(1 - \zeta_3) + 1 + 2\zeta_3)}{(\zeta_3 + v_0'v_2' + v_0')(1 + v_0'v_2'\zeta_3 + v_0')(v_0'v_1'{v_2'}^2(\zeta_3 + 2) + 1 - \zeta_3)},\\
G_0 &= \frac{v_0v_1v_2 + v_0\zeta_3 + \zeta_3}{v_0v_1v_2\zeta_3 + v_0 + \zeta_3},\\
G_1 &= \frac{(v_0v_1v_2 + v_0\zeta_3 + \zeta_3)(v_0v_1v_2\zeta_3 + v_0 + \zeta_3)(v_0v_1^2v_2(2\zeta_3 + 1) + \zeta_3 - 1)}{(v_0v_1v_2\zeta_3 + v_0\zeta_3 + 1)^2(2v_0v_1^2v_2(\zeta_3 - 1) + 2 + \zeta_3)},\\
G_2 &= \frac{v_0v_1v_2\zeta_3 + v_0\zeta_3 + 1}{v_0v_1v_2 + v_0\zeta_3 + \zeta_3}.
\end{aligned}$$

Thus we have rational maps $G : U_3^{(1,2)} \to U_4^{(\zeta_3,0)}$, $(v_0, v_1, v_2) \mapsto (G_0, G_1, G_2)$ and $F : U_4^{(\zeta_3,0)} \to U_3^{(1,2)}$, $(v_0', v_1', v_2') \mapsto (F_0, F_1, F_2)$. Let W_G (resp. W_F) be the union of the zero sets of the denominators of G_k (resp. F_k). Then we have an isomorphism

$$G : U_3^{(1,2)} \setminus \left(W_G \cup G^{-1}(W_F)\right) \longrightarrow U_4^{(\zeta_3,0)} \setminus \left(W_F \cup F^{-1}(W_G)\right).$$

We regard these sets as $U_3^{(1,2)} \cap U_4^{(\zeta_3,0)}$. Note that we have

$$\begin{aligned}
G^{-1}(W_F) &= \{(v_0, v_1, v_2) \in U_3^{(1,2)} \mid G(v_0, v_1, v_2) \in \bigcup V(\mathrm{denom}(F_k))\}\\
&= \{(v_0, v_1, v_2) \in U_3^{(1,2)} \mid \mathrm{denom}(F_k)(G_0, G_1, G_2) = 0 \text{ for some } k\}\\
&= V(v_0v_1v_2)
\end{aligned}$$

by a simple computation, where $\mathrm{denom}(F_k)$ is the denominator of F_k. Similarly, we have $F^{-1}(W_G) = V(v_0'v_2')$. Hence matrices $M_{1,2,3}$ and $M_{\zeta_3,0,4}$ has nonzero determinants $v_0^2v_1v_2^{(1/2)}$, ${v_0'}^2v_2'$ on $U_3^{(1,2)} \cap U_4^{(\zeta_3,0)}$, respectively. Thus for any $(v_0, v_1, v_2) = (G_0, G_1, G_2) \in U_3^{(1,2)} \cap U_4^{(\zeta_3,0)}$, we have

$$\begin{aligned}
\varphi_3^{(1,2)}(v_0, v_1, v_2) &= \left(M_{1,2,3} \cdot \sigma_{(1)}\right)^{-1} \cdot \varphi_3^{(1,2)}(v_0, v_1, v_2)\\
&= \left(\mathfrak{C}_{(J)}, \mathfrak{L}_{(J)}\right) = \left(\mathfrak{C}_{(J')}, \mathfrak{L}_{(J')}\right)\\
&= \left(M_{\zeta_3,0,4} \cdot \sigma_{(\zeta_3)}\right)^{-1} \cdot \varphi_4^{(\zeta_3,0)}(G_0, G_1, G_2)\\
&= \varphi_4^{(\zeta_3,0)}(G_0, G_1, G_2)
\end{aligned}$$

in $\overline{AP}_{1,3}$.

Remark 5.3.4. By the map $\varphi : \tilde{X} \to \overline{AP}_{1,3}$, we have a rough structure of $\overline{AP}_{1,3}$ in the following sense. Let $\tilde{U}^{(a,i)} := \bigcup_{j=1}^{7} U_j^{(a,i)}$, where $a = 0$ or $a^3 = 1$, $0 \leq i \leq 2$. Recall that $\tilde{U}^{(a,i)}$ is given by the blowing-up of $U^{(a,i)}$ three times:

$$U^{(a,i)} \xleftarrow{\pi_1} U'^{(a,i)} \xleftarrow{\pi_2} U''^{(a,i)} \xleftarrow{\pi_3} \tilde{U}^{(a,i)}.$$

Let $E_\ell^{(a,i)}$ be the exceptional sets of the blowing-up π_ℓ and let $\tilde{E}_\ell^{(a,i)} \subset \tilde{U}^{(a,i)}$ be the strict transform of $E_\ell^{(a,i)}$ under the above blowing-ups. Then we have

$$\begin{aligned}
\tilde{E}_1^{(a,i)} &= \bigcup_{j=1}^{7} V\left(v_0^{(a,i,j)}\right), \\
\tilde{E}_2^{(a,i)} &= V\left(v_1^{(a,i,3)}\right) \cup V\left(v_2^{(a,i,4)}\right) \cup V\left(v_1^{(a,i,5)}\right) \cup V\left(v_2^{(a,i,6)}\right), \\
\tilde{E}_3^{(a,i)} &= V\left(v_1^{(a,i,2)}\right) \cup V\left(v_2^{(a,i,3)}\right) \cup V\left(v_1^{(a,i,6)}\right) \cup V\left(v_2^{(a,i,7)}\right),
\end{aligned}$$

where $v_0^{(a,i,j)}$, $v_1^{(a,i,j)}$, $v_2^{(a,i,j)}$ are coordinates of $U_j^{(a,i)}$ defined in (5.9), …, (5.15). Let $\tilde{E}_0^{(a,i)} := V\left(v_1^{(a,i,4)}\right) \cup V\left(v_2^{(a,i,5)}\right)$ and

$$\tilde{E}_\ell := \bigcup_{\substack{a=0 \text{ or } a^3=1, \\ 0 \leq i \leq 2}} \tilde{E}_\ell^{(a,i)} \subset \tilde{X}, \quad 0 \leq \ell \leq 3.$$

For any $v \in \tilde{X}$, we put $\varphi(v) = (C, L) \in \overline{AP}_{1,3}$. Then we have

$$\begin{gathered}
C \text{ is a 3-gon} \Longleftrightarrow v \in \tilde{E}_0, \\
C \text{ is an irreducible conic } Q \text{ plus a line } L' \Longleftrightarrow v \in \tilde{E}_2 \cup \tilde{E}_3 \setminus \tilde{E}_0, \\
C \text{ is a nodal curve} \Longleftrightarrow v \in \tilde{E}_1 \setminus \left(\tilde{E}_0 \cup \tilde{E}_2 \cup \tilde{E}_3\right), \\
C \text{ is a smooth cubic curve} \Longleftrightarrow v \in \tilde{X} \setminus \left(\tilde{E}_0 \cup \tilde{E}_1 \cup \tilde{E}_2 \cup \tilde{E}_3\right).
\end{gathered}$$

5.4 A map from a blowing-up of $SQ_{1,3} \times \mathbb{P}(V)$ to $\overline{P}_{1,3}$

Similarly to the previous section, we want to define a map $\psi : SQ_{1,3} \times \mathbb{P}(V) = \mathbb{P}^1 \times \mathbb{P}^2 \to \overline{P}_{1,3}$,

$$\begin{gathered}
\psi : (\mu_0 : \mu_1) \times (b_0 : b_1 : b_2) \longmapsto (C, L), \\
C : \mu_0(x_0^3 + x_1^3 + x_2^3) = 3\mu_1 x_0 x_1 x_2, \quad L : b_0 x_0 + b_1 x_1 + b_2 x_2 = 0.
\end{gathered}$$

Then (C, L) is not contained in $\overline{P}_{1,3}$ if and only if L passes through a singular point of C or L is 3-tangent to C. Thus ψ is not well-defined on 3-gons $V^{(a)}$ ($a = 0$ or $a^3 = 1$) and lines $A_i^{(j)}$ ($0 \leq i, j \leq 2$) defined in the previous section (see (5.5), (5.6) and Theorem 4.1.2). Recall that $\pi_1 : X' \to X$ is the blowing-up at 12 points $P_\ell^{(a)}$ ($a = 0$ or $a^3 = 1$, $0 \leq \ell \leq 2$). Then each line $A_i^{(j)}$ passes through only 4 points $P_i^{(0)}$, $P_{2j+(2-i)k}^{(k)}$ ($0 \leq k \leq 2$) in these 12 points. Conversely, each $P_\ell^{(a)}$ is contained in only 3 lines $A_0^{(j)}$ ($0 \leq j \leq 2$) if $a = 0$, $A_m^{(2k(m+1)+2\ell)}$ ($0 \leq m \leq 2$) if $a = \zeta_3^k$. Moreover $A_i^{(j)}$'s intersect only at $P_\ell^{(a)}$'s. In particular, the strict transforms $\tilde{A}_i^{(j)}$ of $A_i^{(j)}$ under the blowing-up π_1 are disjoint each other and do not intersect with the center of the blowing-up $\pi_2 : X'' \to X'$. Thus the strict transforms of $A_i^{(j)}$ under the composition of three blowing-ups $\pi := \pi_1 \circ \pi_2 \circ \pi_3 : \tilde{X} \longrightarrow X$ are isomorphic to $\tilde{A}_i^{(j)}$, so we denote by $\tilde{A}_i^{(j)}$ the strict transforms. By Definition 5.3.1, we can define a rational map

$$\psi : \tilde{X} \longrightarrow \overline{P}_{1,3} \quad \text{with base locus} \quad \bigcup_{0 \leq i,j \leq 2} \tilde{A}_i^{(j)}. \tag{5.28}$$

We will construct a scheme $\hat{X}$ such that ψ can be extended on it. $\hat{X}$ will be given by the blowing-up of $\tilde{X}$ three times.

The first step is to retake an affine open covering of $\tilde{X}$. In the previous section, we take an affine open covering

$$\tilde{X} = \bigcup_{0 \leq i \leq 2} U^{(\infty,i)} \cup \bigcup_{\substack{a=0 \text{ or } a^3=1, \\ 0 \leq i \leq 2,\ 1 \leq \ell \leq 7}} U_\ell^{(a,i)}.$$

For $a = 0$ or $a^3 = 1$, $0 \leq i, j \leq 2$, let

$$\alpha_1^{(a,i,j)} = \frac{s_{i+1,i}^{(a)}}{u^{(a)}} - \zeta_3^{2j} \frac{s_{i+2,i}^{(a)}}{u^{(a)}}, \quad \alpha_2^{(a,i,j)} = \frac{s_{i+2,i}^{(a)}}{u^{(a)}} - \zeta_3^{2j},$$
$$\alpha_1^{(\infty,i,j)} = s_{i+1,i}^{(\infty)} - \zeta_3^j, \quad \alpha_2^{(\infty,i,j)} = s_{i+2,i}^{(\infty)} - \zeta_3^{2j} u^{(\infty)}.$$

Then $U_1^{(a,i)}$ has local coordinates $u^{(a)}$, $\alpha_1^{(a,i,j)}$, $\alpha_2^{(a,i,j)}$, and $U^{(\infty,i)}$ has local coordinates $u^{(\infty)}$, $\alpha_1^{(\infty,i,j)}$, $\alpha_2^{(\infty,i,j)}$. We define open subsets $V^{(a,i,j)}$ ($a \in \{0, 1, \zeta_3, \zeta_3^2, \infty\}$, $0 \leq i, j \leq 2$) and $V_\ell^{(a,i)}$ ($a \in \{0, 1, \zeta_3, \zeta_3^2\}$, $0 \leq i \leq 2$, $2 \leq \ell \leq 7$) of $\tilde{X}$ as follows:

$$V^{(a,i,j)} = \operatorname{Spec}\left(k\left[u^{(a)}, \alpha_1^{(a,i,j)}, \alpha_2^{(a,i,j)}, \frac{1}{\left(u^{(a)}\right)^3 - 1}\right]\right) \setminus \mathcal{C}_{i'}^{(j')},$$
$$V_\ell^{(a,i)} = U_\ell^{(a,i)} \setminus \bigcup_{0 \leq s,t \leq 2} \tilde{A}_s^{(t)},$$

where, $\mathcal{C}_{i'}^{(j')} := \bigcup_{(s,t)\neq(i',j')} \tilde{A}_s^{(t)} \subset \tilde{X}$ and (i',j') is defined by

$$(i',j') = \begin{cases} (i,j) & (a=0), \\ (j, 2k(j+1)+2i) & (a=\zeta_3^k,\ 0\leq k\leq 2), \\ (i+2,j) & (a=\infty). \end{cases} \tag{5.29}$$

Then $\tilde{X}$ has an affine open covering

$$\tilde{X} = \bigcup_{\substack{a\in\{0,1,\zeta_3,\zeta_3^2,\infty\},\\ 0\leq i,j\leq 2}} V^{(a,i,j)} \cup \bigcup_{\substack{a=0 \text{ or } a^3=1,\\ 0\leq i\leq 2,\ 2\leq \ell\leq 7}} V_\ell^{(a,i)},$$

and ψ is not well-defined on only

$$V^{(a,i,j)} \cap \tilde{A}_{i'}^{(j')} = V\left(\alpha_1^{(a,i,j)}, \alpha_2^{(a,i,j)}\right) \subset V^{(a,i,j)},$$

where (i',j') is defined by (5.29). Note that these subsets are glued as follows:

$$\begin{aligned} \tilde{A}_i^{(j)} = V\left(\alpha_1^{(0,i,j)}, \alpha_2^{(0,i,j)}\right) \ &\cup \bigcup_{0\leq k\leq 2} V\left(\alpha_1^{(\zeta_3^k, 2j+(2-i)k, i)}, \alpha_2^{(\zeta_3^k, 2j+(2-i)k, i)}\right) \\ &\cup \ V\left(\alpha_1^{(\infty,i+1,j)}, \alpha_2^{(\infty,i+1,j)}\right). \end{aligned} \tag{5.30}$$

Next we give $\hat{X}$ as the blowing-up of $\tilde{X}$ three times:

$$\tilde{X} \xleftarrow{p_1} \tilde{X}' \xleftarrow{p_2} \tilde{X}'' \xleftarrow{p_3} \hat{X}, \tag{5.31}$$

where p_1 is the blowing-up along 9 lines $\tilde{A}_i^{(j)}$ $(0\leq i,j\leq 2)$, p_2 is the blowing-up along 9 lines $L'_{i,j}$ in the exceptional sets of π_1, and p_3 is the blowing-up along 9 lines $L''_{i,j}$ in the exceptional sets of π_2. Then $L'_{i,j}$ and $L''_{i,j}$ are defined as follows.

We denote by $\mathcal{C}_i'^{(j)}$ the strict transform of $\mathcal{C}_i^{(j)}$ under the blowing-up p_1. For each $\tilde{A}_i^{(j)}$, we know that the affine open subsets which intersect with $\tilde{A}_i^{(j)}$ are $V^{(0,i,j)}$, $V^{(\zeta_3^k, 2j+(2-i)k, i)}$ $(0\leq k\leq 2)$ and $V^{(\infty,i+1,j)}$ by the above construction. For each $I = (0,i,j)$, $(\zeta_3^k, 2j+(2-i)k, i)$, $(\infty, i+1, j)$, we denote by $p_1 : V'^I \to V^I$ the blowing-up of V^I along $\tilde{A}_i^{(j)} \cap V^I = V(\alpha_1^I, \alpha_2^I)$. Then we have

$$\begin{aligned} V'^I \ = \ & \left(\operatorname{Spec}\left(k\left[u^{(a(I))}, \alpha_1^I, \frac{\alpha_2^I}{\alpha_1^I}, \frac{1}{\left(u^{(a(I))}\right)^3 - 1}\right]\right)\right. \\ & \left.\cup \operatorname{Spec}\left(k\left[u^{(a(I))}, \frac{\alpha_1^I}{\alpha_2^I}, \alpha_2^I, \frac{1}{\left(u^{(a(I))}\right)^3 - 1}\right]\right)\right) \setminus \mathcal{C}_i'^{(j)}, \end{aligned}$$

where $a(I)$ is the first component of I. We define $L_{i,j}^{\prime(a(I))} := V\left(\alpha_1^I/\alpha_2^I, \alpha_2^I\right) \subset V'^I$. By gluing these lines, we get a projective line

$$\tilde{X}' \supset L'_{i,j} := L_{i,j}^{\prime(0)} \cup \bigcup_{0 \leq k \leq 2} L_{i,j}^{\prime(\zeta_3^k)} \cup L_{i,j}^{\prime(\infty)}.$$

We denote by $\mathcal{C}_i^{\prime\prime(j)}$ the strict transform of $\mathcal{C}_i^{\prime(j)}$ under the blowing-up p_2. We consider locally again. For each $I = (0, i, j)$, $(\zeta_3^k, 2j + (2-i)k, i)$, $(\infty, i+1, j)$, we denote by $p_2 : V''^I \to V'^I$ the blowing-up of V'^I along $L_{i,j}^{\prime(a(I))} = V\left(\alpha_1^I/\alpha_2^I, \alpha_2^I\right)$. Then we have

$$\begin{aligned} V''^I \quad = \quad & \left(\operatorname{Spec}\left(k\left[u^{(a(I))}, \alpha_1^I, \frac{\alpha_2^I}{\alpha_1^I}, \frac{1}{\left(u^{(a(I))}\right)^3 - 1}\right]\right)\right. \\ & \cup \operatorname{Spec}\left(k\left[u^{(a(I))}, \frac{\alpha_1^I}{\alpha_2^I}, \frac{\left(\alpha_2^I\right)^2}{\alpha_1^I}, \frac{1}{\left(u^{(a(I))}\right)^3 - 1}\right]\right) \\ & \left.\cup \operatorname{Spec}\left(k\left[u^{(a(I))}, \frac{\alpha_1^I}{\left(\alpha_2^I\right)^2}, \alpha_2^I, \frac{1}{\left(u^{(a(I))}\right)^3 - 1}\right]\right)\right) \setminus \mathcal{C}_i^{\prime\prime(j)}. \end{aligned}$$

We define $L_{i,j}^{\prime\prime(a(I))} := V\left(\alpha_1^I/\alpha_2^I, \left(\alpha_2^I\right)^2/\alpha_1^I\right) \subset V''^I$. By gluing these lines, we get a projective line

$$\tilde{X}'' \supset L''_{i,j} := L_{i,j}^{\prime\prime(0)} \cup \bigcup_{0 \leq k \leq 2} L_{i,j}^{\prime\prime(\zeta_3^k)} \cup L_{i,j}^{\prime\prime(\infty)}.$$

We denote by $\hat{\mathcal{C}}_i^{(j)}$ the strict transform of $\mathcal{C}_i^{\prime\prime(j)}$ under the blowing-up p_3. Then $\hat{X}$ has an affine open covering

$$\hat{X} = \bigcup_{\substack{a \in \{0,1,\zeta_3,\zeta_3^2,\infty\}, \\ 0 \leq i,j \leq 2,\ 1 \leq r \leq 4}} \hat{V}_r^{(a,i,j)} \cup \bigcup_{\substack{a=0 \text{ or } a^3=1, \\ 0 \leq i \leq 2,\ 2 \leq \ell \leq 7}} V_\ell^{(a,i)},$$

where affine open subsets $\hat{V}_r^{(a,i,j)}$ are given by

$$\hat{V}_1^{(a,i,j)} = \operatorname{Spec}\left(k\left[u^{(a)}, \alpha_1^{(a,i,j)}, \frac{\alpha_2^{(a,i,j)}}{\alpha_1^{(a,i,j)}}, \frac{1}{\left(u^{(a)}\right)^3 - 1}\right]\right) \setminus \hat{\mathcal{C}}_{i'}^{(j')}, \tag{5.32}$$

$$\hat{V}_2^{(a,i,j)} = \operatorname{Spec}\left(k\left[u^{(a)}, \frac{\alpha_1^{(a,i,j)}}{\alpha_2^{(a,i,j)}}, \frac{\left(\alpha_2^{(a,i,j)}\right)^3}{\left(\alpha_1^{(a,i,j)}\right)^2}, \frac{1}{\left(u^{(a)}\right)^3 - 1}\right]\right) \setminus \hat{\mathcal{C}}_{i'}^{(j')}, \tag{5.33}$$

$$\hat{V}_3^{(a,i,j)} = \operatorname{Spec}\left(k\left[u^{(a)}, \frac{\left(\alpha_1^{(a,i,j)}\right)^2}{\left(\alpha_2^{(a,i,j)}\right)^3}, \frac{\left(\alpha_2^{(a,i,j)}\right)^2}{\alpha_1^{(a,i,j)}}, \frac{1}{\left(u^{(a)}\right)^3 - 1}\right]\right) \setminus \hat{\mathcal{C}}_{i'}^{(j')}, \tag{5.34}$$

$$\hat{V}_4^{(a,i,j)} = \operatorname{Spec}\left(k\left[u^{(a)}, \frac{\alpha_1^{(a,i,j)}}{\left(\alpha_2^{(a,i,j)}\right)^2}, \alpha_2^{(a,i,j)}, \frac{1}{\left(u^{(a)}\right)^3 - 1}\right]\right) \setminus \hat{\mathcal{C}}_{i'}^{(j')} \tag{5.35}$$

and (i', j') is defined by (5.29). In what follows, we denote by $w_0^{(a,i,j,r)}$, $w_1^{(a,i,j,r)}$, $w_2^{(a,i,j,r)}$ the above coordinates of $\hat{V}_r^{(a,i,j)}$.

Next we define a map ψ from $\hat{X} = \bigcup \hat{V}_r^{(a,i,j)} \cup \bigcup V_\ell^{(a,i)}$ to $\overline{P}_{1,3}$. Recall that we define already a map from $\bigcup V_\ell^{(a,i)}$ to $\overline{P}_{1,3}$ (see (5.28)). We will define a map $\psi_r^{(a,i,j)}$ from $\hat{V}_r^{(a,i,j)}$ to $\overline{P}_{1,3}$, and we will glue these maps. Let

$$\left(C_{(\infty)}, L_{(\infty)}\right) = \left(V(F_{(\infty)}(x)), V(S_{(\infty)}(x))\right), \tag{5.36}$$

$$F_{(\infty)}(x) := x_0^3 + x_1^3 + x_2^3 - 3u^{(\infty)}x_0x_1x_2,\ S_{(\infty)}(x) := b_0x_0 + b_1x_1 + b_2x_2,$$

and we define matrices $N_{a,i,j,r}$ as follows. When $i = 2$, we set

$$N_{a,2,j,r} = \begin{cases} \begin{bmatrix} \alpha_1^{(a,2,j)} & 0 & 0 \\ \left(\alpha_1^{(a,2,j)}\right)^{2/3}\zeta_3^{2j} & \left(\alpha_1^{(a,2,j)}\right)^{2/3} & 0 \\ \zeta_3^j & \zeta_3^{2j} & 1/u^{(a)} \end{bmatrix} & (r = 1, 2), \\[2em] \begin{bmatrix} \left(\alpha_2^{(a,2,j)}\right)^{1/2} & 0 & 0 \\ \zeta_3^{2j} & 1 & 0 \\ \zeta_3^j/\alpha_2^{(a,2,j)} & \zeta_3^{2j}/\alpha_2^{(a,2,j)} & 1/\alpha_2^{(a,2,j)}u^{(a)} \end{bmatrix} & (r = 3, 4), \end{cases}$$

$$N_{\infty,2,j,r} = \begin{cases} \begin{bmatrix} \alpha_1^{(\infty,2,j)} & 0 & 0 \\ 0 & \left(\alpha_1^{(\infty,2,j)}\right)^{2/3} & 0 \\ \zeta_3^j & u^{(\infty)}\zeta_3^{2j} & 1 \end{bmatrix} & (r = 1, 2), \\[2em] \begin{bmatrix} \left(\alpha_2^{(\infty,2,j)}\right)^{1/2} & 0 & 0 \\ 0 & 1 & 0 \\ \zeta_3^j/\alpha_2^{(\infty,2,j)} & u^{(\infty)}\zeta_3^{2j}/\alpha_2^{(\infty,2,j)} & 1/\alpha_2^{(\infty,2,j)} \end{bmatrix} & (r = 3, 4). \end{cases}$$

When $i = 0$ or 1, we define $N_{a,i,j,r}$ as the matrices given by replacing $\alpha_i^{(a,2,j)}$ with $\alpha_i^{(a,i,j)}$ in $\tau^{i+1} \cdot N_{a,2,j,r} \cdot \tau^{2(i+1)}$, where $\tau = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Then for $a \in \{0, 1, \zeta_3, \zeta_3^2, \infty\}$, we define the image of $\psi_r^{(a,i,j)}$ as the semistable pair given by

$$N_{a,i,j,r} \cdot \left(C_{(a)}, L_{(a)}\right) = \left(V\left(F_{(a)}\left(N_{a,i,j,r}^{-1} \cdot x\right)\right), V\left(S_{(a)}\left(N_{a,i,j,r}^{-1} \cdot x\right)\right)\right),$$

where $\left(C_{(a)}, L_{(a)}\right)$ is defined by (5.17) or (5.36). Specifically, we can write $\psi_r^{(a,i,j)}$ as follows.

Definition 5.4.1. For each affine open subsets $\hat{V}_r^{(a,i,j)}$ ($a = 0$, $a^3 = 1$ or $a = \infty$, $0 \le i, j \le 2$, $1 \le r \le 4$), we put $I = (a, i, j, r)$. Then we define $\psi_r^{(a,i,j)} : \hat{V}_r^{(a,i,j)} \longrightarrow \overline{P}_{1,3}$ as follows. When $i = 2$ and $a = 0$ or $a^3 = 1$,

$$\psi_1^{(a,2,j)}\left(w_0^I, w_1^I, w_2^I\right) := \begin{cases} C : \left(\left(w_0^I\right)^3 - 1\right) y_1^3 = 3\zeta_3^{2j} y_0^2 y_2 - 3\left(w_1^I\right)^{1/3} y_0 y_1 y_2 \\ \quad + \left(w_0^I\right)^3 y_2 \left(\left(w_1^I\right)^2 y_2^2 - 3\zeta_3^{2j}\left(w_1^I\right)^{4/3} y_1 y_2 \right. \\ \qquad\qquad\qquad \left. +3\zeta_3^j \left(w_1^I\right)^{2/3} y_1^2\right), \\ L : y_0 + \left(w_1^I\right)^{1/3} w_2^I y_1 + y_2 = 0, \end{cases} \tag{5.37}$$

$$\psi_2^{(a,2,j)}\left(w_0^I, w_1^I, w_2^I\right) := \begin{cases} C : \left(\left(w_0^I\right)^3 - 1\right) y_1^3 = 3\zeta_3^{2j} y_0^2 y_2 - 3 w_1^I \left(w_2^I\right)^{1/3} y_0 y_1 y_2 \\ \quad + \left(w_0^I\right)^3 \left(w_1^I\right)^2 y_2 \left(\left(w_1^I\right)^4 \left(w_2^I\right)^2 y_2^2 \right. \\ \qquad \left. -3\zeta_3^{2j}\left(w_1^I\right)^2 \left(w_2^I\right)^{4/3} y_1 y_2 + 3\zeta_3^j \left(w_2^I\right)^{2/3} y_1^2\right), \\ L : y_0 + \left(w_2^I\right)^{1/3} y_1 + y_2 = 0, \end{cases} \tag{5.38}$$

$$\psi_3^{(a,2,j)}\left(w_0^I, w_1^I, w_2^I\right) := \begin{cases} C : \left(\left(w_0^I\right)^3 - 1\right) y_1^3 = 3\zeta_3^{2j} y_0^2 y_2 - 3\left(w_1^I\right)^{1/2} w_2^I y_0 y_1 y_2 \\ \quad + \left(w_0^I\right)^3 w_1^I \left(w_2^I\right)^2 y_2 \left(\left(w_1^I\right)^2 \left(w_2^I\right)^4 y_2^2 \right. \\ \qquad \left. -3\zeta_3^{2j} w_1^I \left(w_2^I\right)^2 y_1 y_2 + 3\zeta_3^j y_1^2\right), \\ L : \left(w_1^I\right)^{1/2} y_0 + y_1 + y_2 = 0, \end{cases} \tag{5.39}$$

$$\psi_4^{(a,2,j)}\left(w_0^I, w_1^I, w_2^I\right) := \begin{cases} C : \left(\left(w_0^I\right)^3 - 1\right) y_1^3 = 3\zeta_3^{2j} y_0^2 y_2 - 3\left(w_2^I\right)^{1/2} y_0 y_1 y_2 \\ \quad + \left(w_0^I\right)^3 w_2^I y_2 \left(\left(w_2^I\right)^2 y_2^2 - 3\zeta_3^{2j} w_2^I y_1 y_2 + 3\zeta_3^j y_1^2\right), \\ L : w_1^I \left(w_2^I\right)^{1/2} y_0 + y_1 + y_2 = 0. \end{cases} \tag{5.40}$$

When $i=2$ and $a=\infty$,

$$
\begin{aligned}
&\psi_1^{(\infty,2,j)}\left(w_0^I,w_1^I,w_2^I\right)\\
&:=\left\{\begin{array}{l}
C:\left(\left(w_0^I\right)^3-1\right)y_1^3=3\zeta_3^{2j}y_0^2y_2+3w_0^I\left(w_1^I\right)^{1/3}y_0y_1y_2-3\zeta_3^jw_1^Iy_0y_2^2\\
+y_2\left(\left(w_1^I\right)^2y_2^2-3\zeta_3^{2j}w_0^I\left(w_1^I\right)^{4/3}y_1y_2+3\zeta_3^j\left(w_0^I\right)^2\left(w_1^I\right)^{2/3}y_1^2\right),\\
L:y_0+\left(w_1^I\right)^{1/3}w_2^Iy_1+y_2=0,
\end{array}\right.
\end{aligned}
\tag{5.41}
$$

$$
\begin{aligned}
&\psi_2^{(\infty,2,j)}\left(w_0^I,w_1^I,w_2^I\right)\\
&:=\left\{\begin{array}{l}
C:\left(\left(w_0^I\right)^3-1\right)y_1^3=3\zeta_3^{2j}y_0^2y_2+3w_0^Iw_1^I\left(w_2^I\right)^{1/3}y_0y_1y_2\\
\quad-3\zeta_3^j\left(w_1^I\right)^3w_2^Iy_0y_2^2+\left(w_1^I\right)^2y_2\left(\left(w_1^I\right)^4\left(w_2^I\right)^2y_2^2\right.\\
\qquad\left.-3\zeta_3^{2j}w_0^I\left(w_1^I\right)^2\left(w_2^I\right)^{4/3}y_1y_2+3\zeta_3^j\left(w_0^I\right)^2\left(w_2^I\right)^{2/3}y_1^2\right),\\
L:y_0+\left(w_2^I\right)^{1/3}y_1+y_2=0,
\end{array}\right.
\end{aligned}
\tag{5.42}
$$

$$
\begin{aligned}
&\psi_3^{(\infty,2,j)}\left(w_0^I,w_1^I,w_2^I\right)\\
&:=\left\{\begin{array}{l}
C:\left(\left(w_0^I\right)^3-1\right)y_1^3=3\zeta_3^{2j}y_0^2y_2+3w_0^I\left(w_1^I\right)^{1/2}w_2^Iy_0y_1y_2\\
\quad-3\zeta_3^j\left(w_1^I\right)^{3/2}\left(w_2^I\right)^3y_0y_2^2+\left(w_2^I\right)^2y_2\left(\left(w_1^I\right)^3\left(w_2^I\right)^4y_2^2\right.\\
\qquad\left.-3\zeta_3^{2j}w_0^I\left(w_1^I\right)^2\left(w_2^I\right)^2y_1y_2+3\zeta_3^j\left(w_0^I\right)^2w_1^Iy_1^2\right),\\
L:\left(w_1^I\right)^{1/2}y_0+y_1+y_2=0,
\end{array}\right.
\end{aligned}
\tag{5.43}
$$

$$
\begin{aligned}
&\psi_4^{(\infty,2,j)}\left(w_0^I,w_1^I,w_2^I\right)\\
&:=\left\{\begin{array}{l}
C:\left(\left(w_0^I\right)^3-1\right)y_1^3=3\zeta_3^{2j}y_0^2y_2+3w_0^I\left(w_2^I\right)^{1/2}w_2^Iy_0y_1y_2\\
\quad-3\zeta_3^j\left(w_2^I\right)^{3/2}y_0y_2^2+w_2^Iy_2\left(\left(w_2^I\right)^2y_2^2\right.\\
\qquad\left.-3\zeta_3^{2j}w_0^Iw_2^Iy_1y_2+3\zeta_3^j\left(w_0^I\right)^2y_1^2\right),\\
L:w_1^I\left(w_2^I\right)^{1/2}y_0+y_1+y_2=0.
\end{array}\right.
\end{aligned}
\tag{5.44}
$$

When $i=0$ or 1, we define the images of $\psi_r^{(a,i,j)}$ as semistable pairs given by replacing $\left(y_k,w_k^{(a,2,j,r)}\right)$ with $\left(y_{k+i+1},w_k^{(a,i,j,r)}\right)$ in (5.37), …, (5.44).

The above definitions are independent of a choice of double or cubic roots of w_1^I, w_2^I. In fact, for (5.37), (5.38), (5.41), (5.42) (resp. (5.39), (5.40), (5.43), (5.44)), we have same semistable pairs in $\overline{P}_{1,3}$ under the transformation $y_1\mapsto\beta y_1$, $\beta^3=1$ (resp. $y_1\mapsto -y_1$).

We can show that (5.37), …, (5.44) are samistable pairs. Here we only consider in the cases of (5.39), (5.41). The other cases can be checked by the same arguments. For (5.39), that is, $I=(a,2,j,3)$, if $w_1^Iw_2^I=0$, then we

have
$$C:\left(\left(w_0^I\right)^3-1\right)y_1^3=3\zeta_3^{2j}y_0^2y_2.$$
By definition, we have $\left(w_0^I\right)^3\neq 1$, hence C is a cuspidal curve with the cusp point $P_2=(0:0:1)$. Then L does not pass through P_2 and L is not 3-tangent to C. Hence (C,L) is a semistable pair. Suppose $w_1^Iw_2^I\neq 0$. Then by the transformation
$$y_0=\left(w_1^I\right)^{1/2}w_2^Ix_0,\ y_1=x_1+\zeta_3^{2j}x_0,\ y_2=\left(x_2+\zeta_3^{2j}x_1+\zeta_3^jx_0\right)/\left(w_1^I(w_2^I)^2\right),$$
we obtain $C:x_0^3+x_1^3+\left(w_0^I\right)^3x_2^3=3x_0x_1x_2$ and
$$L:\left(w_1^I(w_2^I)^2\left(w_1^Iw_2^I+\zeta_3^{2j}\right)+\zeta_3^j\right)x_0+\left(w_1^I(w_2^I)^2+\zeta_3^{2j}\right)x_1+x_2=0.$$
If $w_0^I=0$, then C is a nodal curve with the nodal point P_2. However L does not pass through the point. If $w_0^I\neq 0$, then C is a smooth cubic curve, since $\left(w_0^I\right)^3\neq 1$. Recall that L is 3-tangent to C if and only if $(w_0^I,w_1^I,w_2^I)\in\hat{\mathcal{C}}_{i'}^{(j')}$ (where i', j' are defined by (5.29)). By construction, we have $\hat{V}_3^{(a,2,j)}\cap\hat{\mathcal{C}}_{i'}^{(j')}=\emptyset$, hence L is not 3-tangent to C. Thus (C,L) is a semistable pair.

For (5.41), that is, $I=(\infty,2,j,1)$, if $w_1^I=0$, then we have $L:y_0+y_2=0$ and
$$C:\left(\left(w_0^I\right)^3-1\right)y_1^3=3\zeta_3^{2j}y_0^2y_2.$$
Since $\left(w_0^I\right)^3\neq 1$, C is a cuspidal curve with the cusp point $P_2=(0:0:1)$. Then L does not pass through P_2 and L is not 3-tangent to C. Hence (C,L) is a semistable pair. Suppose $w_1^I\neq 0$. Then by the transformation
$$y_0=w_1^Ix_0,\ y_1=\left(w_1^I\right)^{2/3}x_1,\ y_2=x_2+\zeta_3^jx_0+\zeta_3^{2j}w_0^Ix_1,$$
we obtain
$$\begin{cases} C:x_0^3+x_1^3+x_2^3=3w_0^Ix_0x_1x_2,\\ L:\left(w_1^I+\zeta_3^j\right)x_0+\left(w_0^I\zeta_3^{2j}+w_1^Iw_2^I\right)x_1+x_2=0.\end{cases}$$
Then similarly to above, C is a smooth cubic curve, and L is not 3-tangent to C. Hence (C,L) is a semistable pair.

Theorem 5.4.2. *We can glue the maps ψ on $\bigcup V_\ell^{(a,i)}$ and $\psi_r^{(a,i,j)}$ on $\hat{V}_r^{(a,i,j)}$, and we obtain a map*
$$\psi:\hat{X}=\bigcup_{\substack{a\in\{0,1,\zeta_3,\zeta_3^2,\infty\},\\ 0\le i,j\le 2,\ 1\le r\le 4}}\hat{V}_r^{(a,i,j)}\cup\bigcup_{\substack{a\in\{0,1,\zeta_3,\zeta_3^2\},\\ 0\le i\le 2,\ 2\le\ell\le 7}}V_\ell^{(a,i)}\longrightarrow\overline{P}_{1,3}.$$
This map is an extension of (5.28) and $\psi^{-1}(W_{\mathrm{Cusp}})$ is the exceptional sets of the composition $p:=p_1\circ p_2\circ p_3:\hat{X}\to\tilde{X}$.

Proof. The second assertion is clear by the construction (see Remark 5.4.4). We show the first assertion. We denote by $\psi_\ell^{(a,i)}$ the restriction of ψ to $V_\ell^{(a,i)}$. By the definition, we have $\psi_\ell^{(a,i)} = \varphi_\ell^{(a,i)}$ on $V_\ell^{(a,i)} \subset U_\ell^{(a,i)}$. We have to prove that the gluing of maps $\psi_r^{(a,i,j)}$, $\varphi_\ell^{(b,k)}$ is well-defined. Recall that the images of $\psi_r^{(a,i,j)}$, $\varphi_\ell^{(b,k)}$ are given by

$$\begin{aligned} N_{a,i,j,r} \cdot \left(C_{(a)}, L_{(a)}\right) &= \left(N_{a,i,j,r} \cdot \sigma_{(a)}\right) \cdot \left(C_{(0)}, L_{(0)}\right), \\ M_{b,k,\ell} \cdot \left(C_{(b)}, L_{(b)}\right) &= \left(M_{b,k,\ell} \cdot \sigma_{(b)}\right) \cdot \left(C_{(0)}, L_{(0)}\right), \end{aligned}$$

respectively, where $\sigma_{(a)}$ ($a = 0$ or $a^3 = 1$) is defined in the proof of Theorem 5.3.2 and $\sigma_{(\infty)} := \mathrm{id}$. In the proof of Lemma 5.3.2, for $J = (b, k, \ell)$, we write formally

$$\left(M_{b,k,\ell} \cdot \sigma_{(b)}\right)^{-1} \cdot \psi_\ell^{(b,k)} \left(v_0^J, v_1^J, v_2^J\right) = \left(\mathfrak{C}_{(J)}, \mathfrak{L}_{(J)}\right),$$
$$\mathfrak{C}_{(J)} : \mathfrak{u}^J(x_0^3 + x_1^3 + x_2^3) = 3x_0x_1x_2, \ \ \mathfrak{L}_{(J)} : \mathfrak{b}_0^J x_0 + \mathfrak{b}_1^J x_1 + \mathfrak{b}_2^J x_2 = 0.$$

Then $\mathfrak{u}^J$, $\mathfrak{b}_k^J$ are given by (5.25), (5.26), respectively. Similarly, for $I = (a, i, j, r)$, we can write formally

$$\left(N_{a,i,j,r} \cdot \sigma_{(a)}\right)^{-1} \cdot \psi_r^{(a,i,j)} \left(w_0^I, w_1^I, w_2^I\right) = \left(\mathfrak{C}_{(I)}, \mathfrak{L}_{(I)}\right),$$

where $\mathfrak{L}_{(I)}$ is given by $\mathfrak{b}_0^I x_0 + \mathfrak{b}_1^I x_1 + \mathfrak{b}_2^I x_2 = 0$ and $\mathfrak{C}_{(I)}$ is given by

$$\begin{cases} \mathfrak{u}^I(x_0^3 + x_1^3 + x_2^3) = 3x_0x_1x_2 & (a = 0 \text{ or } a^3 = 1), \\ x_0^3 + x_1^3 + x_2^3 = 3\mathfrak{u}^I x_0x_1x_2 & (a = \infty), \end{cases}$$

and $\mathfrak{u}^I$, $\mathfrak{b}_k^I$ are functions on $\hat{V}_r^{(a,i,j)}$, which give $u^{(a)}$, b_k, respectively. Then by constructions, we have

$$\begin{aligned} \mathfrak{u}^I &= \begin{cases} w_0^I & (a = 0 \text{ or } a = \infty), \\ \frac{1-w_0^I}{1+2w_0^I} a & (a^3 = 1), \end{cases} \\ \mathfrak{b}_k^I &= \begin{cases} b_k^I & (a = 0 \text{ or } a = \infty), \\ \left(b_0^I + \zeta_3^{2k} b_1^I + \zeta_3^k b_2^I\right) a^{k(k+2)} & (a^3 = 1), \end{cases} \end{aligned}$$

where b_k^I are defined by

$$\begin{aligned} b_k^I &= \begin{cases} 1 & (k = i, a = 0 \text{ or } a^3 = 1), \\ w_0^I \left(\mathfrak{a}_1^I + \zeta_3^{2j} \mathfrak{a}_2^I + \zeta_3^j\right) & (k = i + 1, a = 0 \text{ or } a^3 = 1), \\ w_0^I \left(\mathfrak{a}_2^I + \zeta_3^{2j}\right) & (k = i + 2, a = 0 \text{ or } a^3 = 1), \end{cases} \\ b_k^I &= \begin{cases} 1 & (k = i, a = \infty), \\ \mathfrak{a}_1^I + \zeta_3^j & (k = i + 1, a = \infty), \\ \mathfrak{a}_2^I + \zeta_3^{2j} w_0^I & (k = i + 2, a = \infty), \end{cases} \end{aligned}$$

and $\mathfrak{a}_m^I$ are defined by

$$\mathfrak{a}_1^I = \begin{cases} w_1^I & (r=1), \\ \left(w_1^I\right)^3 w_2^I & (r=2), \\ \left(w_1^I\right)^2 \left(w_2^I\right)^3 & (r=3), \\ w_1^I \left(w_2^I\right)^2 & (r=4), \end{cases} \quad \mathfrak{a}_2^I = \begin{cases} w_1^I w_2^I & (r=1), \\ \left(w_1^I\right)^2 w_2^I & (r=2), \\ w_1^I \left(w_2^I\right)^2 & (r=3), \\ w_2^I & (r=4). \end{cases}$$

Note that $\mathfrak{a}_m^I$ are functions on $\hat{V}_r^{(a,i,j)}$, which give $\alpha_m^{(a,i,j)}$.

By the same arguments in the proof of Lemma 5.3.2, we obtain that $\psi_\ell^{(b,k)} = \psi_{\ell'}^{(b',k')}$ on $V_\ell^{(b,k)} \cap V_{\ell'}^{(b',k')}$. Similarly, we obtain that $\psi_r^{(a,i,j)} = \psi_{r'}^{(a',i',j')}$ (resp. $\psi_r^{(a,i,j)} = \psi_\ell^{(b,k)}$) on $\tilde{V}_r^{(a,i,j)} \cap \tilde{V}_{r'}^{(a',i',j')}$ (resp. $\tilde{V}_r^{(a,i,j)} \cap V_\ell^{(b,k)}$). $\square$

Example 5.4.3. We here consider the case where $I = (1,2,0,1)$ and $I' = (\infty,0,1,3)$. For simplicity, we put $(w_0,w_1,w_2) = (w_0^I, w_1^I, w_2^I)$, $(w_0', w_1', w_2') = (w_0^{I'}, w_1^{I'}, w_2^{I'})$. Then we have

$$\begin{aligned} \mathfrak{u}^I &= \frac{1-w_0}{1+2w_0}, \\ \mathfrak{b}_k^I &= w_0(w_1 + w_1w_2 + 1) + \zeta_3^{2k} w_0(w_1w_2+1) + \zeta_3^k \quad (0 \le k \le 2), \\ \mathfrak{u}^{I'} &= w_0', \\ \left(\mathfrak{b}_0^{I'} : \mathfrak{b}_1^{I'} : \mathfrak{b}_2^{I'}\right) &= \left(1 : {w_1'}^2 {w_2'}^3 : w_1' {w_2'}^2\right). \end{aligned}$$

By solving equations $\mathfrak{u}^I = 1/\mathfrak{u}^{I'}$ and $\left(\mathfrak{b}_0^I : \mathfrak{b}_1^I : \mathfrak{b}_2^I\right) = \left(\mathfrak{b}_0^{I'} : \mathfrak{b}_1^{I'} : \mathfrak{b}_2^{I'}\right)$, we get rational functions $F_k(w_0', w_1', w_2')$ and $G_k(w_0, w_1, w_2)$ $(0 \le k \le 2)$ such that $w_k = F_k(G_0, G_1, G_2)$ and $w_k' = G_k(F_0, F_1, F_2)$. In fact, we can write

$$\begin{aligned} F_0 &= \frac{1-w_0'}{1+2w_0'}, \\ F_1 &= \frac{(1+2w_0')(w_1'{w_2'}^2(1-\zeta_3) - {w_1'}^2{w_2'}^3(1+2\zeta_3))}{(1-w_0')(w_1'{w_2'}^2(1+\zeta_3) - {w_1'}^2{w_2'}^3 - \zeta_3)}, \\ F_2 &= \frac{{w_1'}^2{w_2'}^3(w_0'(1+2\zeta_3)+2+\zeta_3) + w_1'{w_2'}^2(w_0'(\zeta_3-1)-2-\zeta_3) - 3w_0'\zeta_3}{(1+2w_0')(w_1'{w_2'}^2(1-\zeta_3) - {w_1'}^2{w_2'}^3(1+2\zeta_3))}, \\ G_0 &= \frac{1-w_0}{1+2w_0}, \\ G_1 &= \frac{(2w_0w_1w_2 + w_0w_1 + 2w_0 + 1)(w_0w_1w_2\zeta_3 - w_0w_1 + w_0\zeta_3 - \zeta_3)^2}{(-w_0w_1w_2\zeta_3^2 + w_0w_1 - w_0\zeta_3^2 + \zeta_3^2)^3}, \\ G_2 &= \frac{(-w_0w_1w_2\zeta_3^2 + w_0w_1 - w_0\zeta_3^2 + \zeta_3^2)}{(2w_0w_1w_2 + w_0w_1 + 2w_0 + 1)(-w_0w_1w_2\zeta_3 + w_0w_1 - w_0\zeta_3 + \zeta_3)}. \end{aligned}$$

Then two rational maps $G : \hat{V}_1^{(1,2,0)} \to \hat{V}_3^{(\infty,0,1)}$, $(w_0, w_1, w_2) \mapsto (G_0, G_1, G_2)$ and $F : \hat{V}_3^{(\infty,0,1)} \to \hat{V}_1^{(1,2,0)}$, $(w'_0, w'_1, w'_2) \mapsto (F_0, F_1, F_2)$ induce an isomorphism

$$G : \tilde{V}_1^{(1,2,0)} \setminus \left(W_G \cup G^{-1}(W_F)\right) \to \tilde{V}_3^{(\infty,0,1)} \setminus \left(W_F \cup F^{-1}(W_G)\right), \quad (5.45)$$

where W_G (resp. W_F) is the union of the zero sets of the denominators of G_k (resp. F_k), and we get $G^{-1}(W_F) = V(w_0w_1)$ and $F^{-1}(W_G) = V(w'_1w'_2)$ by simple computations. We regard (5.45) as $\hat{V}_1^{(1,2,0)} \cap \hat{V}_3^{(\infty,0,1)}$. Then $N_{1,2,0,1}$ and $N_{\infty,0,1,3}$ have nonzero determinants $w_1^{5/3}/w_0$ and $1/{w'_1}^{1/2}w'_2$ on $\hat{V}_1^{(1,2,0)} \cap \hat{V}_3^{(\infty,0,1)}$, respectively. Thus for any $(w_0, w_1, w_2) = (G_0, G_1, G_2) \in \hat{V}_1^{(1,2,0)} \cap \hat{V}_3^{(\infty,0,1)}$, we have

$$\begin{aligned} \psi_1^{(1,2,0)}(w_0, w_1, w_2) &= \left(N_{1,2,0,1} \cdot \sigma_{(1)}\right)^{-1} \cdot \psi_1^{(1,2,0)}(w_0, w_1, w_2) \\ &= \left(\mathfrak{C}_{(I)}, \mathfrak{L}_{(I)}\right) = \left(\mathfrak{C}_{(I')}, \mathfrak{L}_{(I')}\right) \\ &= \left(N_{\infty,0,1,3} \cdot \sigma_{(\infty)}\right)^{-1} \cdot \psi_3^{(\infty,0,1)}(G_0, G_1, G_2) \\ &= \psi_3^{(\infty,0,1)}(G_0, G_1, G_2) \end{aligned}$$

in $\overline{P}_{1,3}$.

Remark 5.4.4. By the map $\psi : \hat{X} \to \overline{P}_{1,3}$, we have a rough structure of $\overline{P}_{1,3}$ in the following sense. Let $\hat{V}^{(a,i,j)} := \bigcup_{r=1}^4 \hat{V}_r^{(a,i,j)}$, where $a = 0$, $a^3 = 1$ or $a = \infty$, and $0 \le i, j \le 2$. Recall that $\hat{V}^{(a,i,j)}$ is given by the blowing-up of $V^{(a,i,j)}$ three times:

$$V^{(a,i,j)} \xleftarrow{p_1} V'^{(a,i,j)} \xleftarrow{p_2} V''^{(a,i,j)} \xleftarrow{p_3} \hat{V}^{(a,i,j)}.$$

Let $E_\ell^{(a,i,j)}$ be the exceptional sets of the blowing-up p_ℓ, and let $\hat{E}_\ell^{(a,i,j)} \subset \hat{V}^{(a,i,j)}$ be the strict transform of $E_\ell^{(a,i,j)}$ under the above blowing-ups. Then we have

$$\begin{aligned} \hat{E}_1^{(a,i,j)} &= V\left(w_1^{(a,i,j,1)}\right) \cup V\left(w_2^{(a,i,j,2)}\right), \\ \hat{E}_2^{(a,i,j)} &= V\left(w_1^{(a,i,j,3)}\right) \cup V\left(w_2^{(a,i,j,4)}\right), \\ \hat{E}_3^{(a,i,j)} &= V\left(w_1^{(a,i,j,2)}\right) \cup V\left(w_2^{(a,i,j,3)}\right), \end{aligned}$$

where $w_0^{(a,i,j,r)}$, $w_1^{(a,i,j,r)}$, $w_2^{(a,i,j,r)}$ are coordinates of $\hat{V}_r^{(a,i,j)}$ defined in (5.32), …, (5.35). Let

$$\hat{E}_\ell := \bigcup_{\substack{a\in\{0,1,\zeta_3,\zeta_3^2,\infty\},\\ 0\le i,j\le 2}} \hat{E}_\ell^{(a,i,j)} \subset \hat{X}, \quad 1 \le \ell \le 3.$$

For any $w \in \hat{X}$, we put $\psi(w) = (C, L) \in \overline{P}_{1,3}$. Then we have

$$C \text{ is a cuspidal curve} \Longleftrightarrow w \in \hat{E} := \bigcup_{\ell=1}^{3} \hat{E}_\ell.$$

Hence $\psi^{-1}(W_{\text{Cusp}})$ is the exceptional set $\hat{E}$ of the composition of blowing-ups $p = p_1 \circ p_2 \circ p_3 : \hat{X} \to \tilde{X}$. Since $\hat{X} \setminus \hat{E} \simeq \tilde{X} \setminus \left(\bigcup \tilde{A}_i^{(j)} \right)$ and $\psi = \varphi$ on $\hat{X} \setminus \hat{E}$, we have same results on $w \in \hat{X} \setminus \hat{E}$ as Remark 5.3.4.

In summary, we have the following proposition:

Proposition 5.4.5. *We have a commutative diagram*

$$\begin{tikzcd}
\hat{X} \arrow[r, "\psi"] \arrow[d, "p"'] & \overline{P}_{1,3} \arrow[d, "f"] \\
\tilde{X} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \overline{AP}_{1,3} \\
X &
\end{tikzcd}$$

where f is the birational map defined in Section 4.1, π and p are compositions of blowing-ups (5.8) and (5.31), respectively, and φ and ψ are defined by Definition 5.3.1 and Definition 5.4.1, respectively.

Then $\varphi^{-1}(W_{\mathrm{T}})$ (resp. $\psi^{-1}(W_{\text{Cusp}})$) is the center $\bigcup \tilde{A}_i^{(j)}$ (resp. the exceptional set $\hat{E}$) of the composition of blowing-ups $p = p_1 \circ p_2 \circ p_3 : \hat{X} \to \tilde{X}$.

Proof. Clear from Theorem 5.3.2 and Theorem 5.4.2. □

Acknowledgments

The author would like to thank Professor Masanori Asakura and Professor Iku Nakamura for helpful comments and suggestions, and constant encouragement.

Bibliography

[A02] V. Alexeev, *Complete moduli in the presence of semiabelian group action*, Ann. of Math. **155** (2002), 611–708.

[B91] A. Borel, *Linear Algebraic Groups (Second Enlarged Edition)*, Springer-Verlag, (1991).

[D03] I. V. Dolgachev, *Lectures on invariant theory*, volume 296 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, (2003).

[Ha75] W. J. Haboush, *Reductive groups are geometrically reductive*, Ann. of Math. **102**, 67 (1975).

[Hu86] D. Husemöller, *Elliptic Curves*, Springer-Verlag, (1986).

[MFK94] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 34, Springer-Verlag, (1994).

[Na99] I. Nakamura, *Stability of degenerate abelian varieties*, Invent. Math. **136** (1999), 659–715.

[NaT01] I. Nakamura and T. Terasoma, *Moduli space of elliptic curves with Heisenberg level structure*, Moduli of Abelian Varieties, Proceeding of Texel conference 1999, Progress in Math., **195**, Birkhäuser (2001) 299-324.

[Na10] I. Nakamura, *Another canonical compactification of the moduli space of abelian varieties*, Algebraic and arithmetic structures of moduli spaces (Sapporo, 2007), Advanced Studies in Pure Math., **58** (2010), 69-135.

[Ne78] P. E. Newstead, *Lectures on Introduction to Moduli Problems and Orbit Spaces*, Tata Inst. Lecture Notes, Springer-Verlag, (1978).

[S86] J. H. Silverman, *The Arithmetic of elliptic curves*, GTM 106, Springer-Verlag, New York, (1986).