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Time-ordered Exponential for Unbounded Operators with Applications to Quantum Field Theory

(非有界作用素に対する time-ordered exponential と
場の量子論への応用)

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Abstract

Time-ordered exponential is a fundamental tool in theoretical physics and mathematical physics, and often used in quantum theory to give the perturbative expansion of significant objects, such as the time evolution, the n -point correlation functions, and the scattering amplitudes. The time-ordered exponential generated by a bounded operator-valued function has already been well researched, but the one generated by an unbounded operator-valued function has not fully investigated so far; the mathematical theory which is applicable to the analysis of concrete models of quantum field theory has been inadequately studied. The first main purpose of this work is to provide a general mathematical theory on time-ordered exponential.

The second main purpose is to construct concrete quantum field models and to analyze them rigorously. In this paper, we study the following: (I) construction of dynamics for non-symmetric Hamiltonians, (II) Gupta-Bleuler formalism, (III) Gell-Mann–Low formula, (IV) criteria for essential self-adjointness.

(I) In general, to quantize canonically a gauge theory in a Lorentz-covariant gauge such as the Lorenz gauge, it is necessary to adopt a vector space with an indefinite metric as a vector space of quantum mechanical state vectors, in order to realize the canonical commutation relations. The most important examples that cause this situation contain the mathematical model of quantum electrodynamics (QED) quantized in the Lorenz gauge. In this paper, we adopt a method called *η -formalism* that defines an indefinite metric through a Hilbert space with an inner product. Then the Hamiltonian is self-adjoint with respect to the indefinite metric but not symmetric and not even normal with respect to the inner product. By using the time-ordered exponential, we prove under certain assumptions that there exists a solution of the Schrödinger or the Heisenberg equation of motion generated by a linear operator acting in a Hilbert space, which may be unbounded, not symmetric, or not normal.

(II) As mentioned above, to quantize QED in the Lorenz gauge, the metric is indefinite. Then the Gupta-Bleuler formalism is used to restrict the total state space to the vector space of physical states which is a non-negative inner product space. Using the results obtained in (I), we apply the Gupta-Bleuler formalism to the Dirac-Maxwell model quantized in the Lorenz gauge, which describes a quantum system of Dirac particles and a gauge field minimally interacting each other.

(III) The Gell-Mann–Low formula plays an essential role in quantum field theory by generating the Feynman diagram expansion of the n -point correlation function with respect to the coupling constant. In this thesis, we give a mathematical formulation of Gell-Mann–Low formula and the proof of it by using the complex time evolution under certain abstract conditions. Furthermore, we apply the abstract results to QED with cutoffs.

(IV) In a Hilbert space, there is a one-to-one correspondence between self-adjoint operators and strongly continuous one-parameter unitary groups. In other words, the self-adjointness of the Hamiltonian is equivalent to the existence of the dynamics in the Schrödinger picture. Furthermore, if the dynamics in the interaction picture exists, then we can translate it into the Schrödinger picture (or the Heisenberg picture). Therefore it is expected that, if the time-ordered exponential representation of the time evolution in the interaction picture exists, then the self-adjointness of the total Hamiltonian follows. Based on this consideration, we present a new theorem concerning a sufficient condition for the essential self-adjointness of a symmetric operator acting in a Hilbert space. By applying the theorem, we prove the essential self-adjointness of the Hamiltonian of the Dirac-Maxwell model in the Coulomb gauge, which is not semi-bounded.

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1 Introduction

In Subsection 1.1, we give a brief and general introduction of time-ordered exponential. Subsections 1.2, 1.3, 1.4 and 1.5 are intended to give the relevant background knowledge to Sections 3, 4, 5 and 6, respectively. These subsections require more technical knowledge of mathematics and physics.

1.1 Introduction to time-ordered exponential

In this subsection, we give a formal introduction of time-ordered exponential. The rigorous definition and the properties of time-ordered exponential will be discussed in Section 2.

Time-ordered exponential is an infinite series defined as follows:

$$\mathcal{T}\left\{\exp\left[\int_{t'}^t d\tau A(\tau)\right]\right\} := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{t'}^t \cdots \int_{t'}^t d\tau_1 \cdots d\tau_n \mathcal{T}\{A(\tau_1) \cdots A(\tau_n)\}, \quad t, t' \in \mathbb{R} \quad (1.1)$$

Here A is a function from $\mathbb{R}$ into an algebra. The symbol $\mathcal{T}$ is the *time-ordering*, that is, $\mathcal{T}$ denotes the procedure that orders the product $A(\tau_1) \cdots A(\tau_n)$ according to the value of parameter:

$$\mathcal{T}\{A(\tau_1) \cdots A(\tau_n)\} := \sum_{\sigma \in \mathfrak{S}_n} \chi_{P_n}(\tau_{\sigma(1)}, \dots, \tau_{\sigma(n)}) A(\tau_{\sigma(1)}) \cdots A(\tau_{\sigma(n)}), \quad (1.2)$$

where

$$P_n := \{(\tau_1, \dots, \tau_n) \in \mathbb{R}^n \mid \tau_1 > \cdots > \tau_n\}. \quad (1.3)$$

Here $\mathfrak{S}_n$ denotes the symmetric group of order n , and χ_J is the characteristic function of the set J . For example,

$$\mathcal{T}\{A(1)A(3)A(2)\} = A(3)A(2)A(1). \quad (1.4)$$

Note that the time-ordering $\mathcal{T}$ should not be regarded as the operation which acts on the operator $A(\tau_1) \cdots A(\tau_n)$ itself. In fact, if we define $\mathcal{T}$ as the operation which acts on the operator, then when $A(\tau_1)A(\tau_2) = A(\tau'_1)A(\tau'_2)$ for some $\tau_1, \tau_2, \tau'_1, \tau'_2$ with $\tau_1 > \tau_2, \tau'_1 < \tau'_2$, the results of the operation of $\mathcal{T}$ differ:

$$\mathcal{T}\{A(\tau_1)A(\tau_2)\} = A(\tau_1)A(\tau_2), \quad (1.5)$$

$$\mathcal{T}\{A(\tau'_1)A(\tau'_2)\} = A(\tau'_2)A(\tau'_1), \quad (1.6)$$

even though $\mathcal{T}$ acts on the same operator.

The time-ordered exponential (1.1) can be rewritten without the time-ordering $\mathcal{T}$ as follows:

$$\begin{aligned} & \mathcal{T}\left\{\exp\left[\int_{t'}^t d\tau A(\tau)\right]\right\} \\ &= 1 + \int_{t'}^t d\tau_1 A(\tau_1) + \int_{t'}^t d\tau_1 \int_{t'}^{\tau_1} d\tau_2 A(\tau_1)A(\tau_2) + \cdots, \end{aligned} \quad (1.7)$$

This series is known as the *Dyson series*.

If $A(\tau)$'s commute each other with respect to the product, then (1.1) can be easily rewritten as

$$\begin{aligned}
& \mathcal{T}\left\{\exp\left[\int_{t'}^t d\tau A(\tau)\right]\right\} \\
&= 1 + \int_{t'}^t d\tau_1 A(\tau_1) + \frac{1}{2!} \int_{t'}^t d\tau_1 \int_{t'}^t d\tau_2 A(\tau_1)A(\tau_2) + \cdots \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{t'}^t d\tau A(\tau)\right)^n \\
&= \exp\left(\int_{t'}^t d\tau A(\tau)\right),
\end{aligned} \tag{1.8}$$

and reduces to the ordinary exponential series. Thus the time-ordered exponential has an importance when $A(\tau)$ ($\tau \in \mathbb{R}$) are non-commutative. In this paper, we particularly deal with the case where $A(\tau)$'s are linear operators acting in a Hilbert space. In applications to quantum theory, one of the most important cases is when the function A is given by $A(\tau) = -iH_1(\tau)$, $H_1(\tau) := e^{itH_0}H_1e^{-itH_0}$ for linear operators H_0, H_1 on a Hilbert space $\mathcal{H}$. The operator H_0 and H_1 physically mean the free and the interaction Hamiltonian, respectively. Then we see that the time-ordered exponential $\mathcal{T}\{\exp[-i\int_{t'}^t d\tau H_1(\tau)]\}$ is a formal solution of the following differential equations:

$$\frac{\partial}{\partial t}U(t, t') = -iH_1(t)U(t, t'), \quad U(t, t) = 1, \tag{1.9}$$

and

$$\frac{\partial}{\partial t'}U(t, t') = iU(t, t')H_1(t'), \quad U(t, t) = 1. \tag{1.10}$$

In fact, the time-ordered exponential (1.7) can be obtained by the iteration of the following integral equations:

$$U(t, t') = 1 - i \int_{t'}^t d\tau H_1(\tau)U(\tau, t'), \tag{1.11}$$

which is formally equivalent to (1.9), and

$$U(t, t') = 1 - i \int_{t'}^t d\tau U(t, \tau)H_1(\tau). \tag{1.12}$$

which is formally equivalent to (1.10). Now we set

$$H := H_0 + H_1. \tag{1.13}$$

Then it is easy to see that the operator-valued function $(t, t') \mapsto e^{itH_0}e^{-i(t-t')H}e^{-it'H_0}$ is also a formal solution of (1.9) and (1.10). Hence at least formally, it follows that

$$\mathcal{T}\left\{\exp\left[-i\int_{t'}^t d\tau H_1(\tau)\right]\right\} = e^{itH_0}e^{-i(t-t')H}e^{-it'H_0}. \tag{1.14}$$

(See Theorems 3.2 and 5.2 for rigorous results on this equation). Since Equation (1.7) holds, the time-ordered exponential is quite useful. Equation (1.7) is considered to give an asymptotic expansion of the left-hand side with respect to the interaction H_1 .

The series expansion of the left-hand side of (1.14) has already been rigorously analyzed in the case where H_1 is a bounded operator (See, e.g., Refs. [9], [19], [20], [33], [34], [53, Section X.12]). However, in the case where H_1 is not bounded, we have to handle (1.7) with $A(\tau) = H_1(\tau)$ very cautiously, because we need to care about the domain of the relevant unbounded operators, and examine the convergence conditions of (1.7) carefully; it seems that there have been few mathematically rigorous studies of the series expansion (1.14) in an abstract or a general form in the unbounded case (e.g., [17], [26]). The aim of Section 2 is to prove in a mathematically rigorous manner with certain assumptions that there exists a time evolution operator $U(t, t')$ satisfying (1.9) and (1.10), and the equality (1.14) on a certain dense subspace, including the case where H_1 is unbounded.

In addition, we study an extension of (1.14) to the complex plane:

$$\mathcal{T}\left\{\exp\left[-i\int_{z'}^z d\zeta H_1(\zeta)\right]\right\} = e^{izH_0} e^{-i(z-z')H} e^{-iz'H_0}, \quad z, z' \in \mathbb{C}, \quad (1.15)$$

in anticipation of the application to the Gell-Mann–Low formula in Section 5.

1.2 Construction of dynamics for non-symmetric Hamiltonians

Let $\mathcal{H}$ be a complex Hilbert space and H be a linear operator on $\mathcal{H}$. In Section 3, we consider the initial value problem for the Schrödinger equation

$$\frac{\partial \xi(t)}{\partial t} = -iH\xi(t), \quad \xi(0) = \xi \in \mathcal{H}, \quad (1.16)$$

or for the Heisenberg equation

$$\frac{dB(t)}{dt} = [iH, B(t)], \quad B(0) = B, \quad (1.17)$$

where B is a possibly unbounded linear operator on $\mathcal{H}$, and $[X, Y] := XY - YX$. In the context of quantum mechanics, the parameter $t \in \mathbb{R}$ represents time, H is regarded as the Hamiltonian of the quantum system under consideration and B denotes an observable. Therefore $\xi(t)$ and the operator $B(t)$ describe the state and the observable at time t , respectively. Then the mathematical study of the initial value problems (1.16) or (1.17) in a general framework is of great interest since it will bring us on the knowledge of a general class of quantum systems.

In the ordinary formulation of quantum mechanics, a Hamiltonian H is assumed to be a self-adjoint operator. In this case, the solutions of these equations are given by

$$\xi(t) = e^{-itH}\xi, \quad (1.18)$$

$$B(t) = e^{itH} B e^{-itH}, \quad (1.19)$$

with some suitable conditions for operator domains in the Heisenberg case (see [6], for details). However, in some models, the Hamiltonian H may not be self-adjoint or not even normal. When H is unbounded and not normal, the above time evolution operator e^{-itH} does not immediately make sense since for unbounded H it is usually defined through operational calculus. In such cases, it is not obvious at all that there exist solutions of these equations.

The most important realistic examples that can cause this difficulty include a mathematical model of Quantum Electrodynamics (QED) when it is quantized in a Lorentz covariant gauge such as Lorenz gauge [32, 35, 36, 60]. In the Lorenz-gauge QED, we have to adopt a

vector space with an indefinite metric as a vector space of quantum mechanical state vectors, in order to realize the canonical commutation relations. The indefinite metric formulation makes the Hamiltonian not-symmetric and even non-normal. Thus it is far from trivial that dynamics of the Lorenz-gauge QED really exists. To show the existence of dynamics for such models, one may apply the general theory of evolution equations or Cauchy problems by estimating the resolvent operators [16, 21], but we will take another way to avoid hard resolvent estimates. The first motivation of the present study is to establish a general theory as to the existence of dynamics with Hamiltonians which is not symmetric and not even normal.

In quantum mechanics, one can adopt several “pictures” of time evolution, depending on which variable, a state vector or an observable, should evolve in time. All pictures certainly produce physically equivalent predictions. In the Schrödinger picture, for instance, the state vector is considered to evolve in time. In the Heisenberg picture, an observable (which is represented by a self-adjoint operator) is considered to depend on time. Our existence theorem will be proved via another picture — so called interaction picture —, in which both a state vector and an observable evolve in time. Now we consider a system with a Hamiltonian of the type

$$H = H_0 + H_1, \quad (1.20)$$

where H_0 is a self-adjoint operator denoting a Hamiltonian and H_1 is an interaction Hamiltonian. The evolution operator in the interaction picture from time t' to time t — which is usually denoted by $U(t, t')$ — is a solution of the differential equations (1.9) and (1.10). As mentioned in Subsection 1.1, the time-ordered exponential $\mathcal{T}\{\exp[-i \int_{t'}^t d\tau H_1(\tau)]\}$ forms a solution of the differential equations (1.9) and (1.10), and is formally equivalent to $e^{itH_0}e^{-i(t-t')H}e^{-it'H_0}$. Hence if $\mathcal{T}\{\exp[-i \int_{t'}^t d\tau H_1(\tau)]\}$ converges, then the operator

$$W(t) := e^{-itH_0}\mathcal{T}\left\{\exp\left[-i \int_0^t d\tau H_1(\tau)\right]\right\} \quad (1.21)$$

is expected to correspond to e^{-itH} . Using this operator $W(t)$, we construct solutions of (1.16) and (1.17).

1.3 Gupta-Bleuler formalism for the Dirac-Maxwell model

The purpose of this study is to treat the Dirac-Maxwell model in the Lorenz gauge and to apply the Gupta-Bleuler formalism to this model. Mathematically rigorous study of concrete models of QED in the Lorenz gauge was given only for solvable models so far; the Dirac-Maxwell model is *not solvable*, that is, an explicit representation of the time evolution of the gauge field is not easily found.

As mentioned in Subsection 1.2, to quantize gauge theories such as QED in the Lorenz covariant gauge, we inevitably adopt an indefinite metric space as a state space. In such cases, we have to restrict the total state space to a suitable positive definite subspace which is called *physical subspace*. The Gupta-Bleuler formalism [14, 28] is one of such schemes. The most general scheme to identify the physical subspace of quantized non-abelian gauge theories was given by Kugo and Ojima [39, 40]. Their formulation reduces to Nakanishi and Lautrup’s B -field formalism [41, 45, 46, 47] in the case where the gauge field is abelian; after some procedures, it reduces to the Gupta-Bleuler formalism.

The Gupta-Bleuler formalism is performed by imposing the following condition:

$$(\partial_\mu A^\mu)^{(+)}(t, \mathbf{x})\Psi = 0 \quad (1.22)$$

called *Gupta's subsidiary condition* on the state vectors, where $(\partial_\mu A^\mu)^{(+)}$ denotes the positive frequency part of the field $\partial_\mu A^\mu$ satisfying the Klein-Gordon equation: $\square \partial_\mu A^\mu = 0$. However, to perform this procedure rigorously, we have to solve some problems. Firstly, how to define the fields at time $t \in \mathbb{R}$ in an indefinite metric space? Secondly, is it possible to identify the positive frequency part even in an indefinite metric space? We solve the first problem by using the methods we developed in Section 3. As to the second one, a general definition of positive frequency part of a quantum field satisfying Klein-Gordon equation is given in [32].

The Dirac-Maxwell model is expected to describe a quantum system consisting of a Dirac particle and a quantized electromagnetic field with the minimal interaction. Informal perturbation methods show that this model derives the Klein-Nishina formula for the cross section of the Compton scattering of an electron and a photon, which agrees with the experimental results very well [50]. Hence it is strongly suggested that the Dirac-Maxwell model describes a class of natural phenomena where the quantized electromagnetic field plays an essential role. The mathematically rigorous study of this model was initiated by Arai in Ref. [3], and several mathematical aspects of the model was analyzed so far (see, e.g., [4], [5], [7], [55], and [59]). All of these studies are discussed in the Coulomb gauge. We emphasize that

For the reader's convenience, we give a formal outline of the Gupta-Bleuler formalism for the Dirac-Maxwell model without mathematical rigor. Let us consider a quantum system consisting of N Dirac particles and the quantized electromagnetic field minimally interacting with each other, and construct the quantized electromagnetic field A^μ ($\mu = 0, 1, 2, 3$) as an operator-valued distribution satisfying the following equation:

$$\square A^\mu(t, \mathbf{x}) = j^\mu(t, \mathbf{x}), \quad (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3, \quad (1.23)$$

where the current density j^μ is conserved:

$$\partial_\mu j^\mu(t, \mathbf{x}) = 0. \quad (1.24)$$

By this, together with (1.23) and (1.24), we find that $\partial_\mu A^\mu$ is a free field, that is, $\partial_\mu A^\mu$ satisfies the Klein-Gordon equation:

$$\square \partial_\mu A^\mu(t, \mathbf{x}) = 0. \quad (1.25)$$

Hence we can write $\partial_\mu A^\mu$ as

$$\partial_\mu A^\mu(t, \mathbf{x}) = \int d\mathbf{k} (a(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx}), \quad (1.26)$$

where $kx := k_\mu x^\mu$ with $k^0 = |\mathbf{k}|$ ($\mathbf{k} \in \mathbb{R}^3$), and $a(\cdot)$ and $a^\dagger(\cdot)$ are operator-valued distributions. The term which has the factor e^{-ikx} (resp. e^{ikx}) is called the positive frequency part (resp. negative frequency part) of $\partial_\mu A^\mu(t, \mathbf{x})$ and written as $(\partial_\mu A^\mu)^{(+)}(t, \mathbf{x})$ (resp. $(\partial_\mu A^\mu)^{(-)}(t, \mathbf{x})$). Now, for the state vectors Ψ , the following condition is postulated:

$$(\partial_\mu A^\mu)^{(+)}(t, \mathbf{x})\Psi = 0. \quad (1.27)$$

The state vectors satisfying (1.27) are called the physical states, and (1.27) is called the Gupta subsidiary condition. The set of physical states is denoted by $\mathcal{V}_{\text{phys}}$ and is called the physical subspace. Then the Lorenz condition $\partial_\mu A^\mu(t, \mathbf{x}) = 0$ is not valid in the sense of the operator equality, but it holds in the sense of the expectation value on the physical subspace:

$$\langle \Psi | \partial_\mu A^\mu(t, \mathbf{x}) | \Psi \rangle = 0, \quad \Psi \in \mathcal{V}_{\text{phys}}. \quad (1.28)$$

In Section 4, we perform these procedures in a mathematically rigorous manner.

1.4 Gell-Mann–Low formula

In Section 5, we consider a formula in quantum field theories of the type

$$\begin{aligned} & \left\langle \Omega, \mathcal{T} \{ \phi^{(1)}(x_1) \cdots \phi^{(n)}(x_n) \} \Omega \right\rangle \\ &= \lim_{t \rightarrow \infty} \frac{\left\langle \Omega_0, \mathcal{T} \{ \phi_1^{(1)}(x_1) \cdots \phi_1^{(n)}(x_n) \exp[-i \int_{-t}^t d\tau H_1(\tau)] \} \Omega_0 \right\rangle}{\left\langle \Omega_0, \mathcal{T} \{ \exp[-i \int_{-t}^t d\tau H_1(\tau)] \} \Omega_0 \right\rangle}, \end{aligned} \quad (1.29)$$

called the *Gell-Mann – Low formula* [24]. The meaning of each symbol in the formula (1.29) is as follows: the symbol $\langle \cdot, \cdot \rangle$ denotes the inner product of a Hilbert space of quantum state vectors, $\phi^{(k)}(x_k)$ and $\phi_1^{(k)}(x_k)$ ($k = 1, \dots, n$, $x_k \in \mathbb{R}^4$) denote field operators in the Heisenberg and the interaction picture, respectively. For instance, in quantum electrodynamics (QED), each $\phi^{(k)}$ denotes the Dirac field ψ_l , its conjugate $\psi_l^\dagger$, or the gauge field A_μ . The symbol $\mathcal{T}$ denotes the time-ordering and Ω and Ω_0 the vacuum states of the interacting and the free theory, respectively. The operator

$$\mathcal{T} \{ \exp[-i \int_{-t}^t d\tau H_1(\tau)] \}$$

is the time-ordered exponential for $H_1(\tau) := e^{i\tau H_0} H_1 e^{-i\tau H_0}$ ($\tau \in \mathbb{R}$), where H_0 and H_1 are the free and the interaction Hamiltonians.

This formula is used in physics literatures as a very useful tool to generate a perturbative expansion of the *n-point correlation function*

$$\left\langle \Omega, \mathcal{T} \{ \phi^{(1)}(x_1) \cdots \phi^{(n)}(x_n) \} \Omega \right\rangle \quad (1.30)$$

with respect to the coupling constant, which plays a quite important role in quantum field theory. However, derivation of (1.29) given in physics literatures is very heuristic and informal, and the mathematical proof of (1.29) is far from trivial. In fact, even the Hamiltonian is not easily given a mathematical meaning.

Even when the *n*-point correlation function (1.30) does not mathematically make sense, we can formally compute (after a renormalization procedure) this quantity via formal perturbation series given by (1.29). In other words, we can define the *n*-point correlation function not as an $\mathbb{C}$ -valued function but a formal power series-valued function via Feynman diagrams. To give such formal perturbation series, we no longer need the Hamiltonian and the field operators. When the coupling is small enough (for QED, this seems valid), the first few terms of the formal perturbation series is expected to be a good approximation of the correlation function which gives quantitative predictions for observable variables such as scattering cross section. In QED, these predictions agree with experimental results to eight significant figures, the most accurate predictions in all of natural sciences. Such formal computations can not be regarded as an approximation of the *n*-point function unless it has a well-defined mathematical meaning. Hence what we should do is to investigate what quantity is approximated by the perturbation series and to study the relation between the ordinary Hilbert-space formulation of quantum theory and the perturbation series. In other words, we have to clarify in what sense a perturbative formulation of quantum field theory is indeed a “quantum” theory. Thus it is very important in mathematical and *physical* point of view to study under what conditions the Gell-Mann – Low formula (1.29) is indeed true as a mathematical theorem within a Hilbert space formulation of quantum theory. The purpose of the present paper is to construct a mathematically rigorous setup in which the Gell-Mann – Low formula (1.29) is adequately formulated and proved.

In the 1960s, Wightman and Gårding [67] formulated a set of axioms in the framework of quantum mechanics which requires minimum properties that relativistic quantum field theory should satisfy. However, it is extremely difficult to construct a non-trivial model in the four-dimensional space-time which is physically acceptable and fulfills the axioms, and no such model has been found so far. We do not intend to construct such ideal models but abandon some of the axioms by introducing several regularizations so that each object is easily given mathematical meaning (of course, regularizations are employed in such a way that all the objects heuristically tends to the ideal ones in the limit where the regularizations are removed). In this way, field operators and a Hamiltonian are rigorously defined as linear operators acting in some Hilbert space. Furthermore, the vacuum states Ω and Ω_0 are realized as the eigenvectors corresponding to the infimum of the spectrum of the total and free Hamiltonians, if these exist. The existence of the ground state Ω , on which the validity of the Gell-Mann – Low formula crucially depends, is far from trivial, because it needs to analyze the perturbation of eigenvalues embedded in the continuous spectrum, to which regular perturbation theory [38] can not be applied. From the late 1990s to the 2000s, several important methods to prove the existence of ground states were developed in the study of a quantum system consisting of quantum particles and a Bose field (for example, see [8, 11, 25, 27, 58]). These methods have been improved by many authors to be also applicable to systems of interacting quantum fields [1, 12, 13, 18, 42, 61, 62, 66]. Once field operators and the ground state are given, we can define the n -point correlation function

$$\left\langle \Omega, \mathcal{T}\{\phi^{(1)}(x_1) \cdots \phi^{(n)}(x_n)\} \Omega \right\rangle$$

non-perturbatively. The proof of the Gell-Mann – Low formula is the first step to reveal the relation between the series expansion (which may be divergent asymptotic series) of the non-perturbatively defined objects in this way and the formal perturbation series given in physics literatures.

In the heuristic proof of (1.29), Murray Gell-Mann and Francis Low [24] introduced *adiabatic switching* of the interaction through the time-dependent Hamiltonian of the form $H_0 + e^{-\varepsilon|t|}H_1$, where $\varepsilon > 0$ is the small parameter which eventually vanishes. Mathematical studies of this scheme can be found in the literature [10, 15, 29, 30, 31, 37, 43, 44, 48, 49, 51, 57, 64] and references therein. We take an alternative way by sending the time t to ∞ in the imaginary direction: $t \rightarrow \infty(1 - i\varepsilon)$. The same method can be found in physics literatures (see, for example, [22, 52, 68]).

1.5 Self-adjointness of Hamiltonian

One of the most important mathematical studies of quantum systems is to prove the self-adjointness of the Hamiltonians. A self-adjoint Hamiltonian generates a unique time evolution operator, while symmetric but not self-adjoint Hamiltonians may generate no natural time evolution or may generate a lot of different dynamics, because they have, in general, no self-adjoint extensions or infinitely many ones. Moreover, the “probability interpretation” in quantum theory crucially depends upon the existence of a spectral measure supported on the real line, which belongs *only* to self-adjoint operators. In these viewpoints, proving the self-adjointness of a Hamiltonian is not just a problem on a mathematical technicality but also of *physical* importance, and therefore developing general mathematical theorems for the self-adjointness would contribute both to mathematics and to physics.

Many criterion for self-adjointness are known and well-researched (e.g., the Kato-Rellich theorem, Nelson’s commutator theorem, Nelson’s analytic vector theorem, etc...), but there are a lot of symmetric operators that we are not sure whether it is self-adjoint or not by

using existing methods. The Dirac-Maxwell Hamiltonian with the Coulomb potential in the Coulomb gauge is one of such symmetric operators. This operator has a certain singularity coming from the fact that the free Hamiltonian is not bounded below, which is quite special among Hamiltonians of realistic quantum systems. The essential self-adjointness of the Dirac-Maxwell Hamiltonian has been analyzed in Refs. [3] and [59], but, to our best knowledge, the proof of the essential self-adjointness in the case where the Dirac particle lies in the Coulomb type potential, is still missing, although this is one of the most important situations in physics. The main goal of Section 6 is to give a proof of it.

1.6 A note on notation

Throughout this paper, we employ the following notations. For a complex Hilbert space $\mathcal{H}$, the inner product and the norm of $\mathcal{H}$ are denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ (anti-linear in the first variable) and $\| \cdot \|_{\mathcal{H}}$ respectively. When there can be no danger of confusion, then the subscript $\mathcal{H}$ in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$ is omitted. For a linear operator T in $\mathcal{H}$, we denote its domain (resp. range) by $D(T)$ (resp. $R(T)$). We also denote the adjoint of T by T^* and the closure by $\bar{T}$ if these exist. For a self-adjoint operator T , $E_T(\cdot)$ denotes the spectral measure of T . The symbol $T|_D$ denotes the restriction of a linear operator T to the subspace D . For a linear operators S and T on a Hilbert space, $D(S+T) := D(S) \cap D(T)$, $D(ST) := \{\Psi \in D(T) \mid T\Psi \in D(S)\}$ unless otherwise stated.

We use the unit system in which the speed of light and $\hbar$, the Planck constant divided by 2π , are set to be unity

2 Abstract construction of time-ordered exponential

In Section 2.1 and 2.2, we develop an abstract theory of convergent time-ordered exponential on the real axis and the complex plane, respectively. In Section 2.3, we discuss time-ordered exponential as an asymptotic expansion.

2.1 Time-ordered exponential on the real axis

We begin by introducing a class of operators which plays a crucial role in the following analyses.

Definition 2.1 ($\mathcal{C}_0(A)$ -class). Let $\mathcal{H}$ be a complex Hilbert space, and A be a non-negative self-adjoint operator on $\mathcal{H}$. We say that a linear operator T is in $\mathcal{C}_0(A)$ -class if T satisfies the following (I)-(III):

- (I) T and T^* are densely defined and closed.
- (II) T and T^* are $A^{1/2}$ -bounded, where $A^{1/2}$ defined through the functional calculus.
- (III) There exists a constant $b \geq 0$ such that, for all $L \geq 0$, T and T^* map $R(E_A([0, L]))$ into $R(E_A([0, L+b]))$.

Remark 2.1. The above condition (III) comes from the following physical consideration. In quantum mechanics, a self-adjoint operator represents a physical observable quantity, and the spectrum of the self-adjoint operator is considered to be the set of all possible values obtained when the corresponding observable quantity is measured. The above self-adjoint operator A is expected to be an observable quantity of the quantum system under consideration,

typically a number operator or a free Hamiltonian in application to quantum field theories (see application in Section 3.). Roughly speaking, the condition (III) says that the value of the observable A increase at most b by one action of T .

Remark 2.2. In the above condition (II), we chose $1/2$ as the exponent of A for simplicity, but we can extend it to a real number a satisfying $0 \leq a < 1$. All the theorems stated in this thesis by using $\mathcal{C}_0(A)$ -class hold even if we replace $1/2$ by a in the condition (II).

Hereafter, we use the following notations:

$$V_L(A) := R(E_A([0, L])), \quad L \geq 0 \quad (2.1)$$

$$D_{\text{fin}}(A) := \bigcup_{L \geq 0} V_L(A), \quad (2.2)$$

and denote the set consisting of all the $\mathcal{C}_0(A)$ -class operators also by $\mathcal{C}_0(A)$. Note that the subspace $D_{\text{fin}}(A)$ is dense in $\mathcal{H}$ since A is self-adjoint. For $T \in \mathcal{C}_0(A)$, we denote

$$T(\tau) := e^{i\tau H_0} T e^{-i\tau H_0}, \quad \tau \in \mathbb{R}. \quad (2.3)$$

Note that $T(\tau)$ is closed and its adjoint is given by $T^*(\tau)$.

Let H_0 be a self-adjoint operator on $\mathcal{H}$ and H_1 be a densely defined closed operator on $\mathcal{H}$. Let $\mathcal{SA}_+(H_0)$ denote the set of non-negative self-adjoint operators which strongly commute with H_0 .

The goal of the present subsection is to prove following Theorems 2.1-2.4.

Theorem 2.1. *Let $A \in \mathcal{SA}_+(H_0)$ and $t, t' \in \mathbb{R}$ ($t > t'$). Suppose that H_1 is in $\mathcal{C}_0(A)$ -class. Then for each $\xi \in D_{\text{fin}}(A)$, the series:*

$$U_A(H_1; t, t')\xi := \xi + (-i) \int_{t'}^t d\tau_1 H_1(\tau_1)\xi + (-i)^2 \int_{t'}^t d\tau_1 \int_{t'}^{\tau_1} d\tau_2 H_1(\tau_1)H_1(\tau_2)\xi + \cdots \quad (2.4)$$

converges absolutely, where each of the integrals is taken in the sense of strong integral. Furthermore, the following (i) and (ii) hold.

- (i) *For each fixed $t' \in \mathbb{R}$ and $\xi \in D_{\text{fin}}(A)$, the vector-valued function $\mathbb{R} \ni t \mapsto U_A(H_1; t, t')\xi$ is strongly continuously differentiable, and $U_A(H_1; t, t')\xi \in D(H_1(t))$. Moreover, $U_A(H_1; t, t')\xi$ satisfies*

$$\frac{\partial}{\partial t} U_A(H_1; t, t')\xi = -iH_1(t)U_A(H_1; t, t')\xi. \quad (2.5)$$

- (ii) *For each fixed $t \in \mathbb{R}$ and $\xi \in D_{\text{fin}}(A)$, the vector valued function $\mathbb{R} \ni t' \mapsto U_A(H_1; t, t')\xi$ is strongly continuously differentiable, and satisfies*

$$\frac{\partial}{\partial t'} U_A(H_1; t, t')\xi = iU_A(H_1; t, t')H_1(t')\xi. \quad (2.6)$$

- (iii) *The operator $U_A(H_1; t, t')$ with the domain $D(U_A(H_1; t, t')) := D_{\text{fin}}(A)$ is closable, and satisfies the following inclusion relation:*

$$U_A(H_1; t, t')^* \supset \overline{U_A(H_1^*; t', t)}. \quad (2.7)$$

By the definition of $\mathcal{C}_0(A)$ -class, it is clear that $H_1(t)(A+1)^{-1/2}$ is bounded and there exists a constant $C \geq 0$ independent of $t \in \mathbb{R}$ such that

$$\|H_1(t)(A+1)^{-1/2}\| \leq C, \quad t \in \mathbb{R}. \quad (2.8)$$

It follows that $H_1(t)(A+1)^{1/2}$ is strongly continuous in $t \in \mathbb{R}$. Define

$$L_\xi := \inf\{L \geq 0 \mid \xi \in V_L\}, \quad \xi \in D_{\text{fin}}(A). \quad (2.9)$$

Then the definition of $\mathcal{C}_0(A)$ -class ensures that the vector

$$H_1(\tau_1) \dots H_1(\tau_n)\xi \quad (2.10)$$

is well-defined for all $\tau_1, \dots, \tau_n \in \mathbb{R}$ ($n \in \mathbb{N}$), $\xi \in D_{\text{fin}}(A)$, and belongs to the closed subspace $V_{L_\xi + nb}(A)$.

Lemma 2.1. *Let $A \in \mathcal{SA}_+(H_0)$ and $H_1 \in \mathcal{C}_0(A)$. Then for all $n \in \mathbb{N}$ and $\xi \in D_{\text{fin}}(A)$, the mapping*

$$(\tau_1, \dots, \tau_n) \mapsto H_1(\tau_1) \dots H_1(\tau_n)\xi \quad (2.11)$$

is strongly continuous in $(\tau_1, \dots, \tau_n) \in \mathbb{R}^n$ and the estimate

$$\|H_1(\tau_1) \dots H_1(\tau_n)\xi\| \leq C^n (L_\xi + (n-1)b + 1)^{1/2} \dots (L_\xi + 1)^{1/2} \|\xi\| \quad (2.12)$$

holds.

Proof. Since $H_1(\tau_k) \dots H_1(\tau_n)\xi$ belongs to $V_{L_\xi + (n-k+1)b}(A)$ ($k = 1, 2, \dots, n$) as is just remarked above, we can insert the identity operator into the right side of the factor $H_1(\tau_k)$ in the form

$$(A+1)^{-1/2}(A+1)^{1/2}E_A([0, L_\xi + (n-k)b]). \quad (2.13)$$

Since the operators $H_1(\tau_k)(A+1)^{-1/2}$ and $(A+1)^{1/2}E_A([0, L_\xi + (n-k)b])$ are bounded for $k = 1, 2, \dots, n$, it suffices to prove that, for bounded operator-valued functions $A_k(\cdot)$ ($k = 1, 2, \dots, n$) which are strongly continuous, the vector-valued function

$$(\tau_1, \dots, \tau_n) \mapsto A_1(\tau_1) \dots A_n(\tau_n)\psi, \quad \psi \in \mathcal{H} \quad (2.14)$$

is strongly continuous in $(\tau_1, \dots, \tau_n)$ under the condition that

$$\sup_{\tau \in \mathbb{R}} \|A_k(\tau)\| < \infty, \quad k = 1, 2, \dots, n. \quad (2.15)$$

But this is an easy exercise.

The estimate (2.12) follows from (2.8) and the inequalities

$$\|(A+1)^{1/2}E_A([0, L_\xi + (n-k)b + 1])\| \leq (L_\xi + (n-k)b + 1)^{1/2}, \quad k = 1, 2, \dots, n. \quad (2.16)$$

□

For $t, t' \in \mathbb{R}$, we define a sequence of operators $\{U_{A,n}(H_1; t, t')\}_{n=0}^\infty$ by

$$D(U_{A,n}(H_1; t, t')) := D_{\text{fin}}(A), \quad (2.17)$$

$$U_{A,0}(H_1; t, t')\xi := \xi, \quad (2.18)$$

$$U_{A,n}(H_1; t, t')\xi := \begin{cases} (-i)^n \int_{t' \leq \tau_n \leq \dots \leq \tau_1 \leq t} d\tau_1 \dots d\tau_n H_1(\tau_1) \dots H_1(\tau_n)\xi & (t' \leq t), \\ i^n \int_{t \leq \tau_1 \leq \dots \leq \tau_n \leq t'} d\tau_1 \dots d\tau_n H_1(\tau_1) \dots H_1(\tau_n)\xi & (t < t'). \end{cases} \quad (2.19)$$

Note that Lemma 2.1 ensures that $U_{A,n}(H_1; t, t')$ is indeed a well defined strong integral in $\mathcal{H}$. By definition of $U_{A,n}(H_1; t, t')$, one obtains

$$U_{A,n}(H_1; t, t')\xi = (-i)^n \int_{t'}^t d\tau_1 \int_{t'}^{\tau_1} d\tau_2 \dots \int_{t'}^{\tau_{n-1}} d\tau_n H_1(\tau_1) \dots H_1(\tau_n)\xi \quad (2.20)$$

$$= i^n \int_t^{t'} d\tau_n \int_t^{\tau_n} d\tau_{n-1} \dots \int_t^{\tau_2} d\tau_1 H_1(\tau_1) \dots H_1(\tau_n)\xi. \quad (2.21)$$

We remark that these representations (2.20) and (2.21) are valid independent of the sign of $t - t'$ and show that $U_{A,n}(H_1; t, t')\xi$ ($n \geq 0$) is strongly continuously differentiable in t and t' .

Lemma 2.2. *Let $A \in \mathcal{SA}_+(H_0)$ and $H_1 \in \mathcal{C}_0(A)$. Then the operators $U_{A,n}(H_1; t, t')$ satisfy the recursion formulae*

$$U_{A,n+1}(H_1; t, t')\xi = -i \int_{t'}^t d\tau H_1(\tau) U_{A,n}(H_1; \tau, t')\xi, \quad (2.22)$$

$$U_{A,n+1}(H_1; t, t')\xi = -i \int_{t'}^t d\tau U_{A,n}(H_1; t, \tau) H_1(\tau)\xi \quad (2.23)$$

for all $n \geq 0$ and $\xi \in D_{\text{fin}}(A)$.

Proof. Since H_1 is closed and (2.20) is valid independent of the sign of $t - t'$, we obtain for all $n \geq 1$ and $\xi \in D$,

$$\begin{aligned} U_{A,n}(H_1; t, t')\xi &= (-i)^n \int_{t'}^t d\tau_1 \int_{t'}^{\tau_1} d\tau_2 \dots \int_{t'}^{\tau_{n-1}} d\tau_n H_1(\tau_1) \dots H_1(\tau_n)\xi \\ &= -i \int_{t'}^t d\tau_1 H_1(\tau_1) (-i)^{n-1} \int_{t'}^{\tau_1} d\tau_2 \dots \int_{t'}^{\tau_{n-1}} d\tau_n H_1(\tau_2) \dots H_1(\tau_n)\xi \\ &= -i \int_{t'}^t d\tau_1 H_1(\tau_1) U_{A,n-1}(H_1; \tau, t')\xi. \end{aligned} \quad (2.24)$$

This proves (2.22). The equation (2.23) is proved similarly from (2.21). $\square$

Lemma 2.3. *Let $A \in \mathcal{SA}_+(H_0)$ and $H_1 \in \mathcal{C}_0(A)$. Let $t, t' \in \mathbb{R}$ and $\xi \in D_{\text{fin}}(A)$. Then we can estimate*

$$\|U_{A,n}(H_1; t, t')\xi\| \leq \frac{|t - t'|^n}{n!} C^n (L_\xi + (n-1)b + 1)^{1/2} \dots (L_\xi + 1)^{1/2} \|\xi\|, \quad (2.25)$$

for all $n \geq 0$.

Proof. This follows from Lemma 2.1 and the fact that the Lebesgue measure of the set

$$\{(\tau_1, \dots, \tau_n) \in \mathbb{R}^n \mid t' \leq \tau_n \leq \dots \leq \tau_1 \leq t\} \quad (2.26)$$

is given by $|t - t'|^n / n!$. $\square$

Lemma 2.4. *Let $A \in \mathcal{SA}_+(H_0)$ and $H_1 \in \mathcal{C}_0(A)$. Then for all $t, t' \in \mathbb{R}$ and $\xi \in D$, the followings hold.*

$$\sum_{n=0}^{\infty} \|U_{A,n}(H_1; t, t')\xi\| < \infty, \quad (2.27)$$

$$\sum_{n=0}^{\infty} \|H_1(t) U_{A,n}(H_1; t, t')\xi\| < \infty, \quad (2.28)$$

$$\sum_{n=0}^{\infty} \|U_{A,n}(H_1; t, t') H_1(t')\xi\| < \infty, \quad (2.29)$$

Furthermore, these convergences are uniform in (t, t') on any compact subset in $\mathbb{R}^2$.

Proof. From Lemma 2.3, we know

$$\sum_{n=0}^{\infty} \|U_{A,n}(H_1; t, t')\xi\| \leq \sum_{n=0}^{\infty} \frac{|t - t'|^n}{n!} C^n (L_\xi + (n-1)b + 1)^{1/2} \cdots (L_\xi + 1)^{1/2} \|\xi\|. \quad (2.30)$$

Let $a_n(t, t')$ be the n -th term of the summation in the right-hand side of (2.30). One can see that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(t, t')}{a_n(t, t')} \right| = 0,$$

uniformly in (t, t') on any compact subset in the plane. By using d'Alembert's ratio test, the right hand side converges uniformly in (t, t') on any compact subset, and obtain (2.27).

The convergence of the other two series' (2.28) and (2.29) are also proved in a similar way, and we omit the proof. $\square$

By the definition of $\mathcal{C}_0(A)$ -class, we have in addition to (2.8)

$$\|H_1(t)^*(A+1)^{-1/2}\| \leq C', \quad t \in \mathbb{R}, \quad (2.31)$$

for some constant $C' \geq 0$ independent of $t \in \mathbb{R}$. We can derive the following lemmas in the same manner as before.

Lemma 2.5. *Let $A \in \mathcal{SA}_+(H_0)$ and $H_1 \in \mathcal{C}_0(A)$. Then for all $n \geq 0$ and $\xi \in D_{\text{fin}}(A)$, the mapping*

$$(\tau_1, \dots, \tau_n) \mapsto H_1(\tau_1)^* \cdots H_1(\tau_n)^* \xi$$

is strongly continuous in $(\tau_1, \dots, \tau_n) \in \mathbb{R}^n$ and the estimate

$$\|H_1(\tau_1)^* \cdots H_1(\tau_n)^* \xi\| \leq C'^n (L_\xi + (n-1)b + 1)^{1/2} \cdots (L_\xi + 1)^{1/2} \|\xi\| \quad (2.32)$$

holds.

Lemma 2.5 ensures the existence of the strong Bochner integral

$$\int_A d\tau_1 \cdots d\tau_n H_1(\tau_1)^* \cdots H_1(\tau_n)^* \xi, \quad \xi \in D_{\text{fin}}(A), \quad (2.33)$$

for any bounded Borel set $A \subset \mathbb{R}^n$.

Lemma 2.6. *Let $A \in \mathcal{SA}_+(H_0)$ and $H_1 \in \mathcal{C}_0(A)$. Then $D_{\text{fin}}(A) \subset D(U_{A,n}(H_1; t, t')^*)$ and for all $\xi \in D_{\text{fin}}(A)$,*

$$\begin{aligned} U_{A,n}(H_1; t, t')^* \xi &= i^n \int_{t'}^t d\tau_1 \int_{t'}^{\tau_1} d\tau_2 \cdots \int_{t'}^{\tau_{n-1}} d\tau_n H_1(\tau_n)^* \cdots H_1(\tau_2)^* H_1(\tau_1)^* \xi \\ &= (-i)^n \int_t^{t'} d\tau_n \int_t^{\tau_n} d\tau_{n-1} \cdots \int_t^{\tau_2} d\tau_1 H_1(\tau_n)^* H_1(\tau_{n-1})^* \cdots H_1(\tau_1)^* \xi \end{aligned} \quad (2.34)$$

$$= U_{A,n}(H_1^*; t', t)\xi. \quad (2.35)$$

In particular, $U_{A,n}(H_1; t, t')^* \xi$ is strongly continuously differentiable with respect to t and t' .

Proof. The second equality is obvious. Choose arbitrary $\xi, \eta \in D$. Then

$$\begin{aligned}
\langle U_{A,n}(H_1; t, t') \eta, \xi \rangle &= i^n \int_{t'}^t d\tau_1 \dots \int_{t'}^{\tau_{n-1}} d\tau_n \langle H_1(\tau_1) \dots H_1(\tau_n) \eta, \xi \rangle \\
&= i^n \int_{t'}^t d\tau_1 \dots \int_{t'}^{\tau_{n-1}} d\tau_n \langle \eta, H_1(\tau_n)^* \dots H_1(\tau_1)^* \xi \rangle \\
&= \left\langle \eta, i^n \int_{t'}^t d\tau_1 \dots \int_{t'}^{\tau_{n-1}} d\tau_n H_1(\tau_n)^* \dots H_1(\tau_1)^* \xi \right\rangle \\
&= \left\langle \eta, i^n \int_{t'}^t d\tau_n \int_{\tau_n}^t d\tau_{n-1} \dots \int_{\tau_2}^t d\tau_1 H_1(\tau_n)^* \dots H_1(\tau_1)^* \xi \right\rangle \\
&= \left\langle \eta, (-i)^n \int_t^{t'} d\tau_n \int_t^{\tau_n} d\tau_{n-1} \dots \int_t^{\tau_2} d\tau_1 H_1(\tau_n)^* \dots H_1(\tau_1)^* \xi \right\rangle \quad (2.36) \\
&= \langle U_{A,n}(H_1^*; t', t) \eta, \xi \rangle. \quad (2.37)
\end{aligned}$$

This implies $\xi \in D(U_{A,n}(H_1; t, t')^*)$ and (2.34). $\square$

Proof of Theorem 2.1. We prove (i) and (ii). Let $\xi \in D_{\text{fin}}(A)$ and define

$$S_{A,n}(H_1; t, t') \xi := \sum_{j=0}^n U_{A,j}(H_1; t, t') \xi, \quad n \geq 0, \quad t, t' \in \mathbb{R}.$$

By Lemma 2.2, the recursion formulae for $S_{A,n}(H_1; t, t')$ are

$$S_{A,n+1}(H_1; t, t') \xi = \xi - i \int_{t'}^t d\tau H_1(\tau) S_{A,n}(H_1; \tau, t') \xi, \quad (2.38)$$

$$S_{A,n+1}(H_1; t, t') \xi = \xi - i \int_{t'}^t d\tau S_{A,n}(H_1; t, \tau) H_1(\tau) \xi. \quad (2.39)$$

It is clear from Lemma 2.4 (2.27) that $\{S_{A,n}(H_1; t, t') \xi\}_n$ is Cauchy in $\mathcal{H}$. Thus we can define

$$D(U_A(H_1; t, t')) = D_{\text{fin}}(A), \quad U_A(H_1; t, t') \xi = \lim_{n \rightarrow \infty} S_{A,n}(H_1; t, t') \xi, \quad \xi \in D_{\text{fin}}(A), \quad t, t' \in \mathbb{R}. \quad (2.40)$$

This convergence is locally uniform with respect to $t, t' \in \mathbb{R}$. From Lemma 2.4 (2.28) and (2.29), we find that the sequences $\{H_1(t) S_{A,n}(H_1; t, t') \xi\}_n$ and $\{S_{A,n}(H_1; t, t') H_1(t') \xi\}_n$ are Cauchy locally uniformly in $t, t' \in \mathbb{R}$ and thus converge locally uniformly in $t, t' \in \mathbb{R}$. Since H_1 is closed, $U_A(H_1; t, t') \xi \in D(H_1(t))$ and

$$H_1(t) S_{A,n}(H_1; t, t') \xi \rightarrow H_1(t) U_A(H_1; t, t') \xi, \quad (2.41)$$

$$S_{A,n}(H_1; t, t') H_1(t') \xi \rightarrow U_A(H_1; t, t') H_1(t') \xi \quad (2.42)$$

locally uniformly in $t, t' \in \mathbb{R}$, as n tends to infinity. This implies that the limits $H_1(t) U_A(H_1; t, t') \xi$ and $U_A(H_1; t, t') H_1(t') \xi$ are strongly continuous in both t and t' . Hence by taking the limit $n \rightarrow \infty$ in both sides of (2.38), (2.39), we obtain

$$U_A(H_1; t, t') = \xi - i \int_{t'}^t d\tau H_1(\tau) U_A(H_1; \tau, t') \xi, \quad (2.43)$$

$$U_A(H_1; t, t') = \xi - i \int_{t'}^t d\tau U_A(H_1; t, \tau) H_1(\tau) \xi, \quad (2.44)$$

which verify the assertions of the Theorem 2.1.

We prove (iii). From Lemmas 2.6 and 2.4, one finds

$$\sum_{n=0}^N U_{A,n}(H_1; t, t')^* \xi, \quad \xi \in D_{\text{fin}}(A) \quad (2.45)$$

absolutely converges uniformly in (t, t') on any compact set. For all $\xi, \eta \in D_{\text{fin}}(A)$, we obtain

$$\begin{aligned} \langle \eta, U_A(H_1; t, t') \xi \rangle &= \sum_{n=0}^{\infty} \langle \eta, U_{A,n}(H_1; t, t') \xi \rangle \\ &= \sum_{n=0}^{\infty} \langle U_{A,n}(H_1^*; t', t) \eta, \xi \rangle \\ &= \left\langle \sum_{n=0}^{\infty} U_A(H_1^*; t', t) \eta, \xi \right\rangle, \end{aligned} \quad (2.46)$$

since $\eta \in D(U_{A,n}(H_1^*; t', t))$ for all n . Thus we obtain $D_{\text{fin}}(A) \subset D(U_A(H_1; t, t')^*)$ and (2.7). $\square$

The following theorem ensures that $\overline{U_A(H_1; t, t')}$ does not depend on the choice of $A \in \mathcal{SA}_+(H_0)$ under suitable conditions.

Theorem 2.2. *Let $A, B \in \mathcal{SA}_+(H_0)$ and $H_1 \in \mathcal{C}_0(A) \cap \mathcal{C}_0(B)$. If A and B strongly commute, then for all $t, t' \in \mathbb{R}$, the operator equality*

$$\overline{U_A(H_1; t, t')} = \overline{U_B(H_1; t, t')} \quad (2.47)$$

holds.

Proof. Let $L \geq 0$ and $\Psi \in V_L(A)$ be fixed arbitrarily. Since A and B strongly commute, there exist a sequence $\{\Psi_k\}_{k=1}^{\infty} \subset V_L(A) \cap D_{\text{fin}}(B)$ such that $\Psi_k \rightarrow \Psi$ ($k \rightarrow \infty$). Let us note that

$$U_A(H_1; t, t') \Psi_k = U_B(H_1; t, t') \Psi_k, \quad k = 1, 2, \dots \quad (2.48)$$

from the definition of $U_A(H_1; t, t')$ and $U_B(H_1; t, t')$. Then we see that $\{U_A(H_1; t, t') \Psi_k\}_{k=1}^{\infty}$ converges since $U_A(H_1; t, t') \upharpoonright V_L(A)$ is a bounded operator from Lemma 2.3. Hence (2.48) and the closability of $U_B(H_1; t, t')$ yield $\Psi \in D(\overline{U_B(H_1; t, t')})$ and $U_A(H_1; t, t') \Psi = \overline{U_B(H_1; t, t')} \Psi$. Since $L \geq 0$ and $\Psi \in V_L(A)$ are arbitrary, we have $U_A(H_1; t, t') \subset \overline{U_B(H_1; t, t')}$, and thus

$$\overline{U_A(H_1; t, t')} \subset \overline{U_B(H_1; t, t')}. \quad (2.49)$$

Repeating the same argument replacing A with B , we have

$$\overline{U_A(H_1; t, t')} \supset \overline{U_B(H_1; t, t')}. \quad (2.50)$$

Therefore we obtain the desired result. $\square$

In what follows, we fix an operator $A \in \mathcal{SA}_+(H_0)$ arbitrarily, and shortly denote

$$U(t, t') := U_A(H_1; t, t'), \quad U_n(t, t') := U_{A,n}(H_1; t, t'), \quad S_n(t, t') := S_{A,n}(H_1; t, t'). \quad (2.51)$$

The time evolution operator $U_A(H_1; t, t')$ has the following properties.

Theorem 2.3. Let $A \in \mathcal{SA}_+(H_0)$ and $H_1 \in \mathcal{C}_0(A)$. Then the following (i) and (ii) hold.

(i) For all $\xi \in D_{\text{fin}}(A)$, $t, t', t'' \in \mathbb{R}$, $U_A(H_1; t, t)\xi = \xi$ and the operator equality

$$\overline{U_A(H_1; t, t')}U_A(H_1; t', t'') = U_A(H_1; t, t''). \quad (2.52)$$

holds.

(ii) For any $s, t, t' \in \mathbb{R}$, the operator equality

$$e^{isH_0}\overline{U_A(H_1; t, t')}e^{-isH_0} = \overline{U_A(H_1; t + s, t' + s)} \quad (2.53)$$

holds.

Proposition 2.1. Let $A \in \mathcal{SA}_+(H_0)$ and $H_1 \in \mathcal{C}_0(A)$. If $\xi \in D_{\text{fin}}(A)$, then for each $t, t', s, s' \in \mathbb{R}$, $U(s, s')\xi \in D(\overline{U(t, t')})$ and

$$\overline{U(t, t')}U(s, s')\xi = \sum_{m,n=0}^{\infty} U_m(t, t')U_n(s, s')\xi, \quad (2.54)$$

where the right hand side converges absolutely, and does not depend upon the summation order.

Proof. For all $\xi \in D$ and all $(t, t'), (s, s') \in \mathbb{R}^2$, it is clear that $S_n(s, s')\xi \in D(U(t, t'))$. Since $S_n(s, s')\xi$ converges to $U(s, s')\xi$ as n tends to infinity, it suffices to prove that $U(t, t')S_n(s, s')\xi$ converges as $n \rightarrow \infty$. We have already know that

$$U(t, t')S_n(s, s')\xi = \sum_{m=0}^{\infty} \sum_{j=0}^n U_m(t, t')U_j(s, s')\xi,$$

therefore it is sufficient to derive

$$\sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \|U_m(t, t')U_j(s, s')\xi\| < \infty.$$

By using (2.25),

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \|U_m(t, t')U_j(s, s')\xi\| \\ & \leq \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{|t - t'|^m |s - s'|^j}{m!j!} C^{m+j} (L_{\xi} + (m + j - 1)b + 1)^{1/2} \cdots (L_{\xi} + 1)^{1/2} \|\xi\| \\ & = \sum_{N=0}^{\infty} \sum_{m=0}^N \frac{|t - t'|^m |s - s'|^{N-m}}{m!(N - m)!} C^N (L_{\xi} + (N - 1)b + 1)^{1/2} \cdots (L_{\xi} + 1)^{1/2} \|\xi\| \\ & = \sum_{N=0}^{\infty} \frac{1}{N!} (C(|t - t'| + |s - s'|))^N (L_{\xi} + (N - 1)b + 1)^{1/2} \cdots (L_{\xi} + 1)^{1/2} \|\xi\| \end{aligned} \quad (2.55)$$

From the d'Alembert's ratio test, this is finite, which proves (2.54). $\square$

Proof of Theorem 2.3. We first prove (i). Take arbitrary $\xi, \eta \in D_{\text{fin}}(A)$. $U(t, t)\xi = \xi$ is obvious. By Theorems 2.1, the function $t' \mapsto \langle \eta, \overline{U(t, t')}U(t', t'')\xi \rangle = \langle U(t, t')^*\eta, U(t', t'')\xi \rangle$ is differentiable and

$$\begin{aligned} & \frac{\partial}{\partial t'} \langle \eta, \overline{U(t, t')}U(t', t'')\xi \rangle \\ &= \langle -iH_1(t')^*U(t, t')^*\eta, U(t', t'')\xi \rangle + \langle U(t, t')^*\eta, -iH_1(t')U(t', t'')\xi \rangle \\ &= 0. \end{aligned} \tag{2.56}$$

Thus $\langle \eta, \overline{U(t, t')}U(t', t'')\xi \rangle$ is independent of t' , which implies

$$\begin{aligned} \langle \eta, \overline{U(t, t')}U(t', t'')\xi \rangle &= \langle \eta, \overline{U(t, t'')}U(t'', t'')\xi \rangle \\ &= \langle \eta, U(t, t'')\xi \rangle. \end{aligned} \tag{2.57}$$

Since $\eta \in D_{\text{fin}}(A)$ is arbitrary, it follows that

$$\overline{U(t, t')}U(t', t'')\xi = U(t, t'')\xi. \tag{2.58}$$

hence we obtain (2.52) because $\xi \in D_{\text{fin}}(A)$ is arbitrary and $D_{\text{fin}}(A) = D(\overline{U(t, t')}U(t', t''))$.

Next, we prove (ii). Observe $e^{isH_0}H_1(t)e^{-isH_0} = H_1(t+s)$ by definition. Suppose $t \geq t'$. Then for each $n \in \mathbb{N}$, $\xi \in D$, we obtain

$$\begin{aligned} & e^{isH_0}U_n(t, t')e^{-isH_0}\xi \\ &= \int_{t \geq \tau_1 \geq \dots \geq \tau_n \geq t'} d^n \tau H_1(\tau_1 + s) \dots H_1(\tau_n + s)\xi \\ &= \int_{t+s \geq \tau_1 \geq \dots \geq \tau_n \geq t'+s} d^n \tau H_1(\tau_1) \dots H_1(\tau_n)\xi \\ &= U_n(t+s, t'+s)\xi. \end{aligned} \tag{2.59}$$

The relation (2.59) remains valid in the case where $t < t'$. Thus we have for all $(t, t') \in \mathbb{R}^2$,

$$\begin{aligned} e^{isH_0}U(t, t')e^{-isH_0}\xi &= \sum_{n=0}^{\infty} e^{isH_0}U_n(t, t')e^{-isH_0}\xi \\ &= \sum_{n=0}^{\infty} U_n(t+s, t'+s)\xi \\ &= U(t+s, t'+s)\xi. \end{aligned}$$

Since $D_{\text{fin}}(A)$ is common core of $\overline{U(t, t')}$ and $\overline{U(t+s, t'+s)}$, we obtain the desired result. $\square$

If we assume in addition that H_1 is symmetric, then stronger results follow:

Theorem 2.4. *Let $A \in \mathcal{SA}_+(H_0)$ and let $H_1 \in \mathcal{C}_0(A)$ be a closed symmetric operator. Then $\overline{U(t, t')}$ is unitary and the following properties hold.*

(i) *The operator $U(t, t')$ satisfies the following operator equalities:*

$$\overline{U(t, t)} = I, \quad \overline{U(t, t')} \overline{U(t', t'')} = \overline{U(t, t'')}, \tag{2.60}$$

where I denotes the identity operator.

(ii) $U_A(H_1; t, t')$ is unique in the following sense. If there exist a dense subspace $\tilde{D}$ in $\mathcal{H}$ and an operator valued function $V(t, t')$ ($t, t' \in \mathbb{R}$) such that $\tilde{D} \subset D(V(t, t'))$ for all $t, t' \in \mathbb{R}$ and for $\xi \in \tilde{D}$, $V(t, t')\xi$ is strongly differentiable with respect to t , and $V(t, t')\xi \in D(H_1(t))$, which satisfies

$$V(t, t)\xi = \xi, \quad \frac{\partial}{\partial t}V(t, t')\xi = -iH_1(t)V(t, t')\xi, \quad \xi \in \tilde{D}, \quad t, t' \in \mathbb{R}, \quad (2.61)$$

then $V(t, t') \upharpoonright \tilde{D}$ is closable and $\overline{V(t, t') \upharpoonright \tilde{D}} = \overline{U(t, t')}$. In particular, if $D(V(t, t')) = \mathcal{H}$ and $V(t, t')$ is bounded for all $t, t' \in \mathbb{R}$, then $V(t, t') = \overline{U(t, t')}$.

Proof of Theorem 2.4. We prove the unitarity. Since H_1 is symmetric, one obtains for all $\xi \in D_{\text{fin}}(A)$

$$\begin{aligned} \frac{\partial}{\partial t} \|U(t, t')\xi\|^2 &= \langle -iH_1(t)U(t, t')\xi, U(t, t')\xi \rangle + \langle U(t, t')\xi, -iH_1(t)U(t, t')\xi \rangle \\ &= 0. \end{aligned} \quad (2.62)$$

therefore $\overline{U(t, t')}$ is isometry, in particular, bounded. By using Theorem 2.3, one finds the operator equality

$$\overline{U(t, t')} \overline{U(t', t)} = I, \quad t, t' \in \mathbb{R}, \quad (2.63)$$

which implies that $U(t, t')$ is surjective. Hence it is unitary.

The statement (i) is directly follows from Theorem 2.3.

We prove (ii). For each $\xi \in D_{\text{fin}}(A)$, $\eta \in \tilde{D}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} \frac{\partial}{\partial t} \langle V(t, t')\eta, U(t, t')\xi \rangle &= \langle -iH_1(t)V(t, t')\eta, U(t, t')\xi \rangle + \langle V(t, t')\eta, -iH_1(t)U(t, t')\xi \rangle \\ &= 0. \end{aligned} \quad (2.64)$$

thus we obtain

$$\langle V(t, t')\eta, U(t, t')\xi \rangle = \langle \eta, \xi \rangle, \quad \xi \in D, \quad \eta \in \tilde{D}. \quad (2.65)$$

Since D is dense in $\mathcal{H}$ and since $\overline{U(t, t')}$ is unitary and satisfies $\overline{U(t, t')} \overline{U(t', t)} = I$, (2.65) yields for all $\eta \in \tilde{D}$,

$$V(t, t')\eta = \overline{U(t, t')}\eta, \quad \eta \in \tilde{D}. \quad (2.66)$$

Suppose that a sequence $\{\eta_n\}_n \subset \tilde{D}$ satisfies that $\eta_n \rightarrow 0$ as n tends to infinity. Then (2.66) shows that $V(t, t')\eta_n$ converges to 0, which means that $V(t, t')$ is closable. Take arbitrary $\psi \in \mathcal{H}$. Then there is a sequence $\{\eta_n\}_n$ which converges to ψ as $n \rightarrow \infty$, since $\tilde{D}$ is dense in $\mathcal{H}$. Then (2.66) implies $\psi \in D(\overline{V(t, t') \upharpoonright \tilde{D}})$ and $\overline{V(t, t') \upharpoonright \tilde{D}}\psi = \overline{U(t, t')}\psi$ for all $\psi \in \mathcal{H}$. $\square$

2.2 Time-ordered exponential on the complex plane

By the usual total order on the real line $\mathbb{R}$, we can naturally define the time-ordering on $\mathbb{R}$. However, on the complex plane, we have to pay attention to how we define time-ordering. It is desirable that the complex time-ordered exponential $T\left\{\exp\left[\int_{z'}^z d\tau H_1(\zeta)\right]\right\}\Big|_{D_{\text{fin}}(A)}$ is analytic and depends only on $z, z' \in \mathbb{C}$. Hence we introduce a time-ordering on the complex plane as follows.

Let $z, z' \in \mathbb{C}$ and Γ be a piecewisely continuously differentiable simple curve in $\mathbb{C}$ from z' to z . That is, Γ is a map from a closed interval $I = [\alpha, \beta]$ in $\mathbb{R}$ into $\mathbb{C}$, which is piecewisely continuously differentiable and injective, satisfying

$$\Gamma(\alpha) = z', \quad \Gamma(\beta) = z. \quad (2.67)$$

We define a linear order $\succ$ on $\Gamma(I) = \{\Gamma(t) \mid t \in I\} \subset \mathbb{C}$ as follows. For $\zeta_1, \zeta_2 \in \Gamma(I)$, there exist $t_1, t_2 \in I$ with $\Gamma(t_1) = \zeta_1$ and $\Gamma(t_2) = \zeta_2$. then $\zeta_1 \succ \zeta_2$ if and only if $t_1 > t_2$.

In what follows, we denote $\Gamma(I)$ simply by Γ . Let $\mathfrak{S}_n$ be the symmetric group of order $n \in \mathbb{N}$ and $L(\mathcal{H})$ be (not necessarily bounded) linear operators in $\mathcal{H}$. For mappings $F_1, F_2, \dots, F_k$ ($k \in \mathbb{N}$) from Γ into $L(\mathcal{H})$, we define a map $T[F_1 \dots F_k]$ from Γ^k into $L(\mathcal{H})$ by

$$T[F_1 \dots F_k](\zeta_1, \dots, \zeta_k) := \sum_{\sigma \in \mathfrak{S}_k} \chi_{P_\sigma}(\zeta_1, \dots, \zeta_k) F_{\sigma(1)}(\zeta_{\sigma(1)}) \dots F_{\sigma(k)}(\zeta_{\sigma(k)}), \quad (2.68)$$

whenever the right-hand side makes sense, where χ_J denotes the characteristic function of the set J , and

$$P_\sigma = \{(\zeta_1, \dots, \zeta_k) \in \Gamma^k \mid \zeta_{\sigma(1)} \succ \dots \succ \zeta_{\sigma(k)}\}, \quad \sigma \in \mathfrak{S}_k. \quad (2.69)$$

In what follows, we sometimes adopt a little bit confusing notation

$$T(F_1(\zeta_1) \dots F_k(\zeta_k)) := T[F_1 \dots F_k](\zeta_1, \dots, \zeta_k), \quad (2.70)$$

and call it a *time-ordered product* of $F_1(\zeta_1), \dots, F_k(\zeta_k)$, even though the operation T does *not* act on the product of operators $F_1(\zeta_1), \dots, F_k(\zeta_k)$ but on the product of mappings $F_1, \dots, F_k$.

Next, we define time-ordered exponential of an operator-valued function. Let $F : \Gamma \rightarrow L(\mathcal{H})$ and let $C(F) \subset \mathcal{H}$ be a linear subspace spanned by all the vectors $\Psi \in \mathcal{H}$ such that the mapping

$$(\zeta_1, \dots, \zeta_n) \mapsto F(\zeta_1) \dots F(\zeta_n) \Psi \quad (2.71)$$

is strongly continuous on some region containing Γ^n . We define a time-ordered exponential operator by

$$D \left(T \exp \left(\int_{\Gamma} d\zeta F(\zeta) \right) \right) \quad (2.72)$$

$$:= \left\{ \Psi \in C(F) \mid \sum_{n=0}^{\infty} \frac{1}{n!} \left\| \int_{\Gamma^n} d\zeta_1 \dots d\zeta_n T(F(\zeta_1) \dots F(\zeta_n)) \Psi \right\| < \infty \right\}, \quad (2.73)$$

$$T \exp \left(\int_{\Gamma} d\zeta F(\zeta) \right) \Psi := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma^n} d\zeta_1 \dots d\zeta_n T(F(\zeta_1) \dots F(\zeta_n)) \Psi, \quad (2.74)$$

where the integration is understood in the strong sense.

We also define a more general time-ordered exponential operator. Let $F_1, F_2, \dots, F_k, \dots, F_{k+n}$ be the mappings from Γ into (not necessarily bounded) liner operators in $\mathcal{H}$. We define a map from Γ^n into $L(\mathcal{H})$, which is labeled by $(\zeta_1, \dots, \zeta_k) \in \Gamma^k$,

$$T[F_1(\zeta_1) F_2(\zeta_2) \dots F_k(\zeta_k) F_{k+1} \dots F_{k+n}] : \Gamma^n \rightarrow L(\mathcal{H}) \quad (2.75)$$

by the relation

$$\begin{aligned} & T[F_1(\zeta_1) F_2(\zeta_2) \dots F_k(\zeta_k) F_{k+1} \dots F_{k+n}](\zeta_{k+1}, \dots, \zeta_{k+n}) \\ &:= \sum_{\sigma \in \mathfrak{S}_{k+n}} \chi_{P'_{n,\sigma}}(\zeta_{k+1}, \dots, \zeta_{k+n}) F_{\sigma(1)}(\zeta_{\sigma(1)}) \dots F_{\sigma(k+n)}(\zeta_{\sigma(k+n)}), \end{aligned} \quad (2.76)$$

whenever the operator products in the right-hand side makes sense. Here we denote

$$P'_{n,\sigma} := \{(\zeta_{k+1}, \dots, \zeta_{k+n}) \in \Gamma^n \mid \zeta_{\sigma(1)} \succ \dots \succ \zeta_{\sigma(k+n)}\} \quad (2.77)$$

for $\sigma \in \mathfrak{S}_{k+n}$. In this case, we also employ a confusing notation (really confusing in the case)

$$\begin{aligned} T(F_1(\zeta_1) \dots F_{k+n}(\zeta_{k+n})) \\ := T[F_1(\zeta_1)F_2(\zeta_2) \dots F_k(\zeta_k)F_{k+1} \dots F_{k+n}](\zeta_{k+1}, \dots, \zeta_{k+n}), \end{aligned} \quad (2.78)$$

and call it a *time-ordered product* of $F_1(\zeta_1), \dots, F_{k+n}(\zeta_{k+n})$, following physics literatures. We never use this notation unless it can be clearly understood from a context which variables of $(\zeta_1, \dots, \zeta_{k+n})$ are fixed and which variables are function argument.

Using this notation, we can define more general time-ordered exponential operator. Let $F_1, \dots, F_k, F$ be operator-valued functions from Γ into $L(\mathcal{H})$ and $F_{k+1} = \dots = F_{k+n} = F$. Let $C(F_1, \dots, F_k, F)$ be a linear subspace spanned by all the vectors Ψ for which the mappings

$$(\zeta_{k+1}, \dots, \zeta_{k+n}) \mapsto F_{\sigma(1)}(\zeta_{\sigma(1)}) \dots F_{\sigma(k+n)}(\zeta_{\sigma(k+n)})\Psi \quad (2.79)$$

are continuous for all fixed $(\zeta_1, \dots, \zeta_k)$ and all $\sigma \in \mathfrak{S}_{n+k}$. Then on the domain

$$\begin{aligned} D \left(T F_1(\zeta_1) \dots F_k(\zeta_k) \exp \left(\int_{\Gamma} d\zeta F(\zeta) \right) \right) \\ := \left\{ \Psi \in C(F_1, \dots, F_k, F) \mid \right. \\ \left. \sum_{n=0}^{\infty} \frac{1}{n!} \left\| \int_{\Gamma^n} d\zeta_{k+1} \dots d\zeta_{k+n} T(F_1(\zeta_1) \dots F_k(\zeta_k) F(\zeta_{k+1}) \dots F(\zeta_{k+n})) \Psi \right\| < \infty \right\}, \end{aligned} \quad (2.80)$$

We define

$$\begin{aligned} T F_1(\zeta_1) \dots F_k(\zeta_k) \exp \left(\int_{\Gamma} d\zeta F(\zeta) \right) \Psi \\ := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma^n} d\zeta_{k+1} \dots d\zeta_{k+n} T(F_1(\zeta_1) \dots F_k(\zeta_k) F(\zeta_{k+1}) \dots F(\zeta_{k+n})) \Psi. \end{aligned} \quad (2.81)$$

We remark that for all $\sigma \in \mathfrak{S}_k$,

$$\begin{aligned} T F_1(\zeta_1) \dots F_k(\zeta_k) \exp \left(\int_{\Gamma} d\zeta F(\zeta) \right) \\ = T F_{\sigma(1)}(\zeta_{\sigma(1)}) \dots F_{\sigma(k)}(\zeta_{\sigma(k)}) \exp \left(\int_{\Gamma} d\zeta F(\zeta) \right). \end{aligned} \quad (2.82)$$

Let H_0 be a non-negative self-adjoint operator on $\mathcal{H}$. Hereafter, we use the following abbreviated notations:

$$V_E := R(E_{H_0}([0, E])), \quad (2.83)$$

$$D_{\text{fin}} := \bigcup_{E \geq 0} V_E. \quad (2.84)$$

For $T \in \mathcal{C}_0(H_0)$, we denote

$$T(z) := e^{izH_0} T e^{-izH_0}, \quad z \in \mathbb{C}. \quad (2.85)$$

Note that $T(z)$ is closable since its adjoint includes the operator $e^{iz^*H_0}T^*e^{-iz^*H_0}$ which is densely defined. We denote the closure of $T(z)$ by the same symbol. In this notation, one obtains

$$T(z)^* \supset T^*(z^*). \quad (2.86)$$

The goal of the present section is to prove following Theorems 2.5-2.9.

Theorem 2.5. *Let A be in $\mathcal{C}_0(H_0)$ class and $z, z' \in \mathbb{C}$.*

- (i) *Take a piecewisely continuously differentiable simple curve $\Gamma_{z,z'}$ which starts at z' and ends at z with $\text{Im } z' \leq \text{Im } z$. then*

$$D_{\text{fin}} \subset D \left(T \exp \left(-i \int_{\Gamma_{z,z'}} d\zeta A(\zeta) \right) \right) \quad (2.87)$$

and the restriction

$$T \exp \left(-i \int_{\Gamma_{z,z'}} d\zeta A(\zeta) \right) \Big|_{D_{\text{fin}}} \quad (2.88)$$

does not depend upon the simple curve from z' to z and depends only on z and z' , justifying the notation

$$U(A; z, z') := T \exp \left(-i \int_{\Gamma_{z,z'}} d\zeta A(\zeta) \right) \Big|_{D_{\text{fin}}}. \quad (2.89)$$

- (ii) *$U(A; z, z')$ is closable, and satisfies the following inclusion relation:*

$$U(A; z, z')^* \supset \overline{U(A^*; z'^*, z^*)}. \quad (2.90)$$

Lemma 2.7. *Let $A_1, \dots, A_n$ be in $\mathcal{C}_0(H_0)$ -class. Then for all $\Psi \in D_{\text{fin}}$ and all $n \in \mathbb{N}$, the mapping*

$$\mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto A_1(z_1) \dots A_n(z_n) \Psi \in \mathcal{H} \quad (2.91)$$

is strongly analytic in $\mathbb{C}^n$.

Proof. Each vector in D_{fin} is an entire analytic vector of H_0 , and each $A_j \in \mathcal{C}_0(H_0)$ ($j = 1, 2, \dots, n$) preserves the subspace of all the entire analytic vectors of H_0 . therefore $A_1(z_1) \dots A_n(z_n) \Psi$ permits an absolutely converging power series expansion in $z_1, \dots, z_n$ and thus is strongly analytic. $\square$

From Lemma 2.7, we can define a liner operator $V_n(A; z, z')$ with the domain $D(V_n(A; z, z')) = D_{\text{fin}}$ for $A \in \mathcal{C}_0(H_0)$, $z, z' \in \mathbb{C}$, $n \in \mathbb{N}$, and $\Psi \in D_{\text{fin}}$,

$$V_n(A; z, z') \Psi := \frac{(-i)^n}{n!} \int_{\Gamma_{z,z'}^n} d\zeta_1 \dots d\zeta_n T(A(\zeta_1) \dots A(\zeta_n)) \Psi, \quad (2.92)$$

where Γ denotes a piecewisely continuously differentiable simple curve from z' to z . We regard $V_0(A; z, z') = 1$.

Lemma 2.8. (i) *If $\Psi \in V_E$, then $V_n(A; z, z') \Psi \in V_{E+nb}$, where $b \geq 0$ is a constant stated in Definition 2.1 (III) for $A \in \mathcal{C}_0(H_0)$.*

(ii) The operator $V_n(A; z, z')$ has the following representation

$$V_n(A; z, z') = (-i)^n \int_{z'}^z d\zeta_1 \int_{z'}^{\zeta_1} d\zeta_2 \dots \int_{z'}^{\zeta_{n-1}} d\zeta_n A(\zeta_1) A(\zeta_2) \dots A(\zeta_n) \quad (2.93)$$

$$= (-i)^n \int_{z'}^z d\zeta_n \int_{\zeta_n}^z d\zeta_{n-1} \dots \int_{\zeta_2}^z d\zeta_1 A(\zeta_1) A(\zeta_2) \dots A(\zeta_n), \quad (2.94)$$

where the above integrations denote the indefinite integral of an analytic function which depends only on the start and the end point.

(iii) $V_n(A; z, z')$ is analytic in $z \in \mathbb{C}$ and $z' \in \mathbb{C}$, and independent of the choice of a simple curve $\Gamma_{z, z'}$ from z' to z .

(iv) $V_n(A; z, z')$ satisfies the formulae for $n = 0, 1, \dots$,

$$V_{n+1}(A; z, z') = (-i) \int_{z'}^z d\zeta A(\zeta) V_n(A; \zeta, z') \quad (2.95)$$

$$= (-i) \int_{z'}^z d\zeta V_n(A; z, \zeta) A(\zeta). \quad (2.96)$$

Proof. The assertion (i) follows from the fact that

$$T(A(\zeta_1) \dots A(\zeta_n)) \Psi \in V_{E+nb} \quad (2.97)$$

and V_{E+nb} is closed. Since (iii) and (iv) are simple corollaries of (ii), it suffices to prove (ii). We prove only the case where $\Gamma : [\alpha, \beta] \rightarrow \mathbb{C}$ is continuously differentiable. A general case is straightforward. By definition of the time-ordering operation T (2.68), one finds on D_{fin}

$$\begin{aligned} V_n(A; z, z') &= \frac{(-i)^n}{n!} \int_{\Gamma^n} d\zeta_1 \dots d\zeta_n T(A(\zeta_1) \dots A(\zeta_n)) \\ &= \frac{(-i)^n}{n!} \sum_{\sigma \in \mathfrak{S}_n} \int_{\{\beta \geq t_{\sigma(1)} > \dots > t_{\sigma(n)} \geq \alpha\}} dt_1 \dots dt_n \Gamma'(t_1) \dots \Gamma'(t_n) \\ &\quad \times A(\Gamma(t_{\sigma(1)})) \dots A(\Gamma(t_{\sigma(n)})). \end{aligned} \quad (2.98)$$

The above integration does not depend on $\sigma \in \mathfrak{S}_n$ and is equal to

$$\begin{aligned} &\int_{\{\beta \geq t_1 > \dots > t_n \geq \alpha\}} dt_1 \dots dt_n \Gamma'(t_1) \dots \Gamma'(t_n) A(\Gamma(t_1)) \dots A(\Gamma(t_n)) \\ &= \int_{\alpha}^{\beta} dt_1 \Gamma'(t_1) \int_{\alpha}^{t_1} dt_2 \Gamma'(t_2) \dots \int_{\alpha}^{t_{n-1}} dt_n \Gamma'(t_n) A(\Gamma(t_1)) \dots A(\Gamma(t_n)) \end{aligned} \quad (2.99)$$

$$= \int_{\alpha}^{\beta} dt_n \Gamma'(t_n) \int_{t_n}^{\beta} dt_{n-1} \Gamma'(t_{n-1}) \dots \int_{t_2}^{\beta} dt_1 \Gamma'(t_1) A(\Gamma(t_1)) \dots A(\Gamma(t_n)). \quad (2.100)$$

The expression (2.99) and (2.100) can be rewritten

$$\int_{z'}^z d\zeta_1 \int_{z'}^{\zeta_1} d\zeta_2 \dots \int_{z'}^{\zeta_{n-1}} d\zeta_n A(\zeta_1) \dots A(\zeta_n) \quad (2.101)$$

and

$$\int_{z'}^z d\zeta_n \int_{\zeta_n}^z d\zeta_{n-1} \dots \int_{\zeta_2}^z d\zeta_1 A(\zeta_1) \dots A(\zeta_n) \quad (2.102)$$

respectively. Since the summation over σ gives $n!$, the assertion (ii) follows. $\square$

In the following, we employ the notation

$$\frac{(-i)^n}{n!} \sum_{\sigma \in \mathfrak{S}_n} \int_{\{\zeta_{\sigma(1)} \succ \dots \succ \zeta_{\sigma(n)}\}} d\zeta_1 \dots d\zeta_n A(\zeta_{\sigma(1)}) \dots A(\zeta_{\sigma(n)}). \quad (2.103)$$

to denote the integration such as (2.98).

Lemma 2.9. *For all $n \geq 0$, $A \in \mathcal{C}_0(H_0)$, $E \geq 0$ and $\Psi \in V_E$, the following estimate holds for all $z, z' \in \mathbb{C}$ with $\text{Im } z \leq \text{Im } z'$.*

$$\|V_n(A; z, z')\Psi\| \leq C^n e^{|\text{Im } z'| (2E+nb)} \frac{|z - z'|^n}{n!} (E + (n-1)b + 1)^{1/2} \dots (E + 1)^{1/2} \|\Psi\|, \quad (2.104)$$

where $b \geq 0$ is a constant stated in Definition 2.1 (III) and $C = \|A(H_0 + 1)^{-1/2}\|$. In the case where $n = 0$, we regard the right-hand side as $\|\Psi\|$.

Proof. First, we prove for $\text{Im } z_1 \leq \text{Im } z_2 \leq \dots \leq \text{Im } z_n$,

$$\|A(z_1) \dots A(z_n)\Psi\| \leq C^n e^{|\text{Im } z_n| (2E+nb)} (E + (n-1)b + 1)^{1/2} \dots (E + 1)^{1/2} \|\Psi\|. \quad (2.105)$$

In fact, the identity

$$A(z_1) \dots A(z_n)\Psi \quad (2.106)$$

$$\begin{aligned} &= e^{iz_1 H_0} A e^{-i(z_1 - z_2) H_0} \dots e^{-i(z_{n-1} - z_n) H_0} A_n e^{iz_n H_0} \Psi \\ &= e^{iz_1 H_0} E_{H_0}([0, E + nb]) A(H_0 + 1)^{-1/2} (H_0 + 1)^{1/2} E_{H_0}([0, E + (n-1)b]) e^{-i(z_1 - z_2) H_0} \times \dots \\ &\quad \times e^{-i(z_{n-1} - z_n) H_0} A(H_0 + 1)^{-1/2} (H_0 + 1)^{1/2} E_{H_0}([0, E]) e^{iz_n H_0} \Psi \end{aligned} \quad (2.107)$$

implies (2.105), because $e^{-i(z_j - z_{j+1}) H_0}$ ($j = 1, 2, \dots, n-1$) are bounded with operator norms less than 1. From Lemma 2.8 (iii), to estimate $\|V_n(A; z, z')\|$ we can choose the path C from z' to z as

$$C(t) = z' + (z - z')t, \quad t \in [0, 1]. \quad (2.108)$$

then we have $C'(t) := (d/dt)C(t) = z - z'$ and by (2.105)

$$\begin{aligned} &\|V_n(A; z, z')\Psi\| \\ &\leq \frac{1}{n!} C^n |z - z'|^n \int_{[0, 1]^n} dt_1 \dots dt_n e^{|\text{Im } z_n| (2E+nb)} (E + (n-1)b + 1)^{1/2} \dots (E + 1)^{1/2} \|\Psi\| \\ &\leq C^n e^{|\text{Im } z'| (2E+nb)} \frac{|z - z'|^n}{n!} (E + (n-1)b + 1)^{1/2} \dots (E + 1)^{1/2} \|\Psi\|. \end{aligned} \quad (2.109)$$

This completes the proof. $\square$

For $\zeta, \zeta' \in \mathbb{C}$ and $T \in \mathcal{C}_0(H_0)$, we denote

$$T(\zeta, \zeta') := e^{i\zeta H_0} T e^{i\zeta' H_0}. \quad (2.110)$$

Note that

$$T(\zeta) = T(\zeta, -\zeta). \quad (2.111)$$

Lemma 2.10. *Let T_k, A_k ($k = 1, \dots, m$, $m \geq 1$) be $\mathcal{C}_0(H_0)$ -class operators. Then for all $\Psi \in D_{\text{fin}}$, $z_k, z'_k \in \mathbb{C}$ ($k = 1, \dots, m$) with $\text{Im } z_k \leq \text{Im } z'_k$ and $\zeta_k, \zeta'_k \in \mathbb{C}$, it follows that*

$$\sum_{n_1, \dots, n_m=0}^{\infty} \|T_m(\zeta_m, \zeta'_m)V_{n_m}(A_m; z_m, z'_m) \cdots T_1(\zeta_1, \zeta'_1)V_{n_1}(A_1; z_1, z'_1)\Psi\| < \infty. \quad (2.112)$$

Furthermore, the convergence is locally uniform in $\zeta_1, \zeta'_1, z_1, z'_1, \dots, \zeta_m, \zeta'_m, z_m, z'_m$.

Proof. Let $\Psi \in V_E$ and put for $k = 1, 2, \dots, m$,

$$\Psi_k = T_k(\zeta_k, \zeta'_k)V_{n_k}(A_k; z_k, z'_k) \cdots T_1(\zeta_1, \zeta'_1)V_{n_1}(A_1; z_1, z'_1)\Psi. \quad (2.113)$$

Let $a_k, b_k \geq 0$ ($k = 1, 2, \dots, m$) be constants stated in Definition 2.1 (III) regarding T_k, A_k , respectively. We denote

$$a = \max_k \{a_k\}, \quad b = \max_k \{b_k\}. \quad (2.114)$$

then we see from Lemma 2.8 (i) that

$$\Psi_k \in V_{E+(n_1+\dots+n_k)b+ka}. \quad (2.115)$$

Put

$$\begin{aligned} K &= \max_k \{|\text{Im } \zeta_k|, |\text{Im } \zeta'_k|, |\text{Im } z'_k|\}, \\ N &= n_1 + \dots + n_m, \\ C &= \max_k \left\{ \left\| T_k(H_0 + 1)^{-1/2} \right\|, \left\| A_k(H_0 + 1)^{-1/2} \right\| \right\}. \end{aligned}$$

then from Lemma 2.9, we have

$$\begin{aligned} & \|T_m(\zeta_m, \zeta'_m)V_{n_m}(A_m; z_m, z'_m) \cdots T_1(\zeta_1, \zeta'_1)V_{n_1}(A_1; z_1, z'_1)\Psi\| \\ &= \|T_m(\zeta_m, \zeta'_m)V_{n_m}(A_m; z_m, z'_m)\Psi_{m-1}\| \\ &\leq e^{2K(E+Nb+ma)}(E+Nb+(m-1)a+1)^{1/2} \left\| T_m(H_0 + 1)^{-1/2} \right\| \|V_{n_m}(A_m; z_m, z'_m)\Psi_{m-1}\| \\ &\leq e^{2K(E+Nb+ma)} C^{n_m+1} e^{|\text{Im } z'_m|(2E+2(n_1+\dots+n_{m-1})b+2(m-1)a+n_mb)} \frac{|z_m - z'_m|^{n_m}}{n_m!} \times \\ &\quad \times (E+Nb+(m-1)a+1)^{1/2} \cdots (E+(N-n_m)b+(m-1)a+1)^{1/2} \|\Psi_{m-1}\| \\ &\leq e^{4K(E+Nb+ma)} C^{n_m+1} \frac{|z_m - z'_m|^{n_m}}{n_m!} (E+Nb+(m-1)a+1)^{1/2} \times \cdots \\ &\quad \times (E+(N-n_m)b+(m-1)a+1)^{1/2} \|\Psi_{m-1}\|. \end{aligned} \quad (2.116)$$

Repeating this estimate, we arrive at

$$\begin{aligned} & \|T_m(\zeta_m, \zeta'_m)V_{n_m}(A_m; z_m, z'_m) \cdots T_1(\zeta_1, \zeta'_1)V_{n_1}(A_1; z_1, z'_1)\Psi\| \\ &\leq e^{4mK(E+Nb+ma)} C^{N+m} \frac{|z_m - z'_m|^{n_m} \cdots |z_1 - z'_1|^{n_1}}{n_m! \cdots n_1!} \\ &\quad \times (E+Nb+(m-1)a+1)^{1/2} \cdots (E+(m-1)a+1)^{1/2} \|\Psi\|. \end{aligned} \quad (2.117)$$

therefore we obtain

$$\begin{aligned}
& \sum_{n_1, \dots, n_m=0}^{\infty} \|T_m(\zeta_m, \zeta'_m) V_{n_m}(A_m; z_m, z'_m) \cdots T_1(\zeta_1, \zeta'_1) V_{n_1}(A_1; z_1, z'_1) \Psi\| \\
&= \sum_{N=0}^{\infty} \sum_{n_1 + \dots + n_m = N} \|T_m(\zeta_m, \zeta'_m) V_{n_m}(A_m; z_m, z'_m) \cdots T_1(\zeta_1, \zeta'_1) V_{n_1}(A_1; z_1, z'_1) \Psi\| \\
&\leq \sum_{N=0}^{\infty} \frac{(|z_1 - z'_1| + \dots + |z_m - z'_m|)^N}{N!} e^{4mK(E+Nb+ma)} C^{N+m} \\
&\quad \times (E + Nb + (m-1)a + 1)^{1/2} \cdots (E + (m-1)a + 1)^{1/2} \|\Psi\|. \tag{2.118}
\end{aligned}$$

By d'Alembert's ratio test, the final expression in (2.118) converges locally uniformly in $\zeta_1, \zeta'_1, z_1, z'_1, \dots, \zeta_m, \zeta'_m, z_m, z'_m$. $\square$

Proof of Theorem 2.5. Let $\text{Im } z \leq \text{Im } z'$. Lemma 2.10 (2.112) shows that for all $\Psi \in D_{\text{fin}}$,

$$\begin{aligned}
U(A; z, z') \Psi &:= \sum_{n=0}^{\infty} V_n(A; z, z') \Psi \\
&= T \exp \left(-i \int_{\Gamma_{z, z'}} d\zeta A(\zeta) \right) \Psi \tag{2.119}
\end{aligned}$$

exists and is independent of $\Gamma_{z, z'}$. This proves (i).

We prove (ii). Inductively, we see for all integer $n \geq 0$,

$$V_n(A; z, z')^* \Psi = V_n(A^*; z'^*, z^*) \Psi, \quad \Psi \in D_{\text{fin}}. \tag{2.120}$$

The case $n = 0$ is trivial. Assume that (2.120) holds for some n . Let $\Gamma : [0, 1] \rightarrow \mathbb{C}$ be a continuously differentiable simple curve from z' to z . Then we have for all $\Psi, \Phi \in D_{\text{fin}}$,

$$\begin{aligned}
\langle \Psi, V_{n+1}(A; z, z') \Phi \rangle &= -i \int_{z'}^z d\zeta \langle \Psi, A(\zeta) V_n(A; \zeta, z') \Phi \rangle \\
&= -i \int_0^1 dt \Gamma'(t) \langle \Psi, A(\Gamma(t)) V_n(A; \Gamma(t), z') \Phi \rangle \\
&= \left\langle i \int_0^1 \Gamma'(t)^* V_n(A^*; z'^*, \Gamma(t)^*) A^*(\Gamma(t)^*) \Psi, \Phi \right\rangle \\
&= \left\langle i \int_{z'^*}^{z^*} d\zeta V_n(A^*; z'^*, \zeta) A^*(\zeta) \Psi, \Phi \right\rangle \\
&= \langle V_{n+1}(A^*; z'^*, z^*) \Psi, \Phi \rangle,
\end{aligned}$$

where we have used Lemma 2.8 (iii) in the first and the last equality, and the induction hypothesis in the third equality. Thus (2.120) holds for $n + 1$, so the induction step is complete. then by (2.120), we have for all $\Psi, \Phi \in D_{\text{fin}}$,

$$\begin{aligned}
\langle \Psi, U(A; z, z') \Phi \rangle &= \sum_{n=0}^{\infty} \langle \Psi, V_n(A; z, z') \Phi \rangle \\
&= \sum_{n=0}^{\infty} \langle V_n(A^*; z'^*, z^*) \Psi, \Phi \rangle \\
&= \langle U(A^*; z'^*, z^*) \Psi, \Phi \rangle.
\end{aligned}$$

This yields the inclusion relation

$$U(A; z, z')^* \supset U(A^*; z'^*, z^*), \quad (2.121)$$

implying that $U(A; z, z')$ is closable. Therefore we can take the closure of the both sides of (2.121), and the desired result follows. $\square$

Theorem 2.6. *Let T_k, A_k ($k = 1, \dots, m$, $m \geq 1$) be $\mathcal{C}_0(H_0)$ -class operators. Then for all $z_k, z'_k \in \mathbb{C}$ ($k = 1, \dots, m$) with $\text{Im } z_k \leq \text{Im } z'_k$ and $\zeta_k, \zeta'_k \in \mathbb{C}$, it follows that*

$$D_{\text{fin}} \subset D(T_m(\zeta_m, \zeta'_m) \overline{U(A_m; z_m, z'_m)} \cdots T_1(\zeta_1, \zeta'_1) \overline{U(A_1; z_1, z'_1)}). \quad (2.122)$$

Moreover, for all $\Psi \in D_{\text{fin}}$,

$$\begin{aligned} & T_m(\zeta_m, \zeta'_m) \overline{U(A_m; z_m, z'_m)} \cdots T_1(\zeta_1, \zeta'_1) \overline{U(A_1; z_1, z'_1)} \Psi \\ &= \sum_{n_1, \dots, n_m=0}^{\infty} T_m(\zeta_m, \zeta'_m) V_{n_m}(A_m; z_m, z'_m) \cdots T_1(\zeta_1, \zeta'_1) V_{n_1}(A_1; z_1, z'_1) \Psi, \end{aligned} \quad (2.123)$$

where the right-hand side converges absolutely, and does not depend upon the summation order. Furthermore, this convergence is locally uniform in the complex variables $z_1, z'_1, \zeta_1, \zeta'_1, \dots, z_m, z'_m, \zeta_m, \zeta'_m$.

By Theorem 2.6, it is natural to introduce the algebra $\mathfrak{A}$ generated by

$$\left\{ T, \overline{U(A; z, z')}, e^{i\zeta H_0} \mid T, A \in \mathcal{C}_0(H_0), z, z', \zeta \in \mathbb{C}, \text{Im } z \leq \text{Im } z' \right\}. \quad (2.124)$$

It is clear that all $a \in \mathfrak{A}$ is closable since they have densely defined adjoints and the subspace D_{fin} is a common domain of $\mathfrak{A}$. We define a dense subspace $\mathcal{D}$ by

$$\mathcal{D} := \mathfrak{A} D_{\text{fin}}. \quad (2.125)$$

Theorem 2.6 shows that $\mathcal{D}$ is also a common domain of $\mathfrak{A}$. Moreover, for all $\Psi \in \mathcal{D}$, there exists a sequence $\{\Psi_N\}_N \subset D_{\text{fin}}$ such that

$$\Psi_N \rightarrow \Psi, \quad a\Psi_N \rightarrow a\Psi \quad (a \in \mathfrak{A}) \quad (2.126)$$

as N tends to infinity. This implies that if an equality $a = b$ ($a, b \in \mathfrak{A}$) holds on D_{fin} , then $a = b$ on $\mathcal{D}$ and the convergence is locally uniform in all the complex variables included in a and b . From this observation, we immediately have

Corollary 2.1. *Let A be in $\mathcal{C}_0(H_0)$ class and $z, z' \in \mathbb{C}$ with $\text{Im } z \leq \text{Im } z'$. then*

$$\mathcal{D} \subset D \left(T \exp \left(-i \int_{\Gamma_{z, z'}} d\zeta A(\zeta) \right) \right) \quad (2.127)$$

and for $\Psi \in \mathcal{D}$,

$$T \exp \left(-i \int_{\Gamma_{z, z'}} d\zeta A(\zeta) \right) \Psi = \overline{U(A; z, z')} \Psi. \quad (2.128)$$

In particular,

$$T \exp \left(-i \int_{\Gamma_{z, z'}} d\zeta A(\zeta) \right) \Psi \quad (2.129)$$

is independent of the simple curve $\Gamma_{z, z'}$ and depends only on z, z' if $\Psi \in \mathcal{D}$.

Proof of Theorem 2.6. We prove the claim by induction on $m \geq 1$. Let $m = 1$, and let $\Psi \in D_{\text{fin}}$. By Lemma 2.10,

$$\sum_{n=0}^{\infty} \|T_1(\zeta_1)V_n(A_1; z_1, z'_1)\Psi\| < \infty. \quad (2.130)$$

then since $T_1(\zeta_1)$ is closed, we get $U(A_1; z_1, z'_1)\Psi \in D(T_1(\zeta_1))$ and (2.123) for $m = 1$.

Suppose that the claim is true for some $m \geq 1$. Let $\Psi \in D_{\text{fin}}$. By Lemma 2.10, one sees

$$\sum_{n_1, \dots, n_m=0}^{\infty} \|V_{n_{m+1}}(A_{m+1}; z_{m+1}, z'_{m+1}) \cdots T_1(\zeta_1)V_{n_1}(A_1; z_1, z'_1)\Psi\| < \infty, \quad (2.131)$$

$$\sum_{n_1, \dots, n_m=0}^{\infty} \|T_{m+1}(\zeta_{m+1})V_{n_{m+1}}(A_{m+1}; z_{m+1}, z'_{m+1}) \cdots T_1(\zeta_1)V_{n_1}(A_1; z_1, z'_1)\Psi\| < \infty. \quad (2.132)$$

hence we have using induction hypothesis

$$T_m(\zeta_m)\overline{U(A_m; z_m, z'_m)} \cdots T_1(\zeta_1)\overline{U(A_1; z_1, z'_1)}\Psi \in D(T_{m+1}(\zeta_{m+1})\overline{U(A_{m+1}; z_{m+1}, z'_{m+1})}) \quad (2.133)$$

and (2.123) for $m + 1$ since T_{m+1} is closed and $U(A_{m+1}; z_{m+1}, z'_{m+1})$ are closable. thus the assertion holds also for $m + 1$. The local uniformity of the convergence follows the fact that the series in Lemma 2.10 (2.112) converges locally uniformly. $\square$

Theorem 2.7. *Let A be in $\mathcal{C}_0(H_0)$ class and $z, z' \in \mathbb{C}$.*

(i) *For all $\Psi \in \mathcal{D}$, the vector valued function*

$$\{(z, z') \mid \text{Im } z \leq \text{Im } z'\} \ni (z, z') \mapsto \overline{U(A; z, z')}\Psi \in \mathcal{H}$$

is analytic on the region $\{\text{Im } z < \text{Im } z'\}$ and continuous on $\{\text{Im } z \leq \text{Im } z'\}$. Moreover, it is a solution of differential equations

$$\frac{\partial}{\partial z} \overline{U(A; z, z')}\Psi = -iA(z)\overline{U(A; z, z')}\Psi, \quad (2.134)$$

$$\frac{\partial}{\partial z'} \overline{U(A; z, z')}\Psi = i\overline{U(A; z, z')}A(z')\Psi, \quad (2.135)$$

on $\{\text{Im } z < \text{Im } z'\}$.

(ii) *For all $\Psi \in \mathcal{D}$, the vector valued function $\mathbb{R}^2 \ni (t, t') \mapsto \overline{U(A; t, t')}\Psi$ is continuously differentiable on the region $\mathbb{R}^2$, satisfying the differential equations*

$$\frac{\partial}{\partial t} \overline{U(A; t, t')}\Psi = -iA(t)\overline{U(A; t, t')}\Psi, \quad (2.136)$$

$$\frac{\partial}{\partial t'} \overline{U(A; t, t')}\Psi = i\overline{U(A; t, t')}A(t')\Psi. \quad (2.137)$$

Proof. We prove (i). Since the convergence in (2.119) is locally uniform in z, z' and each $V_n(A; z, z')$ are analytic on all $z, z' \in \mathbb{C}$, we conclude that $U(A; z, z')$ is analytic on the region $\{\text{Im } z < \text{Im } z'\}$ and continuous on $\{\text{Im } z \leq \text{Im } z'\}$. Due to the fact that the convergences are uniform in (2.112), one finds

$$\sum_{n=0}^{\infty} A(z)V_n(A; z, z')\Psi = A(z)U(A; z, z')\Psi, \quad (2.138)$$

$$\sum_{n=0}^{\infty} V_n(A; z, z')A(z')\Psi = U(A; z, z')A(z')\Psi, \quad (2.139)$$

absolutely and locally uniformly in z, z' when n tends to infinity. By taking $n \rightarrow \infty$ in (2.95) and (2.96), we obtain

$$U(A; z, z') = 1 - i \int_{z'}^z d\zeta A(\zeta) U(A; \zeta, z'), \quad (2.140)$$

$$= 1 - i \int_{z'}^z d\zeta U(A; z, \zeta) A(\zeta), \quad (2.141)$$

on D_{fin} . By the remark just below the statement of Theorem 2.6, integral equations (2.140) and (2.141) can be extended to $\mathcal{D}$ in the form

$$\overline{U(A; z, z')} = 1 - i \int_{z'}^z d\zeta A(\zeta) \overline{U(A; \zeta, z')}, \quad (2.142)$$

$$= 1 - i \int_{z'}^z d\zeta \overline{U(A; z, \zeta)} A(\zeta). \quad (2.143)$$

Differentiating these expression with respect to z or z' , one finds (2.134) and (2.135).

Considering the case where z, z' are real, we obtain (ii) in the same manner. $\square$

Theorem 2.8. *Let $A \in \mathcal{C}_0(H_0)$ and $z, z', z'' \in \mathbb{C}$. Then the following properties hold.*

(i) *If $\text{Im } z \leq \text{Im } z' \leq \text{Im } z''$, the equalities*

$$\overline{U(A; z, z)} = I, \quad \overline{U(A; z, z')} \overline{U(A; z', z'')} = \overline{U(A; z, z'')} \quad (2.144)$$

hold on the subspace $\mathcal{D}$, where I is the identity operator.

(ii) *Let $\text{Im } z \leq \text{Im } z'$. Then $U(A; z, z')$ is translationally invariant in the sense that the equality*

$$e^{izH_0} \overline{U(A; z', z'')} e^{-izH_0} \Psi = \overline{U(A; z' + z, z'' + z)} \Psi \quad (2.145)$$

holds on the subspace $\mathcal{D}$.

(iii) *For all $t, t' \in \mathbb{R}$, $\overline{U(A; t, t')}$ is unitary. Moreover, for all $t, t', t'' \in \mathbb{R}$, the operator equality*

$$\overline{U(A; t, t')} \overline{U(A; t', t'')} = \overline{U(A; t, t'')} \quad (2.146)$$

holds.

Proof. (i) Fix z, z'' so that $\text{Im } z < \text{Im } z''$. then by Theorem 2.5, for all $\Psi, \Phi \in \mathcal{D}$ and $z' \in \mathbb{C}$ with $\text{Im } z' \in (\text{Im } z, \text{Im } z'')$,

$$\begin{aligned} & \frac{d}{dz'} \left\langle \Phi, \overline{U(A; z, z')} \overline{U(A; z', z'')} \Psi \right\rangle \\ &= \frac{d}{dz'} \left\langle \overline{U(A^*; z'^*, z^*)} \Phi, \overline{U(A; z', z'')} \Psi \right\rangle \\ &= \left\langle -iA^*(z'^*) \overline{U(A^*; z'^*, z^*)} \Phi, \overline{U(A; z', z'')} \Psi \right\rangle + \left\langle \overline{U(A; z'^*, z^*)} \Phi, -iA(z') \overline{U(A; z', z'')} \Psi \right\rangle \\ &= 0. \end{aligned}$$

This yields that

$$z' \rightarrow \left\langle \Phi, \overline{U(A; z, z')} \overline{U(A; z', z'')} \Psi \right\rangle \quad (2.147)$$

is constant on the region $\{z' \mid \text{Im } z' \in (\text{Im } z, \text{Im } z'')\}$. But this function is continuous on its closure, implying that it must be constant on the closed region $\text{Im } z \leq \text{Im } z' \leq \text{Im } z''$. Taking $z' = z$ we have

$$\left\langle \Phi, \overline{U(A; z, z')} \overline{U(A; z', z'')} \Psi \right\rangle = \left\langle \Phi, \overline{U(A; z, z'')} \Psi \right\rangle \quad (2.148)$$

for all $\text{Im } z \leq \text{Im } z' \leq \text{Im } z''$ with $\text{Im } z < \text{Im } z''$. Fix $z, z' \in \mathbb{C}$ so that $\text{Im } z = \text{Im } z'$ and regard both sides of (2.148) as a function of z'' . Since these functions are continuous on $\{z'' \mid \text{Im } z \leq \text{Im } z''\}$ and coincide on $\{z'' \mid \text{Im } z < \text{Im } z''\}$, they must coincide on $\{z'' \mid \text{Im } z \leq \text{Im } z''\}$. This completes the proof.

(ii) We first show by induction on $n \geq 0$ that

$$e^{izH_0} V_n(A; z', z'') e^{-izH_0} \Psi = V_n(A; z' + z, z'' + z) \Psi, \quad \Psi \in D_{\text{fin}}. \quad (2.149)$$

The case $n = 0$ is trivial. Assume that (2.149) holds for some n . Then we have for all $\Psi \in D_{\text{fin}}$,

$$\begin{aligned} e^{izH_0} V_{n+1}(A; z', z'') e^{-izH_0} \Psi &= -i \int_{z''}^{z'} d\zeta e^{izH_0} A(\zeta) V_n(A; \zeta, z'') e^{-izH_0} \Psi \\ &= -i \int_{z''}^{z'} d\zeta A(\zeta + z) V_n(A; \zeta + z, z'' + z) \Psi \\ &= -i \int_{z''+z}^{z'+z} d\zeta A(\zeta) V_n(A; \zeta, z'' + z) \Psi \\ &= V_{n+1}(A; z' + z, z'' + z) \Psi, \end{aligned}$$

where we have used the basic property $e^{izH_0} A(\zeta) e^{-izH_0} \Phi = A(\zeta + z) \Phi$ ($\Phi \in D_{\text{fin}}$) in the second equality and the induction hypothesis in the third. This completes the induction. Summing up the both sides of (2.149) over all $n \geq 0$, and using the closedness of e^{izH_0} , we obtain

$$e^{izH_0} U(A; z', z'') e^{-izH_0} = U(A; z' + z, z'' + z) \quad (2.150)$$

on D_{fin} . But both sides belong to $\mathfrak{A}$, this equality holds on $\mathcal{D}$ in the form

$$e^{izH_0} \overline{U(A; z', z'')} e^{-izH_0} = \overline{U(A; z' + z, z'' + z)}. \quad (2.151)$$

(iii) Similar to the proof of [23, Theorem 2.4]. □

Theorem 2.9. Let $A_1, \dots, A_k, B \in \mathcal{C}_0(H_0)$, and $z, z' \in \mathbb{C}$ with $\text{Im } z \leq \text{Im } z'$. Let $\Gamma_{z, z'}$ be a simple curve from z' to z and $\zeta_1, \dots, \zeta_k \in \Gamma$ be different from each other. Then we have

$$\mathcal{D} \subset D \left(T A_1(\zeta_1) \dots A_k(\zeta_k) \exp \left(-i \int_{\Gamma_{z, z'}} d\zeta B(\zeta) \right) \right) \quad (2.152)$$

and

$$\begin{aligned} T A_1(\zeta_1) \dots A_k(\zeta_k) \exp \left(-i \int_{\Gamma_{z, z'}} d\zeta B(\zeta) \right) \Psi \\ = \overline{U(B; z, \zeta_{j_1})} A_{j_1}(\zeta_{j_1}) \overline{U(B; \zeta_{j_1}, \zeta_{j_2})} \dots \overline{U(B; \zeta_{k-1}, \zeta_k)} A_{j_k}(\zeta_{j_k}) \overline{U(B; \zeta_{j_k}, z')} \Psi \end{aligned} \quad (2.153)$$

for all $\Psi \in \mathcal{D}$, where $(j_1, \dots, j_k)$ is the permutation of $(1, 2, \dots, k)$ with $\zeta_{j_1} \succ \dots \succ \zeta_{j_k}$.

Proof. Put

$$A_{k+1} = \cdots = A_{k+n} = B. \quad (2.154)$$

We can assume that

$$\zeta_1 \succ \cdots \succ \zeta_k \quad (2.155)$$

without loss of generality. Take $\Psi \in \mathcal{D}$. For all $n \in \mathbb{N}$ and all $\sigma \in \mathfrak{S}_{k+n}$, it is clear that the mapping

$$(\zeta_{k+1}, \dots, \zeta_{k+n}) \mapsto A_{\sigma(k)}(\zeta_{\sigma(k)}) \cdots A_{\sigma(k+n)}(\zeta_{\sigma(k+n)}) \Psi \quad (2.156)$$

is analytic and thus the strong integral

$$\begin{aligned} & \frac{(-i)^n}{n!} \int_{\Gamma_{z, z'}^n} d\zeta_{k+1} \cdots d\zeta_{k+n} T A_1(\zeta_1) \cdots A_k(\zeta_k) B(\zeta_{k+1}) \cdots B(\zeta_{k+n}) \Psi \\ &= \frac{(-i)^n}{n!} \sum_{\sigma \in \mathfrak{S}_{k+n}} \int_{P'_{n, \sigma}} d\zeta_{k+1} \cdots d\zeta_{k+n} A_{\sigma(1)}(\zeta_{\sigma(1)}) \cdots A_{\sigma(k+n)}(\zeta_{\sigma(k+n)}) \Psi \end{aligned} \quad (2.157)$$

exists. The integral in the right-hand side vanishes unless σ is of the following form: There are $l_1, \dots, l_{k+1}$ satisfying

$$l_1, \dots, l_{k+1} \geq 0, \quad l_1 + \cdots + l_{k+1} = n \quad (2.158)$$

and

$$\sigma(l_1 + 1) = 1, \quad \sigma(l_1 + l_2 + 2) = 2, \quad \dots, \quad \sigma(l_1 + \cdots + l_k + k) = k. \quad (2.159)$$

If we denote such permutation σ by $\sigma_{l_1, \dots, l_{k+1}}$, the summation over σ can be performed by summing up all σ 's of the form $\sigma = \sigma_{l_1, \dots, l_{k+1}}$ for some $l_1, \dots, l_{k+1}$ (there are $n!$ such σ 's for each fixed $l_1, \dots, l_{k+1}$), and then summing over all $l_1, \dots, l_{k+1}$ satisfying (2.158):

$$\sum_{\sigma \in \mathfrak{S}_{k+n}} = \sum_{\substack{l_1, \dots, l_{k+1} \geq 0 \\ l_1 + \cdots + l_{k+1} = n}} \sum_{\sigma = \sigma_{l_1, \dots, l_{k+1}}} . \quad (2.160)$$

The integration in (2.157) depends only upon $l_1, \dots, l_{k+1}$, but not upon the concrete form of $\sigma = \sigma_{l_1, \dots, l_{k+1}}$, and thus the summation over $\sigma = \sigma_{l_1, \dots, l_{k+1}}$ just gives the factor $n!$. Then we

have

$$\begin{aligned}
& \frac{(-i)^n}{n!} \sum_{\sigma \in \mathfrak{S}_{k+n}} \int_{P'_{n,\sigma}} d\zeta_{k+1} \dots d\zeta_{k+n} A_{\sigma(1)}(\zeta_{\sigma(1)}) \dots A_{\sigma(k+n)}(\zeta_{\sigma(k+n)}) \Psi \\
&= \frac{(-i)^n}{n!} \sum_{\substack{l_1, \dots, l_{k+1} \geq 0 \\ l_1 + \dots + l_{k+1} = n}} \sum_{\sigma = \sigma_{l_1, \dots, l_{k+1}}} \int_{z \succ \tau_1^{(1)} \succ \dots \succ \tau_{l_1}^{(1)} \succ \zeta_1 \succ \dots \succ \zeta_k \succ \tau_1^{(k+1)} \succ \dots \succ \tau_{l_{k+1}}^{(k+1)} \succ z'} d\tau_1^{(1)} \dots d\tau_{l_1}^{(1)} \dots d\tau_1^{(k+1)} \dots d\tau_{l_{k+1}}^{(k+1)} \\
&\quad B(\tau_1^{(1)}) \dots B(\tau_{l_1}^{(1)}) A_1(\zeta_1) \dots A_k(\zeta_k) B(\tau_1^{(k+1)}) \dots B(\tau_{l_{k+1}}^{(k+1)}) \Psi \\
&= \sum_{\substack{l_1, \dots, l_{k+1} \geq 0 \\ l_1 + \dots + l_{k+1} = n}} \left(\frac{(-i)^{l_1}}{l_1!} \int_{\Gamma_{z, \zeta_1}^{l_1}} d\tau_1^{(1)} \dots d\tau_{l_1}^{(1)} TB(\tau_1^{(1)}) \dots B(\tau_{l_1}^{(1)}) \right) A_1(\zeta_1) \dots \\
&\quad \dots A_k(\zeta_k) \left(\frac{(-i)^{l_{k+1}}}{l_{k+1}!} \int_{\Gamma_{\zeta_k, z'}^{l_{k+1}}} d\tau_1^{(k+1)} \dots d\tau_{l_{k+1}}^{(k+1)} TB(\tau_1^{(k+1)}) \dots B(\tau_{l_{k+1}}^{(k+1)}) \right) \Psi \\
&= \sum_{\substack{l_1, \dots, l_{k+1} \geq 0 \\ l_1 + \dots + l_{k+1} = n}} V_{l_1}(B; z, \zeta_1) A_1(\zeta_1) \dots A_k(\zeta_k) V_{l_{k+1}}(B; \zeta_k, z') \Psi. \tag{2.161}
\end{aligned}$$

The final expression in (2.161) is absolutely summable with respect to $n = 0, 1, 2, \dots$ to give

$$\overline{U(B; z, \zeta_1)} A_1(\zeta_1) \dots A_k(\zeta_k) \overline{U(B; \zeta_k, z')} \Psi \tag{2.162}$$

by Theorem 2.6, which means that Ψ belongs to the subspace

$$D \left(TA_1(\zeta_1) \dots A_k(\zeta_k) \exp \left(-i \int_{\Gamma_{z, z'}} d\zeta B(\zeta) \right) \right), \tag{2.163}$$

and

$$TA_1(\zeta_1) \dots A_k(\zeta_k) \exp \left(-i \int_{\Gamma_{z, z'}} d\zeta B(\zeta) \right) \Psi = \overline{U(B; z, \zeta_1)} A_1(\zeta_1) \dots A_k(\zeta_k) \overline{U(B; \zeta_k, z')} \Psi. \tag{2.164}$$

This completes the proof. $\square$

Under the condition that A is in $\mathcal{C}_0(H_0)$ -class, we have to restrict the arguments $z, z' \in \mathbb{C}$ to $\{\text{Im} z' \leq \text{Im} z\}$ in order to prove the convergence of the time-ordered exponential as shown in Theorem 2.5. If a stronger condition holds, Then the time-ordered exponential converge for all $z, z' \in \mathbb{C}$:

Theorem 2.10. *Let A be in $\mathcal{C}_0(H_0)$. Suppose that there exists a locally bounded function $F : \mathbb{C} \rightarrow [0, \infty)$ such that, for all $\Psi \in D_{\text{fin}}$,*

$$\|A(z)\Psi\| \leq F(z) \|(H_0 + 1)^{1/2} \Psi\|, \quad z \in \mathbb{C}. \tag{2.165}$$

then for all $z, z' \in \mathbb{C}$, the followings hold.

- (i) Take a piecewisely continuously differentiable simple curve $\Gamma_{z,z'}$ which starts at z' and ends at z . Then for all $\Psi \in D_{\text{fin}}$, the series

$$U(A; z, z')\Psi := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma_{z,z'}^n} d\zeta_1 \dots d\zeta_n \mathcal{T}(A(\zeta_1) \cdots A(\zeta_n))\Psi \quad (2.166)$$

converges absolutely.

- (ii) For all $\Psi \in D_{\text{fin}}$, the vector-valued function $\mathbb{C}^2 \ni (z, z') \mapsto U(A; z, z')\Psi$ is analytic, and satisfies the following differential equations:

$$\frac{\partial}{\partial z} U(A; z, z')\Psi = -iA(z)U(A; z, z')\Psi, \quad (2.167)$$

$$\frac{\partial}{\partial z'} U(A; z, z')\Psi = iU(A; z, z')A(z')\Psi. \quad (2.168)$$

Proof. (i) Without loss of generality, we can assume that $\Psi \in V_E$ for some $E \geq 0$. In a similar way as in the proof of Lemma 2.9, we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \left\| \frac{1}{n!} \int_{\Gamma_{z,z'}^n} d\zeta_1 \dots d\zeta_n \mathcal{T}(A(\zeta_1) \cdots A(\zeta_n))\Psi \right\| \\ & \leq \sum_{n=0}^{\infty} \frac{|z - z'|^n}{n!} \left(\sup_{\zeta \in L(z, z')} F(\zeta) \right) (E + (n-1)b + 1)^{1/2} \cdots (E + 1)^{1/2} \|\Psi\| \\ & < \infty, \end{aligned} \quad (2.169)$$

by using the condition (2.165). Here $L(z, z') \subset \mathbb{C}$ is the line segment from z' to z .

- (ii) Similar to the proof of Theorem 2.7 (i). □

2.3 Time-ordered exponential as an asymptotic expansion

Let H_0 be a non-negative self-adjoint operator, and H_1 a symmetric operator on a Hilbert space $\mathcal{H}$. Set

$$H(\lambda) := H_0 + \lambda H_1 \quad (\lambda \in \mathbb{R}). \quad (2.170)$$

Assumptions we employ here is:

- Assumption 2.1.** (I) There exists a sufficiently small interval $J \subset \mathbb{R}$ ($0 \in J$) such that each $H(\lambda)$ ($\lambda \in J$) is essentially self-adjoint and bounded from below with $\inf_{\lambda \in J} (\inf \sigma(\overline{H(\lambda)})) > -\infty$.
- (II) There exists a constant $b \geq 0$ such that for all $E \geq 0$, H_1 maps $R(E_{H_0}([0, E]))$ into $R(E_{H_0}([0, E + b]))$.
- (III) H_1 is H_0^r -bounded for a $r \geq 0$, and there exists a constant $C \geq 0$ such that

$$\|H_1 \Psi\| \leq C(H_0 + 1)^r \Psi, \quad \Psi \in D(H_0^r). \quad (2.171)$$

We denote the closure of $H(\lambda)$ ($\lambda \in J$) by the same symbol. Set $H_1(\zeta) := e^{i\zeta H_0} H_1 e^{-i\zeta H_0}$ ($\zeta \in \mathbb{C}$).

Theorem 2.11. *Suppose that Assumption 2.1 holds. Then, for all $z, z' \in \mathbb{C}$ with $\text{Im} z \leq \text{Im} z'$ and $\Psi, \Phi \in D_{\text{fin}} := \bigcup_{E \geq 0} R(E_{H_0}([0, E]))$, the following asymptotic expansion:*

$$\begin{aligned} & \left\langle e^{-iz^* H_0} \Phi, e^{-i(z-z')H(\lambda)} e^{-iz' H_0} \Psi \right\rangle \\ & \sim \langle \Phi, \Psi \rangle + \sum_{n=1}^{\infty} (-i\lambda)^n \int_{z'}^z d\zeta_1 \int_{z'}^z d\zeta_2 \dots \int_{z'}^z d\zeta_n \langle \Phi, \mathcal{T}(H_1(\zeta_1)H_1(\zeta_2) \dots H_1(\zeta_n)) \Psi \rangle \\ & \quad (\lambda \rightarrow 0) \end{aligned} \quad (2.172)$$

holds, that is, for each $N = 0, 1, 2, \dots$,

$$\begin{aligned} & \left\langle e^{-iz^* H_0} \Phi, e^{-i(z-z')H(\lambda)} e^{-iz' H_0} \Psi \right\rangle \\ & = \langle \Phi, \Psi \rangle + \sum_{n=1}^N (-i\lambda)^n \int_{z'}^z d\zeta_1 \int_{z'}^z d\zeta_2 \dots \int_{z'}^z d\zeta_n \langle \Phi, \mathcal{T}(H_1(\zeta_1)H_1(\zeta_2) \dots H_1(\zeta_n)) \Psi \rangle \\ & \quad + o(\lambda^N) \quad (\lambda \rightarrow 0), \end{aligned} \quad (2.173)$$

where the symbol $\mathcal{T}$ denotes the time-ordering defined in (2.70).

Remark 2.3. Theorem 2.5 says that the right-hand side of (2.172) becomes convergent series if the constant r stated in Assumption 2.1 (III) satisfies $0 \leq r < 1$.

Lemma 2.11. *Suppose that Assumption 2.1 holds. Let $w \in \mathbb{C}$ with $\text{Im} w \geq 0$. Then for all $\Psi \in D_{\text{fin}}$ and $N = 0, 1, 2, \dots$,*

$$\begin{aligned} & e^{iwH(\lambda)} e^{-iwH_0} \Psi \\ & = \Psi + \sum_{n=1}^N (-i\lambda)^n \int_w^0 d\zeta_1 \int_w^{\zeta_1} d\zeta_2 \dots \int_w^{\zeta_{n-1}} d\zeta_n H_1(\zeta_1) \dots H_1(\zeta_n) \Psi + o(\lambda^N)_{\mathcal{H}} \quad (\lambda \rightarrow 0), \end{aligned} \quad (2.174)$$

where $o(\lambda^N)_{\mathcal{H}} (\lambda \rightarrow 0)$ denotes a vector-valued function $\lambda \mapsto f(\lambda) \in \mathcal{H}$ satisfying $\lim_{\lambda \rightarrow 0} \|f(\lambda)/\lambda^N\|_{\mathcal{H}} = 0$.

Proof. Note that $e^{iwH(\lambda)} (\lambda \in J)$ is a bounded operator, and $D_{\text{fin}} \subset D(e^{iwH(\lambda)} e^{-iwH_0})$. Let $\Psi \in R(E_{H_0}([0, E]))$, and set $\mathbb{H} := \{w \in \mathbb{C} \mid \text{Im} w > 0\}$. Using Assumption 2.1 (III), we see that the vector-valued function $\mathbb{H} \ni w \mapsto e^{iwH(\lambda)} e^{-iwH_0} \Psi$ is strongly differentiable and

$$\frac{d}{dw} e^{iwH(\lambda)} e^{-iwH_0} \Psi = e^{iwH(\lambda)} (i\lambda H_1) e^{-iwH_0} \Psi, \quad (2.175)$$

hence we have

$$\begin{aligned} e^{iwH(\lambda)} e^{-iwH_0} \Psi & = \Psi + i\lambda \int_0^w d\zeta_1 e^{i\zeta_1 H(\lambda)} H_1 e^{-i\zeta_1 H_0} \Psi \\ & = \Psi + i\lambda \int_0^w d\zeta_1 e^{i\zeta_1 H(\lambda)} e^{-i\zeta_1 H_0} H_1(\zeta_1) \Psi. \end{aligned} \quad (2.176)$$

Here we used Assumption 2.1 (II) in the second equality. Note that the point $0 \in \mathbb{C}$ does not belong to $\mathbb{H}$ but the integrals in (2.176) are independent of the choice of the path from 0 to w . Since $H_1(\zeta_1) \Psi \in D_{\text{fin}}$, it follows from (2.176) that

$$\begin{aligned} & e^{iwH(\lambda)} e^{-iwH_0} \Psi \\ & = \Psi + \sum_{n=1}^N (i\lambda)^n \int_0^w d\zeta_1 \int_0^{\zeta_1} d\zeta_2 \dots \int_0^{\zeta_{n-1}} d\zeta_n H_1(\zeta_n) \dots H_1(\zeta_2) H_1(\zeta_1) \Psi + R_{N+1}(w) \end{aligned} \quad (2.177)$$

with

$$R_{N+1}(w) := (i\lambda)^{N+1} \int_0^w d\zeta_1 \int_0^{\zeta_1} d\zeta_2 \dots \int_0^{\zeta_N} d\zeta_{N+1} e^{i\zeta_{N+1}H(\lambda)} e^{-i\zeta_{N+1}H_0} H_1(\zeta_{N+1}) \dots H_1(\zeta_2) H_1(\zeta_1) \Psi.$$

We show $R_{N+1}(w) = o(\lambda^N)_{\mathcal{H}}$. For $\zeta_1, \zeta_2, \dots, \zeta_{N+1}$ in the line segment from 0 to w satisfying $\text{Im}\zeta_1 \geq \text{Im}\zeta_2 \geq \dots \geq \text{Im}\zeta_{N+1}$, we have

$$\begin{aligned} & \|e^{i\zeta_{N+1}H(\lambda)} e^{-i\zeta_{N+1}H_0} H_1(\zeta_{N+1}) \dots H_1(\zeta_2) H_1(\zeta_1) \Psi\| \\ & \leq e^{\gamma \text{Im}w} C^{N+1} (E + Nb + 1)^r \dots (E + b + 1)^r (E + 1)^r e^{E \text{Im}w} \|\Psi\|, \end{aligned} \quad (2.178)$$

where $\gamma := -\inf_{\lambda \in J} \sigma(H(\lambda))$, and $b \geq 0$ and $C \geq 0$ are the constants stated in Assumption 2.1 (II) and (III), respectively. Thus we have

$$\begin{aligned} \left\| \frac{R_{N+1}(w)}{\lambda^N} \right\| & \leq |\lambda| \frac{|w|^{N+1} C^{N+1}}{(N+1)!} e^{\gamma \text{Im}w} (E + Nb + 1)^r \dots (E + b + 1)^r (E + 1)^r e^{E \text{Im}w} \|\Psi\| \\ & \rightarrow 0 \quad (\lambda \rightarrow 0) \end{aligned} \quad (2.179)$$

which implies $R_{N+1}(w) = o(\lambda^N)_{\mathcal{H}}$. On the other hand, by a direct calculation,

$$\begin{aligned} & \sum_{n=1}^N (i\lambda)^n \int_0^w d\zeta_1 \int_0^{\zeta_1} d\zeta_2 \dots \int_0^{\zeta_{n-1}} d\zeta_n H_1(\zeta_n) \dots H_1(\zeta_2) H_1(\zeta_1) \Psi \\ & = \sum_{n=1}^N (-i\lambda)^n \int_w^0 d\zeta_1 \int_w^{\zeta_1} d\zeta_2 \dots \int_w^{\zeta_{n-1}} d\zeta_n H_1(\zeta_1) H_1(\zeta_2) \dots H_1(\zeta_n) \Psi. \end{aligned} \quad (2.180)$$

Therefore we obtain (2.174) for $\text{Im}w > 0$.

In the same manner as above, we obtain (2.174) for $\text{Im}w = 0$. $\square$

Proof of Theorem 2.11. Using Lemma 2.11 in the case $w = -z + z'$, we have

$$\begin{aligned} & e^{-i(z-z')H(\lambda)} e^{-iz'H_0} \Psi \\ & = e^{-i(z-z')H(\lambda)} e^{i(z-z')H_0} e^{-izH_0} \Psi \\ & = e^{-izH_0} \Psi + \sum_{n=1}^N (-i\lambda)^n \int_{-z+z'}^0 d\zeta_1 \int_{-z+z'}^{\zeta_1} d\zeta_2 \dots \int_{-z+z'}^{\zeta_{n-1}} d\zeta_n H_1(\zeta_1) H_1(\zeta_2) \dots H_1(\zeta_n) e^{-izH_0} \Psi \\ & \quad + o(\lambda^N)_{\mathcal{H}} \quad (\lambda \rightarrow 0), \end{aligned}$$

hence

$$\begin{aligned} & \left\langle e^{-iz^*H_0} \Phi, e^{-i(z-z')H(\lambda)} e^{-iz'H_0} \Psi \right\rangle \\ & \sim \langle \Phi, \Psi \rangle + \sum_{n=1}^{\infty} (-i\lambda)^n \int_{z'}^z d\zeta_1 \int_{z'}^{\zeta_1} d\zeta_2 \dots \int_{z'}^{\zeta_{n-1}} d\zeta_n \langle \Phi, H_1(\zeta_1) H_1(\zeta_2) \dots H_1(\zeta_n) \Psi \rangle \\ & \quad (\lambda \rightarrow 0). \end{aligned} \quad (2.181)$$

In the same manner as in the proof of 2.8 (ii), we see that the right-hand side of (2.181) is equal to that of (2.173). $\square$

3 Construction of dynamics for non-symmetric Hamiltonians

We construct a time-evolution for a non-normal Hamiltonian via the time ordered exponential.

3.1 Schrödinger and Heisenberg equations of motion

In this subsection, we use the notations which have already been introduced in Subsection 2.1.

Let H_0 be a self-adjoint operator, and H_1 a densely defined closed operator on a Hilbert space $\mathcal{H}$. Set

$$H := H_0 + H_1. \quad (3.1)$$

We emphasize that H_1 is not necessary to be symmetric or normal, and hence so is H .

By Theorem 2.1, when there exists a self-adjoint operator $A \in \mathcal{SA}_+(H_0)$ such that H_1 is in $\mathcal{C}_0(A)$ -class, we can construct the time-evolution operator in the interaction picture $U_A(H_1; t, t')$. We shortly denote $U_A(H_1; t, t')$ by $U(t, t')$. Now we set

$$W(t) := e^{-itH_0} \overline{U(t, 0)}, \quad t \in \mathbb{R}. \quad (3.2)$$

Our goal in this subsection is to prove following Theorems 3.1-3.3.

Theorem 3.1. *Let $A \in \mathcal{SA}_+(H_0)$ and $H_1 \in \mathcal{C}_0(A)$. Then for each $\xi \in D(H_0) \cap D$, the vector valued function $t \mapsto \xi(t) := W(t)\xi$ is a solution of the initial value problem for the Schrödinger equation:*

$$\frac{d}{dt}\xi(t) = -iH\xi(t), \quad \xi(0) = \xi. \quad (3.3)$$

If H is symmetric, then we obtain a stronger result:

Theorem 3.2. *Let $A \in \mathcal{SA}_+(H_0)$ and $H_1 \in \mathcal{C}_0(A)$. If H_1 is a symmetric operator, then there exists a unique self-adjoint operator $\tilde{H}$ such that*

$$W(t) = e^{-it\tilde{H}}, \quad t \in \mathbb{R}. \quad (3.4)$$

Moreover

$$\overline{U(t, t')} = e^{itH_0} e^{-i(t-t')\tilde{H}} e^{-it'H_0}, \quad t, t' \in \mathbb{R} \quad (3.5)$$

and

$$\overline{H \upharpoonright D \cap D(H_0)} \subset \tilde{H}, \quad (3.6)$$

In particular, if H is essentially self-adjoint on $D \cap D(H_0)$, then we have

$$\overline{H} = \tilde{H}. \quad (3.7)$$

Theorem 3.3. *Let $A \in \mathcal{SA}_+(H_0)$ and $H_1 \in \mathcal{C}_0(A)$. Then for all $B \in \mathcal{C}_0(A)$, it follows that $D_{\text{fin}}(A) \subset D(W(-t)BW(t))$, and the operator valued function $B(t)$ defined as*

$$D(B(t)) := D_{\text{fin}}(A), \quad B(t)\xi := W(-t)BW(t)\xi, \quad \xi \in D_{\text{fin}}(A), \quad t \in \mathbb{R}, \quad (3.8)$$

is a solution of weak Heisenberg equation:

$$\frac{d}{dt} \langle \eta, B(t)\xi \rangle = \langle (iH)^*\eta, B(t)\xi \rangle - \langle B(t)^*\eta, iH\xi \rangle, \quad \xi, \eta \in D(H_0) \cap D_{\text{fin}}(A). \quad (3.9)$$

In what follows, we denote the closure of $U(t, t')$ by the same symbol. Put

$$D' := D_{\text{fin}}(A) \cap D(H_0).$$

We remark that D' is dense in $\mathcal{H}$ under the assumption that the self-adjoint operators H_0 and A strongly commute. This can be seen as follows. Let $\psi \in D(H_0)$ and, for $n \in \mathbb{N}$,

$$\psi_n := E_A([0, n])\psi.$$

Then $\psi_n \in D_{\text{fin}}(A)$. Since H_0 and A strongly commute, $\psi_n \in D(H_0)$. Hence $\psi_n \in D'$. It is clear that $\psi_n \rightarrow \psi$ as n tends to infinity. Thus D' is dense in $D(H_0)$ and then also in $\mathcal{H}$.

Proof of Theorem 3.1. For all $\eta \in D(H_0)$,

$$\begin{aligned} \frac{d}{dt} \langle \eta, W(t)\xi \rangle &= \frac{d}{dt} \langle e^{itH_0}\eta, U(t, 0)\xi \rangle \\ &= \langle iH_0\eta, W(t)\xi \rangle + \langle \eta, -iH_1W(t)\xi \rangle. \end{aligned} \quad (3.10)$$

By Theorem 2.3 (ii), $W(t)$ can be rewritten as $U(0, -t)e^{-itH_0}$. Since, for all $\xi \in D'$, the functions $e^{-itH_0}\xi$ and $U(0, -t)\xi$ are strongly differentiable and since $H_0D' \subset D_{\text{fin}}(A)$, it follows that the function $W(t)\xi$ is also strongly differentiable and the derivative becomes

$$\begin{aligned} \frac{d}{dt} W(t)\xi &= U(0, -t)(-iH_1(-t) - iH_0)e^{-itH_0}\xi \\ &= W(t)(-iH)\xi. \end{aligned} \quad (3.11)$$

hence by (3.10) and (3.11), we have

$$\langle iH_0\eta, W(t)\xi \rangle + \langle \eta, -iH_1W(t)\xi \rangle = \langle \eta, W(t)(-iH)\xi \rangle, \quad \eta \in D(H_0), \quad (3.12)$$

which implies that $W(t)\xi \in D(H_0)$

$$-iW(t)H\xi = -iHW(t)\xi \quad (3.13)$$

therefore we obtain (3.3). $\square$

Proof of Theorem 3.2. By Theorems 2.1 and 2.4, for all $t \in \mathbb{R}$, $W(t)$ is unitary with $W(0) = I$ and strongly continuous in $t \in \mathbb{R}$. By Theorem 2.3 (ii), we have

$$W(t)W(s) = W(t+s), \quad s, t \in \mathbb{R}.$$

Thus $\{W(t)\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group. hence by Stone's theorem, the first statement of the theorem holds.

By (3.4), we have for all $t \in \mathbb{R}$

$$\overline{U(t, 0)} = e^{itH_0}e^{-it\tilde{H}}.$$

By this equation and Theorem 2.3 (2.53), and Theorem 2.4 (i), we obtain (3.5).

It follows from Theorem 3.1 that $D' = D_{\text{fin}}(A) \cap D(H_0) \subset D(\tilde{H})$ and $H\xi = \tilde{H}\xi$, $\xi \in D'$. Hence (3.6) follows.

If H is essentially self-adjoint on the subspace D' , then one finds

$$\overline{H} \subset \tilde{H}.$$

But since both $\overline{H}$ and $\tilde{H}$ are self-adjoint, we have the equality. $\square$

Next, we prepare a lemma to prove Theorem 3.3.

Lemma 3.1. *Let $A \in \mathcal{SA}_+(H_0)$, $H_1, B \in \mathcal{C}_0(A)$. Then the followings hold.*

(i) *For all $\xi \in D'$, the function $W(t)^*\xi$ is strongly differentiable and satisfies*

$$\frac{d}{dt}W(t)^*\xi = iH^*W(t)^*\xi = iW(t)^*H^*\xi. \quad (3.14)$$

(ii) $D_{\text{fin}}(A) \subset D(W(-t)BW(t))$.

(iii) $D_{\text{fin}}(A) \subset D(W(t)^*B^*W(-t)^*)$ and

$$B(t)^*\xi = W(t)^*B^*W(-t)^*\xi, \quad \xi \in D_{\text{fin}}(A), \quad (3.15)$$

hold.

(iv) *For all $\xi \in D'$, the function $BW(t)\xi$ is strongly differentiable and satisfies*

$$\frac{d}{dt}BW(t)\xi = -iBW(t)H\xi. \quad (3.16)$$

Proof. (i) Note that $W(t)^*$ can be rewritten as

$$W(t)^* = U(t, 0)^*e^{itH_0} = e^{itH_0}U(0, -t)^* \quad (3.17)$$

since e^{itH_0} is unitary. By using Theorem 2.1, for all $\eta \in D(H)$ and $\xi \in D'$, we have

$$\frac{d}{dt}\langle \eta, W(t)^*\xi \rangle = \langle -iH\eta, W(t)^*\xi \rangle. \quad (3.18)$$

On the other hand, we can see that $W(t)^*\xi$ is strongly differentiable and the derivative becomes

$$\frac{d}{dt}W(t)^*\xi = U(t, 0)^*(iH_1(t)^* + iH_0)e^{itH_0}\xi = iW(t)^*H^*\xi, \quad (3.19)$$

in the same way as (3.11). Hence by (3.18) and (3.19), we have

$$\langle \eta, iW(t)^*H^*\xi \rangle = \langle -iH\eta, W(t)^*\xi \rangle, \quad (3.20)$$

which implies that $W(t)^*\xi \in D(H^*)$ and $iW(t)^*H^*\xi = iH^*W(t)^*\xi$. Therefore we obtain (3.14).

(ii) First, we show that $W(t)\xi \in D(B)$ for each $\xi \in D_{\text{fin}}(A)$. By using Lemma 2.3, one finds

$$\begin{aligned} & \|Be^{-itH_0}S_n(t, 0)\xi\| \\ & \leq \sum_{j=0}^n \|Be^{-itH_0}U_j(t, 0)\xi\| \\ & \leq \sum_{j=0}^{\infty} C_0(L_\xi + jb_0 + 1)^{1/2} \frac{|t|^j}{j!} C^j(L_\xi + (j-1)b + 1)^{1/2} \cdots (L_\xi + 1)^{1/2} \|\xi\| \\ & < \infty, \end{aligned} \quad (3.21)$$

where the convergence is uniform in t on any compact interval. Hence it follows that $W(t)\xi \in D(B)$ and that

$$BW(t)\xi = \sum_{n=0}^{\infty} Be^{-itH_0}U_n(t, 0)\xi = \lim_{n \rightarrow \infty} Be^{-itH_0}S_n(t, 0)\xi \quad (3.22)$$

from the closedness of B . Next, we show that $BW(t)\xi \in D(W(-t))$. Since $W(-t)$ is closed, it is sufficient to prove that the sequence

$$W(-t)Be^{-itH_0}S_n(t, 0)\xi$$

converges. But, this follows because

$$\begin{aligned} & \|W(-t)Be^{-itH_0}S_n(t, 0)\xi\| \\ &= \|e^{itH_0}U(-t, 0)Be^{-itH_0}S_n(t, 0)\xi\| \\ &\leq \sum_{m,j=0}^{\infty} \|e^{itH_0}U_m(-t, 0)Be^{-itH_0}U_j(t, 0)\xi\| \\ &\leq \sum_{m,j=0}^{\infty} \frac{|t|^m}{m!} C^m (L_\xi + (m+j-1)b + b_0 + 1)^{1/2} \cdots (L_\xi + jb + b_0 + 1)^{1/2} \\ &\quad \times C_0 (L_\xi + jb + 1)^{1/2} \frac{|t|^j}{j!} C^j (L_\xi + (j-1)b + 1)^{1/2} \cdots (L_\xi + 1)^{1/2} \|\xi\| \\ &\leq \sum_{N=0}^{\infty} \frac{(2|t|)^N}{N!} C^N (L_\xi + (N-1)b + b_0 + 1)^{1/2} \cdots (L_\xi + b_0 + 1)^{1/2} C_0 (L_\xi + Nb + 1)^{1/2} \|\xi\| \\ &< \infty. \end{aligned} \tag{3.23}$$

hence it follows that $BW(t)\xi \in D(W(-t))$ and

$$W(-t)BW(t)\xi = \sum_{m,j=0}^{\infty} e^{itH_0}U_m(-t, 0)Be^{-itH_0}U_j(t, 0)\xi, \tag{3.24}$$

where the right hand side converges absolutely. This means that $D \subset D(W(-t)BW(t))$.

- (iii) By using (3.17) and Theorem 2.1 (2.7), we get the desired conclusion in the same manner as in the proof of (ii), since $B(t)^* \supset W(-t)^*B^*W(t)^*$ in general.
- (iv) Using (2.39), we have

$$\frac{d}{dt}BS_n(0, -t)e^{-itH_0}\xi = -iBS_{n-1}(0, -t)e^{-itH_0}H\xi. \tag{3.25}$$

From the estimation (3.21), one finds that $\{(d/dt)BS_n(0, -t)e^{-itH_0}\xi\}_n$ is locally uniformly Cauchy for all $\xi \in D'$. Hence the locally uniform limit $\lim_{n \rightarrow \infty} (d/dt)BS_n(0, -t)e^{-itH_0}\xi = -i \lim_{n \rightarrow \infty} BS_{n-1}(0, -t)e^{-itH_0}H\xi$ exists. Due to the fact that B is closed, this implies $W(t)H\xi \in D(B)$ and

$$\frac{d}{dt}BS_n(0, -t)e^{-itH_0}\xi \rightarrow -iBW(t)H\xi, \quad (n \rightarrow \infty), \tag{3.26}$$

uniformly in $t \in \mathbb{R}$. On the other hand, we have the convergence (3.22). Thus the assertion follows. □

Proof of Theorem 3.3. By Lemma 3.1, for each $\xi, \eta \in D'$, we know that the functions $W(-t)^*\eta$ and $BW(t)\xi$ are strongly differentiable. Hence we have

$$\begin{aligned} \frac{d}{dt} \langle \eta, B(t)\xi \rangle &= \frac{d}{dt} \langle W(-t)^*\eta, BW(t)\xi \rangle \\ &= \langle -iW(-t)^*H^*\eta, BW(t)\xi \rangle + \langle W(-t)^*\eta, -iBW(t)H\xi \rangle \\ &= \langle (iH)^*\eta, W(-t)BW(t)\xi \rangle - \langle W(t)^*B^*W(-t)^*\eta, iH\xi \rangle. \end{aligned} \tag{3.27}$$

therefore by (3.15), we obtain (3.9). □

3.2 N -derivatives and Taylor expansion

In this subsection, we develop a general theory for N -times strong differentiability of the operator $B(t) := W(-t)BW(t)$ on a subspace.

Lemma 3.2. *Let $B \in \mathcal{C}_0$. Then the mapping*

$$B(t)\xi = W(-t)BW(t)\xi \quad (3.28)$$

is strongly continuous in $t \in \mathbb{R}$ for all $\xi \in D_{\text{fin}}(A)$.

Proof. By Lemma 2.3 and the assumption that B is in $\mathcal{C}_0$ -class, it is straightforward to check the mapping

$$t \mapsto U_n(0, t)e^{itH_0}Be^{-itH_0}U_m(t, 0)\xi, \quad \xi \in D_{\text{fin}}(A) \quad (3.29)$$

is strongly continuous. Since $W(-t)BW(t)$ can be expanded in a series converging absolutely and locally uniformly in $t \in \mathbb{R}$:

$$W(-t)BW(t)\xi = \sum_{n,m=0}^{\infty} U_n(0, t)e^{itH_0}Be^{-itH_0}U_m(t, 0)\xi, \quad (3.30)$$

the limit

$$t \mapsto W(-t)BW(t)\xi, \quad \xi \in D_{\text{fin}}(A) \quad (3.31)$$

is also strongly continuous. $\square$

Definition 3.1. *Let A and B be densely defined linear operators on a Hilbert space $\mathcal{H}$, and $\mathcal{D}$ be a subspace of $\mathcal{H}$. Suppose that $\mathcal{D} \subset D(A) \cap D(B) \cap D(A^*) \cap D(B^*)$, and there exists a linear operator C such that $D(C) = \mathcal{D}$ and*

$$\langle A^*\psi, B\phi \rangle - \langle B^*\psi, A\phi \rangle = \langle \psi, C\phi \rangle, \quad (3.32)$$

for all $\psi, \phi \in \mathcal{D}$. Then we say that the operators A and B have a weak commutator C on $\mathcal{D}$ and write

$$C = [A, B]_{\text{w}}^{\mathcal{D}}. \quad (3.33)$$

For two densely defined linear operators S, T on a Hilbert space $\mathcal{H}$, and a subspace $\mathcal{D} \subset \mathcal{H}$, we inductively define $\text{w-ad}_S^n(T)$ ($n = 0, 1, 2, \dots$) by

$$\text{w-ad}_S^0(T) := T, \quad (3.34)$$

$$\text{w-ad}_S^n(T) := [S, \text{w-ad}_S^{n-1}(T)]_{\text{w}}^{\mathcal{D}} \quad n \geq 1, \quad (3.35)$$

if these exist.

Definition 3.2. *We say an operator B is in $\mathcal{C}_1(A)$ -class if it satisfies*

(I) *B is in $\mathcal{C}_0(A)$ -class.*

(II) *iH and B have a weak commutator on $D' = D_{\text{fin}} \cap D(H_0)$, and the weak commutator $\text{w-ad}_{iH}^1(B)$ is in $\mathcal{C}_0(A)$ -class.*

The strong Heisenberg equation is satisfied if B is in $\mathcal{C}_1(A)$ -class:

Theorem 3.4. For $B \in \mathcal{C}_1(A)$ and $\xi \in D'$, the mapping

$$\mathbb{R} \ni t \mapsto B(t) = W(-t)BW(t)\xi \in \mathcal{H} \quad (3.36)$$

is strongly continuously differentiable in $t \in \mathbb{R}$ and satisfies the Heisenberg equation of motion:

$$\frac{d}{dt}B(t)\xi = W(-t)\text{ad}(B)W(t)\xi. \quad (3.37)$$

Proof. Since $B \in \mathcal{C}_0(A)$, we can use Theorem 3.3, and obtain

$$\begin{aligned} \frac{d}{dt} \langle \eta, B(t)\xi \rangle &= \langle (iH)^*\eta, B(t)\xi \rangle - \langle B(t)^*\eta, iH\xi \rangle \\ &= \langle (iH)^*W(-t)^*\eta, BW(t)\xi \rangle - \langle B^*W(-t)^*\eta, iHW(t)\xi \rangle \\ &= \langle W(-t)^*\eta, \text{w-ad}_{iH}^1(B)W(t)\xi \rangle. \end{aligned} \quad (3.38)$$

Since $\text{w-ad}_{iH}^1(B)$ is in $\mathcal{C}_0(A)$, $\text{w-ad}_{iH}^1(B)W(t)\xi \in D(W(-t))$ and one gets

$$\frac{d}{dt} \langle \eta, B(t)\xi \rangle = \langle \eta, W(-t)\text{w-ad}_{iH}^1(B)W(t)\xi \rangle. \quad (3.39)$$

hence we have

$$\begin{aligned} \langle \eta, B(t)\xi \rangle &= \langle \eta, \text{w-ad}_{iH}^1(B)\xi \rangle + \int_0^t ds \langle \eta, W(-s)\text{w-ad}_{iH}^1(B)W(s)\xi \rangle \\ &= \langle \eta, \text{w-ad}_{iH}^1(B)\xi \rangle + \left\langle \eta, \int_0^t ds W(-s)\text{w-ad}_{iH}^1(B)W(s)\xi \right\rangle, \end{aligned} \quad (3.40)$$

and thus

$$B(t)\xi = \text{w-ad}_{iH}^1(B)\xi + \int_0^t W(-s)\text{w-ad}_{iH}^1(B)W(s)\xi. \quad (3.41)$$

This equation shows that $B(t)\xi$ is continuously strongly differentiable and

$$\frac{d}{dt}B(t)\xi = W(-t)\text{w-ad}_{iH}^1(B)W(t)\xi, \quad \xi \in D'. \quad (3.42)$$

This completes the proof. $\square$

One of the merit of the present formulation of the strong Heisenberg equation is that it is easy to extend for n -times differentiability.

Definition 3.3. For $n = 0, 1, 2, \dots$, we inductively define $\mathcal{C}_n(A)$ -class as follows. We say that an operator B is in $\mathcal{C}_n(A)$ -class if B is in $\mathcal{C}_{n-1}(A)$ -class and $\text{w-ad}_{iH}^1(B)$ is in $\mathcal{C}_{n-1}(A)$ -class.

It is clear that $\mathcal{C}_n(A) \subset \mathcal{C}_{n-1}(A)$ for $n = 1, 2, \dots$, and that if $B \in \mathcal{C}_n(A)$, then $\text{w-ad}_{iH}^1(B) \in \mathcal{C}_{n-1}(A)$, $\text{w-ad}_{iH}^2(B) \in \mathcal{C}_{n-2}(A), \dots, \text{w-ad}_{iH}^n(B) \in \mathcal{C}_0(A)$.

An operator B is said to be in $\mathcal{C}_\infty(A)$ -class if B is in $\mathcal{C}_n(A)$ for all $n \in \mathbb{N}$. Namely,

$$\mathcal{C}_\infty(A) := \bigcap_{n=0}^{\infty} \mathcal{C}_n(A). \quad (3.43)$$

The following theorem is important and useful but the proof is almost trivial by induction.

Theorem 3.5. *Let B be in $\mathcal{C}_n(A)$ -class. Then for all $\xi \in D'$, $B(t)\xi$ is n -times strongly continuously differentiable in $t \in \mathbb{R}$ and*

$$\frac{d^k}{dt^k} B(t)\xi = W(-t) \text{w-ad}_{iH}^k(B) W(t)\xi, \quad (3.44)$$

for all $k = 0, 1, 2, \dots, n$. In particular, if $B \in \mathcal{C}_\infty(A)$, then (3.44) holds for all $k \geq 0$.

From Theorem 3.5, we immediately have

Theorem 3.6. *Let $B \in \mathcal{C}_n(A)$ and $\xi \in D'$. Then there is a $\theta \in (0, 1)$ such that*

$$B(t)\xi = \sum_{k=0}^{n-1} \frac{t^k}{k!} \text{w-ad}_{iH}^k(B)\xi + \frac{t^n}{n!} W(-\theta t) \text{w-ad}_{iH}^n(B) W(\theta t)\xi. \quad (3.45)$$

To obtain a Taylor series expansion for $B(t)\xi$ for $\xi \in D'$, we need one more concept.

Definition 3.4. *We say that an operator B is in $\mathcal{C}_\omega(A)$ -class if*

- (i) $B \in \mathcal{C}_\infty(A)$.
- (ii) There exist constants $R_0, R_1 \geq 0$ such that for all $n = 0, 1, 2, \dots$,

$$\|\text{w-ad}_{iH}^n(B)(A+1)^{1/2}\| \leq R_0 R_1^n. \quad (3.46)$$

- (iii) There exists a constant $b > 0$ such that for all $n \geq 0$ and $L \geq 0$, $\text{w-ad}_{iH}^n(B)$ maps V_L into V_{L+b} .

We finally arrive at the following result.

Theorem 3.7. *Suppose that $B \in \mathcal{C}_\omega(A)$. Then for each $\xi \in D'$, $B(t)\xi$ has the norm-converging power series expansion formula*

$$B(t)\xi = \sum_{n=0}^{\infty} \frac{t^n}{n!} \text{w-ad}_{iH}^n(B)\xi, \quad t \in \mathbb{R}. \quad (3.47)$$

Proof. By Theorem 3.5, all we have to show is that the norm of the reminder term in (3.45) vanishes.

Since B is in $\mathcal{C}_\omega$ -class, there is a constant $b' > 0$, which is independent of n , such that $\xi \in V_L$ implies $\text{w-ad}_{iH}^n(B)\xi \in V_{B+b'}$. Choose $b > 0$ such that $\xi \in V_L$ implies $H_1\xi \in V_{B+b}$ and $H_1^*\xi \in V_{B+b}$. Put

$$C := \|H_1(A+1)^{-1/2}\|. \quad (3.48)$$

Let $\xi \in V_L$ for some. Then using and Lemma 2.3, for all $L \geq 0$ and $\xi \in V_L$, we have

$$\begin{aligned} & \left\| \frac{t^n}{n!} W(-\theta t) \text{w-ad}_{iH}^n(B) W(\theta t)\xi \right\| \\ & \leq \frac{|t|^n}{n!} \sum_{k,l=0}^{\infty} \|U_k(0, \theta t) e^{i\theta t H_0} \text{w-ad}_{iH}^n(B) e^{i\theta t H_0} U_l(\theta t, 0)\xi\| \\ & \leq \frac{|t|^n}{n!} \sum_{k,l=0}^{\infty} \frac{C^{k+l} R_0 R_1^n |\theta t|^{k+l}}{k!l!} (L + (k+l-1)b + b' + 1)^{1/2} \times \dots \\ & \quad \times (L + (l-1)b + b' + 1)^{1/2} (L + (l-1)b + 1)^{1/2} \dots (L+1)^{1/2} \|\xi\| \\ & \rightarrow 0. \end{aligned}$$

therefore we obtain the desired result. □

3.3 Application to QED in the Lorenz gauge

In this subsection, we apply the general theory obtained in the preceding sections to a mathematical model of QED, quantized in the Lorenz gauge. As we emphasized in Introduction, our construction of $U(t, t')$ does not require that H be self-adjoint, and Theorems 3.1, and 3.3 are independent of the self-adjointness of H . This method is particularly valid for analyzing Lorenz-gauge QED, whose Hamiltonian is not self-adjoint and not even normal.

QED describes a system in which the quantum electromagnetic field and the quantum Dirac field are minimally interacting. It is well known that, in the Coulomb gauge, one can employ a state space constructed by usual Fock spaces, which equip a positive definite metric, at the cost of the Lorentz covariance. In this formulation, the Hamiltonian H is self-adjoint [61] under some regularizations, hence there clearly exists the time evolution operator e^{-itH} such that $\xi(t) = e^{-itH}\xi$ and $B(t) = e^{itH}Be^{-itH}$ are the unique solutions of the initial value problems (1.16) and (1.17), respectively. In contrast to the case of Coulomb gauge, in the Lorenz gauge, the Hamiltonian is neither self-adjoint nor normal in consequence of the inevitability of introducing an indefinite metric [60], and hence the time evolution operator e^{-itH} does not necessarily exist. As a result, even the existence of solutions of (1.16) and (1.17) becomes a highly nontrivial problem. It does not seem to be easy to apply the general theory of evolution operators through hard analyses of the resolvent of the Hamiltonian of Lorenz-gauge QED. But our general theory works well to construct an appropriate time evolution as we will see in the present section.

3.3.1 Gauge fields

We introduce the photon field quantized in the Lorenz gauge.

We adopt as the one-photon Hilbert space

$$\mathcal{H}_{\text{ph}}^{(\text{L})} := L^2(\mathbb{R}_{\mathbf{k}}^3; \mathbb{C}^4). \quad (3.49)$$

The above $\mathbb{R}_{\mathbf{k}}^3 := \{\mathbf{k} = (k^1, k^2, k^3) \mid k^j \in \mathbb{R}, j = 1, 2, 3\}$ physically represents the momentum space of photons. If there is no danger of confusion, we omit the subscript $\mathbf{k}$ in $\mathbb{R}_{\mathbf{k}}^3$. $\mathcal{H}_{\text{ph}}^{(\text{L})}$ can be identified as $\oplus^4 L^2(\mathbb{R}_{\mathbf{k}}^3)$. We freely use this identification. The Hilbert space for the quantized photon field in the Lorenz gauge is given by

$$\mathcal{F}_{\text{ph}}^{(\text{L})} := \mathcal{F}_{\text{b}}(\mathcal{H}_{\text{ph}}^{(\text{L})}), \quad (3.50)$$

the Boson Fock space over $\mathcal{H}_{\text{ph}}^{(\text{L})}$. For the definition of the Boson Fock space, see, Appendix A.

The main obstacle for the mathematical treatment of the Lorentz covariant gauge is that in order to realize the canonical commutation relations in a Lorentz covariant manner, we have to employ an indefinite metric vector space as a space of state vectors. In the above Hilbert space $\mathcal{H}_{\text{ph}}^{(\text{L})}$ for photon fields, the ordinary positive definite inner product

$$\langle f, h \rangle := \sum_{\mu=0}^3 \int_{\mathbb{R}^3} f^\mu(\mathbf{k})^* h^\mu(\mathbf{k}) d\mathbf{x} \quad (3.51)$$

for $f(\mathbf{k}) = (f^0(\mathbf{k}), \dots, f^3(\mathbf{k}))$ and $h(\mathbf{k}) = (h^0(\mathbf{k}), \dots, h^3(\mathbf{k}))$ in $\mathcal{H}_{\text{ph}}^{(\text{L})}$, plays no physical role and just define the Hilbert space topology on the vector space $\mathcal{H}_{\text{ph}}^{(\text{L})}$. The “physical” inner product is given by the mapping

$$\langle f|h \rangle := - \int_{\mathbb{R}^3} g_{\mu\nu} f^\mu(\mathbf{k})^* h^\nu(\mathbf{k}) d\mathbf{x}, \quad (3.52)$$

with the metric tensor

$$g = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (3.53)$$

where the summation over $\mu, \nu = 0, 1, 2, 3$ is understood (In what follows, we omit the summation symbol whenever the summation is taken with respect to one upper and one lower Lorentz indices). Note that $\langle f|f \rangle$ may become negative and the mapping naturally defines indefinite metric on $\mathcal{F}_{\text{ph}}^{(\text{L})}$, which is denoted by the same symbol $\langle \cdot | \cdot \rangle$. Again we emphasize that the former ordinary positive definite inner product has nothing to do with the physical result. For instance, the probability amplitude that a state $\Psi \in \mathcal{F}_{\text{ph}}^{(\text{L})}$ is observed in a state $\Phi \in \mathcal{H}_{\text{ph}}^{(\text{L})}$ is not given by $\langle \Psi, \Phi \rangle$ but by $\langle \Psi | \Phi \rangle$. Therefore the self-adjointness with respect to the indefinite metric $\langle \cdot | \cdot \rangle$ should be required for the Hamiltonian, in stead of the ordinary self-adjointness.

First of all, we have to make precise the statement that a linear operator is self-adjoint with respect to the indefinite metric $\langle \cdot | \cdot \rangle$. The 4×4 matrix g naturally defines the unitary operator acting on $\mathcal{H}_{\text{ph}}^{(\text{L})}$, and we denote it by the same symbol g . We define η by the second quantization of $-g$, i.e.,

$$\eta := \Gamma_{\text{b}}(-g) := \bigoplus_{n=0}^{\infty} \otimes^n (-g) : \mathcal{F}_{\text{ph}}^{(\text{L})} \rightarrow \mathcal{F}_{\text{ph}}^{(\text{L})}. \quad (3.54)$$

Then η is unitary and satisfies $\eta^* = \eta$, $\eta^2 = I$. By using η we introduce an indefinite metric on $\mathcal{F}_{\text{ph}}^{(\text{L})}$ by

$$\langle \Psi | \Phi \rangle := \langle \Psi, \eta \Phi \rangle_{\mathcal{F}_{\text{ph}}^{(\text{L})}}, \quad \Psi, \Phi \in \mathcal{F}_{\text{ph}}^{(\text{L})}. \quad (3.55)$$

In order to define the adjoint with respect to indefinite metric (3.55), we introduce the η -adjoint. For a densely defined linear operator T on $\mathcal{F}_{\text{ph}}^{(\text{L})}$, the adjoint operator $T^\dagger$ with respect to the metric $\langle \cdot | \cdot \rangle$ is defined by

$$T^\dagger := \eta T^* \eta. \quad (3.56)$$

then clearly it follows that

$$\langle \Psi | T \Phi \rangle = \langle T^\dagger \Psi | \Phi \rangle, \quad \Psi \in D(T^\dagger), \quad \Phi \in D(T). \quad (3.57)$$

We introduce notions of η -symmetry, η -self-adjointness and η -unitarity [32, 60] below.

Definition 3.5. (i) A densely defined linear operator T is η -symmetric if $T \subset T^\dagger$.

(ii) A densely defined linear operator T is η -self-adjoint if $T^\dagger = T$.

(iii) A densely defined linear operator T is essentially η -self-adjoint if $\overline{T}$ is η -self-adjoint.

(iv) A densely defined linear operator T is η -unitary if T is injective and $T^\dagger = T^{-1}$.

Lemma 3.3. (i) T is η -symmetric if and only if ηT is symmetric.

(ii) T is η -self-adjoint if and only if ηT is self-adjoint.

(iii) T is essentially η -self-adjoint if and only if ηT is essentially self-adjoint.

(iv) If T is η -symmetric then T is closable.

(v) Let T be η -self-adjoint and ηT is essentially self-adjoint on a subspace D , Then D is a core of T .

Proof. See [32]. □

One-photon Hamiltonian in $\mathcal{H}_{\text{ph}}^{(\text{L})}$ is the multiplication operator by the function $\omega(\mathbf{k}) := |\mathbf{k}|$ ($\mathbf{k} \in \mathbb{R}^3$). We also denote by ω the matrix valued function

$$\mathbf{k} \mapsto \begin{pmatrix} \omega(\mathbf{k}) & 0 & 0 & 0 \\ 0 & \omega(\mathbf{k}) & 0 & 0 \\ 0 & 0 & \omega(\mathbf{k}) & 0 \\ 0 & 0 & 0 & \omega(\mathbf{k}) \end{pmatrix} \quad (3.58)$$

and the multiplication operator by it. Then the free Hamiltonian of the quantum electromagnetic field is given by

$$H_{\text{ph}}^{(\text{L})} := d\Gamma_{\text{b}}(\omega), \quad (3.59)$$

the second quantization of ω . For the definition of the second quantization operator, see, Appendix A.

Note that the free Hamiltonian $H_{\text{ph}}^{(\text{L})}$ is self-adjoint and η -self-adjoint.

We denote by $c(F)$ ($F \in \mathcal{H}_{\text{ph}}$) the annihilation operator on $\mathcal{F}_{\text{ph}}^{(\text{L})}$. For each $f \in L^2(\mathbb{R}_{\mathbf{k}}^3)$, we introduce the components of $c(\cdot)$ with *upper* indices by

$$c^0(f) := c(f, 0, 0, 0), \quad c^1(f) := c(0, f, 0, 0), \quad (3.60)$$

$$c^2(f) := c(0, 0, f, 0), \quad c^3(f) := c(0, 0, 0, f). \quad (3.61)$$

then the operator equalities

$$c_0(f)^\dagger = -c_0(f)^*, \quad (3.62)$$

$$c_j(f)^\dagger = c_j(f)^*, \quad j = 1, 2, 3, \quad (3.63)$$

hold.

Let us introduce photon polarization vectors $\{e_\lambda\}_{\lambda=0,1,2,3}$. Photon polarization vectors are $\mathbb{R}_{\mathbf{k}}^4$ -valued measurable functions defined on $\mathbb{R}^4$, $e_\lambda(\cdot)$ ($\lambda = 0, 1, 2, 3$), such that, for all $\mathbf{k} \in M_0 := \mathbb{R}^3 \setminus \{(0, 0, k^3) \mid k^3 \in \mathbb{R}\}$,

$$e_\lambda(\mathbf{k}) \cdot e_{\lambda'}(\mathbf{k}) = g_{\lambda\lambda'}, \quad e_\lambda(\mathbf{k}) \cdot \mathbf{k} = 0, \quad \lambda = 1, 2, \quad (3.64)$$

where the above $\cdot$ means the Minkowski inner product defined by

$$e_\lambda(\mathbf{k}) \cdot e_{\lambda'}(\mathbf{k}) = g_{\mu\nu} e_\lambda^\mu(\mathbf{k}) e_{\lambda'}^\nu(\mathbf{k}), \quad e_\lambda(\mathbf{k}) \cdot \mathbf{k} = g_{\mu\nu} e_\lambda^\mu(\mathbf{k}) k^\nu, \quad \mathbf{k} = (|\mathbf{k}|, \mathbf{k}) \in \mathbb{R}^4,$$

with e_λ^μ being the μ -th component of e_λ with respect to the standard basis in $\mathbb{R}^4$. Note that such vector valued functions can be chosen so that they are continuous on M_0 , for instance, we may choose

$$e_0(\mathbf{k}) = (1, \mathbf{0}), \quad e_1(\mathbf{k}) = (0, \mathbf{e}_1(\mathbf{k})), \quad e_2(\mathbf{k}) = (0, \mathbf{e}_2(\mathbf{k})), \quad e_3(\mathbf{k}) = (0, \mathbf{k}/|\mathbf{k}|) \quad (3.65)$$

by using $\{\mathbf{e}_r(\mathbf{k})\}_{r=1,2}$ satisfying the relations

$$\mathbf{e}_r(\mathbf{k}) \cdot \mathbf{e}_{r'}(\mathbf{k}) = \delta_{rr'}, \quad \mathbf{e}_r(\mathbf{k}) \cdot \mathbf{k} = 0, \quad r, r' = 1, 2.$$

In this paper, we assume the photon polarization vectors are chosen in this way.

For each $f \in L^2(\mathbb{R}_{\mathbf{k}}^3)$ and $\mu = 0, 1, 2, 3$, we define

$$\begin{aligned} a^\mu(f) &:= c(fe^\mu_0, fe^\mu_1, fe^\mu_2, fe^\mu_3) \\ &= c^\nu(e^\mu_\nu f), \end{aligned} \quad (3.66)$$

and

$$a_\mu(f) := g_{\mu\nu} a^\nu(f). \quad (3.67)$$

We often use the notation $a_\mu^\dagger(f) = a_\mu(f)^\dagger$.

The operators $a_\mu(f)$ and $a_\mu^\dagger(f)$ are closed, and satisfy the Lorentz covariant canonical commutation relations:

$$\begin{aligned} [a_\mu(f), a_\nu^\dagger(g)] &= -g_{\mu\nu} \langle f, g \rangle_{L^2(\mathbb{R}^3)}, \\ [a_\mu(f), a_\nu(g)] &= [a_\mu^\dagger(f), a_\nu^\dagger(g)] = 0, \end{aligned}$$

on $\mathcal{F}_{b,0}(\mathcal{H}_{\text{ph}})$.

For all $f \in L^2(\mathbb{R}_{\mathbf{x}}^3)$ satisfying $\widehat{f}/\sqrt{\omega} \in L^2(\mathbb{R}_{\mathbf{k}}^3)$, we set

$$A_\mu(0, f) := a_\mu\left(\frac{\widehat{f^*}}{\sqrt{2\omega}}\right) + a_\mu^\dagger\left(\frac{\widehat{f}}{\sqrt{2\omega}}\right), \quad (3.68)$$

where $\widehat{f}$ denotes the Fourier transform of f , and f^* denotes the complex conjugate of f . The functional $\mathcal{S}(\mathbb{R}_{\mathbf{x}}^3) \ni f \mapsto A_\mu(0, f)$ gives an operator-valued distribution (Cf. [2] Definition 7-1) acting on $(\mathcal{F}_{\text{ph}}^{(\text{L})}, \mathcal{F}_{b,0}(\mathcal{H}_{\text{ph}}^{(\text{L})}))$ and it is called the quantized photon field at time $t = 0$. Now, fix $\chi_{\text{ph}} \in L^2(\mathbb{R}_{\mathbf{x}}^3)$ such that it is real and satisfies $\widehat{\chi_{\text{ph}}}/\sqrt{\omega} \in L^2(\mathbb{R}_{\mathbf{k}}^3)$. We set

$$A_\mu(0, \mathbf{x}) := A_\mu(0, \chi_{\text{ph}}^{\mathbf{x}}), \quad (3.69)$$

$$\chi_{\text{ph}}^{\mathbf{x}}(\mathbf{y}) := \chi_{\text{ph}}(\mathbf{y} - \mathbf{x}), \quad \mathbf{y} \in \mathbb{R}^3. \quad (3.70)$$

$A_\mu(0, \mathbf{x})$ is called the point-like quantized photon field with momentum cutoff $\widehat{\chi_{\text{ph}}}$ at time $t = 0$. As will be seen later, for real-valued f , the closures of $A_\mu(0, f)$, $\mu = 0, 1, 2, 3$, are η -self-adjoint but not even normal.

We assume the following condition.

Hypothesis 3.1. $\widehat{\chi_{\text{ph}}}/\omega \in L^2(\mathbb{R}_{\mathbf{k}}^3)$.

Let $N_{\text{b}} := d\Gamma_{\text{b}}(I_{\mathcal{H}_{\text{ph}}^{(\text{L})}})$ be the photon number operator.

Lemma 3.4. (i) For all $f \in L^2(\mathbb{R}_{\mathbf{k}}^3)$ and $\Psi \in D(N_{\text{b}}^{1/2})$,

$$\|a_\mu(f)\Psi\|_{\mathcal{F}_{\text{ph}}^{(\text{L})}} \leq 4\|f\|_{L^2(\mathbb{R}_{\mathbf{k}}^3)} \left\| N_{\text{b}}^{1/2}\Psi \right\|_{\mathcal{F}_{\text{ph}}^{(\text{L})}}, \quad (3.71)$$

$$\|a_\mu^\dagger(f)\Psi\|_{\mathcal{F}_{\text{ph}}^{(\text{L})}} \leq 4\|f\|_{L^2(\mathbb{R}_{\mathbf{k}}^3)} \left\| N_{\text{b}}^{1/2}\Psi \right\|_{\mathcal{F}_{\text{ph}}^{(\text{L})}} + 4\|f\|_{L^2(\mathbb{R}_{\mathbf{k}}^3)} \|\Psi\|_{\mathcal{F}_{\text{ph}}^{(\text{L})}}. \quad (3.72)$$

(ii) For all $f \in D(\omega^{-1/2})$ and $\Psi \in D(H_{\text{ph}}^{1/2})$,

$$\|a_\mu(f)\Psi\|_{\mathcal{F}_{\text{ph}}} \leq 4\|f/\sqrt{\omega}\|_{L^2(\mathbb{R}_{\mathbf{k}}^3)} \left\| H_{\text{ph}}^{(\text{L})1/2}\Psi \right\|_{\mathcal{F}_{\text{ph}}^{(\text{L})}}, \quad (3.73)$$

$$\|a_\mu^\dagger(f)\Psi\|_{\mathcal{F}_{\text{ph}}} \leq 4\|f/\sqrt{\omega}\|_{L^2(\mathbb{R}_{\mathbf{k}}^3)} \left\| H_{\text{ph}}^{(\text{L})1/2}\Psi \right\|_{\mathcal{F}_{\text{ph}}^{(\text{L})}} + 4\|f\|_{L^2(\mathbb{R}_{\mathbf{k}}^3)} \|\Psi\|_{\mathcal{F}_{\text{ph}}^{(\text{L})}}. \quad (3.74)$$

Proof. These estimates (i) and (ii) are easily proved by applying Lemma A.4, and we omit the proof. $\square$

Lemma 3.5. (i) For all $\mu = 0, 1, 2, 3$, $f \in L^2(\mathbb{R}_x^3)$ satisfying $\widehat{f}/\sqrt{\omega} \in L^2(\mathbb{R}_k^3)$, and $\Psi \in D(N_b^{1/2})$,

$$\|A_\mu(0, f)\Psi\| \leq \left(\sqrt{2}\|\widehat{f}/\sqrt{\omega}\|_{L^2(\mathbb{R}_k^3)} + \frac{1}{\sqrt{2}}\|\widehat{f}\|_{L^2(\mathbb{R}_k^3)} \right) \|(N_b + 1)^{1/2}\Psi\|. \quad (3.75)$$

(ii) For all $\mu = 0, 1, 2, 3$, $f \in L^2(\mathbb{R}_x^3)$ satisfying $\widehat{f}/\omega \in L^2(\mathbb{R}_k^3)$, and $\Psi \in D(H_{\text{ph}}^{(L)1/2})$,

$$\|A_\mu(0, f)\Psi\| \leq \left(\sqrt{2}\|\widehat{f}/\omega\|_{L^2(\mathbb{R}_k^3)} + \frac{1}{\sqrt{2}}\|\widehat{f}/\sqrt{\omega}\|_{L^2(\mathbb{R}_k^3)} \right) \|(H_{\text{ph}}^{(L)} + 1)^{1/2}\Psi\|. \quad (3.76)$$

Proof. This immediately follows from Lemma 3.4. $\square$

Remark 3.1. If the momentum cutoff function $\widehat{\chi}_{\text{ph}}$ is taken to be the characteristic function of the set $\{\mathbf{k} \in \mathbb{R}^3 \mid |\mathbf{k}| \leq \Lambda_0\}$ with $\Lambda_0 \geq 0$, then this satisfies Hypothesis 3.1.

Lemma 3.6. Let $f \in L^2(\mathbb{R}_x^3)$. If $\text{supp } \widehat{f} \subset \{|\mathbf{k}| \leq \Lambda\}$ for some $\Lambda \geq 0$, then for all $\mu = 0, 1, 2, 3$ and $E \geq 0$, $A_\mu(0, f)$ maps $R(E_{H_{\text{ph}}}([0, E]))$ into $R(E_{H_{\text{ph}}}([0, E + \Lambda_f]))$.

Proof. It is easy to see that $(\widehat{f}e_{\mu 0}/\sqrt{\omega}, \widehat{f}e_{\mu 1}/\sqrt{\omega}, \widehat{f}e_{\mu 2}/\sqrt{\omega}, \widehat{f}e_{\mu 3}/\sqrt{\omega}) \in R(E_\omega([0, \Lambda])) \subset \mathcal{H}_{\text{ph}}^{(L)}$. Hence using Lemma A.8, the assertion follows. $\square$

3.3.2 Dirac fields

Next, we define the quantized Dirac field. We adopt as the one-electron Hilbert space

$$\mathcal{H}_{\text{el}} := L^2(\mathbb{R}_p^3; \mathbb{C}^4), \quad (3.77)$$

where $\mathbb{R}_p^3 := \{\mathbf{p} = (p^1, p^2, p^3) \mid p^j \in \mathbb{R}, j = 1, 2, 3\}$ physically represents the momentum space of electrons. The Hilbert space for the quantized Dirac field is given by

$$\mathcal{F}_{\text{el}} := \mathcal{F}_{\text{f}}(\mathcal{H}_{\text{el}}), \quad (3.78)$$

the Fermion Fock space over $\mathcal{H}_{\text{el}}$. For the definition of the Fermion Fock space, see, Appendix A.

We denote the mass of the Dirac particle by $M > 0$. One-electron Hamiltonian in $\mathcal{H}_{\text{el}}$ is the multiplication operator by the function $E_M(\mathbf{p}) := \sqrt{\mathbf{p}^2 + M^2}$ ($\mathbf{p} \in \mathbb{R}^3$). The Hamiltonian of the free quantum Dirac field is given by

$$H_{\text{el}} := d\Gamma_{\text{f}}(E_M), \quad (3.79)$$

the second quantization of E_M . The operator H_{el} is self-adjoint and non-negative.

Let γ^μ ($\mu = 0, 1, 2, 3$) be 4×4 gamma matrices, i.e., γ^0 is hermitian and γ^j ($j = 1, 2, 3$) are anti-hermitian, satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3, \quad (3.80)$$

where $\{X, Y\} := XY + YX$. Let $\alpha^\mu := \gamma^0\gamma^\mu$, $\beta := \gamma^0$, and let $s_1 := \frac{i}{2}\gamma^2\gamma^3$, $s_2 := \frac{i}{2}\gamma^3\gamma^1$, $s_3 := \frac{i}{2}\gamma^1\gamma^2$. Let $u_s(\mathbf{p}) = (u_s^l(\mathbf{p}))_{l=1}^4 \in \mathbb{C}^4$ describe the positive energy part with spin $s = \pm 1/2$ and $v_s(\mathbf{p}) = (v_s^l(\mathbf{p}))_{l=1}^4 \in \mathbb{C}^4$ the negative energy part with spin s , that is,

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta M)u_s(\mathbf{p}) = E_M(\mathbf{p})u_s(\mathbf{p}), \quad (\mathbf{s} \cdot \mathbf{p})u_s(\mathbf{p}) = s|\mathbf{p}|u_s(\mathbf{p}), \quad (3.81)$$

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta M)v_s(\mathbf{p}) = -E_M(\mathbf{p})v_s(\mathbf{p}), \quad (\mathbf{s} \cdot \mathbf{p})v_s(\mathbf{p}) = s|\mathbf{p}|v_s(\mathbf{p}), \quad \mathbf{p} \in \mathbb{R}^3. \quad (3.82)$$

These form an orthogonal base of $\mathbb{C}^4$,

$$u_s(\mathbf{p})^* u_{s'}(\mathbf{p}) = v_s(\mathbf{p})^* v_{s'}(\mathbf{p}) = \delta_{ss'}, \quad u_s(\mathbf{p})^* v_{s'}(\mathbf{p}) = 0, \quad (3.83)$$

and satisfy the completeness,

$$\sum_s (u_s^l(\mathbf{p}) u_s^{l'}(\mathbf{p})^* + v_s^l(\mathbf{p}) v_s^{l'}(\mathbf{p})^*) = \delta_{ll'}.$$

We denote by $B(F)$ ($F \in \mathcal{H}_{\text{el}}$) the annihilation operator on $\mathcal{F}_{\text{el}}$. For each $g \in L^2(\mathbb{R}_{\mathbf{p}}^3)$, we use the notation

$$\begin{aligned} b_{1/2}(g) &:= B(g, 0, 0, 0), & b_{-1/2}(g) &:= B(0, g, 0, 0), \\ d_{1/2}(g) &:= B(0, 0, g, 0), & d_{-1/2}(g) &:= B(0, 0, 0, g), \end{aligned}$$

and $b_s^*(g) := b_s(g)^*$, $d_s^*(g) := d_s(g)^*$, ($s = \pm 1/2$). Then we have the canonical anti-commutation relations:

$$\{b_s(g), b_{s'}^*(g')\} = \{d_s(g), d_{s'}^*(g')\} = \delta_{ss'} \langle g, g' \rangle_{L^2(\mathbb{R}_{\mathbf{p}}^3)}, \quad (3.84)$$

$$\{b_s(g), b_{s'}(g')\} = \{d_s(g), d_{s'}(g')\} = \{b_s(g), d_{s'}^*(g')\} = \{b_s(g), d_{s'}^*(g')\} = 0. \quad (3.85)$$

For all $g \in L^2(\mathbb{R}_{\mathbf{x}}^3)$, we set

$$\psi_l(0, g) := \sum_{s=\pm 1/2} \left(b_s \left(\widehat{g}^* \cdot u_s^{l*} \right) + d_s^* \left(\widehat{g} \cdot \widetilde{v}_s^l \right) \right), \quad (3.86)$$

where $\widetilde{v}_s^l(\mathbf{p}) := v_s^l(-\mathbf{p})$. The functional $\mathcal{S}(\mathbb{R}_{\mathbf{x}}^3) \ni g \mapsto \psi_l(0, g)$ is called the quantized Dirac field at time $t = 0$. Now, fix $\chi_{\text{el}} \in L^2(\mathbb{R}_{\mathbf{x}}^3)$. We set

$$\psi_l(0, \mathbf{x}) := \psi_l(0, \chi_{\text{el}}^{\mathbf{x}}), \quad (3.87)$$

$$\chi_{\text{el}}^{\mathbf{x}}(\mathbf{y}) := \chi_{\text{el}}(\mathbf{y} - \mathbf{x}), \quad \mathbf{y} \in \mathbb{R}^3. \quad (3.88)$$

$\psi_l(\mathbf{x})$ is called the point-like quantized Dirac field with momentum cutoff $\widehat{\chi}_{\text{el}}$ at time $t = 0$. For each $\mathbf{x} \in \mathbb{R}^3$ and $\mu = 0, 1, 2, 3$, we define the current operator $j^\mu(0, \mathbf{x})$ by

$$j^\mu(0, \mathbf{x}) := \sum_{l, l'=1}^4 \psi_l(\mathbf{x})^* \alpha_{ll'}^\mu \psi_{l'}(\mathbf{x}). \quad (3.89)$$

Then $j^\mu(\mathbf{x})$ is bounded and self-adjoint.

Lemma 3.7. *For all $\mu = 0, 1, 2, 3$ and $\mathbf{x} \in \mathbb{R}^3$,*

$$M_{\text{cu}} := \sup_{\mathbf{x} \in \mathbb{R}^3, \mu=0,1,2,3} \| : j^\mu(0, \mathbf{x}) : \| < \infty, \quad (3.90)$$

where the symbol $:O:$ consisting of two colons with an operator O is the normal ordering (see, Definition A.1).

Proof. A simple application of (A.43). □

Lemma 3.8. *Let $g \in L^2(\mathbb{R}_{\mathbf{x}}^3)$. If $\text{supp } \widehat{g} \subset \{|\mathbf{k}| \leq \Lambda\}$ for some $\Lambda \geq 0$, then for all $l = 1, 2, 3, 4$ and $E \geq 0$, $\psi_l(0, g)$ maps $R(E_{H_{\text{el}}}([0, E]))$ into $R(E_{H_{\text{el}}}([0, E + \Lambda_g]))$.*

Proof. Similar to the proof of Lemma 3.6 □

3.3.3 Total Hamiltonian

The Hilbert space of state vectors for QED in Lorenz gauge is taken to be

$$\mathcal{F}_{\text{tot}} := \mathcal{F}_{\text{el}} \otimes \mathcal{F}_{\text{ph}}^{(\text{L})}. \quad (3.91)$$

The free Hamiltonian is

$$H_{\text{fr}} := H_{\text{el}} \otimes I + I \otimes H_{\text{ph}}^{(\text{L})}, \quad (3.92)$$

where the subscript fr in H_{fr} means *free*.

We introduce an indefinite metric on $\mathcal{F}_{\text{tot}}$ by

$$\langle \Psi | \Phi \rangle := \langle \Psi, I \otimes \eta \Phi \rangle_{\mathcal{F}_{\text{tot}}}, \quad \Psi, \Phi \in \mathcal{F}_{\text{tot}}. \quad (3.93)$$

then η -adjointness, η -symmetricity, η -self-adjointness and η -unitarity are defined on $\mathcal{F}_{\text{tot}}$, in the same way as in subsection 3.3.1, by replacing η with $I \otimes \eta$.

We introduce the minimal interaction between the quantized Dirac field and the quantized electromagnetic field. We denote the charge of the Dirac particle by $e \in \mathbb{R}$. Let $\chi_{\text{sp}} \in L^1(\mathbb{R}^3)$ be a real-valued function on $\mathbb{R}^3$ playing the role of a spacial cut-off.

Our interaction Hamiltonian H_{int} is defined as

$$D(H_{\text{int}}) = D(I \otimes N_{\text{b}}^{\frac{1}{2}}), \quad (3.94)$$

$$H_{\text{int}}\Psi = e \int_{\mathbb{R}^3} d\mathbf{x} \chi_{\text{sp}}(\mathbf{x}) : j^\mu(\mathbf{x}) : \otimes A_\mu(\mathbf{x})\Psi, \quad \Psi \in D(H_{\text{int}}), \quad (3.95)$$

where the integral in the right hand side is taken in the sense of strong Bochner integral, and we used the standard Einstein notation in which the summation over repeated indices with one upper and the other lower is understood. As will be seen in later, the operator H_{int} is well-defined since

$$\int_{\mathbb{R}^3} d\mathbf{x} |\chi_{\text{sp}}(\mathbf{x})| \| : j^\mu(\mathbf{x}) : \otimes A_\mu(\mathbf{x})\Psi \| < \infty, \quad \Psi \in D(I \otimes N_{\text{b}}^{\frac{1}{2}}). \quad (3.96)$$

Moreover, H_{int} is essentially η -self-adjoint.

The quantum system under consideration is described by the Hamiltonian

$$H_{\text{QED}} := H_{\text{fr}} + \overline{H_{\text{int}}} \quad \text{in } \mathcal{F}_{\text{tot}}. \quad (3.97)$$

The time evolution of the quantum fields A_μ, ψ_l is generated by the Heisenberg equations:

$$\frac{d}{dt} A^\mu(t, f) = [iH_{\text{QED}}, A^\mu(t, f)], \quad (3.98)$$

$$\frac{d}{dt} \psi_l(t, g) = [iH_{\text{QED}}, \psi_l(t, g)]. \quad (3.99)$$

It is easy to find *formal* solutions of these equations:

$$A^\mu(t, f) = e^{itH_{\text{QED}}} A^\mu(0, f) e^{-itH_{\text{QED}}}, \quad (3.100)$$

$$\psi_l(t, g) = e^{itH_{\text{QED}}} \psi_l(0, g) e^{-itH_{\text{QED}}}. \quad (3.101)$$

However, this does not immediately make sense because our QED Hamiltonian in Lorenz gauge is neither self-adjoint nor normal, and therefore we cannot define the time evolution operational $e^{-itH_{\text{QED}}}$ through the operational calculus.

3.3.4 η -self-adjointness

In this subsection, we prove the η -self-adjointness of $A_\mu(0, f)$, H_{int} , and H_{QED} under some suitable conditions.

First, note that the spectrum of the photon number operator N_{b} is a purely discrete set $\{0, 1, 2, \dots\}$, and that for all integer $N \geq 0$,

$$R(E_{N_{\text{b}}}([0, N])) = \bigoplus_{n=0}^N \bigotimes_s \mathcal{H}_{\text{ph}}^{(\text{L})} \subset \mathcal{F}_{\text{ph}}^{(\text{L})}. \quad (3.102)$$

For each $\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_{\text{b},0}(\mathcal{H}_{\text{ph}}^{(\text{L})})$, we denote by N_Ψ the maximum photon number of Ψ , that is,

$$N_\Psi := \max\{n \geq 0 \mid \Psi^{(n)} \neq 0\} < \infty. \quad (3.103)$$

Lemma 3.9. *Let f be a real-valued function satisfying $f \in L^2(\mathbb{R}_{\mathbf{x}}^3)$ and $\widehat{f}/\sqrt{\omega} \in L^2(\mathbb{R}_{\mathbf{k}}^3)$. Then the quantized electromagnetic field $A_\mu(0, f)$ is essentially η -self-adjoint, i.e., $\overline{A_\mu(0, f)} = \left(\overline{A_\mu(0, f)}\right)^\dagger$.*

Proof. By Lemma 3.3 (iii), it is sufficient to prove that $\eta A_\mu(0, f)$ is essentially self-adjoint.

Put $f_- := \widehat{f}/\sqrt{2\omega}$. Since $\eta^* = \eta$, we have

$$\begin{aligned} (\eta A_\mu(0, f))^* &= A_\mu(0, f)^* \eta \\ &\supset (a_\mu(f_-)^* + a_\mu^\dagger(f_-)^*) \eta \\ &= \eta \eta (a_\mu(f_-)^* + (\eta a_\mu^*(f_-) \eta)^*) \eta \\ &= \eta (a_\mu^\dagger(f_-) + a_\mu(f_-)) \\ &= \eta A_\mu(0, f), \end{aligned} \quad (3.104)$$

which means that $A_\mu(0, f)$ is η -symmetric.

We prove the η -self-adjointness by Nelson's analytic vector theorem ([53], Theorem X.39 and its corollaries). Clearly, $A_\mu(0, f)$ and η leave $\mathcal{F}_{\text{b},0}(\mathcal{H}_{\text{ph}})$ invariant. By the fact that η and N_{b} are strongly commuting, and by Lemma 3.5 (i), one finds for each $\Psi \in \mathcal{F}_{\text{b},0}(\mathcal{H}_{\text{ph}})$ and $n = 1, 2, \dots$,

$$\|(\eta A_\mu(0, f))^n \Psi\| \leq M_f^n (N_\Psi + n)^{1/2} \dots (N_\Psi + 1)^{1/2} \|\Psi\|, \quad (3.105)$$

where $M_f := \sqrt{2} \|\widehat{f}/\sqrt{\omega}\| + \frac{1}{\sqrt{2}} \|\widehat{f}\|$. Thus one obtains for all $t > 0$,

$$\begin{aligned} \sum_{n=0}^\infty \frac{t^n}{n!} \|(\eta A_\mu(0, f))^n \Psi\| &\leq \sum_{n=0}^\infty \frac{t^n}{n!} M_f^n (N_\Psi + n)^{1/2} \dots (N_\Psi + 1)^{1/2} \|\Psi\| \\ &< \infty, \end{aligned} \quad (3.106)$$

by using d'Alembert's ratio test. Therefore $\mathcal{F}_{\text{b},0}(\mathcal{H}_{\text{ph}})$ is a $\eta A_\mu(0, f)$ -invariant analytic vector space, and we have the essentially self-adjointness of $\eta A_\mu(0, f)$ by Nelson's analytic vector theorem. By Lemma 3.3 (v), the assertion follows. $\square$

We denote the closure of $A_\mu(0, f)$ by the same symbol.

Lemma 3.10. *Let $\Psi \in D((I \otimes N_{\text{b}})^{1/2})$. Then the followings hold.*

(i)

$$\| :j^\mu(0, \mathbf{x}): \otimes A_\mu(0, \mathbf{x}) \Psi \| \leq M_{\text{cu}} M_{\text{ph}} \| (I \otimes N_{\text{b}} + 1)^{1/2} \Psi \|, \quad (3.107)$$

where M_{el} is a constant defined in (3.90), and $M_{\text{ph}} := \sqrt{2} \|\widehat{\chi}_{\text{ph}} / \sqrt{\omega}\|_{L^2(\mathbb{R}_{\mathbf{k}}^3)} + \frac{1}{\sqrt{2}} \|\widehat{\chi}_{\text{ph}}\|_{L^2(\mathbb{R}_{\mathbf{k}}^3)}$.

(ii) The vector-valued function $\mathbf{x} \mapsto j^\mu(0, \mathbf{x}) \otimes A_\mu(0, \mathbf{x}) \Psi$ is strongly continuous.

(iii) Let χ_{sp} be a real-valued function satisfying $\chi_{\text{sp}} \in L^1(\mathbb{R}_{\mathbf{x}}^3)$. then

$$\int_{\mathbb{R}^3} d\mathbf{x} |\chi_{\text{sp}}(\mathbf{x})| \| :j^\mu(0, \mathbf{x}): \otimes A_\mu(0, \mathbf{x}) \Psi \| \leq \|\chi_{\text{sp}}\|_{L^1(\mathbb{R}^3)} M_{\text{el}} M_{\text{ph}} \| (I \otimes N_{\text{b}} + 1)^{1/2} \Psi \| < \infty. \quad (3.108)$$

Proof. First of all, notice that the operator identity $(I \otimes N_{\text{b}})^{1/2} = I \otimes N_{\text{b}}^{1/2}$ follows from operational calculus.

(i) By using Lemma 3.5 (i), one finds for all $\Psi \in \mathcal{F}_{\text{el}} \widehat{\otimes} D(N_{\text{b}}^{1/2})$,

$$\begin{aligned} \| j^\mu(0, \mathbf{x}) \otimes A_\mu(0, \mathbf{x}) \Psi \| &= \| (j^\mu(0, \mathbf{x}) \otimes I) (I \otimes A_\mu(0, \mathbf{x})) \Psi \| \\ &\leq M_{\text{el}} M_{\text{ph}} \| (I \otimes N_{\text{b}} + 1)^{1/2} \Psi \|. \end{aligned} \quad (3.109)$$

Since $\mathcal{F}_{\text{el}} \widehat{\otimes} D(N_{\text{b}}^{1/2})$ is a core of $(I \otimes N_{\text{b}})^{1/2}$, we obtain (3.107).

(ii) Let $\mathbf{P} := (P^1, P^2, P^3)$, $P^j := \overline{\text{d}\Gamma_{\text{f}}(p^j) \otimes I + I \otimes \text{d}\Gamma_{\text{b}}(k^j)}$, where p^j and k^j are the multiplication operators in $\mathcal{H}_{\text{el}}$ and $\mathcal{H}_{\text{ph}}$ respectively. Then P^j , $j = 1, 2, 3$ are self-adjoint. Note that the operator $j^\mu(0, \mathbf{x}) \otimes A_\mu(0, \mathbf{x})$ can be rewritten as $e^{-i\mathbf{P} \cdot \mathbf{x}} j^\mu(0, \mathbf{0}) \otimes A_\mu(0, \mathbf{0}) e^{i\mathbf{P} \cdot \mathbf{x}}$. Since P^j and $I \otimes N_{\text{b}}$ are strongly commuting, we have for all $\Psi \in D((I \otimes N_{\text{b}})^{1/2})$,

$$j^\mu(0, \mathbf{x}) \otimes A_\mu(0, \mathbf{x}) \Psi = e^{-i\mathbf{P} \cdot \mathbf{x}} j^\mu(0, \mathbf{0}) \otimes A_\mu(0, \mathbf{0}) (I \otimes N_{\text{b}})^{-1/2} e^{i\mathbf{P} \cdot \mathbf{x}} (I \otimes N_{\text{b}})^{1/2} \Psi, \quad (3.110)$$

and the right hand side is strongly continuous since $j^\mu(0, \mathbf{0}) \otimes A_\mu(0, \mathbf{0}) (I \otimes N_{\text{b}})^{-1/2}$ is a bounded operator by (3.107).

(iii) The inequality (3.108) immediately follows from (3.107). □

The total space $\mathcal{H}_{\text{tot}}$ can naturally be identified as

$$\mathcal{H}_{\text{tot}} = \bigoplus_{n=0}^{\infty} (\mathcal{F}_{\text{el}} \otimes (\bigotimes_s^n \mathcal{H}_{\text{ph}}^{(\text{L})})). \quad (3.111)$$

Hereafter, we freely use this identification. Set

$$\mathcal{F}_{\text{b},0} := \widehat{\bigoplus}_{n=0}^{\infty} (\mathcal{F}_{\text{el}} \otimes (\bigotimes_s^n \mathcal{H}_{\text{ph}}^{(\text{L})})) \quad (3.112)$$

where $\widehat{\bigoplus}_{n=0}^{\infty}$ denotes the algebraic direct sum. Let N_{Ψ} denote the maximum number of photons of $\Psi \in \mathcal{F}_{\text{b},0}$, namely, for $\Psi = \{\Psi^{(N)}\}_{N=0}^{\infty} \in \bigoplus_{n=0}^{\infty} (\mathcal{F}_{\text{el}} \otimes (\bigotimes_s^n \mathcal{H}_{\text{ph}}^{(\text{L})}))$, N_{Ψ} is the largest natural number N satisfying $\Psi^{(N)} \neq 0$. Define

$$\mathcal{F}_N := \bigoplus_{n=0}^N (\mathcal{F}_{\text{el}} \otimes (\bigotimes_s^n \mathcal{H}_{\text{ph}}^{(\text{L})})), \quad N = 0, 1, 2, \dots \quad (3.113)$$

then for all integer $N \geq 0$, one finds

$$R(E_{I \otimes N_b}([0, N])) = \mathcal{F}_N. \quad (3.114)$$

Since the photon field $A_\mu(\mathbf{x})$ creates at most one photon, it follows that if $\Psi \in \mathcal{F}_{b,0}$ belongs to $\mathcal{F}_N$, then $A_\mu(\mathbf{x})\Psi \in \mathcal{F}_{N+1}$ for all $\mathbf{x} \in \mathbb{R}^3$. Thus since $\mathcal{F}_{N+1}$ is a closed subspace, we find that $H_{\text{int}}\mathcal{F}_N$ is contained in $\mathcal{F}_{N+1}$.

Lemma 3.11. *The interaction Hamiltonian H_{int} is essentially η -self-adjoint.*

Proof. Firstly, we show the η -symmetricity of H_{int} . By a direct calculation, one finds for each $\Psi, \Phi \in \mathcal{F}_{\text{el}} \widehat{\otimes} D(N_b^{1/2})$,

$$\langle \Psi, \eta H_{\text{int}} \Phi \rangle = \langle \eta H_{\text{int}} \Psi, \Phi \rangle. \quad (3.115)$$

Since $\mathcal{F}_{\text{el}} \widehat{\otimes} D(N_b^{1/2})$ is a core of $(I \otimes N_b)^{1/2}$, and since H_{int} is $(I \otimes N_b)^{1/2}$ -bounded, (3.115) holds for all $\Psi, \Phi \in D((I \otimes N_b)^{1/2})$, thus H_{int} is η -symmetric.

Next, we show the η -self-adjointness by Nelson's analytic vector theorem, similarly to the proof of Lemma 3.9. Note that H_{int} and $I \otimes \eta$ leaves $\mathcal{F}_{b,0}$ invariant. Put

$$M_{\text{int}} := |e| \|\chi_{\text{sp}}\|_{L^1(\mathbb{R}_{\mathbf{x}}^3)} M_{\text{el}} M_{\text{ph}}. \quad (3.116)$$

By Lemma 3.10 (iii), and by the fact that $I \otimes \eta$ and $I \otimes N_b$ are strongly commuting, one finds for each $\Psi \in \mathcal{F}_{b,0}$ and $n = 1, 2, \dots$,

$$\|(\eta H_{\text{int}})^n \Psi\| \leq (N_\Psi + n)^{1/2} \dots (N_\Psi + 1)^{1/2} M_{\text{int}}^n \|\Psi\|. \quad (3.117)$$

thus one obtains for all $t > 0$,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} \|(\eta H_{\text{int}})^n \Psi\| &\leq \sum_{n=0}^{\infty} \frac{t^n}{n!} (N_\Psi + n)^{1/2} \dots (N_\Psi + 1)^{1/2} M_{\text{int}}^n \|\Psi\| \\ &< \infty, \end{aligned} \quad (3.118)$$

by using d'Alembert's ratio test. Therefore $\mathcal{F}_{b,0}$ is an ηH_{int} -invariant analytic vector space, and the essentially self-adjointness of ηH_{int} follows. $\square$

We denote the closure of H_{int} by the same symbol.

Proposition 3.1. *Under Hypothesis 3.1, H_{QED} is η -self-adjoint.*

Proof. By the Kato-Rellich theorem and Lemma 3.3, it is sufficient to show that the symmetric operator ηH_{int} is ηH_{fr} -bounded with a relative bound less than 1.

By Lemma 3.5, we have for all $\Psi \in \mathcal{F}_{\text{el}} \widehat{\otimes} D(H_{\text{ph}}^{(L)1/2})$, one finds

$$\begin{aligned} \|\cdot j^\mu(0, \mathbf{x}) : \otimes A_\mu(0, \mathbf{x}) \Psi\| &= \|(j^\mu(0, \mathbf{x}) \otimes I)(I \otimes A_\mu(0, \mathbf{x})) \Psi\| \\ &\leq M_{\text{el}} \left(\sqrt{2} \|\widehat{\chi_{\text{ph}}}/\omega\| \|I \otimes H_{\text{ph}}^{(L)1/2} \Psi\| + \frac{1}{\sqrt{2}} \|\widehat{\chi_{\text{ph}}}/\sqrt{\omega}\| \|\Psi\| \right). \end{aligned} \quad (3.119)$$

By (3.119), we find that for all $\Psi \in \mathcal{F}_{\text{el}} \widehat{\otimes} (D(H_{\text{ph}}^{(L)1/2}) \cap D(N_b^{1/2}))$ ($\subset D(H_{\text{int}})$),

$$\|H_{\text{int}} \Psi\| \leq |e| \|\chi_{\text{sp}}\|_{L^1(\mathbb{R}_{\mathbf{x}}^3)} M_{\text{el}} \left(\sqrt{2} \|\widehat{\chi_{\text{ph}}}/\omega\| \|I \otimes H_{\text{ph}}^{1/2} \Psi\| + \frac{1}{\sqrt{2}} \|\widehat{\chi_{\text{ph}}}/\sqrt{\omega}\| \|\Psi\| \right). \quad (3.120)$$

Since $\mathcal{F}_{\text{el}} \widehat{\otimes} (D(H_{\text{ph}}^{(L)1/2}) \cap D(N_b^{1/2}))$ is a core of $I \otimes H_{\text{ph}}^{(L)1/2}$, we conclude that H_{int} is $I \otimes H_{\text{ph}}^{(L)1/2}$ -bounded. As is easily proved, one finds $I \otimes H_{\text{ph}}^{(L)1/2}$ is infinitesimally small with respect to H_{fr} . therefore it follows that ηH_{int} is infinitesimally small with respect to ηH_{fr} , hence we get the desired result. $\square$

3.3.5 Existence of dynamics

In order to prove that there exists dynamics of these quantum fields, we have only to see that Theorem 3.3 can be applied to our case by checking that H_{int} is in $\mathcal{C}_0(A)$ -class for an operator A . We see in what follows that this is indeed the case where (i) $H_0 = H_{\text{fr}}$, $H_1 = H_{\text{int}}$, $A = I \otimes N_{\text{b}}$, (ii) $H_0 = H_{\text{fr}}$, $H_1 = H_{\text{int}}$, $A = N_{\text{f}} \otimes I + I \otimes N_{\text{b}}$, and (iii) $H_0 = H_{\text{fr}}$, $H_1 = H_{\text{int}}$, $A = H_{\text{fr}}$. Here N_{f} is the electron number operator:

$$N_{\text{f}} := d\Gamma_{\text{f}}(I_{\mathcal{H}_{\text{el}}}) : \mathcal{F}_{\text{el}} \rightarrow \mathcal{F}_{\text{el}}. \quad (3.121)$$

Hereafter, we omit trivial tensor product like $\otimes I$ or $I \otimes$ when no confusion may occur and just write, for instance, H_{el} instead of $H_{\text{el}} \otimes I$ and so forth.

Lemma 3.12. (I) N_{b} is self-adjoint and non-negative.

(II) H_{int} and its adjoint H_{int}^* are $N_{\text{b}}^{1/2}$ -bounded.

(III) For all $L \geq 0$, H_{int} and H_{int}^* map $R(E_{N_{\text{b}}}([0, L]))$ into $R(E_{N_{\text{b}}}([0, L+1]))$.

(IV) H_{int} is in $\mathcal{C}_0(N_{\text{b}})$ -class.

Proof. (I) is well known fact.

We prove (II). As in the proof of Lemma 3.10, it can be seen that H_{int} is $N_{\text{b}}^{1/2}$ -bounded. We prove that H_{int}^* is also $H_{\text{b}}^{1/2}$ -bounded. Take arbitrary $\Psi \in \mathcal{F}_{\text{b},0}$. By Lemma 3.11, we obtain

$$\begin{aligned} \|H_{\text{int}}^* \Psi\| &= \|H_{\text{int}} \eta \Psi\| \\ &\leq M_{\text{int}} \|(N_{\text{b}} + 1)^{\frac{1}{2}} \eta \Psi\| \\ &\leq M_{\text{int}} \|(N_{\text{b}} + 1)^{\frac{1}{2}} \Psi\|. \end{aligned} \quad (3.122)$$

Since $\mathcal{F}_{\text{b},0}$ is a core of $N_{\text{b}}^{1/2}$, we have the assertion (II).

We prove (III). Notice that the spectrum of the photon number operator N_{b} is a purely discrete set $\{0, 1, 2, \dots\}$, and that for all $L \geq 0$ and $[L]$, the largest integer satisfying $[L] \leq L$,

$$R(E_{N_{\text{b}}}([0, L])) = R(E_{N_{\text{b}}}([0, [L]])) = \mathcal{F}_{[L]}. \quad (3.123)$$

Suppose Ψ belongs to $R(E_{N_{\text{b}}}([0, L]))$. Then it is clear that Ψ belongs to $\mathcal{F}_{[L]+1} = R(E_{N_{\text{b}}}([0, L+1]))$, since the interaction Hamiltonian creates at most one photon. By Lemma 3.11, and the fact that η and N_{b} are strongly commuting and thus η preserves photon number, it immediately follows that $H_{\text{int}}^* \Psi \in \mathcal{F}_{[L]+1} = R(E_{N_{\text{b}}}([0, L+1]))$.

(IV) follows from (I)-(III). $\square$

Lemma 3.13. (I) $N_{\text{f}} + N_{\text{b}}$ is self-adjoint and non-negative.

(II) H_{int} and its adjoint H_{int}^* are $(N_{\text{f}} + N_{\text{b}})^{1/2}$ -bounded.

(III) For all $L \geq 0$, H_{int} and H_{int}^* map $R(E_{N_{\text{f}}+N_{\text{b}}}([0, L]))$ into $R(E_{N_{\text{f}}+N_{\text{b}}}([0, L+3]))$.

(IV) H_{int} is in $\mathcal{C}_0(N_{\text{f}} + N_{\text{b}})$ -class.

Proof. (I) follows from a general theory of tensor products of self-adjoint operators.

(II) follows from Lemma 3.12 (II) since $N_{\text{b}}^{1/2}$ is $(N_{\text{f}} + N_{\text{b}})^{1/2}$ -bounded,

We prove (III). Note that for all $L \geq 0$,

$$R(E_{N_{\text{f}}+N_{\text{b}}}([0, L])) = \bigcup_{\substack{p,q \geq 0 \\ p+q \leq [L]}} \left(\left(\bigwedge^p \mathcal{H}_{\text{el}} \right) \otimes \left(\bigotimes_s^p \mathcal{H}_{\text{ph}}^{(L)} \right) \right) \subset \mathcal{F}_{\text{tot}}. \quad (3.124)$$

then we see that H_{int} maps $R(E_{N_f+N_b}([0, L]))$ into $R(E_{N_f+N_b}([0, L+3]))$ since the interaction Hamiltonian creates at most one photon and two electrons (positrons). Since η and $N_f + N_b$ are strongly commutes, we obtain (III) in a same manner as in the proof of Lemma 3.12 (III).

(IV) follows from (I)-(III). $\square$

We prove under a stronger assumption that H_{int} is in $\mathcal{C}_0(H_{\text{fr}})$ -class.

Hypothesis 3.2 (Ultraviolet cutoff). *There exist constants $\Lambda_{\text{el}}, \Lambda_{\text{ph}} \geq 0$ such that $\text{supp } \widehat{\chi}_{\text{el}} \subset \{|\mathbf{p}| \leq \Lambda_{\text{el}}\}$, $\text{supp } \widehat{\chi}_{\text{ph}} \subset \{|\mathbf{k}| \leq \Lambda_{\text{ph}}\}$.*

Lemma 3.14. *Assume Hypothesis 3.2. Then the following (i) and (ii) hold.*

- (i) *For all $L \geq 0$, $\mathbf{x} \in \mathbb{R}^3$ and $\mu = 0, 1, 2, 3$, $A_\mu(0, \mathbf{x})$ maps $R(E_{H_{\text{ph}}^{(\text{L})}}([0, L]))$ into $R(E_{H_{\text{ph}}^{(\text{L})}}([0, L + \Lambda_{\text{ph}}]))$.*
- (ii) *For all $L \geq 0$, $\mathbf{x} \in \mathbb{R}^3$ and $l = 1, 2, 3, 4$, $\psi_l(0, \mathbf{x})$ and $\psi_l(0, \mathbf{x})^*$ map $R(E_{H_{\text{el}}}([0, L]))$ into $R(E_{H_{\text{el}}}([0, L + \sqrt{\Lambda_{\text{el}}^2 + M^2}]))$.*

Proof. (i) This follows from Lemma 3.6.

(ii) This follows from Lemma 3.8. $\square$

Lemma 3.15. *Assume Hypotheses 5.1 and 5.2. Then the following (i)-(iii) hold:*

- (i) *H_{int} and H_{int}^* are $H_{\text{fr}}^{1/2}$ -bounded.*
- (ii) *For all $L \geq 0$, H_{int} and H_{int}^* map $R(E_{H_{\text{fr}}}([0, L]))$ into $R(E_{H_{\text{fr}}}([0, L + 2\sqrt{\Lambda_{\text{el}}^2 + M^2} + \Lambda_{\text{ph}}]))$.*
- (iii) *H_{int} is in $\mathcal{C}_0(H_{\text{fr}})$ -class.*

Proof. (i) As seen in the proof of Proposition 3.1, under Hypothesis 3.1, H_{int} is $H_{\text{fr}}^{1/2}$ -bounded. Similarly as in the proof of Lemma 3.12 (III), we see that H_{int}^* is also $H_{\text{fr}}^{1/2}$ -bounded, hence (I) follows.

(ii) One can see that, for all $L \geq 0$, $\mathbf{x} \in \mathbb{R}^3$ and $\mu = 0, 1, 2, 3$, $j^\mu(0, \mathbf{x})$ maps $R(E_{H_{\text{el}}}([0, L]))$ into $R(E_{H_{\text{el}}}([0, L + 2\sqrt{\Lambda_{\text{el}}^2 + M^2}]))$ in the same manner as Lemma 3.14 (ii). Now, fix $L \geq 0$ arbitrarily, and let $\Psi \in R(E_{H_{\text{fr}}}([0, L]))$. Applying Lemmas B.1 and 3.14, we see that $j^\mu(0, \mathbf{x}) \otimes A_\mu(0, \mathbf{x}) \Psi \in R(E_{H_{\text{fr}}}([0, L + 2\sqrt{\Lambda_{\text{el}}^2 + M^2 + \Lambda_{\text{ph}}}])))$ for all $\mathbf{x} \in \mathbb{R}^3$ and $\mu = 0, 1, 2, 3$. Hence we have $H_{\text{int}} \Psi \in R(E_{H_{\text{fr}}}([0, L + 2\sqrt{\Lambda_{\text{el}}^2 + M^2 + \Lambda_{\text{ph}}}])))$ because $R(E_{H_{\text{fr}}}([0, L + 2\sqrt{\Lambda_{\text{el}}^2 + M^2 + \Lambda_{\text{ph}}}])))$ is closed subspace. Furthermore, by Lemma 3.11, and the fact that η and H_{fr} are strongly commuting, we have $H_{\text{int}}^* \Psi \in R(E_{H_{\text{fr}}}([0, L + 2\sqrt{\Lambda_{\text{el}}^2 + M^2 + \Lambda_{\text{ph}}}])))$. Thus the assertion follows.

(iii) This follows from (i), (ii). $\square$

Lemma 3.16. *N_b , $N_f + N_b$ and H_{fr} are strongly commuting.*

From Lemmas 3.12, 3.13, 3.15 and 3.16, we can apply the general theory constructed in the earlier sections to obtain:

Theorem 3.8. *For all $\Psi \in D_{\text{fin}}(N_{\text{b}}) \cup D_{\text{fin}}(N_{\text{f}} + N_{\text{b}})$, the series*

$$\Psi + (-i) \int_{t'}^t d\tau_1 H_{\text{int}}(\tau_1) \Psi + (-i)^2 \int_{t'}^t d\tau_1 \int_{t'}^{\tau_1} d\tau_2 H_{\text{int}}(\tau_1) H_{\text{int}}(\tau_2) \Psi + \dots \quad (3.125)$$

converges absolutely, where each of integrals is strong integral. Let $U_{N_{\text{b}}}(H_1; t, t')$ and $U_{N_{\text{f}}+N_{\text{b}}}(H_1; t, t')$ be operators which are defined by the convergent series (3.125) with the domain $D(U_{N_{\text{b}}}(H_1; t, t')) = D_{\text{fin}}(N_{\text{b}})$ and $D(U_{N_{\text{f}}+N_{\text{b}}}(H_1; t, t')) D_{\text{fin}}(N_{\text{f}}+N_{\text{b}})$, respectively. Then these operators have properties stated in Theorems 2.1 and 2.3, and the operator equality

$$\overline{U_{N_{\text{b}}}(H_1; t, t')} = \overline{U_{N_{\text{f}}+N_{\text{b}}}(H_1; t, t')} \quad (3.126)$$

holds. Moreover, if Hypotheses 5.1 and 5.2 are valid, then for all $\Psi \in D_{\text{fin}}(H_{\text{fr}})$, the series (3.125) converges absolutely; for the operator $U_{H_{\text{fr}}}(H_1; t, t')$ defined by the convergent series (3.125) with the domain $D(U_{H_{\text{fr}}}(H_1; t, t')) D_{\text{fin}}(H_{\text{fr}})$, it follows that

$$\overline{U_{N_{\text{b}}}(H_1; t, t')} = \overline{U_{N_{\text{f}}+N_{\text{b}}}(H_1; t, t')} = \overline{U_{H_{\text{fr}}}(H_1; t, t')}. \quad (3.127)$$

Theorem 3.8 ensures that the closure of the time-evolution operator in the interaction picture $U(t, t')$ which is given by the convergent series (3.125) is independent of the choice of the strongly commuting self-adjoint operators $A = N_{\text{b}}, N_{\text{f}} + N_{\text{b}}, H_{\text{fr}}$. Hence we consider only the case $A = N_{\text{b}}$ in what follows.

Hereafter, we shortly denote

$$U(t, t') := \overline{U_{N_{\text{b}}}(H_1; t, t')}. \quad (3.128)$$

As we have already seen in the general theory, once $U(t, t')$ is constructed, we immediately obtain a time evolution which is generated by $W(t) := e^{-itH_{\text{fr}}} U(t, 0)$. In the present application to physics, it should be made sure that this time evolution is physically acceptable, that is, the probability amplitude is conserved

$$\langle \Psi | \Phi \rangle = \langle W(t) \Psi | W(t) \Phi \rangle,$$

for suitable vectors Ψ, Φ . The following theorem is concerned with this aspect:

Theorem 3.9. *The time evolution operator*

$$W(t) := e^{-itH_{\text{fr}}} U(t, 0) \quad (3.129)$$

satisfies

$$W(t)^\dagger \supset W(t)^{-1}. \quad (3.130)$$

In particular,

$$\langle W(t) \Psi | W(t) \Phi \rangle = \langle \Psi | \Phi \rangle \quad (3.131)$$

for all $\Psi, \Phi \in D(W(t))$.

To prove Theorem 3.9, we prepare some facts.

Lemma 3.17. *For each $\Psi \in \mathcal{F}_{\text{b},0}$,*

$$\eta U(t, t') \eta \Psi = U(t', t)^* \Psi \quad (3.132)$$

holds.

Proof. From Lemma 3.11, the operator identities $\eta H_{\text{int}}^* \eta = H_{\text{int}}$ and $\eta H_{\text{fr}} \eta = H_{\text{fr}}$ hold, which imply

$$\eta H_{\text{int}}(t)^* \eta = H_{\text{int}}(t). \quad (3.133)$$

We claim that (3.133) implies the identity

$$\eta U_n(t, t') \eta \Psi = U_n(t', t)^* \Psi \quad (3.134)$$

for all $n = 0, 1, 2, \dots$ and $\Psi \in \mathcal{F}_{\text{b},0}$. In fact, by Lemma 2.6, one finds

$$\begin{aligned} \eta U_n(t, t') \eta \Psi &= (-i)^n \int_{t'}^t d\tau_1 \dots \int_{t'}^{\tau_{n-1}} d\tau_n H_{\text{int}}(\tau_1)^* \dots H_{\text{int}}(\tau_n)^* \Psi \\ &= (-i)^n \int_{t'}^t d\tau_n \dots \int_{t'}^{\tau_2} d\tau_1 H_{\text{int}}(\tau_n)^* \dots H_{\text{int}}(\tau_1)^* \Psi \\ &= U_n(t', t)^* \Psi. \end{aligned} \quad (3.135)$$

Hence (3.134) is true for all n , and by summing up over $n = 0, 1, 2, \dots$, we obtain (3.132). $\square$

Proposition 3.2. *For all $t, t' \in \mathbb{R}$, $U(t, t')$ is invertible and the operator equality*

$$U(t, t') = U(t', t)^{-1} \quad (3.136)$$

holds.

Proof. Fix t, t' . Firstly, we prove that $U(t, t')$ is injective. Note that if for all $\Psi \in \mathcal{F}_{\text{b},0}$, $\langle \Psi | \Phi \rangle = 0$, then it follows that $\Phi = 0$. From Lemma 3.17, we have the operator relation

$$\eta U(t, t') \subset U(t', t)^* \eta, \quad (3.137)$$

since $\mathcal{F}_{\text{b},0}$ is a core of $U(t, t')$, and since $U(t', t)^*$ is closed. Now, suppose that Φ satisfies $U(t, t')\Phi = 0$. Let $\Psi \in \mathcal{F}_{\text{b},0}$ be arbitrary. It follows that

$$\begin{aligned} 0 &= \langle U(t, t')\Psi | U(t, t')\Phi \rangle \\ &= \langle U(t, t')\Psi, \eta U(t, t')\Phi \rangle \\ &= \langle U(t, t')\Psi, U(t', t)^* \eta \Phi \rangle \\ &= \langle U(t', t)U(t, t')\Psi | \Phi \rangle \\ &= \langle \Phi | \Psi \rangle, \end{aligned} \quad (3.138)$$

where we have used (3.137) in the third equality, and Theorem 2.3 (2.52) in the last equality. Since $\Psi \in \mathcal{F}_{\text{b},0}$ is arbitrarily taken, we find $\Phi = 0$. This proves that $U(t, t')$ is injective.

Secondly, we prove

$$U(t, t') \subset U(t', t)^{-1}. \quad (3.139)$$

Take arbitrary $\Psi \in \mathcal{F}_{\text{b},0}$. Then we have from Theorem 2.3, (2.52)

$$U(t', t)U(t, t')\Psi = \Psi,$$

which implies $\Psi \in D(U(t', t)^{-1})$ and

$$U(t, t')\Psi = U(t', t)^{-1}\Psi, \quad \Psi \in \mathcal{F}_{\text{b},0}.$$

But since $\mathcal{F}_{\text{b},0}$ is a core of $U(t, t')$, the relation (3.139) follows.

Finally, we prove

$$U(t, t') \supset U(t', t)^{-1}. \quad (3.140)$$

Let $\Psi \in D(U(t', t)^{-1}) = R(U(t', t))$. Then there is some $\Phi \in D(U(t', t))$ with $\Psi = U(t', t)\Phi$. Since $\mathcal{F}_{b,0}$ is a core of $U(t', t)$, we can choose a sequence $\{\Phi_n\}_n \subset \mathcal{F}_{b,0}$ satisfying

$$\Phi_n \rightarrow \Phi, \quad U(t', t)\Phi_n \rightarrow U(t', t)\Phi = \Psi, \quad (3.141)$$

as n tends to infinity. Therefore we have

$$U(t, t')U(t', t)\Phi_n = \Phi_n \rightarrow \Phi, \quad n \rightarrow \infty.$$

Since $U(t, t')$ is closed, we conclude that $\Psi = U(t', t)\Phi \in D(U(t, t'))$ and

$$U(t, t')\Psi = U(t, t')U(t', t)\Phi = \Phi = U(t', t)^{-1}\Psi.$$

This proves (3.140). □

Proposition 3.3. *For all $t \in \mathbb{R}$, the operator $W(t)$ is injective and*

$$W(t)^{-1} = W(-t) \quad (3.142)$$

holds as an operator equality.

Proof. Since $e^{-itH_{\text{fr}}}$ is unitary and $U(t, 0)$ is injective by Lemma 3.2, one finds $W(t)$ is injective.

We prove (3.142). Fix $t \in \mathbb{R}$. For all $\Psi \in D(W(t)) = D(U(t, 0))$, we have by Theorem 2.3 (2.52)

$$\begin{aligned} W(t)W(-t)\Psi &= e^{-itH_{\text{fr}}}U(t, 0)e^{itH_{\text{fr}}}U(-t, 0)\Psi \\ &= U(0, -t)U(-t, 0)\Psi \\ &= \Psi. \end{aligned} \quad (3.143)$$

This means $\Psi \in D(W(t)^{-1})$ and

$$W(-t)\Psi = W(t)^{-1}\Psi.$$

Since $\mathcal{F}_{b,0}$ is a core of $W(t)$, we conclude $W(-t) \subset W(t)^{-1}$. The proof of the inverse inclusion is very similar to the proof of (3.140), and we omit it. □

Proof of Theorem 3.9. Let $\Psi \in \mathcal{F}_{b,0}$. Then we have

$$\eta W(t)^*\eta\Psi = W(-t)\Psi. \quad (3.144)$$

thus by taking closure, it follows that

$$\eta W(t)^*\eta \supset W(-t). \quad (3.145)$$

But, we know $W(-t) = W(t)^{-1}$ from Proposition 3.3. This proves (3.130).

Moreover, since $W(t)\Phi \in D(W(t)^{-1}) \subset D(\eta W(t)^*\eta)$, we obtain

$$\begin{aligned} \langle W(t)\Psi | W(t)\Phi \rangle &= \langle W(t)\Psi, \eta W(t)\Phi \rangle \\ &= \langle \Psi, W(t)^*\eta W(t)\Phi \rangle \\ &= \langle \Psi | W(t)^{-1}W(t)\Phi \rangle \\ &= \langle \Psi | \Phi \rangle. \end{aligned} \quad (3.146)$$

This completes the proof. □

We next consider the Heisenberg equations of motion for quantum fields A_μ and ψ_l .

Lemma 3.18. (I) $A_\mu(0, f)$ and $A_\mu(0, f)^*$ are $N_b^{1/2}$ -bounded and closed.

(II) $\psi_l(0, g)$ and $\psi_l(0, g)^*$ are $N_b^{1/2}$ -bounded and closed.

(III) For all $L \geq 0$, $A_\mu(0, f)$, $A_\mu(0, f)^*$, $\psi_l(0, g)$, and $\psi_l(0, g)^*$ map $R(E_{N_b}([0, L]))$ into $R(E_{N_b}([0, L+1]))$.

(IV) $A_\mu(0, f)$ and $\psi_l(0, g)$ are in $\mathcal{C}_0(N_b)$ -class.

Proof. The assertion (I) follows from Lemma 3.4. The closedness is obvious.

The assertion (II) is an immediate consequence of the fact that Dirac fields are bounded operators.

The claim (III) immediately follows from the fact that photon field operators $A_\mu(0, f)$ and $A_\mu(0, f)^*$ create at most one photon and Dirac fields $\psi_l(0, g)$ and $\psi_l(0, g)^*$ create no photon.

(IV) follows from (I)-(III). $\square$

Finally, we have arrived at the existence of strong solutions for the Heisenberg equations of motion for quantum fields, by combining all the results obtained so far. We can define from Lemma 3.12, Lemma 3.18 and Theorem 3.3 the following field operators:

$$D(A_\mu(t, f)) := \mathcal{F}_{b,0}, \quad A_\mu(t, f)\Psi := W(-t)A_\mu(0, f)W(t)\Psi, \quad \Psi \in \mathcal{F}_{b,0}, \quad f \in \mathcal{S}(\mathbb{R}^3), \quad (3.147)$$

$$D(\psi_l(t, g)) := \mathcal{F}_{b,0}, \quad \psi_l(t, g)\Psi := W(-t)\psi_l(0, g)W(t)\Psi, \quad \Psi \in \mathcal{F}_{b,0}, \quad g \in \mathcal{S}(\mathbb{R}^3). \quad (3.148)$$

Theorem 3.10. For all $f, g \in \mathcal{S}(\mathbb{R}^3)$, $\mu = 0, 1, 2, 3$ and $l = 1, 2, 3, 4$, the operator valued functions $\mathbb{R} \ni t \mapsto A_\mu(t, f)$ and $\mathbb{R} \ni t \mapsto \psi_l(t, g)$ are strong solutions of the weak Heisenberg equation:

$$\begin{aligned} \frac{d}{dt} \langle \Phi | A_\mu(t, f) \Psi \rangle &= \left\langle (iH_{\text{QED}})^\dagger \Phi | A_\mu(t, f) \Psi \right\rangle - \left\langle A_\mu(t, f)^\dagger \Phi | iH_{\text{QED}} \Psi \right\rangle, \\ \frac{d}{dt} \langle \Phi | \psi_l(t, g) \Psi \rangle &= \left\langle (iH_{\text{QED}})^\dagger \Phi | \psi_l(t, g) \Psi \right\rangle - \left\langle \psi_l(t, g)^\dagger \Phi | iH_{\text{QED}} \Psi \right\rangle, \\ \Psi, \Phi &\in D(H_{\text{fr}}) \cap \mathcal{F}_{b,0}. \end{aligned} \quad (3.149)$$

We remark that from Theorem 3.9 the above quantum fields are also written as

$$A_\mu(t, f)\Psi = W(t)^\dagger A_\mu(0, f)W(t)\Psi, \quad \Psi \in \mathcal{F}_{b,0}, \quad (3.150)$$

$$\psi_l(t, g)\Psi = W(t)^\dagger \psi_l(0, g)W(t)\Psi, \quad \Psi \in \mathcal{F}_{b,0}. \quad (3.151)$$

4 Gupta-Bleuler formalism for the Dirac-Maxwell model

4.1 The Dirac-Maxwell Hamiltonian in the Lorenz gauge

We introduce the Dirac-Maxwell Hamiltonian quantized in the Lorenz gauge. Dirac-Maxwell model describes a quantum system consisting of a Dirac particle and a gauge field minimally interacting with each other.

4.1.1 Dirac particle sector

First, we consider the Dirac particle sector. Let us denote the mass and the charge of the Dirac particle by $M > 0$ and $q \in \mathbb{R}$, respectively. The Hilbert space of state vectors for the Dirac particle is taken to be

$$\mathcal{H}_D := L^2(\mathbb{R}_x^3; \mathbb{C}^4), \quad (4.1)$$

the square integrable functions on $\mathbb{R}_x^3 = \{\mathbf{x} = (x^1, x^2, x^3) \mid x^j \in \mathbb{R}, j = 1, 2, 3\}$ with values in $\mathbb{C}^4$. The vector space $\mathbb{R}_x^3$ here represents the position space of the Dirac particle. We sometimes omit the subscript $\mathbf{x}$ and just write $\mathbb{R}^3$ instead of $\mathbb{R}_x^3$ when no confusion may occur. The target space $\mathbb{C}^4$ realizes a representation of the four dimensional Clifford algebra accompanied by the four dimensional Minkowski vector space. The generators $\{\gamma^\mu\}_{\mu=0,1,2,3}$ satisfy the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3, \quad (4.2)$$

where the Minkowski metric tensor $g = (g_{\mu\nu})$ is given by

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (4.3)$$

We set $g^{-1} = (g^{\mu\nu})$, the inverse matrix of g . We have $g^{\mu\nu} = g_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$. We take γ^0 to be Hermitian and γ^j 's ($j = 1, 2, 3$) be anti-Hermitian. We use the notations following Dirac:

$$\beta := \gamma^0, \quad (4.4)$$

$$\alpha^\mu := \gamma^0 \gamma^\mu, \quad \mu = 0, 1, 2, 3. \quad (4.5)$$

then α^j 's and β satisfy the anti-commutation relations

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}, \quad i, j = 1, 2, 3, \quad (4.6)$$

$$\{\alpha^j, \beta\} = 0, \quad \beta^2 = 1, \quad j = 1, 2, 3, \quad (4.7)$$

where δ_{ij} is the Kronecker delta. The momentum operator of the Dirac particle is given by

$$\mathbf{p} := (p_1, p_2, p_3) := (-iD_1, -iD_2, -iD_3) \quad (4.8)$$

with D_j being the generalized partial differential operator on $L^2(\mathbb{R}^3; \mathbb{C}^4)$ with respect to the variable x^j , the j -th component of $\mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$. We write in short

$$\boldsymbol{\alpha} \cdot \mathbf{p} := \sum_{j=1}^3 \alpha^j p_j.$$

The Hamiltonian of the Dirac particle is then given by the Dirac operator

$$H_D := \boldsymbol{\alpha} \cdot \mathbf{p} + M\beta \quad (4.9)$$

acting in $\mathcal{H}_D$, with the domain $D(H_D) := H^1(\mathbb{R}^3; \mathbb{C}^4)$, where $H^1(\mathbb{R}^3; \mathbb{C}^4)$ denotes the $\mathbb{C}^4$ -valued Sobolev space of order one.

Suppose that there are N Dirac particles. In this case, the Hilbert space should be

$$\mathcal{H}_{D,N} := \wedge^N \mathcal{H}_D = \bigotimes_{\text{as}}^N L^2(\mathbb{R}^3; \mathbb{C}^4) = L_{\text{as}}^2((\mathbb{R}^3 \times \{1, 2, 3, 4\})^N), \quad (4.10)$$

where $\bigotimes_{\text{as}}^N$ denotes the N -fold anti-symmetric tensor product. The a -th component in

$$\mathbf{X} = (\mathbf{x}_1, l_1; \dots; \mathbf{x}_N, l_N) \in (\mathbb{R}^3 \times \{1, 2, 3, 4\})^N$$

represents the position and the spinor of the a -th Dirac particle. For notational simplicity, we denote the position-spinor space of one electron by $\mathcal{X} = \mathbb{R}^3 \times \{1, 2, 3, 4\}$ in what follows. We regard $\mathcal{X}$ as a topological space with the product topology of the ordinary one on $\mathbb{R}^3$ and the discrete one on $\{1, 2, 3, 4\}$.

The N particle Hamiltonian is then given by

$$H_{D,N} := \sum_{a=1}^N (1 \otimes \dots \otimes \overset{a\text{-th}}{H_D} \otimes \dots \otimes 1), \quad (4.11)$$

which is written as

$$H_{D,N} = \sum_{a=1}^N (\boldsymbol{\alpha}_a \cdot \mathbf{p}_a + \beta_a M), \quad (4.12)$$

with $\mathbf{p}_a$ denoting the generalized differential operator with respect to the a -th coordinate

$$\mathbf{p}_a = (-iD_{a1}, -iD_{a2}, -iD_{a3}), \quad a = 1, 2, \dots, N,$$

$\boldsymbol{\alpha} = (\alpha_a^1, \alpha_a^2, \alpha_a^3)$, α_a^j and β_a denoting the operators acting in $\bigotimes^N \mathbb{C}^4$

$$1 \otimes \dots \otimes \overset{a\text{-th}}{\alpha^j} \otimes \dots \otimes 1 \quad (4.13)$$

and

$$1 \otimes \dots \otimes \overset{a\text{-th}}{\beta} \otimes \dots \otimes 1 \quad (4.14)$$

respectively.

4.1.2 Interaction between the Dirac particles and the gauge field, and the total Hamiltonian

Next, we introduce the interaction Hamiltonian and the total Hamiltonian in the Hilbert space of state vectors for the coupled system. For the photon field quantized into the Lorenz gauge, we use the same notations as in Subsection 3.3.1.

The total state space is taken to be

$$\mathcal{F}_{\text{DM},N} := (\wedge^N \mathcal{H}_D) \otimes \mathcal{F}_{\text{ph}}^{(\text{L})}. \quad (4.15)$$

We remark that this Hilbert space can be naturally identified as

$$\mathcal{F}_{\text{DM},N} = L_{\text{as}}^2(\mathcal{X}^N; \mathcal{F}_{\text{ph}}^{(\text{L})}) = A_N \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \mathcal{F}_{\text{ph}}^{(\text{L})}, \quad (4.16)$$

the Hilbert space of $\mathcal{F}_{\text{ph}}^{(\text{L})}$ -valued functions on $\mathcal{X}^N = \mathcal{X} \times \dots \times \mathcal{X}$ which are square integrable with respect to the Borel measure $d\mathbf{X}$ (the product measure of Lebesgue measure on $\mathbb{R}^3$ and

the counting measure on $\{1, 2, 3, 4\}$) and which are anti-symmetric in the arguments, that is, the exchange of the i -th electron and j -th electron

$$(\mathbf{x}_1, l_1; \dots; \mathbf{x}_i, l_i, \dots, \mathbf{x}_j, l_j, \dots, \mathbf{x}_N, l_N) \mapsto (\mathbf{x}_1, l_1; \dots; \mathbf{x}_j, l_j, \dots, \mathbf{x}_i, l_i, \dots, \mathbf{x}_N, l_N) \quad (4.17)$$

gives a minus sign. We freely use this identification.

The mapping: $\mathbf{X} \mapsto \widehat{\chi_{\text{ph}}^{\mathbf{x}_a}}/\sqrt{\omega}$ ($a = 1, \dots, N$) from $\mathcal{X}^N$ to $\mathcal{H}_{\text{ph}}^{(\text{L})}$ is strongly continuous, and thus we can define a decomposable self-adjoint operator A_a^μ by

$$A_a^\mu := \int_{\mathcal{X}^N}^\oplus d\mathbf{X} A^\mu(0, \mathbf{x}_a), \quad \mu = 0, 1, 2, 3, \quad a = 1, 2, \dots, N, \quad (4.18)$$

acting in $\int_{\mathcal{X}^N}^\oplus d\mathbf{X} \mathcal{F}_{\text{ph}}^{(\text{L})}$.

We have now arrived at the position to define the minimal interaction Hamiltonian H_1 between the Dirac particles and the quantized gauge field with the UV cutoff χ . It is given by

$$H_1 := q \sum_{a=1}^N \alpha_a^\mu A_{a\mu}, \quad \alpha_a^\mu = (1, \boldsymbol{\alpha}) = (1, \alpha_a^1, \alpha_a^2, \alpha_a^3). \quad (4.19)$$

The total Hamiltonian of the coupled system is then given by

$$H := H_0 + H_1, \quad (4.20)$$

$$H_0 := \overline{H_{\text{D},N} \otimes 1 + 1 \otimes H_{\text{ph}}^{(\text{L})}}. \quad (4.21)$$

This is the N -particle *Dirac-Maxwell Hamiltonian* in the Lorenz gauge. We remark that there seems to be no term for the Coulomb interaction between the Dirac particles in the Hamiltonian at a first glance. In the Lorenz gauge, this contribution is included in the photon kinetic term $H_{\text{ph}}^{(\text{L})}$. In fact, as will be seen later, the Coulomb interaction of Dirac particles emerges from the photon kinetic term via a transformation.

4.2 Time evolution of field operator and field equations

We omit trivial tensor products such that $\otimes I$ or $I \otimes$ when no confusion may occur and just write, for instance, H_{D} instead of $H_{\text{D}} \otimes I$ and so forth.

Let $N_{\text{b}} := d\Gamma_{\text{b}}(1)$ be the photon number operator on $\mathcal{F}_{\text{ph}}^{(\text{L})}$.

Lemma 4.1. (I) H_0 and N_{b} are strongly commuting.

(II) H_1 is densely defined and closed.

(III) H_1 and its adjoint H_1^* are $N_{\text{b}}^{1/2}$ -bounded.

(IV) For all $L \geq 0$, H_1 and H_1^* map $R(E_{N_{\text{b}}}([0, L]))$ into $R(E_{N_{\text{b}}}([0, L+1]))$.

(V) H_1 is in $\mathcal{C}_0(N_{\text{b}})$ -class.

Proof. (I) follows from a general theory of the tensor products.

(II) is obvious.

We prove (III). Let $\Psi \in D(N_{\text{b}}^{1/2})$. Then using Lemma 3.5, we have

$$\|H_1 \Psi\| \leq M_{\text{ph}} |q| \|(N_{\text{b}} + 1)^{1/2} \Psi\|, \quad (4.22)$$

with $M_{\text{ph}} = \sqrt{2}\|\widehat{\chi_{\text{ph}}}/\sqrt{\omega}\|_{L^2(\mathbb{R}_k^3)} + \frac{1}{\sqrt{2}}\|\widehat{\chi_{\text{ph}}}\|_{L^2(\mathbb{R}_k^3)}$. Hence H_1 is $N_{\text{b}}^{1/2}$ -bounded. Since $H_1^*\Psi = \eta H_1 \eta \Psi$ and η commutes with N_{b} , H_1^* is also $N_{\text{b}}^{1/2}$ -bounded.

We prove (IV). It is clear by definition that H_1 and H_1^* create at most one photon. Hence the assertion follows.

(V) follows from (II)-(IV). $\square$

The total space $\mathcal{F}_{\text{DM},N}$ can naturally be identified as

$$\mathcal{F}_{\text{DM},N} = \bigoplus_{n=0}^{\infty} (\bigwedge^N \mathcal{H}_D \otimes (\bigotimes_s^n \mathcal{H}_{\text{ph}}^{(\text{L})})). \quad (4.23)$$

We freely use this identification, and set

$$\mathcal{F}_{\text{b},0} := \bigoplus_{n=0}^{\infty} (\bigwedge^N \mathcal{H}_D \otimes (\bigotimes_s^n \mathcal{H}_{\text{ph}}^{(\text{L})})). \quad (4.24)$$

then it follows that

$$\mathcal{F}_{\text{b},0} = \bigcup_{L \geq 0} R(E_{N_{\text{b}}}([0, L])). \quad (4.25)$$

Set

$$D' := \mathcal{F}_{\text{b},0} \cap D(H_0). \quad (4.26)$$

From Lemma 4.1, we can apply Theorem 2.1 and obtain:

Theorem 4.1. *For each $\Psi \in \mathcal{F}_{\text{b},0}$, the series*

$$U(t, t')\Psi := \Psi + (-i) \int_{t'}^t d\tau_1 H_1(\tau_1)\Psi + (-i)^2 \int_{t'}^t d\tau_1 \int_{t'}^{\tau_1} d\tau_2 H_1(\tau_1)H_1(\tau_2)\Psi + \cdots \quad (4.27)$$

converges absolutely, where each of integrals is strong integral, and $H_1(\tau) = e^{i\tau H_0} H_1 e^{-i\tau H_0}$ ($\tau \in \mathbb{R}$). Furthermore, $U(t, t')$ has properties stated in Theorems 2.1, 2.3.

By Theorem 4.1, we can construct the time evolution operator $\{W(t)\}_{t \in \mathbb{R}}$ as follows:

$$W(t) := e^{-itH_0} U(t, 0), \quad t \in \mathbb{R}. \quad (4.28)$$

In the same manner as Lemma 3.18, one can see that $A_{\mu}(0, f)$ is in $\mathcal{C}_0(N_{\text{b}})$ -class. Then using Theorem 3.3, we can set

$$D(A_{\mu}(t, f)) := \mathcal{F}_{\text{b},0}, \quad (4.29)$$

$$A_{\mu}(t, f)\Psi = W(t)^{\dagger} A_{\mu}(0, f) W(t)\Psi, \quad \Psi \in \mathcal{F}_{\text{b},0}, \quad f \in \mathcal{S}(\mathbb{R}^3), \quad (4.30)$$

and obtain:

Theorem 4.2. *For all $\Psi, \Phi \in D'$,*

$$\frac{d}{dt} \langle \Psi | A^{\mu}(t, f) \Phi \rangle = \left\langle (iH)^{\dagger} \Psi | A^{\mu}(t, f) \Phi \right\rangle - \langle A^{\mu}(t, f) \Psi | (iH) \Phi \rangle. \quad (4.31)$$

Lemma 4.2. For each $\mu = 0, 1, 2, 3$ and $f \in \mathcal{S}(\mathbb{R}^3)$, $A_\mu(0, f)$ is in $\mathcal{C}_1(N_b)$ -class, and thus the strong derivative $\dot{A}_\mu(t, f) := \frac{\partial}{\partial t} A_\mu(t, f)$ on D' exists and it follows that

$$\dot{A}_\mu(0, f) = \frac{1}{\sqrt{2}} \left(a_\mu(i\sqrt{\omega}\widehat{f^*}) + a_\mu^\dagger(i\sqrt{\omega}\widehat{f}) \right) \quad (4.32)$$

on D' . Moreover, for all $\mu, \nu = 0, 1, 2, 3$ and $f, g \in \mathcal{S}(\mathbb{R}^3)$, the commutation relations

$$[A_\mu(t, f), A_\nu(t, g)] = [\dot{A}_\mu(t, f), \dot{A}_\nu(t, g)] = 0, \quad (4.33)$$

$$[A_\mu(t, f), \dot{A}_\nu(t, g)] = ig_{\mu\nu} \langle f, g \rangle \quad (4.34)$$

holds on D' .

Proof. By a direct calculation, we have

$$[iH, A_\mu(0, f)]_w^{D'} = \frac{1}{\sqrt{2}} \left(a_\mu(i\sqrt{\omega}\widehat{f^*}) + a_\mu^\dagger(i\sqrt{\omega}\widehat{f}) \right). \quad (4.35)$$

It is easy to check that the operator of the right-hand side is in $\mathcal{C}_0(N_b)$ -class. Hence we have $A_\mu(0, f) \in \mathcal{C}_1(N_b)$ and (4.32). (4.33) follows from a direct calculation. $\square$

Theorem 4.3. $A_\mu(0, f)$ is in $\mathcal{C}_2(N_b)$ -class, and thus for each $f \in \mathcal{S}(\mathbb{R}^3)$ and $\Psi \in D'$, the map $\mathbb{R} \ni t \mapsto A_\mu(t, f)\Psi$ is twice strongly differentiable. Moreover, it follows that

$$\square A_\mu(t, f)\Psi = -q \sum_{a=1}^N \alpha_{a\mu} \int_{\mathcal{X}^N}^\oplus d\mathbf{X} (f * \chi_{\text{ph}})(\mathbf{x}_a) \quad (4.36)$$

on D' . Here $f * \chi_{\text{ph}}$ is the ordinary convolution of f and χ_{ph} :

$$(f * \chi_{\text{ph}})(\mathbf{x}) = \int_{\mathbb{R}^3} d\mathbf{y} f(\mathbf{x} - \mathbf{y}) \chi_{\text{ph}}(\mathbf{y}). \quad (4.37)$$

Proof. We have already seen that $A_\mu(0, f)$ is in $\mathcal{C}_1(N_b)$ -class. To prove, $A_\mu(0, f) \in \mathcal{C}_2(N_b)$, we compute the weak commutator:

$$[iH, \dot{A}_\mu(0, f)]_w^{D'} = \frac{1}{\sqrt{2}} \left(a_\mu\left(\frac{-\omega^2 \widehat{f^*}}{\sqrt{\omega}}\right) + a_\mu^\dagger\left(\frac{-\omega^2 \widehat{f}}{\sqrt{\omega}}\right) \right) - q \sum_{j=1}^N \alpha_{a\mu} \int_{\mathcal{X}^N}^\oplus d\mathbf{X} (f * \chi_{\text{ph}})(\mathbf{x}_a). \quad (4.38)$$

The operator on the right-hand side is in $\mathcal{C}_0(N_b)$ -class. We note that, it follows that

$$\Delta A_\mu(0, f) = A_\mu(0, \Delta f) = \frac{1}{\sqrt{2}} \left(a_\mu\left(\frac{-\omega^2 \widehat{f^*}}{\sqrt{\omega}}\right) + a_\mu^\dagger\left(\frac{-\omega^2 \widehat{f}}{\sqrt{\omega}}\right) \right), \quad (4.39)$$

in the operator-valued distribution sense. Combining this with (4.38), we obtain (4.36). $\square$

4.3 Current conservation

In this subsection, we consider the electro-magnetic current density operator $j^\mu(0, f)$ and its time evolution.

Definition 4.1. The electro-magnetic current density operator at time $t = 0$ is an operator-valued distribution defined by

$$j^\mu(0, f) := q \sum_{a=1}^N \alpha_a^\mu \int_{\mathcal{X}}^\oplus d\mathbf{X} f(\mathbf{x}_a), \quad f \in \mathcal{S}(\mathbb{R}^3). \quad (4.40)$$

Remark 4.1. α_a^μ physically denotes the μ -component of the velocity of the a -th electron. Thus the informal definition of the current density operator should be

$$j^\mu(0, \mathbf{x}) = q \sum_{a=1}^N \alpha_a^\mu \delta(\mathbf{x} - \mathbf{x}_a). \quad (4.41)$$

By smearing $j^\mu(0, \mathbf{x})$ with the smooth f , we arrive at Definition 4.1.

Theorem 4.4. The current density at time $t = 0$: $j^\mu(0, f)$ is in $\mathcal{C}_0(N_b)$ -class, and thus the Heisenberg operator $j^\mu(t, f) := W(-t)j^\mu(0, f)W(t)$ has a dense domain $\mathcal{F}_{b,0}$. The zero-th component $j^0(0, f)$ is in $\mathcal{C}_1(N_b)$ -class, and thus $t \mapsto j^0(t, f)$ satisfies the strong Heisenberg equation of motion. Moreover, the current conservation:

$$\partial_\mu j^\mu(t, f) := \frac{\partial}{\partial t} j^0(t, f) + \sum_{k=1,2,3} \frac{\partial}{\partial x^k} j^k(t, f) = 0 \quad (4.42)$$

holds on D' .

Proof. Since $j^\mu(0, f)$ is bounded and leave V_L invariant, we have $j^\mu(0, f) \in \mathcal{C}_0(N_b)$.

We prove that $j^0(0, f)$ is in $\mathcal{C}_0(N_b)$ -class. By a direct calculation, we have

$$[iH, j^0(0, f)]_w^{D'} = q \sum_{a=1}^N \sum_{k=1}^3 \alpha_a^k \int_{\mathcal{X}}^{\oplus} d\mathbf{X} (\partial_k f)(\mathbf{x}_a) \quad (4.43)$$

$$= \sum_{k=1}^3 j^k(0, \partial_k f). \quad (4.44)$$

Since $\sum_{k=1}^3 j^k(0, \partial_k f) \in \mathcal{C}_0(N_b)$ by the above observation, it follows that $j^0(0, f) \in \mathcal{C}_1(N_b)$. Therefore applying Theorem 3.4, we conclude that the map $t \mapsto j^0(t, f)$ is strongly differentiable on D' and satisfies (4.42). $\square$

4.4 Gupta-Bleular condition

By Lemma 4.2, for all $f \in \mathcal{S}(\mathbb{R}^3)$, we can define

$$\Omega(t, f) := \partial_\mu A^\mu(t, f) \quad (4.45)$$

$$= \frac{\partial}{\partial t} A^0(t, f) + \sum_{k=1}^3 \frac{\partial}{\partial x^k} A^k(0, f). \quad (4.46)$$

then $\Omega(t, f)$ is a free field, and we can write it explicitly:

Theorem 4.5. For all $t \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^3)$,

$$\square \Omega(t, f) = 0 \quad (4.47)$$

on D' . In particular,

$$\begin{aligned} \Omega(t, f) &= a_\mu \left(\frac{ik^\mu e^{it\omega} \widehat{f^*}}{\sqrt{2\omega}} \right) + a_\mu^\dagger \left(\frac{ik^\mu e^{it\omega} \widehat{f}}{\sqrt{2\omega}} \right) \\ &\quad + \sum_{a=1}^N iq \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \left(\left\langle \frac{e^{it\omega} \widehat{f^*}}{\sqrt{2\omega}}, \frac{\widehat{\chi_{\text{ph}}^{\mathbf{x}_a}}}{\sqrt{2\omega}} \right\rangle - \left\langle \frac{\widehat{\chi_{\text{ph}}^{\mathbf{x}_a}}}{\sqrt{2\omega}}, \frac{e^{it\omega} \widehat{f}}{\sqrt{2\omega}} \right\rangle \right) \end{aligned} \quad (4.48)$$

on D' .

Lemma 4.3. (i) For all $f \in \mathcal{S}(\mathbb{R}^3)$ and $n = 0, 1, 2, \dots$,

$$\begin{aligned} \text{w-ad}_{iH,D'}^n(\Omega(0, f)) &= a_\mu \left(\frac{ik^\mu(i\omega)^n \widehat{f^*}}{\sqrt{2\omega}} \right) + a_\mu^\dagger \left(\frac{ik^\mu(i\omega)^n \widehat{f}}{\sqrt{2\omega}} \right) \\ &\quad + \sum_{a=1}^N iq \int_{\mathcal{X}^N}^\oplus d\mathbf{X} \left(\left\langle \frac{(i\omega)^n \widehat{f^*}}{\sqrt{2\omega}}, \frac{\widehat{\chi_{\text{ph}}^{x_a}}}{\sqrt{2\omega}} \right\rangle - \left\langle \frac{\widehat{\chi_{\text{ph}}^{x_a}}}{\sqrt{2\omega}}, \frac{(i\omega)^n \widehat{f}}{\sqrt{2\omega}} \right\rangle \right). \end{aligned} \quad (4.49)$$

(ii) For all $f \in \mathcal{S}(\mathbb{R}^3)$, $\Omega(t, f)$ is in $C_\infty(N_b)$ -class.

(iii) For all $f \in \mathcal{S}(\mathbb{R}^3)$ such that $\widehat{f} \in C_0^\infty(\mathbb{R}^3)$, $\Omega(t, f)$ is in $C_\omega(N_b)$ -class.

Proof. (i) This follows from a direct calculation.

(ii) In the same manner as in the proof of Lemma 3.12, we see that $\text{w-ad}_{iH,D'}^n(\Omega(0, f)) \in C_0(N_b)$. Hence $\Omega(t, f) \in C_\infty(N_b)$.

(iii) We show that $\Omega(0, f)$ satisfies the condition stated in Definition 3.4. From (ii), it follows that $\Omega(0, f) \in C_\infty(N_b)$.

By the assumption $\widehat{f} \in C_0^\infty(\mathbb{R}^3)$, there exists a constant $R_1 \geq 0$ such that $\text{supp } \widehat{f} \subset \{\mathbf{k} \in \mathbb{R}^3 \mid |\mathbf{k}| \leq R_1\}$. Using Lemma 3.4, for all $\Psi \in D(N_b^{1/2})$, we have

$$\|a_\mu \left(\frac{ik^\mu(i\omega)^n \widehat{f^*}}{\sqrt{2\omega}} \right) \Psi\| \leq R_1^n \left\| \frac{k^\mu \widehat{f}}{\sqrt{2\omega}} \right\| \|N_b^{1/2} \Psi\|, \quad (4.50)$$

$$\|a_\mu^\dagger \left(\frac{ik^\mu(i\omega)^n \widehat{f^*}}{\sqrt{2\omega}} \right) \Psi\| \leq R_1^n \left\| \frac{k^\mu \widehat{f}}{\sqrt{2\omega}} \right\| \|N_b^{1/2} \Psi\| + R_1^n \left\| \frac{k^\mu \widehat{f}}{\sqrt{2\omega}} \right\| \|\Psi\|. \quad (4.51)$$

The operator of the third term in the right-hand side of (4.49) is a bounded operator, and its operator norm is estimated as

$$\left\| \sum_{a=1}^N iq \int_{\mathcal{X}^N}^\oplus d\mathbf{X} \left(\left\langle \frac{(i\omega)^n \widehat{f^*}}{\sqrt{2\omega}}, \frac{\widehat{\chi_{\text{ph}}^{x_a}}}{\sqrt{2\omega}} \right\rangle - \left\langle \frac{\widehat{\chi_{\text{ph}}^{x_a}}}{\sqrt{2\omega}}, \frac{(i\omega)^n \widehat{f}}{\sqrt{2\omega}} \right\rangle \right) \right\| \leq 2N|q|R_1^n \left\| \frac{\widehat{f}}{\sqrt{2\omega}} \right\| \left\| \frac{\widehat{\chi_{\text{ph}}}}{\sqrt{2\omega}} \right\|. \quad (4.52)$$

hence we have

$$\|\text{w-ad}_{iH,D'}^n(\Omega(0, f))\Psi\| \leq R_0 R_1^n \|(N_b + 1)^{1/2}\| \quad (4.53)$$

with a constant $R_0 > 0$ which is independent of n .

It is obvious from (4.49) that for all n and $L \geq 0$, $\text{w-ad}_{iH,D'}^n(\Omega(0, f))$ maps V_L into V_{L+1} . therefore $\Omega(t, f)$ is in $C_\omega(N_b)$ -class. $\square$

Proof of Theorem 4.5. Let By Lemma 4.3, the map $t \mapsto \Omega(t, f)$ is strongly differentiable on D' and the n -th strong derivative on D' is then given by

$$\frac{\partial^n}{\partial t^n} \Omega(t, f) = W(-t) \text{w-ad}_{iH,D'}^n(\Omega(0, f_m)) W(t). \quad (4.54)$$

This and (4.49) imply (4.47).

Let $f \in \mathcal{S}(\mathbb{R}^3)$. Then there exists a sequence $f_m \in \mathcal{S}(\mathbb{R}^3)$ such that $\widehat{f_m} \in C_0^\infty(\mathbb{R}^3)$ and $f_m \rightarrow f$ in the distribution sense. By Lemma 4.3, we can apply Theorem 3.7, and obtain

$$\Omega(t, f_m)\Psi = \sum_{n=0}^{\infty} \frac{t^n}{n!} \text{w-ad}_{iH, D'}^n(\Omega(0, f_m))\Psi. \quad (4.55)$$

Combining this with Lemma 4.49, the equation (4.48) holds for f_m . Furthermore, we can take the limit $m \rightarrow \infty$ for the right-hand side of (4.48) by using Lemma A.7, for the left-hand side by using the fact that $\Omega(t, f)$ is operator-valued distribution. Therefore the desired result follows. $\square$

Let us consider the Gupta subsidiary condition. Theorem 4.5 (4.48) means that $\Omega(t, f)$ is formally written as

$$\begin{aligned} \Omega(t, f) = & \int d\mathbf{k} \left(a_\mu(\mathbf{k}) \frac{-ik^\mu e^{-it\omega(\mathbf{k})} \widehat{f^*}(\mathbf{k})^*}{\sqrt{2\omega(\mathbf{k})}} + a_\mu^\dagger(\mathbf{k}) \frac{ik^\mu e^{it\omega(\mathbf{k})} \widehat{f}(\mathbf{k})}{\sqrt{2\omega(\mathbf{k})}} \right) \\ & + \sum_{a=1}^N iq \int_{\mathcal{X}^N}^\oplus d\mathbf{X} \int d\mathbf{k} \left(\frac{e^{-it\omega(\mathbf{k})} \widehat{f^*}(\mathbf{k})^*}{\sqrt{2\omega(\mathbf{k})}} \frac{\widehat{\chi_{\text{ph}}^a}(\mathbf{k})}{\sqrt{2\omega(\mathbf{k})}} - \frac{\widehat{\chi_{\text{ph}}^a}(\mathbf{k})}{\sqrt{2\omega(\mathbf{k})}} \frac{e^{it\omega(\mathbf{k})} \widehat{f}(\mathbf{k})}{\sqrt{2\omega(\mathbf{k})}} \right), \end{aligned} \quad (4.56)$$

where $a_\mu(\mathbf{k})$ and $a_\mu^\dagger(\mathbf{k})$ are the formal distribution kernel of the operator-valued distribution $a_\mu(f)$ and $a_\mu^\dagger(f)$ ($f \in L^2(\mathbb{R}_\mathbf{k}^3)$), respectively. Since *positive frequency part* is defined as the term including the factor $e^{-it\omega(\mathbf{k})}$, we define the positive frequency part $\Omega^{(+)}(t, f)$ of $\Omega(t, f)$ as follows:

$$\Omega^{(+)}(t, f) := a_\mu \left(\frac{ik^\mu e^{it\omega} \widehat{f^*}}{\sqrt{2\omega}} \right) + \sum_{a=1}^N iq \int_{\mathcal{X}^N}^\oplus d\mathbf{X} \left\langle \frac{e^{it\omega} \widehat{f^*}}{\sqrt{2\omega}}, \frac{\widehat{\chi_{\text{ph}}^a}}{\sqrt{2\omega}} \right\rangle. \quad (4.57)$$

then the physical subspace $\mathcal{V}_{\text{phys}}$ is defined by

$$\mathcal{V}_{\text{phys}} := \{ \Psi \in \mathcal{F}_{\text{DM}} \mid \Omega^{(+)}(t, f)\Psi = 0 \text{ for all } t \in \mathbb{R} \text{ and } f \in \mathcal{S}(\mathbb{R}^3) \}. \quad (4.58)$$

Proposition 4.1. $\mathcal{V}_{\text{phys}}$ is a closed subspace.

Proof. By the definition of $\mathcal{V}_{\text{phys}}$,

$$\mathcal{V}_{\text{phys}} = \bigcap_{t \in \mathbb{R}, f \in \mathcal{S}(\mathbb{R}^3)} \ker \Omega^{(+)}(t, f). \quad (4.59)$$

hence the assertion follows from the closedness of the operators $\Omega^{(+)}(t, f)$. $\square$

Theorem 4.6. $\mathcal{V}_{\text{phys}}$ is non-negative, that is, for all $\Psi \in \mathcal{V}_{\text{phys}}$, $\langle \Psi | \Psi \rangle \geq 0$.

To prove Theorem 4.6, we need some preliminaries. Let $W := \Gamma_b([W]) : \mathcal{F}_{\text{ph}} \rightarrow \mathcal{F}_{\text{ph}}$ the second quantization operator of the unitary operator

$$[W] := \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix} : \bigoplus^4 L^2(\mathbb{R}^3) \rightarrow \bigoplus^4 L^2(\mathbb{R}^3). \quad (4.60)$$

W is then unitary and self-adjoint, and satisfies $W^2 = 1$ and

$$Wc(f_0, f_1, f_2, f_3)W = c\left(\frac{-f_0 + f_3}{\sqrt{2}}, f_1, f_2, \frac{f_0 + f_3}{\sqrt{2}}\right) \quad (4.61)$$

for $f_\mu \in L^2(\mathbb{R}^3)$ ($\mu = 0, 1, 2, 3$).

Following the original method of Bleuler [14], we define a linear operator G on $\mathcal{F}_{\text{DM}}(N)$ by

$$G := -q \sum_{a=1}^N \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \frac{1}{\sqrt{2}} \left(c^3 \left(\frac{i \widehat{\chi_{\text{ph}}^{x_a}}}{\omega^{3/2}} \right) + c_3^\dagger \left(\frac{i \widehat{\chi_{\text{ph}}^{x_a}}}{\omega^{3/2}} \right) \right). \quad (4.62)$$

then G is essentially self-adjoint on $\mathcal{F}_{\text{b},0}$. We denote the closure of G by the same symbol.

Lemma 4.4. *For all $t \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^3)$,*

$$e^{-iG} \Omega^{(+)}(t, f) e^{iG} \Psi = \left(c^0 \left(\frac{i\omega e^{it\omega} \widehat{f^*}}{\sqrt{2\omega}} \right) - c^3 \left(\frac{i\omega e^{it\omega} \widehat{f^*}}{\sqrt{2\omega}} \right) \right) \Psi. \quad (4.63)$$

In particular,

$$W e^{-iG} \Omega^{(+)}(t, f) e^{iG} W \Psi = \sqrt{2} c^0 \left(\frac{i\omega e^{it\omega} \widehat{f^*}}{\sqrt{2\omega}} \right) \Psi. \quad (4.64)$$

Proof. Using the definitions (4.57) and (4.62), we have

$$\begin{aligned} e^{-iG} \Omega^{(+)}(t, f) e^{iG} \Psi &= \sum_{m,n=0}^{\infty} \frac{1}{n!m!} (iG)^n \Omega^{(+)}(t, f) (-iG)^m \Psi \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \text{ad}_{iG}^N (\Omega^{(+)}(t, f)) \Psi. \end{aligned} \quad (4.65)$$

for all $\Psi \in \mathcal{F}_{\text{b},0}$. By a direct computation, we have

$$\text{ad}_{iG}^1 (\Omega^{(+)}(t, f)) \Psi = - \sum_{a=1}^N i q \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \left\langle \frac{e^{it\omega} \widehat{f^*}}{\sqrt{2\omega}}, \frac{\widehat{\chi_{\text{ph}}^{x_a}}}{\sqrt{2\omega}} \right\rangle \Psi, \quad (4.66)$$

$$\text{ad}_{iG}^N (\Omega^{(+)}(t, f)) \Psi = 0, \quad N \geq 2. \quad (4.67)$$

Combining (4.66), (4.67) with (4.65), we obtain (4.63).

(4.64) follows from (4.63), (4.61) and the fact that W leave $\mathcal{F}_{\text{b},0}$ invariant. $\square$

To identify the physical subspace $\mathcal{V}_{\text{phys}}$, we introduce some notations. Hereafter, we freely use the following identification:

$$\mathcal{F}_{\text{b}}(\oplus^4 L^2(\mathbb{R}^3)) = \mathcal{F}_{\text{b}}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_{\text{b}}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_{\text{b}}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_{\text{b}}(L^2(\mathbb{R}^3)). \quad (4.68)$$

Let $\mathcal{L}\{\Omega_{\mathcal{F}}\}$ be the Fock vacuum of $\mathcal{F}_{\text{b}}(L^2(\mathbb{R}^3))$. Then we set

$$\mathcal{F}_{\text{TL}}^{(0)} := \mathcal{L}\{\Omega_{\mathcal{F}}\} \otimes \mathcal{F}_{\text{b}}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_{\text{b}}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_{\text{b}}(L^2(\mathbb{R}^3)). \quad (4.69)$$

Lemma 4.5. *The following holds:*

$$\mathcal{V}_{\text{phys}} = e^{iG} W \mathcal{F}_{\text{TL}}^{(0)}. \quad (4.70)$$

Proof. By the definition (4.58) and Lemma 4.4, we have

$$\mathcal{V}_{\text{phys}} = e^{iG} W \bigcap_{t \in \mathbb{R}, f \in \mathcal{S}(\mathbb{R}^3)} \ker c^0 \left(\frac{i\omega e^{it\omega} \widehat{f^*}}{\sqrt{2\omega}} \right). \quad (4.71)$$

Let us note that the subspace

$$\left\{ \frac{i\omega e^{it\omega} \widehat{f^*}}{\sqrt{2\omega}} \mid f \in \mathcal{S}(\mathbb{R}^3), t \in \mathbb{R} \right\} \quad (4.72)$$

is dense in $L^2(\mathbb{R}^3)$. This fact and Lemma A.6 imply that

$$\bigcap_{t \in \mathbb{R}, f \in \mathcal{S}(\mathbb{R}^3)} \ker c^0 \left(\frac{i\omega e^{it\omega} \widehat{f^*}}{\sqrt{2\omega}} \right) = \mathcal{F}_{\text{TL}}^{(0)}. \quad (4.73)$$

hence we obtain (4.70). $\square$

Proof of Theorem 4.6. We provide only a sketch of the proof since it is similar to that of [60, Theorem 2.17].

Let

$$\mathcal{F}_{\text{TL}} := \mathcal{L}\{\Omega_{\mathcal{F}}\} \otimes \mathcal{F}_{\text{b}}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_{\text{b}}(L^2(\mathbb{R}^3)) \otimes \mathcal{L}\{\Omega_{\mathcal{F}}\}. \quad (4.74)$$

then we observe that

$$\mathcal{F}_{\text{TL}}^{(0)} = \mathcal{F}_{\text{TL}} \oplus (\mathcal{F}_{\text{TL}}^\perp \cap \mathcal{F}_{\text{TL}}^{(0)}). \quad (4.75)$$

Here we denote by the $\mathcal{M}^\perp$ the orthogonal component of a set $\mathcal{M}$. By Lemma 4.5, for all $\Psi \in \mathcal{V}_{\text{phys}}$, there exists a vector $\Phi \in \mathcal{F}_{\text{TL}}^{(0)}$ which is decomposed as

$$\Phi = \Phi_{\text{TL}} + \Phi_0, \quad \Phi_{\text{TL}} \in \mathcal{F}_{\text{TL}}, \quad \Phi_0 \in \mathcal{F}_{\text{TL}}^\perp \cap \mathcal{F}_{\text{TL}}^{(0)}, \quad (4.76)$$

and satisfies $U\Psi = \Phi$ with $U := We^{-iG}$. Then it follows that

$$\langle \Psi | \Psi \rangle = \langle U^*(\Phi_{\text{TL}} + \Phi_0), \eta U^*(\Phi_{\text{TL}} + \Phi_0) \rangle \quad (4.77)$$

$$= \langle \Phi, \Phi \rangle + \langle \Phi_0, \Phi_{\text{TL}} \rangle + \langle \Phi_{\text{TL}}, \Phi_0 \rangle + \langle U^* \Phi_{\text{TL}}, \eta U^* \Phi_{\text{TL}} \rangle. \quad (4.78)$$

The first term in the right-hand side is non-negative, and the other terms vanish. Hence the assertion follows. $\square$

Let us consider the Hamiltonian restricted over $\mathcal{V}_{\text{phys}}$. We denote by P_{phys} by the orthogonal projection onto $\mathcal{V}_{\text{phys}}$, and define

$$H_{\mathcal{V}_{\text{phys}}} := P_{\text{phys}} H P_{\text{phys}}. \quad (4.79)$$

In what follows, we use the natural identification

$$\mathcal{F}_{\text{ph}}^{(\text{C})} \cong \mathcal{F}_{\text{TL}}, \quad (4.80)$$

where $\mathcal{F}_{\text{ph}}^{(\text{C})}$ is the state space of the photon field introduced in Subsection 5.2. The point-like photon field quantized in the Coulomb gauge $A_j^{(\text{C})}(0, f)$ ($j = 1, 2, 3$, $f \in \mathcal{S}(\mathbb{R}^3)$) is then identified with

$$1 \otimes A_j^{(\text{C})}(0, f) \otimes 1 \quad (4.81)$$

acting in $\mathcal{F}_{\text{TL}} = \mathcal{L}\{\Omega_{\mathcal{F}}\} \otimes \mathcal{F}_{\text{b}}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}_{\text{b}}(L^2(\mathbb{R}^3)) \otimes \mathcal{L}\{\Omega_{\mathcal{F}}\}$.

Theorem 4.7.

$$\begin{aligned}
e^{iG} H e^{-iG} &= H_0 - q \sum_{a=1}^N \sum_{k=1}^3 \alpha_a^k \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} A_k^{(C)}(0, \mathbf{x}_a) \\
&\quad + q^2 \sum_{a < b}^N \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \int \frac{d\mathbf{k}}{\omega(\mathbf{k})^2} |\widehat{\chi_{\text{ph}}}(\mathbf{k})|^2 e^{i\mathbf{k}(\mathbf{x}_a - \mathbf{x}_b)} + \frac{Nq^2}{2} \|\widehat{\chi_{\text{ph}}}/\omega\|^2 \\
&\quad - q \sum_{a=1}^N \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \frac{1}{\sqrt{2}} \left(c^0 \left(\frac{\widehat{\chi_{\text{ph}}^{\mathbf{x}_a}}}{\sqrt{\omega}} \right) - c^3 \left(\frac{\widehat{\chi_{\text{ph}}^{\mathbf{x}_a}}}{\sqrt{\omega}} \right) + c_0^\dagger \left(\frac{\widehat{\chi_{\text{ph}}^{\mathbf{x}_a}}}{\sqrt{\omega}} \right) - c_3^\dagger \left(\frac{\widehat{\chi_{\text{ph}}^{\mathbf{x}_a}}}{\sqrt{\omega}} \right) \right). \quad (4.82)
\end{aligned}$$

on $\mathcal{D} := (\widehat{\otimes}_a^N C_0^\infty(\mathbb{R}^3; \mathbb{C}^4)) \widehat{\otimes} \mathcal{F}_{\text{b,fin}}(C_0^\infty(\mathbb{R}^3; \mathbb{C}^4))$.

Lemma 4.6. (i) $\mathcal{D}$ forms an entire analytic vector space for G .

(ii) $\mathcal{D} \subset D(e^{iG} H e^{-iG})$ and

$$e^{iG} H e^{-iG} \Psi = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}_{iG}^n H) \Psi, \quad (4.83)$$

for every $\Psi \in \mathcal{D}$.

(iii) It follows that

$$\begin{aligned}
\text{ad}_{iG}^1 H &= q \sum_{a=1}^N \sum_{i=1}^3 \alpha_a^i \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \frac{1}{\sqrt{2}} \left(c^3 \left(\frac{k^i \widehat{\chi_{\text{ph}}^{\mathbf{x}_a}}}{\omega^{3/2}} \right) + c_3^\dagger \left(\frac{k^i \widehat{\chi_{\text{ph}}^{\mathbf{x}_a}}}{\omega^{3/2}} \right) \right) \\
&\quad + q \sum_{a=1}^N \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \frac{1}{\sqrt{2}} \left(c^3 \left(\frac{-\widehat{\chi_{\text{ph}}^{\mathbf{x}_a}}}{\omega^{1/2}} \right) + c_3^\dagger \left(\frac{-\widehat{\chi_{\text{ph}}^{\mathbf{x}_a}}}{\omega^{1/2}} \right) \right), \quad (4.84)
\end{aligned}$$

$$\text{ad}_{iG}^2 H = q^2 N \|\widehat{\chi_{\text{ph}}}/\omega\|^2 + 2q^2 \sum_{a < b}^N \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \int \frac{d\mathbf{k}}{\omega(\mathbf{k})^2} |\widehat{\chi_{\text{ph}}}(\mathbf{k})|^2 e^{i\mathbf{k}(\mathbf{x}_a - \mathbf{x}_b)}. \quad (4.85)$$

on $\mathcal{D}$

Proof. (i) It is easy to see that for all $t > 0$ and $\Psi \in \mathcal{F}_{\text{b},0}$,

$$\sum_{n=0}^{\infty} \frac{\|G^n \Psi\|}{n!} t^n < \infty. \quad (4.86)$$

hence the assertion follows.

(ii) Since e^{iG} is unitary, it is sufficient to prove $e^{-iG} \Psi \in D(H)$. By (i), we have

$$e^{iG} \Psi = \sum_{n=0}^{\infty} \frac{i^n}{n!} G^n \Psi. \quad (4.87)$$

It is easy to see that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|H_1 G^n \Psi\| < \infty, \quad (4.88)$$

by using Lemma 4.1 (IV) and $\|G(N_b+1)^{-1/2}\| < \infty$. Hence it follows from the closedness of H_1 that $e^{-iG}\Psi \in D(H_1)$. Moreover, one can see that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|H_0 G^n \Psi\| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{j=1}^n \|G^{j-1} [H_0, G] G^{n-j} \Psi\| + \|G^n H_0 \Psi\| \right) \quad (4.89)$$

$$< \infty, \quad (4.90)$$

by using calculating the commutator $[H_0, G]$ which is $N_b^{1/2}$ -bounded. Thus $e^{-iG}\Psi \in D(H_0)$ follows from the closedness of H_0 , and the desired result holds.

(iii) This follows from a direct calculation. □

Proof of Theorem 4.7. Note that the interaction Hamiltonian H_1 is decomposed as

$$\begin{aligned} H_1 = & q \sum_{a=1}^N \sum_{i=1}^3 \alpha_a^i \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \sum_{r=1,2} \frac{1}{\sqrt{2}} \left(c^r \left(\frac{\widehat{\chi_{\text{ph}}^{\mathbf{x}_a} e_{ir}}}{\sqrt{\omega}} \right) + c_r^\dagger \left(\frac{\widehat{\chi_{\text{ph}}^{\mathbf{x}_a} e_{ir}}}{\sqrt{\omega}} \right) \right) \\ & + q \sum_{a=1}^N \sum_{i=1}^3 \alpha_a^i \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \frac{1}{\sqrt{2}} \left(c^3 \left(\frac{\widehat{\chi_{\text{ph}}^{\mathbf{x}_a} k_i}}{\sqrt{\omega}} \right) + c_3^\dagger \left(\frac{\widehat{\chi_{\text{ph}}^{\mathbf{x}_a} k_i}}{\sqrt{\omega}} \right) \right) \\ & + q \sum_{a=1}^N \sum_{i=1}^3 \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \frac{1}{\sqrt{2}} \left(c^0 \left(\frac{\widehat{\chi_{\text{ph}}^{\mathbf{x}_a} k_i}}{\sqrt{\omega}} \right) + c_0^\dagger \left(\frac{\widehat{\chi_{\text{ph}}^{\mathbf{x}_a} k_i}}{\sqrt{\omega}} \right) \right) \end{aligned}$$

on $\mathcal{F}_{b,0}$. Combining this with Lemma 4.6 (i) and (ii), we obtain (4.82). □

Corollary 4.1.

$$\begin{aligned} H_{V_{\text{phys}}} = & e^{-iG} \left(H_D(V, N) + H_{\text{ph}} \upharpoonright \mathcal{F}_{\text{TL}}^{(0)} - q \sum_{a=1}^N \sum_{k=1}^3 \alpha_a^k \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} A_k^{(C)}(0, \mathbf{x}_a) \right. \\ & \left. + q^2 \sum_{a < b}^N \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \int \frac{d\mathbf{k}}{\omega(\mathbf{k})^2} |\widehat{\chi_{\text{ph}}}(\mathbf{k})|^2 e^{i\mathbf{k}(\mathbf{x}_a - \mathbf{x}_b)} + \frac{Nq^2}{2} \|\widehat{\chi_{\text{ph}}}/\omega\|^2 \right) e^{iG} \quad (4.91) \end{aligned}$$

on $\mathcal{D}$.

Remark 4.2. We remark that the operator

$$q^2 \sum_{a < b}^N \int_{\mathcal{X}^N}^{\oplus} d\mathbf{X} \int \frac{d\mathbf{k}}{\omega(\mathbf{k})^2} |\widehat{\chi_{\text{ph}}}(\mathbf{k})|^2 e^{i\mathbf{k}(\mathbf{x}_a - \mathbf{x}_b)} \quad (4.92)$$

on the right-hand side of (4.91) is identified as a multiplication operator acting on $\wedge^N \mathcal{H}_D$ defined by the function

$$(\mathbf{x}_1, \dots, \mathbf{x}_N) \mapsto q^2 \sum_{a < b}^N \int \frac{d\mathbf{k}}{\omega(\mathbf{k})^2} |\widehat{\chi_{\text{ph}}}(\mathbf{k})|^2 e^{i\mathbf{k}(\mathbf{x}_a - \mathbf{x}_b)}, \quad (4.93)$$

which converges to the familiar Coulomb potential

$$\sum_{a < b}^N \frac{q^2}{4\pi} \frac{1}{|\mathbf{x}_a - \mathbf{x}_b|}$$

in the distribution sense as the photon ultraviolet cutoff $\widehat{\chi_{\text{ph}}}$ is removed.

The constant term

$$\frac{Nq^2}{2} \|\widehat{\chi_{\text{ph}}}/\omega\|^2 \quad (4.94)$$

on the right-hand side of (4.91) is the self-energy of electrons which diverges as the photon ultraviolet cutoff $\widehat{\chi_{\text{ph}}}$ is removed.

Remark 4.3. The methods which we presented in this section can be applied also to several solvable model. However, our methods cannot be applied to QED with naive cutoffs defined in Section 3.3. In fact, both the ultraviolet and the spatial cutoffs brake the current conservation: $\partial_\mu j^\mu \neq 0$, and hence $\partial_\mu A^\mu$ is not free: $\square \partial_\mu A^\mu \neq 0$.

5 Gell-Mann – Low formula

5.1 Complex time evolution and Gell-Man – Low formula

In this subsection, we use the notations which have already been introduced in Subsection 2.2.

Let H_0 be a non-negative self-adjoint operator on a Hilbert space $\mathcal{H}$. In this subsection, we consider the operator

$$H = H_0 + H_1 \quad (5.1)$$

with $H_1 \in \mathcal{C}_0(H_0)$, and we state and derive the Gell-Mann – Low formula. In what follows, we shortly denote

$$V_n(z, z') := V_n(H_1; z, z'), \quad U(z, z') := U(H_1; z, z'), \quad (5.2)$$

where $V_n(H_1; z, z')$ and $U(H_1; z, z')$ are the operators defined by (2.92) and (2.89), respectively.

We define complex time evolution operator

$$W(z) := e^{-izH_0} \overline{U(z, 0)} \quad (5.3)$$

for $z \in \mathbb{C}$ with $\text{Im } z \leq 0$. The operator $W(z)$ generates the “complex time evolution” in the following sense.

Let $\mathcal{D}$ be the dense subspace defined by (2.125). Then

Theorem 5.1. *For all $\Psi \in \mathcal{D}$, the mapping $z \mapsto W(z)\Psi$ is analytic on the lower half plain and satisfies the “complex Schrödinger equation”*

$$\frac{d}{dz} W(z)\Psi = -iHW(z)\Psi. \quad (5.4)$$

Proof. We first remark that $\mathcal{D} \subset D(H_0)$. This can be seen by noting that $\mathcal{D} \subset D(e^{H_0}) \subset D(H_0)$. By Theorem 2.5, one can easily estimate

$$\left\| \frac{W(z+h)\Psi - W(z)\Psi}{h} - (-iH)W(z)\Psi \right\| \quad (5.5)$$

to know that this vanishes in the limit $h \rightarrow 0$.

□

Theorem 5.2. *Suppose that H_1 is $\mathcal{C}_0(H_0)$ -class symmetric operator. Then H is self-adjoint and bounded below. Moreover, it follows that*

$$\overline{W(z)}\Psi = e^{-izH}, \quad (5.6)$$

for all $z \in \mathbb{C}$ with $\text{Im } z \leq 0$. In particular, it follows that

$$\overline{U(z, z')} = e^{izH_0} e^{-i(z-z')H} e^{-iz'H_0}, \quad \text{Im } z \leq \text{Im } z'. \quad (5.7)$$

Proof. From Assumption 5.1, H_1 is infinitesimal with respect to H_0 and thus H is self-adjoint with $D(H) = D(H_0)$, and bounded below by the Kato-Rellich Theorem.

By Theorem 5.1, we can differentiate for all $\Psi \in \mathcal{D}$, $\Phi \in D_0(H) := \cup_{L \in \mathbb{R}} R(E_H([-L, L]))$, and $z \in \mathbb{C}$ with $\text{Im } z < 0$,

$$\begin{aligned} \frac{d}{dz} \langle e^{-iz^*H} \Phi, W(z)\Psi \rangle &= \langle -iH e^{-iz^*H} \Phi, W(z)\Psi \rangle + \langle e^{-iz^*H} \Phi, -iW(z)\Psi \rangle \\ &= 0. \end{aligned} \quad (5.8)$$

thus one finds

$$\langle \Phi, \Psi \rangle = \langle e^{-iz^*H} \Phi, W(z)\Psi \rangle, \quad (5.9)$$

for all $\Psi \in \mathcal{D}$ and $\Phi \in D_0(H)$. Since $D_0(H)$ is a core of e^{-iz^*H} , we obtain from (5.9) $W(z)\Psi \in D(e^{izH})$ and

$$e^{izH} W(z)\Psi = \Psi. \quad (5.10)$$

hence we arrive at

$$W(z)\Psi = e^{-izH}\Psi, \quad (5.11)$$

for all $z \in \mathbb{C}$ with $\text{Im } z < 0$. But since both sides of (5.11) are continuous on the region $\text{Im } z \leq 0$, (5.11) must hold on $\text{Im } z \leq 0$. Since the both sides are bounded, one has

$$\overline{W(z)} = e^{-izH}, \quad \text{Im } z \leq 0. \quad (5.12)$$

For z, z' satisfying $\text{Im } z \leq \text{Im } z'$, we have from (2.145)

$$\begin{aligned} W(z - z')\Psi &= e^{-i(z-z')H_0} \overline{U(z - z', 0)}\Psi \\ &= e^{-izH_0} \overline{U(z, z')} e^{iz'H_0}\Psi, \quad \Psi \in \mathcal{D}. \end{aligned} \quad (5.13)$$

This implies

$$\overline{U(z, z')}\Psi = e^{izH_0} e^{-i(z-z')H} e^{iz'H_0}\Psi. \quad (5.14)$$

If z, z' are real, the right-hand-side is unitary, and thus the last assertion follows. $\square$

We introduce the assumptions needed to derive the Gell-Mann – Low formula. For a linear operator T , we denote the spectrum of T by $\sigma(T)$. If T is self-adjoint and bounded from below, then we define

$$E_0(T) := \inf \sigma(T). \quad (5.15)$$

We say that T has a ground state if $E_0(T)$ is an eigenvalue of T . In that case, $E_0(T)$ is called the *ground energy* of T , and each non-zero vector in $\ker(T - E_0(T))$ is called a *ground state* of T . If $\dim \ker(T - E_0(T)) = 1$, we say that T has a unique ground state. The following assumption are used to prove the Gell-Mann – Low formula.

Assumption 5.1. (I) H_0 has a unique ground state Ω_0 with $\|\Omega_0\| = 1$, and the ground energy is zero: $E_0(H_0) = 0$.

(II) H has a unique ground state Ω with $\|\Omega\| = 1$.

(III) $\langle \Omega, \Omega_0 \rangle \neq 0$.

Under Assumption 5.1, we define m -point Green's function $G_m(z_1, \dots, z_m)$ by

$$G_m(z_1, \dots, z_m) := e^{i(z_1 - z_m)E_0(H)} \langle \Omega, A_1 W(z_1 - z_2) A_2 \dots A_{m-1} W(z_{m-1} - z_m) \Omega \rangle, \quad (5.16)$$

for $\text{Im } z_1 \leq \dots \leq \text{Im } z_m$ whenever the right-hand-side is well-defined. The Gell-Mann and Low formula is given by:

Theorem 5.3. Suppose that Assumption 5.1 holds and H_1 is symmetric. Let A_k ($k = 1, \dots, m$, $m \geq 1$) be linear operators having the following properties:

(I) Each A_k is in $\mathcal{C}_0(H_0)$ -class.

(II) For each k , there exist integer $r_k \geq 0$ such that, for all $n \in \mathbb{N}$, A_k maps $D(H^{n+r_k})$ into $D(H^n)$.

Fix $w \in \mathbb{C}$ with $\text{Im } w < 0$, and $z_1, \dots, z_m \in \mathbb{C}$ with $\text{Im } z_1 \leq \dots \leq \text{Im } z_m$. Choose a simple curve $\Gamma_{w,T}$ from $-Tw$ to Tw ($T > 0$) on which $z_1 \succ \dots \succ z_m$. Then m -point Green's function $G_m(z_1, \dots, z_m)$ is well-defined and satisfies the formula

$$G_m(z_1, \dots, z_m) = \lim_{T \rightarrow \infty} \frac{\left\langle \Omega_0, \mathcal{T} A_1(z_1) \dots A_m(z_m) \exp \left(-i \int_{\Gamma_{w,T}} d\zeta H_1(\zeta) \right) \Omega_0 \right\rangle}{\left\langle \Omega_0, \mathcal{T} \exp \left(-i \int_{\Gamma_{w,T}} d\zeta H_1(\zeta) \right) \Omega_0 \right\rangle}. \quad (5.17)$$

To prove the Gell-Mann – Low formula (5.17), we prepare some lemmas. We denote $E_0(H)$ simply by E_0 .

Lemma 5.1. For $w \in \mathbb{C}$ with $\text{Im } w < 0$ and a Borel measurable function $f : \mathbb{R} \rightarrow \mathbb{C}$, it follows that

$$\lim_{T \rightarrow \infty} f(H) e^{iTwE_0} W(Tw) \Psi = f(E_0) P_0 \Psi, \quad \Psi \in D(f(H)), \quad (5.18)$$

where P_0 is the Projection onto the closed subspace $\ker(H - E_0)$.

Proof. By the functional calculus and Lebesgue's convergence Theorem, we have

$$\begin{aligned} & \|f(H) e^{iTwE_0} W(Tw) \Psi - f(E_0) P_0 \Psi\|^2 \\ &= \left\| f(H) e^{-iTw(H-E_0)} \Psi - f(E_0) E_H(\{E_0\}) \Psi \right\|^2 \\ &= \int_{[E_0, \infty)} d\|E_H(\lambda) \Psi\|^2 \left| f(\lambda) (e^{T(\text{Im } w)(\lambda - E_0)} \Psi - \chi_{\{E_0\}}(\lambda)) \right|^2 \\ &= \int_{(E_0, \infty)} d\|E_H(\lambda) \Psi\|^2 \left| f(\lambda) e^{T(\text{Im } w)(\lambda - E_0)} \Psi \right|^2 \\ &\rightarrow 0, \end{aligned} \quad (5.19)$$

as T tends to infinity. $\square$

Lemma 5.2. Under the same assumptions of Theorem 5.3, the operators

$$\widetilde{A}_k := (H - \zeta)^{\sum_{j=1}^{k-1} r_j} A_k (H - \zeta)^{-\sum_{j=1}^k r_j}, \quad k = 1, \dots, m, \quad (5.20)$$

are bounded.

Proof. From the assumptions,

$$A_k(H - \zeta)^{-\sum_{j=1}^k r_j} \Psi \in D(H^{\sum_{j=1}^{k-1} r_j}), \quad (5.21)$$

for all $\Psi \in \mathcal{H}$. Thus

$$D(\widetilde{A}_k) = \mathcal{H}.$$

On the other hand, it is easy to check that $\widetilde{A}_k$'s are closed. hence by the closed graph theorem, each $\widetilde{A}_k$'s are bounded. $\square$

Lemma 5.3. *Under the same assumptions of Theorem 5.3, it follows that*

$$\begin{aligned} \lim_{T \rightarrow \infty} A_1 W(z_1 - z_2) A_2 \dots A_{m-1} W(z_{m-1} - z_m) A_m f(H) e^{iT w} W(T w) \Psi \\ = A_1 W(z_1 - z_2) A_2 \dots A_{m-1} W(z_{m-1} - z_m) A_m f(E_0) P_0 \Psi, \\ \Psi \in \bigcap_{n \in \mathbb{N}} D(H^n f(H)). \end{aligned} \quad (5.22)$$

for all $w \in \mathbb{C}$ with $\text{Im} w < 0$ and a Borel measurable function $f : \mathbb{R} \rightarrow \mathbb{C}$.

Proof. Under the present assumptions, we see that each A_k leaves the subspace $\bigcap_{n=1}^{\infty} D(H^n)$ invariant, and thus

$$\Psi \in D(A_1 W(z_1 - z_2) A_2 \dots A_{m-1} W(z_{m-1} - z_m) A_m f(H) e^{iT w} W(T w)). \quad (5.23)$$

Now let $\zeta \in \mathbb{C} \setminus \mathbb{R}$. Then we can rewrite

$$\begin{aligned} A_1 W(z_1 - z_2) A_2 \dots A_{m-1} W(z_{m-1} - z_m) A_m \\ = \widetilde{A}_1 W(z_1 - z_2) \dots \widetilde{A}_m W(z_{m-1} - z_m) (H - \zeta)^{\sum_k r_k} \end{aligned} \quad (5.24)$$

with

$$\widetilde{A}_k := (H - \zeta)^{\sum_{j=1}^{k-1} r_j} A_k (H - \zeta)^{-\sum_{j=1}^k r_j}, \quad k = 1, \dots, m. \quad (5.25)$$

Note that each of $\widetilde{A}_k$'s and $W(z_{k-1} - z_k)$'s is a bounded operator by Theorem 5.2 and Lemma 5.2. then by Lemma 5.1, one sees that for all $n \geq 1$,

$$\lim_{T \rightarrow \infty} (H - \zeta)^n e^{iT w} W(T w) \Psi = (E_0 - \zeta)^n P_0 \Psi = (H - \zeta)^n P_0 \Psi, \quad (5.26)$$

which implies the desired result. $\square$

Proof of Theorem 5.3. Put

$$\mathcal{O}(z_1, \dots, z_m) := A_1 W(z_1 - z_2) A_2 \dots A_{m-1} W(z_{m-1} - z_m) A_m. \quad (5.27)$$

From Assumption 5.1, one finds

$$\Omega = \frac{P_0 \Omega_0}{\|P_0 \Omega_0\|}, \quad (5.28)$$

to obtain

$$G_m(z_1, \dots, z_m) = e^{i(z_1 - z_m) E_0} \frac{\langle P_0 \Omega_0, \mathcal{O}(z_1, \dots, z_m) P_0 \Omega_0 \rangle}{\langle P_0 \Omega_0, P_0 \Omega_0 \rangle}. \quad (5.29)$$

By Lemmas 5.1 and 5.3, we have

$$\begin{aligned} & \frac{\langle P_0 \Omega_0, \mathcal{O}(z_1, \dots, z_m) P_0 \Omega_0 \rangle}{\langle P_0 \Omega_0, P_0 \Omega_0 \rangle} \\ &= \lim_{T \rightarrow \infty} \frac{\langle e^{-iz_1^*(H-E_0)} W(-Tw^*) \Omega_0, \mathcal{O}(z_1, \dots, z_m) e^{-iz_m(H-E_0)} W(Tw) \Omega_0 \rangle}{\langle W(-Tw^*) \Omega_0, W(Tw) \Omega_0 \rangle}. \end{aligned} \quad (5.30)$$

Using Theorem 5.2, we find

$$e^{-iz_1^*(H-E_0)} W(-Tw^*) = e^{iz_1^* E_0} e^{-iz_1^* H_0} \overline{U(z_1^*, Tw^*)} e^{iTw^* H_0} \quad (5.31)$$

$$e^{-iz_m(H-E_0)} W(Tw) = e^{iz_m E_0} e^{-iz_m H_0} \overline{U(z_m, -Tw)} e^{-iTw H_0} \quad (5.32)$$

on $\mathcal{D}$. therefore by Theorem 2.9 the numerator on the right-hand-side of (5.30) can be rewritten as

$$\begin{aligned} & e^{-i(z_1-z_m)E_0} \left\langle \Omega_0, \overline{U(Tw, z_1)} A_1(z_1) \overline{U(z_1, z_2)} \dots \overline{U(z_{m-1}, z_m)} A_m(z_m) \overline{U(z_m, -Tw)} \Omega_0 \right\rangle \\ &= e^{-i(z_1-z_m)E_0} \left\langle \Omega_0, \mathcal{T} A_1(z_1) \dots A_m(z_m) \exp \left(-i \int_{\Gamma_{w,T}} d\zeta H_1(\zeta) \right) \Omega_0 \right\rangle \end{aligned} \quad (5.33)$$

and the denominator as

$$\langle \Omega_0, U(Tw, -Tw) \Omega_0 \rangle = \left\langle \Omega_0, \mathcal{T} \exp \left(-i \int_{\Gamma_{w,T}} d\zeta H_1(\zeta) \right) \Omega_0 \right\rangle. \quad (5.34)$$

Finally, inserting (5.30), (5.33), and (5.34) into (5.29), we arrive at the Gell-Mann – Low formula (5.17). $\square$

5.2 Application to QED

In this section we apply the abstract results obtained in the preceding subsection to QED with several regularizations in the Coulomb gauge. Our main goal here is Theorem 5.6, which shows that QED with regularizations satisfies Gell-Mann – Low formula. To prove this, it is sufficient to see that the conditions of Theorem 5.3 hold. Under suitable hypotheses, it is not difficult to prove that the interaction Hamiltonian and each field operator are in $\mathcal{C}_0$ -class (Lemmas 3.14 and 3.12). However, the existence of the ground state (Assumption 5.1) and the condition (II) of Theorem 5.3 are not obvious at all. The existence of the ground state is discussed in [61]. To check the condition (II), we need some preliminaries (Lemmas 5.10-5.14).

5.2.1 Electromagnetic fields

We introduce the photon field quantized in the Coulomb gauge. We adopt as the one-photon Hilbert space

$$\mathcal{H}_{\text{ph}}^{(\text{C})} := L^2(\mathbb{R}_{\mathbf{k}}^3; \mathbb{C}^2). \quad (5.35)$$

We freely use the identification $L^2(\mathbb{R}_{\mathbf{k}}^3; \mathbb{C}^2) = \oplus^2 L^2(\mathbb{R}_{\mathbf{k}}^3)$. The Hilbert space for the quantized electromagnetic field in the Coulomb gauge is given by

$$\mathcal{F}_{\text{ph}}^{(\text{C})} := \mathcal{F}_{\text{b}}(\mathcal{H}_{\text{ph}}^{(\text{C})}), \quad (5.36)$$

the boson Fock space over $\mathcal{H}_{\text{ph}}^{(\text{C})}$.

The function $\omega(\mathbf{k}) := |\mathbf{k}|$ ($\mathbf{k} \in \mathbb{R}^3$) naturally defines uniquely a multiplication operator on $\mathcal{H}_{\text{ph}}^{(\text{C})}$ which is injective, non-negative and self-adjoint. We denote it by the same symbol ω also. The free Hamiltonian of the quantum electromagnetic field in the Coulomb gauge is given by the second quantization of ω :

$$H_{\text{ph}}^{(\text{C})} := d\Gamma_{\text{b}}(\omega) : \mathcal{F}_{\text{b}}(\mathcal{H}_{\text{ph}}^{(\text{C})}) \rightarrow \mathcal{F}_{\text{b}}(\mathcal{H}_{\text{ph}}^{(\text{C})}). \quad (5.37)$$

We denote by $a(\cdot)$ ($\cdot \in \mathcal{H}_{\text{ph}}^{(\text{C})}$) the annihilation operator on $\mathcal{F}_{\text{ph}}^{(\text{C})}$. For each $f \in L^2(\mathbb{R}_{\mathbf{k}}^3)$, we use the notation:

$$a_1(f) := a(f, 0), \quad a_2(f) := a(0, f). \quad (5.38)$$

For all $f \in L^2(\mathbb{R}_{\mathbf{x}}^3)$ satisfying $\widehat{f}/\sqrt{\omega} \in L^2(\mathbb{R}_{\mathbf{k}}^3)$, we set

$$A^i(0, f) := \sum_{r=1,2} \left(a_r \left(\frac{\widehat{f} e_r^i}{\sqrt{2\omega}} \right) + a_r \left(\frac{\widehat{f} e_r^i}{\sqrt{2\omega}} \right)^* \right). \quad (5.39)$$

$$(5.40)$$

The functions $\mathbf{e}_r(\mathbf{k}) = (e_r^i(\mathbf{k}))_{i=1}^3 \in \mathbb{R}^3$, $r = 1, 2$, are the polarization vectors satisfying

$$\mathbf{e}_r(\mathbf{k}) \cdot \mathbf{e}_{r'}(\mathbf{k}) = \delta_{rr'}, \quad \mathbf{k} \cdot \mathbf{e}_r(\mathbf{k}) = 0, \quad \text{a.e. } \mathbf{k} \in \mathbb{R}^3, \quad r, r' = 1, 2. \quad (5.41)$$

The functional $\mathcal{S}(\mathbb{R}_{\mathbf{x}}^3) \ni f \mapsto A_i(0, f)$ gives an operator-valued distribution (Cf. [2] Definition 7-1) acting on $(\mathcal{F}_{\text{ph}}^{(\text{C})}, \mathcal{F}_{\text{b},0}(\mathcal{H}_{\text{ph}}^{(\text{C})}))$ and it is called the quantized electromagnetic field at time $t = 0$. As is well-known, $A_j(0, \mathbf{x})$ ($j = 1, 2, 3$) are essentially self-adjoint. We denote the closure of $A_j(0, \mathbf{x})$ by the same symbol.

Now, fix $\chi_{\text{ph}} \in L^2(\mathbb{R}_{\mathbf{x}}^3)$ such that it is real and satisfies $\widehat{\chi_{\text{ph}}}/\sqrt{\omega} \in L^2(\mathbb{R}_{\mathbf{k}}^3)$. We set

$$A_i(0, \mathbf{x}) := A_i(0, \chi_{\text{ph}}^{\mathbf{x}}), \quad (5.42)$$

$$\chi_{\text{ph}}^{\mathbf{x}}(\mathbf{y}) := \chi_{\text{ph}}(\mathbf{y} - \mathbf{x}), \quad \mathbf{y} \in \mathbb{R}^3. \quad (5.43)$$

$A_i(0, \mathbf{x})$ is called the point-like quantized electromagnetic field with momentum cutoff $\widehat{\chi_{\text{ph}}}$ at time $t = 0$.

We assume the following condition.

Hypothesis 5.1. $\widehat{\chi_{\text{ph}}}/\omega \in L^2(\mathbb{R}_{\mathbf{k}}^3)$.

Lemma 5.4. Under Hypothesis 5.1, for all $i = 1, 2, 3$, $\mathbf{x} \in \mathbb{R}^3$ and $\Psi \in D(H_{\text{ph}}^{1/2})$,

$$\|A_i(0, \mathbf{x})\Psi\| \leq M_{\text{ph}} \|(H_{\text{ph}} + 1)^{1/2}\Psi\|, \quad (5.44)$$

where $M_{\text{ph}} := 2\sqrt{2}\|\widehat{\chi_{\text{ph}}}/\omega\|_{L^2(\mathbb{R}_{\mathbf{k}}^3)} + \sqrt{2}\|\widehat{\chi_{\text{ph}}}/\sqrt{\omega}\|_{L^2(\mathbb{R}_{\mathbf{k}}^3)}$.

Proof. This is a simple application of Lemma A.4. □

Remark 5.1. If the momentum cutoff function $\widehat{\chi_{\text{ph}}}$ is taken to be the characteristic function of the set $\{\mathbf{k} \in \mathbb{R}^3 \mid |\mathbf{k}| \leq \Lambda_0\}$ with $\Lambda_0 \geq 0$, then this satisfies Hypothesis 5.1.

5.2.2 Dirac fields

For the quantized Dirac field, we use the same notations as in Subsubsection 3.3.2.

Lemma 5.5. *For all $\mu = 0, 1, 2, 3$ and $\mathbf{x} \in \mathbb{R}^3$,*

$$M_{\text{cu},1} := \sup_{\mathbf{x} \in \mathbb{R}^3, \mu=0,1,2,3} \| :j^\mu(0, \mathbf{x}) : \| < \infty, \quad (5.45)$$

$$M_{\text{cu},2} := \sup_{\mathbf{x}, \mathbf{y} \in \mathbb{R}^3, \mu, \nu=0,1,2,3} \| :j^\mu(0, \mathbf{x}) j^\nu(0, \mathbf{y}) : \| < \infty. \quad (5.46)$$

Proof. A simple application of (A.43). $\square$

5.2.3 The total Hamiltonian with cutoffs in the Coulomb gauge

The state space for QED in the Coulomb gauge is taken to be

$$\mathcal{F}_{\text{tot}} := \mathcal{F}_{\text{el}} \otimes \mathcal{F}_{\text{ph}}^{(\text{C})}. \quad (5.47)$$

The free Hamiltonian is

$$H_{\text{fr}} := H_{\text{el}} \otimes I + I \otimes H_{\text{ph}}^{(\text{C})}, \quad (5.48)$$

where the subscript fr in H_{fr} means *free*.

We denote the charge of the Dirac particle by $e \in \mathbb{R}$. Let $\chi_{\text{sp}} \in L^1(\mathbb{R}^3)$ be a real-valued function on $\mathbb{R}^3$ playing the role of spacial cut-off. The first interaction term H_{I} is defined as

$$\begin{aligned} D(H_{\text{I}}) &= D((I \otimes H_{\text{ph}})^{1/2}), \\ H_{\text{I}}\Psi &= e \sum_{i=1}^3 \int_{\mathbb{R}^3} d\mathbf{x} \chi_{\text{sp}}(\mathbf{x}) :j^i(0, \mathbf{x}) : \otimes A_i(0, \mathbf{x})\Psi, \quad \Psi \in D(H_{\text{I}}), \end{aligned} \quad (5.49)$$

where the integral in the right hand side is a strong Bochner integral. We adopt the Coulomb term H_{II} which is given by

$$\begin{aligned} D(H_{\text{II}}) &:= \mathcal{F}_{\text{tot}}, \\ H_{\text{II}} &:= \frac{e^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{x} d\mathbf{y} \chi_{\text{sp}}(\mathbf{x}) \chi_{\text{sp}}(\mathbf{y}) V_C(\mathbf{x} - \mathbf{y}) :j^0(0, \mathbf{x}) j^0(0, \mathbf{y}) : \otimes I, \end{aligned} \quad (5.50)$$

with

$$V_C(\mathbf{x} - \mathbf{y}) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{d\mathbf{k}}{\omega(\mathbf{k})^2} |\widehat{\chi_{\text{ph}}}(\mathbf{k})|^2 e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}, \quad (5.51)$$

where the integral in the right-hand side of (5.50) is a Bochner integral with respect to the operator norm. The well-definedness of H_{I} and H_{II} is proven in later (see Lemma 5.6). Then H_{I} is symmetric, and H_{II} is bounded and self-adjoint. We remark that the interaction potential $V_C(\mathbf{x} - \mathbf{y})$ converges to the familiar Coulomb potential

$$\frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{y}|}$$

in the distribution sense as the photon ultraviolet cutoff $\widehat{\chi_{\text{ph}}}$ is removed. Finally, the interaction Hamiltonian H_{int} and the total Hamiltonian H_{tot} is defined by

$$H_{\text{int}} := \overline{H_{\text{I}}} + H_{\text{II}}, \quad (5.52)$$

$$H_{\text{tot}} := H_{\text{fr}} + H_{\text{int}}. \quad (5.53)$$

5.2.4 Self-adjointness

Lemma 5.6. *Assume Hypothesis 5.1. Then the following (i)-(iii) hold:*

(i) *For all $\Psi \in D((I \otimes H_{\text{ph}})^{1/2})$,*

$$\begin{aligned} & \sum_{i=1}^3 \int_{\mathbb{R}^3} d\mathbf{x} |\chi_{\text{sp}}(\mathbf{x})| \| :j^i(0, \mathbf{x}): \otimes A_i(0, \mathbf{x}) \Psi \| \\ & \leq 3 \|\chi_{\text{sp}}\|_{L^1(\mathbb{R}^3)} M_{\text{cu},1} M_{\text{ph}} \|(I \otimes H_{\text{ph}} + 1)^{1/2} \Psi\| < \infty. \end{aligned}$$

(ii) *It follows that*

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{x} d\mathbf{y} |\chi_{\text{sp}}(\mathbf{x}) \chi_{\text{sp}}(\mathbf{y}) V_C(\mathbf{x} - \mathbf{y})| \| :j^0(\mathbf{x}) j^0(\mathbf{y}): \otimes I \| \leq \|\chi_{\text{sp}}\|_{L^1(\mathbb{R}^3)}^2 M_C M_{\text{cu},2} < \infty,$$

where $M_C := (1/4\pi) \|\widehat{\chi_{\text{ph}}}/\omega\|_{L^2(\mathbb{R}_k^3)}^2$.

(iii) H_{int} *is $H_{\text{fr}}^{1/2}$ -bounded, closed and symmetric.*

(iv) H_{tot} *is self-adjoint on $D(H_{\text{fr}})$, and bounded from below.*

Proof. (i) and (ii) follow from Lemma 5.4 and 5.5.

We prove (iii). It is easy to see that H_{I} and H_{II} are symmetric. By (i), H_{I} is $H_{\text{fr}}^{1/2}$ -bounded. By (ii), H_{II} is bounded. Thus H_{int} is $H_{\text{fr}}^{1/2}$ -bounded, closed and symmetric.

By (iii), H_{int} is infinitesimally H_{fr} -bounded. Thus (iv) follows from the Kato-Rellich theorem. $\square$

5.2.5 Time-ordered exponential on the complex plane

A basic hypothesis to apply our abstract theory is:

Hypothesis 5.2 (Ultraviolet cutoff). *There exist constants $\Lambda_{\text{el}}, \Lambda_{\text{ph}} \geq 0$ such that $\text{supp } \widehat{\chi_{\text{el}}} \subset \{|\mathbf{p}| \leq \Lambda_{\text{el}}\}$, $\text{supp } \widehat{\chi_{\text{ph}}} \subset \{|\mathbf{k}| \leq \Lambda_{\text{ph}}\}$.*

In what follows, we use the following notations:

$$\mathcal{F}_E := R(E_{H_{\text{fr}}}([0, E])), \quad E \geq 0, \quad (5.54)$$

$$\mathcal{F}_{\text{fin}} := \bigcup_{E \geq 0} \mathcal{F}_E. \quad (5.55)$$

In order to construct the time-ordered exponential, it is sufficient to see that Theorem 2.5 can be applied to our case by checking that H_{int} is in $\mathcal{C}_0(H_{\text{fr}})$ -class. The correspondence of the symbols is as follows: $H_0 = H_{\text{fr}}$, $H_1 = H_{\text{int}}$, $V_E = \mathcal{F}_E$, $D_{\text{fin}} = \mathcal{F}_{\text{fin}}$.

Lemma 5.7. *Assume Hypothesis 5.2. Then the following (i) and (ii) hold.*

(i) *For all $E \geq 0$, $\mathbf{x} \in \mathbb{R}^3$ and $j = 1, 2, 3$, $A_j(0, \mathbf{x})$ maps $R(E_{H_{\text{ph}}}([0, E]))$ into $R(E_{H_{\text{ph}}}([0, E + \Lambda_{\text{ph}}]))$.*

(ii) *For all $E \geq 0$, $\mathbf{x} \in \mathbb{R}^3$ and $l = 1, 2, 3, 4$, $\psi_l(0, \mathbf{x})$ and $\psi_l(0, \mathbf{x})^*$ map $R(E_{H_{\text{el}}}([0, E]))$ into $R(E_{H_{\text{el}}}([0, E + \sqrt{\Lambda_{\text{el}}^2 + M^2}]))$.*

(iii) $I \otimes A_j(0, \mathbf{x})$ *is in $\mathcal{C}_0(H_{\text{fr}})$ -class.*

(iv) $\psi_l(0, \mathbf{x}) \otimes I$ *is in $\mathcal{C}_0(H_{\text{fr}})$ -class.*

Proof. (i) Similar to the proof of Lemma 3.6.

(ii) Similar to the proof of Lemma 3.8.

(iii) By Lemma (5.4), $I \otimes A_j(0, \mathbf{x})$ is $H_{\text{fr}}^{1/2}$ -bounded. Combining (i) and Lemma B.1, we see that, for all $E \geq 0$, $I \otimes A_j(0, \mathbf{x})$ maps $R(E_{H_{\text{fr}}}([0, E]))$ into $R(E_{H_{\text{fr}}}([0, E + \Lambda_{\text{ph}}]))$. Therefore the assertion follows.

(iv) Similar to the proof of (iii). □

Lemma 5.8. *Assume Hypotheses 5.1 and 5.2. Then the following (i) and (ii) hold:*

(i) *For all $E \geq 0$, H_{I} maps $\mathcal{F}_E$ into $\mathcal{F}_{E+2\sqrt{\Lambda_{\text{el}}^2+M^2}+\Lambda_{\text{ph}}}$.*

(ii) *For all $E \geq 0$, H_{II} maps $\mathcal{F}_E$ into $\mathcal{F}_{E+4\sqrt{\Lambda_{\text{el}}^2+M^2}}$.*

(iii) *H_{int} is in $\mathcal{C}_0(H_{\text{fr}})$ -class.*

Proof. (i) One can easily see that, for all $E \geq 0$, $\mathbf{x} \in \mathbb{R}^3$ and $\mu = 0, 1, 2, 3$, $:j^\mu(0, \mathbf{x}):$ maps $R(E_{H_{\text{el}}}([0, E]))$ into $R(E_{H_{\text{el}}}([0, E + 2\sqrt{\Lambda_{\text{el}}^2 + M^2}]))$. Now, fix $E \geq 0$ arbitrarily, and let $\Psi \in \mathcal{F}_E$. Applying Lemmas B.1 and 5.7, we see that $:j^i(0, \mathbf{x}): \otimes A_i(0, \mathbf{x})\Psi \in \mathcal{F}_{E+2\sqrt{\Lambda_{\text{el}}^2+M^2}+\Lambda_{\text{ph}}}$ for all $\mathbf{x} \in \mathbb{R}^3$ and $i = 1, 2, 3$. Hence we have $H_{\text{I}}\Psi \in \mathcal{F}_{E+2\sqrt{\Lambda_{\text{el}}^2+M^2}+\Lambda_{\text{ph}}}$ because $\mathcal{F}_{E+2\sqrt{\Lambda_{\text{el}}^2+M^2}+\Lambda_{\text{ph}}}$ is closed subspace. Thus the assertion follows.

(ii) Similar to the proof of (i).

(iii) This follows from (i), (ii) and Lemma 5.6. □

From Lemma 5.8, we can apply the abstract theory constructed in the previous sections to obtain:

Theorem 5.4. *Assume Hypotheses 5.1 and 5.2. Take a piecewisely continuously differentiable simple curve $\Gamma_{z, z'}$ which starts at z' and ends at z with $\text{Im } z' \leq \text{Im } z$. then*

$$\mathcal{F}_{\text{fin}} \subset D \left(T \exp \left(-i \int_{\Gamma_{z, z'}} d\zeta H_{\text{int}}(\zeta) \right) \right), \quad (5.56)$$

where

$$H_{\text{int}}(z) := e^{izH_{\text{fr}}} H_{\text{int}} e^{-izH_{\text{fr}}}, \quad z \in \mathbb{C}. \quad (5.57)$$

Furthermore, $T \exp \left(-i \int_{\Gamma_{z, z'}} d\zeta H_{\text{int}}(\zeta) \right)$ has properties stated in Theorems 2.5-2.9, with H_0 replaced by H_{fr} , H_1 by H_{int} and D_{fin} by $\mathcal{F}_{\text{fin}}$.

Lemma 5.9. *For all $z \in \mathbb{C}$ and $\Psi \in D_{\text{fin}}$, it follows that*

$$\|H_{\text{int}}(z)\Psi\| \leq M_{\text{int}}(z) \|(H_{\text{fr}} + 1)^{1/2}\Psi\|, \quad z \in \mathbb{C}, \quad (5.58)$$

where

$$M_{\text{int}}(z) := 3|e| \|\chi_{\text{sp}}\| M_{\text{cu}}(z) M_{\text{ph}}(z) + \frac{e^2}{2} \|\chi_{\text{sp}}\|^2 M_{\text{C}} M_{\text{cu}}(z)^2, \quad (5.59)$$

$$M_{\text{cu}}(z) := 64(e^{|\text{Im } z| \sqrt{\Lambda_{\text{el}}^2 + M^2}} + e^{-|\text{Im } z| M})^2 \|\widehat{\chi_{\text{el}}}\|^2, \quad (5.60)$$

$$M_{\text{ph}}(z) := 2(e^{|\text{Im } z| \Lambda_{\text{ph}}} + 1) \left\| \frac{\widehat{\chi_{\text{ph}}}}{\sqrt{2\omega}} \right\|. \quad (5.61)$$

Proof. Let $\Psi \in \mathcal{F}_{\text{fin}}$. Then using Lemma 5.8 (i), (ii), we have

$$H_{\text{int}}(z)\Psi = e^{izH_{\text{fr}}E_{H_{\text{fr}}}([0,E])}H_{\text{int}}e^{-izH_{\text{fr}}E_{H_{\text{fr}}}([0,E])}\Psi \quad (5.62)$$

for sufficiently large $E \geq 0$. Note that $e^{izH_{\text{fr}}E_{H_{\text{fr}}}([0,E])}$ is a bounded operator. Using Lemmas A.9 and A.12, we have

$$\begin{aligned} H_{\text{int}}(z)\Psi &= e \sum_{i=1}^3 \int_{\mathbb{R}^3} d\mathbf{x} \chi_{\text{sp}}(\mathbf{x}) :j_{\text{int}}^i(z, \mathbf{x}): \otimes A_{\text{int},i}(z, \mathbf{x})\Psi \\ &+ \frac{e^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{x} d\mathbf{y} \chi_{\text{sp}}(\mathbf{x}) \chi_{\text{sp}}(\mathbf{y}) V_C(\mathbf{x} - \mathbf{y}) :j_{\text{int}}^0(z, \mathbf{x}) j_{\text{int}}^0(z, \mathbf{y}): \otimes I \Psi, \end{aligned} \quad (5.63)$$

where

$$A_{\text{int},i}(z, \mathbf{x}) := \sum_{r=1,2} \left(a_r \left(\frac{e^{iz^* \omega} \widehat{\chi_{\text{ph}}^{\mathbf{x}} e_r^i}}{\sqrt{2\omega}} \right) + a_r^* \left(\frac{e^{iz\omega} \widehat{\chi_{\text{ph}}^{\mathbf{x}} e_r^i}}{\sqrt{2\omega}} \right) \right), \quad (5.64)$$

$$j_{\text{int}}^\mu(z, \mathbf{x}) := \sum_{l,l'=1}^4 \psi_{\text{int},l}(\mathbf{x})^* \alpha_{ll'}^\mu \psi_{\text{int},l'}(\mathbf{x}), \quad (5.65)$$

$$\psi_{\text{int},l}(z, \mathbf{x}) := \sum_{s=\pm 1/2} \left(b_s \left(e^{iz^* E_M} \widehat{\chi_{\text{el}}^{\mathbf{x}} (u_s^l)^*} \right) + d_s^* \left(e^{iz E_M} \widehat{\chi_{\text{el}}^{\mathbf{x}} \tilde{v}_s^l} \right) \right). \quad (5.66)$$

Using (A.43),

$$\| :j_{\text{int}}^i(z, \mathbf{x}): \| \leq M_{\text{cu}}(z), \quad \| :j_{\text{int}}^0(z, \mathbf{x}) j_{\text{int}}^0(z, \mathbf{y}): \| \leq M_{\text{cu}}(z)^2. \quad (5.67)$$

Using Lemma A.4,

$$\| A_{\text{int},i}(z, \mathbf{x}) \Phi \| \leq M_{\text{ph}}(z) \| (H_{\text{ph}} + 1)^{1/2} \Phi \|, \quad \Phi \in D(H_{\text{ph}}^{1/2}). \quad (5.68)$$

Combining these estimates, we obtain (5.58). $\square$

Theorem 5.5. *Take a piecewisely continuously differentiable simple curve $\Gamma_{z,z'}$ ($z, z' \in \mathbb{C}$) which starts at z' and ends at z . Then for all $\Psi \in D_{\text{fin}}$, the series*

$$U(H_{\text{int}}; z, z')\Psi := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma_{z,z'}^n} d\zeta_1 \dots d\zeta_n \mathcal{T}(H_{\text{int}}(\zeta_1) \cdots H_{\text{int}}(\zeta_n))\Psi \quad (5.69)$$

converges absolutely. The operator $U(H_{\text{int}}; z, z')$ satisfies the differential equations (2.167) and 2.168.

5.2.6 Gell-Mann – Low formula for QED

To apply our abstract theory, we need some preliminaries. For two linear operators A and B on a Hilbert space $\mathcal{H}$, we define $\text{ad}_A^k(B)$, ($k = 0, 1, 2, \dots$) by

$$\text{ad}_A^0(B) := B, \quad (5.70)$$

$$\text{ad}_A^k(B) := [A, \text{ad}_A^{k-1}(B)], \quad k \geq 1. \quad (5.71)$$

It is easy to see that, for all integer $n \geq 0$,

$$A^n B \psi = \sum_{k=0}^n n C_k \text{ad}_A^k(B) A^{n-k} \Psi, \quad \Psi \in \bigcap_{k=0}^n D(A^k B A^{n-k}). \quad (5.72)$$

Lemma 5.10. *Let $n_0 \geq 0$ be an integer and $r \geq 0$ a real number. Let T_0 be a self-adjoint operator and T_1 a densely defined closed operator on a Hilbert space $\mathcal{H}$. Suppose that there exists a subspace $D \subset \mathcal{H}$ having the following properties (I)-(III):*

- (I) T_0 and T_1 leave D invariant.
- (II) D is a core of $T_0^{n_0+r}$.
- (III) For all $n = 0, \dots, n_0$, $\text{ad}_{T_0}^n(T_1)$ is T_0^{n+r} -bounded on D , i.e., there exist constants $C_1, C_2 \geq 0$ such that for all $\Psi \in D$,

$$\|\text{ad}_{T_0}^n(T_1)\Psi\| \leq C_1\|T_0^{n+r}\Psi\| + C_2\|\Psi\|. \quad (5.73)$$

then for all $n = 0, \dots, n_0$, T_1 maps $D(T_0^{n+r})$ into $D(T_0^n)$.

Proof. Let $n = 0, \dots, n_0$ be fixed arbitrarily. By the condition (I), for all $\Psi \in D$,

$$T_0^n T_1 \Psi = \sum_{k=0}^n n C_k \text{ad}_{T_0}^k(T_1) T_0^{n-k} \Psi. \quad (5.74)$$

By the condition (III), each of $\text{ad}_{T_0}^k(T_1) T_0^{n-k}$ ($k = 0, \dots, n$) is T_0^{n+r} -bounded on D , and thus so is $T_0^n T_1$. Hence there exist constants $C_1, C_2 \geq 0$ such that

$$\|T_0^n T_1 \Psi\| \leq C_1\|T_0^{n+r}\Psi\| + C_2\|\Psi\|, \quad \Psi \in D. \quad (5.75)$$

Let us note that T_1 is T_0^r -bounded on D from the condition (III). Using the condition (II) and the closedness of T_0^n , we see that the above Ψ can be extended onto $D(T_0^{n+r})$. Thus the assertion follows. $\square$

Lemma 5.11. *Let $n_0 \geq 0$ be an integer. Let T_0 be a self-adjoint operator and T_1 a closed symmetric operator on a Hilbert space $\mathcal{H}$. Suppose that there exists a subspace $D \subset \mathcal{H}$ having the following properties (I)-(III).*

- (I) T_0 and T_1 leave D invariant.
- (II) D is a core of $T_0^{n_0+1}$.
- (III) For all $n = 0, \dots, n_0$, $\text{ad}_{T_0}^n(T_1)$ is infinitesimally T_0^{n+1} -bounded on D , i.e., for all $\varepsilon > 0$, there exists a constant $C_\varepsilon \geq 0$ such that for all $\Psi \in D$,

$$\|\text{ad}_{T_0}^n(T_1)\Psi\| \leq \varepsilon\|T_0^{n+1}\Psi\| + C_\varepsilon\|\Psi\|. \quad (5.76)$$

then $T := T_0 + T_1$ is self-adjoint. Furthermore, for all $n = 1, \dots, n_0 + 1$, $T^n - T_0^n$ is infinitesimally T_0^n -bounded, and it follows that

$$D(T^n) = D(T_0^n). \quad (5.77)$$

Proof. From the conditions (II) and (III) for $n = 0$, T_1 is infinitesimally T_0 -bounded. Hence it follows from the Kato-Rellich theorem that T is self-adjoint and

$$D(T) = D(T_0). \quad (5.78)$$

By Lemma 5.10, for all $n = 1, \dots, n_0$, T_1 maps $D(T_0^{n+1})$ into $D(T_0^n)$. Hence we have

$$D(T^n) \supset D(T_0^n), \quad n = 1, \dots, n_0 + 1. \quad (5.79)$$

We prove the remaining claim by induction. The case $n = 1$ has already been proved. Suppose that the claim is true for some $n < n_0 + 1$. By the condition (I), we have

$$T^{n+1}\Psi - T_0^{n+1}\Psi = T_0^n T_1 \Psi + (T^n - T_0^n)T\Psi, \quad \Psi \in D. \quad (5.80)$$

From the induction hypothesis, for all $\varepsilon > 0$,

$$\|(T^n - T_0^n)T\Psi\| \leq \varepsilon \|T_0^n T\Psi\| + C_\varepsilon \|T\Psi\|, \quad \Psi \in D(T_0^{n+1}), \quad (5.81)$$

where $C_\varepsilon > 0$ is a constant depending on ε and n . In the same manner as in the proof of Lemma 5.10, one can see that $T_0^n T_1$ is infinitesimally T_0^{n+1} -bounded on D . Combining this with (5.79), (5.80), (5.81) and the condition (II), we see that $T^{n+1} - T_0^{n+1}$ is infinitesimally T_0^{n+1} -bounded. Hence it follows from the Kato-Rellich theorem that $T_0^{n+1} + (T^{n+1} - T_0^{n+1})$ is self-adjoint on $D(T_0^{n+1})$. On the other hand, by the definition of the sum operator, we have the inclusion relation $T^{n+1} \supset T_0^{n+1} + (T^{n+1} - T_0^{n+1})$. Since both sides are self-adjoint, we obtain the operator equality

$$T^{n+1} = T_0^{n+1} + (T^{n+1} - T_0^{n+1}), \quad (5.82)$$

which implies (5.77) for $n + 1$. Thus the induction step is complete, and the assertion follows. $\square$

Lemma 5.12. *Assume Hypotheses 5.1 and 5.2. Then the following (i)-(iii) hold:*

- (i) H_{fr} and H_{int} leave $\mathcal{F}_{\text{fin}}$ invariant.
- (ii) For each $n \in \mathbb{N}$, $\mathcal{F}_{\text{fin}}$ is a core of H_{fr}^n .
- (iii) For all $n \in \mathbb{N}$, $\text{ad}_{H_{\text{fr}}}^n(H_{\text{int}})$ is infinitesimally H_{fr} -bounded on $\mathcal{F}_{\text{fin}}$, i.e., for all $\varepsilon > 0$, there exists a constant $C_\varepsilon \geq 0$ such that for all $\Psi \in \mathcal{F}_{\text{fin}}$,

$$\|\text{ad}_{H_{\text{fr}}}^n(H_{\text{int}})\Psi\| \leq \varepsilon \|H_{\text{fr}}\Psi\| + C_\varepsilon \|\Psi\|. \quad (5.83)$$

Proof. (i) It is obvious that H_{fr} leaves $\mathcal{F}_{\text{fin}}$ invariant from the definition of $\mathcal{F}_{\text{fin}}$ (5.55). The remaining claim follows from Lemma 5.8 (i) and (ii).

(ii) This follows from the general theory of the functional calculus.

(iii) For each integer $n \geq 0$, we define linear operators $H_{\text{I}}^{(n)}$ and $H_{\text{II}}^{(n)}$ by

$$\begin{aligned} D(H_{\text{I}}^{(n)}) &:= D((I \otimes H_{\text{ph}})^{1/2}), \\ H_{\text{I}}^{(n)}\Psi &:= e \int_{\mathbb{R}^3} d\mathbf{x} H_{\text{I}}^{(n)}(\mathbf{x})\Psi, \quad \Psi \in D(H_{\text{I}}^{(n)}), \end{aligned} \quad (5.84)$$

$$\begin{aligned} D(H_{\text{II}}^{(n)}) &:= \mathcal{F}_{\text{tot}}, \\ H_{\text{II}}^{(n)} &:= \frac{e^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{x} d\mathbf{y} H_{\text{II}}^{(n)}(\mathbf{x}, \mathbf{y}), \end{aligned} \quad (5.85)$$

with

$$H_I^{(n)}(\mathbf{x}) := \chi_{\text{sp}}(\mathbf{x}) \sum_{i=1}^3 \sum_{\substack{n_1+n_2=n, \\ n_1, n_2 \geq 0}} \frac{n!}{n_1!n_2!} :j^{i(n_1)}(0, \mathbf{x}) : \otimes A_i^{(n_2)}(0, \mathbf{x}), \quad (5.86)$$

$$H_{II}^{(n)}(\mathbf{x}, \mathbf{y}) := \chi_{\text{sp}}(\mathbf{x}) \chi_{\text{sp}}(\mathbf{y}) V_C(\mathbf{x} - \mathbf{y}) \sum_{\substack{n_1+n_2=n, \\ n_1, n_2 \geq 0}} \frac{n!}{n_1!n_2!} :j^{0(n_1)}(0, \mathbf{x}) j^{0(n_2)}(0, \mathbf{y}) : \otimes I, \quad (5.87)$$

$$j^{\mu(n)}(0, \mathbf{x}) := \sum_{\substack{n_1+n_2=n, \\ n_1, n_2 \geq 0}} \frac{n!}{n_1!n_2!} \sum_{l, l'=1}^4 \psi_l^{(n_1)}(0, \mathbf{x})^* \alpha_{ll'}^\mu \psi_{l'}^{(n_2)}(0, \mathbf{x}), \quad (5.88)$$

$$\psi_l^{(n)}(0, \mathbf{x}) := \sum_{s=\pm 1/2} \left(b_s((iE_M)^n \widehat{\chi}_{\text{el}}^{\mathbf{x}}(u_s^l)^*) + d_s^*((iE_M)^n \widehat{\chi}_{\text{el}}^{\mathbf{x}} \widehat{v}_s^l) \right), \quad (5.89)$$

$$A_i^{(n)}(0, \mathbf{x}) := \sum_{r=1,2} \left(a_r \left(\frac{(i\omega)^n \widehat{\chi}_{\text{ph}}^{\mathbf{x}} e_r^i}{\sqrt{2\omega}} \right) + a_r^* \left(\frac{(i\omega)^n \widehat{\chi}_{\text{ph}}^{\mathbf{x}} e_r^i}{\sqrt{2\omega}} \right) \right), \quad (5.90)$$

where the integral in (5.84) is taken in the sense of the strong Bochner integral, and the integral in (5.85) is the Bochner integral with respect to the operator norm. Then $H_I = H_I^{(0)}$, $H_{II} = H_{II}^{(0)}$. In the same way as in Lemma 5.6, one can show that each $\overline{H_I^{(n)}}$ is infinitesimally H_{fr} -bounded, and each $H_{II}^{(n)}$ is bounded.

To prove the claim, it is sufficient to show that

$$\text{ad}_{iH_{\text{fr}}}^n(H_{\text{int}})\Psi = (H_I^{(n)} + H_{II}^{(n)})\Psi, \quad \Psi \in \mathcal{F}_{\text{fin}}. \quad (5.91)$$

The left-hand side can be rewritten as $\text{ad}_{iH_{\text{fr}}E_{H_{\text{fr}}}}^n([0, E]) (H_{\text{int}})\Psi$ for sufficiently large $E \geq 0$; $H_{\text{fr}}E_{H_{\text{fr}}}([0, E])$ is bounded. Hence we have

$$\begin{aligned} & \text{ad}_{iH_{\text{fr}}}^n(H_{\text{int}})\Psi \\ &= e \int_{\mathbb{R}^3} d\mathbf{x} \text{ad}_{iH_{\text{fr}}}^n(H_I^{(0)}(\mathbf{x}))\Psi + \frac{e^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{x} d\mathbf{y} \text{ad}_{iH_{\text{fr}}}^n(H_{II}^{(0)}(\mathbf{x}, \mathbf{y}))\Psi. \end{aligned} \quad (5.92)$$

Using Lemmas A.5 and A.10, we have

$$\text{ad}_{iH_{\text{fr}}}^n(H_I^{(0)}(\mathbf{x}))\Psi = H_I^{(n)}(\mathbf{x})\Psi, \quad (5.93)$$

$$\text{ad}_{iH_{\text{fr}}}^n(H_{II}^{(0)}(\mathbf{x}, \mathbf{y}))\Psi = H_{II}^{(n)}(\mathbf{x}, \mathbf{y})\Psi. \quad (5.94)$$

therefore we obtain (5.91), and the assertion follows. $\square$

Lemma 5.13. *Under Hypotheses 5.1 and 5.2, it follows that*

$$D(H_{\text{tot}}^n) = D(H_{\text{fr}}^n). \quad (5.95)$$

for all $n \in \mathbb{N}$.

Proof. By Lemma 5.12, we can apply Lemma 5.10 to the case $T_0 = H_{\text{fr}}$, $T_1 = H_{\text{int}}$ and $D = \mathcal{F}_{\text{fin}}$, and thus the assertion follows. $\square$

Lemma 5.14. *Under Hypotheses 5.1, the following (i) and (ii) hold.*

- (i) For each integer $n \geq 0$, $I \otimes A_j(0, \mathbf{x})$ maps $D(H_{\text{tot}}^{n+1})$ into $D(H_{\text{tot}}^n)$.
- (ii) For each integer $n \geq 0$, $\psi_l(0, \mathbf{x}) \otimes I$ and $\psi_l(0, \mathbf{x})^* \otimes I$ leave $D(H_{\text{tot}}^n)$ invariant.

Proof. (i) Applying Lemma 5.10 to the case $T_0 = H_{\text{fr}}$, $T_1 = I \otimes A_j(0, \mathbf{x})$ and $D = \mathcal{F}_{\text{fin}}$, we see that $I \otimes A_j(0, \mathbf{x})$ maps $D(H_{\text{fr}}^{n+1/2})$ into $D(H_{\text{fr}}^n)$. Combining this with Lemma 5.13, the assertion follows.

- (ii) Similar to the proof of (i).

□

Now we are ready to prove the Gell-Mann – Low formula. We assume the following:

Hypothesis 5.3. (I) H_{tot} has a unique ground state Ω_{tot} ($\|\Omega_{\text{tot}}\| = 1$).

(II) $\langle \Omega_{\text{tot}}, \Omega_0 \rangle \neq 0$, where $\Omega_0 := \Omega_{\text{f}} \otimes \Omega_{\text{b}}$, $\Omega_{\text{f}} := \{1, 0, 0, \dots\} \in \mathcal{F}_{\text{f}}(\mathcal{H}_{\text{el}})$, and $\Omega_{\text{b}} := \{1, 0, 0, \dots\} \in \mathcal{F}_{\text{b}}(\mathcal{H}_{\text{ph}})$.

For conditions for Hypothesis 5.3 to hold, see [61]. Because of some technical problems, the coupling constant e is currently restricted to a sufficiently small region in order to prove the existence of the ground state.

Let $\phi^{(k)}(0, \mathbf{x})$ ($k = 1, \dots, m$, $m \geq 1$, $\mathbf{x} \in \mathbb{R}^3$) denote the point-like field operators, that is, for each k , $\phi^{(k)}(0, \mathbf{x})$ denotes $I \otimes A_j(0, \mathbf{x})$, $\psi_l(0, \mathbf{x}) \otimes I$, or $\psi_l(0, \mathbf{x})^* \otimes I$. For each $z \in \mathbb{C}$, we set

$$\phi_{\text{int}}^{(k)}(z, \mathbf{x}) := e^{izH_{\text{fr}}} \phi^{(k)}(0, \mathbf{x}) e^{-izH_{\text{fr}}}. \quad (5.96)$$

Theorem 5.6. Assume Hypotheses 5.1-5.3. Let $z_1, \dots, z_m \in \mathbb{C}$ with $\text{Im } z_1 \leq \dots \leq \text{Im } z_m$, and $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^3$. Choose a simple curve Γ_T^ε from $-T(1 - i\varepsilon)$ to $T(1 - i\varepsilon)$ ($T, \varepsilon > 0$) on which $z_1 \succ \dots \succ z_m$. Then the m -point Green's function

$$\begin{aligned} G_m(z_1, \dots, z_m) &:= e^{i(z_1 - z_m)E_0(H_{\text{tot}})} \left\langle \Omega_{\text{tot}}, \phi^{(1)}(0, \mathbf{x}_1) e^{-i(z_1 - z_2)H_{\text{tot}}} \right. \\ &\quad \left. \phi^{(m-1)}(0, \mathbf{x}_{m-1}) e^{-i(z_{m-1} - z_m)H_{\text{tot}}} \phi^{(m)}(0, \mathbf{x}_m) \Omega_{\text{tot}} \right\rangle, \end{aligned} \quad (5.97)$$

is well-defined and satisfies the formula

$$G_m(z_1, \dots, z_m) = \lim_{T \rightarrow \infty} \frac{\left\langle \Omega_0, T \phi_{\text{int}}^{(1)}(z_1, \mathbf{x}_1) \dots \phi_{\text{int}}^{(m)}(z_m, \mathbf{x}_m) \exp \left(-i \int_{\Gamma_T^\varepsilon} d\zeta H_{\text{int}}(\zeta) \right) \Omega_0 \right\rangle}{\left\langle \Omega_0, T \exp \left(-i \int_{\Gamma_T^\varepsilon} d\zeta H_{\text{int}}(\zeta) \right) \Omega_0 \right\rangle}. \quad (5.98)$$

Proof. We have only to see that the conditions of Theorem 5.3 hold when $H_0 = H_{\text{fr}}$, $H_1 = H_{\text{int}}$, $H = H_{\text{tot}}$ and $A_k = \phi^{(k)}(0, \mathbf{x}_k)$.

As is well known, H_{fr} has a unique ground state Ω_0 , and the corresponding eigenvalue is zero. Thus Assumption 5.1 (I) holds. Assumption 5.1 (II) and (III) follow from Hypothesis 5.3.

From Lemma 5.7 (iii) and (iv), each $\phi^{(k)}(0, \mathbf{x}_k)$ is in $\mathcal{C}_0$ -class. The remaining assumptions follow from Lemma 5.14. Therefore the desired result follows. □

Remark 5.2. The above formula (5.98) is more general than the Gell-Mann – Low formula discussed in physics literatures. To obtain the original Gell-Mann – Low formula, we regard the arguments $z_k \in \mathbb{C}$ ($k = 1, \dots, m$) as the time parameters which are usually real numbers, $z_k \in \mathbb{R}$. Then these are naturally time-ordered in $\mathbb{R}$ whenever these are different from each

other. Therefore to derive the original formula, choose a simple curve Γ_T^ε from $-T(1 - i\varepsilon)$ to $T(1 - i\varepsilon)$ in such a way that this natural time-ordering coincides our time-ordering defined above. For instance, take a polyline that passes $-T(1 - i\varepsilon)$, $t_{\min}$, $t_{\max}$, and $T(1 - i\varepsilon)$ in this order, where

$$t_{\min} = \min\{t_1, \dots, t_m\}, \quad t_{\max} = \max\{t_1, \dots, t_m\}.$$

6 A criteria for essential self-adjointness

6.1 Abstract result

In this subsection, we develop a general strategy for proving essential self-adjointness of symmetric operators, by using the results obtained in Section 3.

Let H_0 be a self-adjoint operator in $\mathcal{H}$, and H_1 be a symmetric one in $\mathcal{H}$. Suppose that $D(H_0) \cap D(H_1)$ is dense in $\mathcal{H}$. The main object we consider in the present section is the symmetric operator

$$H := H_0 + H_1, \tag{6.1}$$

defined on the dense subspace $D(H) = D(H_0) \cap D(H_1)$. Assumptions we employ here is:

Assumption 6.1. *There exists an operator A in $\mathcal{H}$ satisfying the following conditions:*

- (I) *A is self-adjoint and non-negative.*
- (II) *A and H_0 are strongly commuting.*
- (III) *H_1 is in $\mathcal{C}_0(A)$ -class.*

For the notational simplicity, we put

$$D := \bigcup_{L \geq 0} R(E_A([0, L])), \tag{6.2}$$

and $D' := D(H_0) \cap D$. Note that D, D' are dense subspaces.

Our goal of the present section is to prove the following theorem :

Theorem 6.1. *Suppose that Assumption 6.1 holds.*

- (i) *Exactly one of the following (a) and (b) holds.*
 - (a) *H has no self-adjoint extension.*
 - (b) *H is essentially self-adjoint.*
- (ii) *If D' is a core of H , then H is essentially self-adjoint on D' .*

To prove Theorem 6.1, we need the following lemmas.

The following fact is well known (see, e.g., Ref. [56]):

Lemma 6.1. *Let T be a symmetric operator in $\mathcal{H}$. If T has a unique self-adjoint extension, then T is essentially self-adjoint.*

Lemma 6.2. *Let T be a symmetric operator in $\mathcal{H}$. If there exists a dense subspace V such that for any $\xi \in V$ the initial value problem*

$$\frac{d}{dt}\xi(t) = -iT\xi(t), \quad \xi(0) = \xi, \tag{6.3}$$

has a strong solution $\mathbb{R} \ni t \mapsto \xi(t) \in D(T)$, then exactly one of the following (a) or (b) holds.

- (a) T has no self-adjoint extension.
(b) T is essentially self-adjoint.

Proof. It is sufficient to prove that, if there exists a self-adjoint extension, then T is essentially self-adjoint. Suppose that T has a self-adjoint extension $\widehat{T}$. Then for each $\eta \in D(\widehat{T})$ and $\xi \in V$, we have

$$\frac{d}{dt} \langle \eta, e^{it\widehat{T}} \xi(t) \rangle = \langle -i\widehat{T}e^{-it\widehat{T}} \eta, \xi(t) \rangle + \langle e^{-it\widehat{T}} \eta, -iT\xi(t) \rangle. \quad (6.4)$$

Since $\xi(t)$ belongs to $D(T)$, the first term in the right hand side is equal to $\langle -ie^{-it\widehat{T}} \eta, T\xi(t) \rangle$. Hence

$$\frac{d}{dt} \langle \eta, e^{it\widehat{T}} \xi(t) \rangle = 0 \quad (6.5)$$

for all $t \in \mathbb{R}$. thus we have

$$\langle \eta, e^{it\widehat{T}} \xi(t) \rangle = \langle \eta, \xi(0) \rangle = \langle \eta, \xi \rangle, \quad t \in \mathbb{R}. \quad (6.6)$$

Since $\eta \in D(\widehat{T})$ is arbitrary, $\xi(t) = e^{-it\widehat{T}} \xi$ ($t \in \mathbb{R}$) for all $\xi \in V$. This implies that, if T has another self-adjoint extension $\widehat{T}'$, then $e^{-it\widehat{T}} = e^{-it\widehat{T}'}$ ($t \in \mathbb{R}$). Hence $\widehat{T} = \widehat{T}'$ by Stone's theorem. This means that the self-adjoint extension of T is unique. Thus T is essentially self-adjoint by Lemma 6.1. $\square$

The next lemma is related to Stone's theorem and the proof can be found in e.g., Ref. [54, p. 267].

Lemma 6.3. *Let T be a symmetric operator in the Hilbert space $\mathcal{H}$. If for any $\xi \in D(T)$ the initial value problem*

$$\frac{d}{dt} \xi(t) = -i\overline{T}\xi(t), \quad \xi(0) = \xi, \quad (6.7)$$

has a strong solution $\mathbb{R} \ni t \mapsto \xi(t) \in D(\overline{T})$, then $\overline{T}$ is self-adjoint.

Proof of Theorem 6.1. We first prove (i). From Theorem 3.1, we find that for all $\xi \in D'$, there is a solution $\xi(t) = e^{-itH_0} \overline{U(t,0)} \xi \in D(H)$ of the initial value problem generated by H ,

$$\frac{d}{dt} \xi(t) = -iH\xi(t), \quad \xi(0) = \xi. \quad (6.8)$$

Since D' is dense, the assertion follows from Lemma 6.2.

Next, we prove (ii). Let H' be a restriction of H to D' . Then H' is a symmetric operator, and $\overline{H'} = \overline{H}$, because D' is a core of H . From Theorem 3.1, we conclude that for all $\xi \in D' = D(H')$ there is a solution $\xi(t) = e^{-itH_0} \overline{U(t,0)} \xi \in D(H)$ of the initial value problem generated by H ,

$$\frac{d}{dt} \xi(t) = -i\overline{H}\xi(t) = -i\overline{H'}\xi(t), \quad \xi(0) = \xi \quad (6.9)$$

From Lemma 6.3, it follows that $\overline{H'}$ is self-adjoint, but this immediately implies that H is essentially self-adjoint, because $\overline{H'} = \overline{H}$. $\square$

6.2 Application to the Dirac-Maxwell Hamiltonian in the Coulomb gauge

In this section, we introduce the Dirac-Maxwell Hamiltonian in the Coulomb gauge and prove its essential self-adjointness under a suitable condition.

As already mentioned, the Dirac-Maxwell model describes the interacting system constituted by Dirac particles and a electromagnetic field. We use the same notations as in Subsubsections 4.1.1 and 5.2.1 for describing the Dirac particles and the electromagnetic field quantized in the Coulomb gauge, respectively.

We consider the case where one Dirac particle under a potential V and the quantized electromagnetic field interact with each other. The potential is represented by a 4×4 Hermitian matrix-valued function V on $\mathbb{R}_x^3$ with each matrix components being Borel measurable. Note that the function V naturally defines a multiplication operator acting in $\mathcal{H}_D$ and we denote it by the same symbol V . The Hamiltonian of the Dirac particle under the influence of this external potential V is then given by the Dirac operator

$$H_D(V) := \boldsymbol{\alpha} \cdot \mathbf{p} + M\beta + V \quad (6.10)$$

acting in $\mathcal{H}_D$, with the domain $D(H_D(V)) := H^1(\mathbb{R}^3; \mathbb{C}^4) \cap D(V)$.

Let C be the conjugation operator in $\mathcal{H}_D$ defined by

$$(Cf)(\mathbf{x}) = f(\mathbf{x})^*, \quad f \in \mathcal{H}_D, \quad \mathbf{x} \in \mathbb{R}^3,$$

where $*$ means the usual complex conjugation. By Pauli's lemma [65], there is a 4×4 matrix U satisfying

$$U^2 = 1, \quad UC = CU, \quad (6.11)$$

$$U^{-1}\alpha^j U = \overline{\alpha^j}, \quad j = 1, 2, 3, \quad U^{-1}\beta U = -\beta, \quad (6.12)$$

where for a matrix A , $\overline{A}$ denotes its complex-conjugated matrix and 1 the identity matrix. We assume that the potential V satisfies the following conditions :

Assumption 6.2. (I) *Each matrix component of V belongs to*

$$L_{\text{loc}}^2(\mathbb{R}^3) := \left\{ f : \mathbb{R}^3 \rightarrow \mathbb{C} \left| \begin{array}{l} \text{Borel measurable and} \\ \int_{|\mathbf{x}| \leq R} |f(\mathbf{x})|^2 < \infty \text{ for all } R > 0. \end{array} \right. \right\}.$$

(II) *V is Charge-Parity (CP) invariant in the following sense:*

$$U^{-1}V(\mathbf{x})U = V(-\mathbf{x})^*, \quad \text{a.e. } \mathbf{x} \in \mathbb{R}^3. \quad (6.13)$$

(III) *$H_D(V)$ is essentially self-adjoint.*

Hereafter, we denote the closure of $H_D(V)$, which is self-adjoint by Assumption 6.2, by the same symbol. The important remark is that the Coulomb type potential

$$V(\mathbf{x}) = -\frac{Zq^2}{|\mathbf{x}|} \quad (6.14)$$

satisfies Assumption 6.2 provided that $Zq^2 < 1/2$, or more concretely, $Z \leq 68$ if we put $q = e$, the elementary charge [65].

The Hilbert space of state vectors for the coupled system is then taken to be

$$\mathcal{F}_{\text{DM}} := \mathcal{H}_D \otimes \mathcal{F}_{\text{ph}}^{(C)}. \quad (6.15)$$

We remark that this Hilbert space can be identified as

$$\mathcal{F}_{\text{DM}} = L^2(\mathcal{X}; \mathcal{F}_{\text{ph}}^{(\text{C})}) = \int_{\mathcal{X}}^{\oplus} d\mathbf{X} \mathcal{F}_{\text{ph}}^{(\text{C})}, \quad \mathbf{X} = (\mathbf{x}, l) \in \mathcal{X} = \mathbb{R}^3 \times \{1, 2, 3, 4\}. \quad (6.16)$$

We freely use this identification. Fix $\chi_{\text{ph}} \in L^2(\mathbb{R}_{\mathbf{x}}^3)$ such that it is real and satisfies $\widehat{\chi_{\text{ph}}}/\sqrt{\omega} \in L^2(\mathbb{R}_{\mathbf{k}}^3)$. Then the mapping $\mathbf{x} \mapsto \widehat{\chi_{\text{ph}}^{\mathbf{x}}}/\sqrt{\omega}$ from $\mathbb{R}^3$ to $\mathcal{H}_{\text{ph}}$ is strongly continuous, and thus we can define a decomposable self-adjoint operator A^j by

$$A^j := \int_{\mathcal{X}}^{\oplus} d\mathbf{X} A^j(0, \mathbf{x}), \quad j = 1, 2, 3, \quad (6.17)$$

acting in $\int_{\mathcal{X}}^{\oplus} d\mathbf{X} \mathcal{F}_{\text{ph}}^{(\text{C})}$. The minimal interaction Hamiltonian H_1 between the Dirac particle and the quantized electromagnetic field with the UV cutoff χ_{ph} is given by

$$H_1 := -q\boldsymbol{\alpha} \cdot \mathbf{A} = -q \sum_{j=1}^3 \alpha^j A^j. \quad (6.18)$$

The total Hamiltonian of the coupled system is then given by

$$H_{\text{DM}}(V) := H_0 + H_1, \quad (6.19)$$

$$H_0 := \overline{H_{\text{D}}(V) + H_{\text{ph}}^{(\text{C})}}. \quad (6.20)$$

This is called the *Dirac-Maxwell Hamiltonian* in the Coulomb gauge. The essential self-adjointness of $H_{\text{DM}}(V)$ is discussed in [3]. However, when the potential V is of the Coulomb type (6.14), the essential self-adjointness of $H_{\text{DM}}(V)$ remains to be proved.

The rest of the present paper is devoted to prove

Theorem 6.2. *Suppose that the potential V satisfies Assumption 6.2, and that the Fourier transformation of the UV cut-off, which we denote by $\widehat{\chi_{\text{ph}}}$, is real-valued. Then $H_{\text{DM}}(V)$ is essentially self-adjoint.*

We emphasize here again that Theorem 6.2 certainly covers the Coulomb potential case (6.14), if $Zq^2 < 1/2$.

To prove Theorem 6.2, we recall the important result obtained in Ref. [3].

Lemma 6.4. *Suppose that Assumption 6.2 is valid and $\widehat{\chi}$ is real-valued. then $H_{\text{DM}}(V)$ has a self-adjoint extension.*

Proof. See Ref. [3], Theorem 1.2. □

Let N_{b} be the photon number operator which is defined by

$$N_{\text{b}} := 1 \otimes d\Gamma_{\text{b}}(1), \quad (6.21)$$

acting in $\mathcal{H}_{\text{D}} \otimes \mathcal{F}_{\text{ph}}^{(\text{C})}$. Note that the operator N_{b} can be identified with the following decomposable operator in the sense of (6.16):

$$N_{\text{b}} = \int_{\mathcal{X}}^{\oplus} d\mathbf{X} d\Gamma_{\text{b}}(1). \quad (6.22)$$

Lemma 6.5. *The Dirac Maxwell Hamiltonian $H_{\text{DM}}(V)$ fulfills Assumption 6.1, where $A = N_{\text{b}}$. Namely,*

- (i) N_b is self-adjoint and non-negative.
- (ii) N_b and H_0 are strongly commuting.
- (iii) H_1 is in $\mathcal{C}_0(N_b)$ -class.

Proof. Similar to the proof of Lemma 4.1 □

Proof of Theorem 6.2. From Lemma 6.5 and Theorem 6.1 (i), we find that if there exists at least one self-adjoint extension of $H_{\text{DM}}(V)$, then $H_{\text{DM}}(V)$ is essentially self-adjoint. But from Lemma 6.4, $H_{\text{DM}}(V)$ indeed has a self-adjoint extension. This completes the proof. □

Finally, we remark that our proof presented here is also applicable to similar particle-field Hamiltonians. For instance, we can prove that the Dirac-Klein-Gordon Hamiltonian $H_{\text{DKG}}(V)$ acting in the Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}_b(L^2(\mathbb{R}^3))$, given by

$$H_{\text{DKG}}(V) = H_{\text{D}}(V) \otimes 1 + 1 \otimes d\Gamma_b(\omega) + \lambda \int_{\mathcal{X}}^{\oplus} d\mathbf{X} \beta \phi(\mathbf{x}), \quad \lambda \in \mathbb{R}, \quad (6.23)$$

with

$$\phi(\mathbf{x}) := \frac{1}{\sqrt{2}} \left(a \left(\frac{\widehat{\chi^{\mathbf{x}}}}{\sqrt{\omega}} \right) + a^* \left(\frac{\widehat{\chi^{\mathbf{x}}}}{\sqrt{\omega}} \right) \right), \quad (6.24)$$

$$\chi^{\mathbf{x}}(\mathbf{y}) := \chi(\mathbf{y} - \mathbf{x}), \quad \chi \in L^2(\mathbb{R}_x^3), \quad (6.25)$$

$a(\cdot)$ and $a^*(\cdot)$ ($\cdot \in \mathcal{F}_b(L^2(\mathbb{R}^3))$) be the annihilation and creation operator on $\mathcal{F}_b(L^2(\mathbb{R}^3))$, respectively, is essentially self-adjoint as long as the above assumptions are satisfied. This Hamiltonian describes a quantum system of a Dirac particle under the potential V interacting with a neutral scalar field.

Moreover, we can prove the essential self-adjointness of the N -body Dirac-Maxwell Hamiltonian in the Coulomb gauge in the same manner as our proof. The Hilbert space of state vectors is taken to be

$$\left(\bigwedge^N \mathcal{H}_{\text{D}} \right) \otimes \mathcal{F}_{\text{ph}}^{(\text{C})} = \int_{\mathcal{X}}^{\oplus} d\mathbf{X} \mathcal{F}_{\text{ph}}^{(\text{C})}, \quad (6.26)$$

and the N -body Dirac-Maxwell Hamiltonian is given by

$$H_{\text{DM}}(V, \widetilde{V}; N) := \overline{(H_{\text{D}}(V, N) + \widetilde{V}) \otimes 1 + 1 \otimes H_{\text{ph}}^{(\text{C})}} - q \sum_{a=1}^N \boldsymbol{\alpha} \cdot \mathbf{A}, \quad (6.27)$$

where $\widetilde{V}$ is a multiplication operator on $L^2(\mathbb{R}^{3N}; \mathbb{C}^{4^N}) = \bigotimes^N \mathcal{H}_{\text{D}}$ which is given by a $4^N \times 4^N$ Hermitian matrix valued function on $\mathbb{R}^{3N}$, and reduced by $\bigwedge^N \mathcal{H}_{\text{D}}$.

Using the unitary matrix U given in subsection 4.1.1, we set

$$U_N := \bigotimes^N U \quad (6.28)$$

the $4^N \times 4^N$ unitary matrix.

The basic hypothesis for the essential self-adjointness of $H_{\text{DM}}(V, \widetilde{V}; N)$ is:

Assumption 6.3. (I) Each matrix component of $\widetilde{V}$ belongs to $L_{\text{loc}}^2(\mathbb{R}^{3N})$.

(II) $\widetilde{V}$ is CP invariant in the following sense:

$$U_N^{-1} \widetilde{V}(\mathbf{X}) U_N = \widetilde{V}(-\mathbf{X})^*, \quad \text{a.e. } \mathbf{X} \in \mathbb{R}^{3N}. \quad (6.29)$$

(III) $H_D(V, N) + \tilde{V}$ is essentially self-adjoint.

One of the most physically important case is where V is the external Coulomb potential (6.14), and $\tilde{V}$ is given by the Coulomb interaction between the electrons:

$$\tilde{V}(\mathbf{X}) = \sum_{a < b}^N \frac{q^2}{|\mathbf{x}_a - \mathbf{x}_b|}, \quad \mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathbb{R}^{3N}. \quad (6.30)$$

In this case, it is easy to check that the external Coulomb potential (6.14) satisfies Assumption 6.2, and the Coulomb interaction (6.30) satisfies Assumption 6.3 (I), (II). However, Assumption 6.3 (III) is not trivial at all. In [63], it is proven that for the case $N = 2$, Assumption 6.3 (III) is valid for sufficiently small $|q|$ where V and $\tilde{V}$ are given by (6.14) and (6.30), respectively. To our best knowledge, the case $N \geq 3$ is still an open problem.

The following theorem can be proven in the same manner as in the proof of Theorem 6.2.

Theorem 6.3. *Suppose that Assumptions 6.2 and 6.3 hold, and the UV cut-off $\widehat{\chi_{\text{ph}}}$ is real-valued. Then $H_{\text{DM}}(V, \tilde{V}; N)$ is essentially self-adjoint.*

7 Future work

As mentioned in Introduction, the Gell-Mann – Low formula (1.29) is used to give a formal asymptotic expansion of the n -point correlation function with respect to the coupling constant, and this formal procedure is performed as follows. First, we give the asymptotic series:

$$\frac{\langle \Omega_0, \mathcal{T}\{\phi_1^{(1)}(x_1) \cdots \phi_1^{(n)}(x_n) \exp[-i\lambda \int_{-t}^t d\tau H_1(\tau)]\} \Omega_0 \rangle}{\langle \Omega_0, \mathcal{T}\{\exp[-i\lambda \int_{-t}^t d\tau H_1(\tau)]\} \Omega_0 \rangle} \sim \sum_{n=0}^{\infty} a_n(t) \lambda^n \quad (\lambda \rightarrow 0). \quad (7.1)$$

Second, we regard the formal series

$$\sum_{n=0}^{\infty} a_n \lambda^n, \quad a_n := \lim_{t \rightarrow \infty} a_n(t) \quad (7.2)$$

as the asymptotic expansion of the n -point correlation function. From the physical point of view, it is strongly expected that this formal series gives the true asymptotic expansion if the n -point correlation function exists, but from the mathematical point of view, it is highly non-trivial and provides a quite interesting problem of the perturbation theory.

A Fock spaces and second quantizations

A.1 Fock spaces

Let $\mathcal{H}$ be a complex separable Hilbert space, and $\otimes^n \mathcal{H}$ ($n \in \mathbb{N}$) the n -fold tensor product of $\mathcal{H}$. Let $\mathfrak{S}_n$ be the symmetric group of order n and U_σ ($\sigma \in \mathfrak{S}_n$) be a unitary operator on $\otimes^n \mathcal{H}$ such that

$$U_\sigma(\psi_1 \otimes \cdots \otimes \psi_n) = \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(n)}, \quad \psi_j \in \mathcal{H}, \quad j = 1, \dots, n. \quad (\text{A.1})$$

then the symmetrization operator S_n and the anti-symmetrization operator A_n are defined by

$$S_n := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} U_\sigma, \quad A_n := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) U_\sigma, \quad (\text{A.2})$$

where $\text{sgn}(\sigma)$ is the signature of the permutation σ . The operators S_n and A_n are orthogonal projections on $\otimes^n \mathcal{H}$. Hence the subspaces

$$\otimes_s^n \mathcal{H} := S_n(\otimes^n \mathcal{H}), \quad \wedge^n \mathcal{H} := A_n(\otimes^n \mathcal{H}) \quad (\text{A.3})$$

are Hilbert spaces. The space $\otimes_s^n \mathcal{H}$ (resp. $\wedge^n \mathcal{H}$) is called the n -fold symmetric (resp. anti-symmetric) tensor product of $\mathcal{H}$. We set $\otimes_s^0 \mathcal{H} := \mathbb{C}$, $\wedge^0 \mathcal{H} := \mathbb{C}$, and define

$$\mathcal{F}_b(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \otimes_s^n \mathcal{H}, \quad \mathcal{F}_f(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \wedge^n \mathcal{H}. \quad (\text{A.4})$$

$\mathcal{F}_b(\mathcal{H})$ (resp. $\mathcal{F}_f(\mathcal{H})$) is called the Boson (resp. Fermion) Fock space over $\mathcal{H}$.

A.2 Second quantization operators

For a densely defined closable operator T on $\mathcal{H}$ and $j = 1, \dots, n$, we define a linear operator $\tilde{T}_j$ on $\otimes^n \mathcal{H}$ by

$$\tilde{T}_j := I \otimes \cdots \otimes I \otimes \overset{j\text{-th}}{T} \otimes I \otimes \cdots \otimes I. \quad (\text{A.5})$$

For each integer $n \geq 0$, we define a linear operator $T^{(n)}$ on $\otimes^n \mathcal{H}$ by

$$T^{(0)} := 0, \quad T^{(n)} := \overline{\sum_{j=1}^n \tilde{T}_j \upharpoonright \otimes^n D(T)}, \quad n \geq 1, \quad (\text{A.6})$$

where $\otimes^n D(T)$ denotes the n -fold algebraic tensor product of $D(T)$. Then the infinite direct sum operator

$$d\Gamma(T) := \bigoplus_{n=0}^{\infty} T^{(n)} \quad (\text{A.7})$$

on $\mathcal{F}(\mathcal{H})$ is called the *second quantization of T* . If T is non-negative self-adjoint, then so is $d\Gamma(T)$. It is easy to see that $T^{(n)}$ is reduced by $\otimes_s^n \mathcal{H}$ and $\wedge^n \mathcal{H}$ respectively. We denote the reduced part of $T^{(n)}$ to $\otimes_s^n \mathcal{H}$ and $\wedge^n \mathcal{H}$ by $T_b^{(n)}$ and $T_f^{(n)}$ respectively. We set

$$d\Gamma_b(T) := \bigoplus_{n=0}^{\infty} T_b^{(n)}, \quad d\Gamma_f(T) := \bigoplus_{n=0}^{\infty} T_f^{(n)}. \quad (\text{A.8})$$

The operator $d\Gamma_b(T)$ (resp. $d\Gamma_f(T)$) is called the boson (resp. fermion) second quantization operator.

For a densely defined closable operator T on $\mathcal{H}$, we define a linear operator $\Gamma(T)$ on $\mathcal{F}(\mathcal{H})$ by

$$\Gamma(T) := \bigoplus_{n=0}^{\infty} \left(\otimes^n T \right). \quad (\text{A.9})$$

We denote the reduced part of $\Gamma(T)$ to $\mathcal{F}_b(\mathcal{H})$ and $\mathcal{F}_f(\mathcal{H})$ by $\Gamma_b(T)$ and $\Gamma_f(T)$ respectively.

Lemma A.1. Let K_j ($j = 1, \dots, n$, $n \geq 1$) be strongly commuting self-adjoint operators on a Hilbert space $\mathcal{H}$, and let $E^n := E_{K_1} \times \dots \times E_{K_n}$ be a product measure. Set

$$P(J) := E^n \left(\left\{ \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \sum_{j=1}^n \lambda_j \in J \right\} \right), \quad J \in \mathcal{B}^1. \quad (\text{A.10})$$

then $\{P(J) \mid J \in \mathcal{B}^1\}$ is the spectral measure of a self-adjoint operator $\overline{\sum_{j=1}^n K_j}$.

Proof. See e.g., [2, Lemma 2-33]. $\square$

Lemma A.2. Let T be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$. Then the following (i) and (ii) hold.

(i) Let $E_T^n := E_{\tilde{T}_1} \times \dots \times E_{\tilde{T}_n}$ be a product measure. then

$$E_{T^{(n)}}(J) = E_T^n \left(\left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \sum_{j=1}^n \lambda_j \in J \right\} \right), \quad J \in \mathcal{B}^1. \quad (\text{A.11})$$

(ii) For all $B_1, B_2 \in \mathcal{B}^1$ and $n \geq 0$, it follows that

$$R(E_T(B_1)) \otimes R(E_{T^{(n)}}(B_2)) \subset R(E_{T^{(n+1)}}(B_1 + B_2)) \quad \text{on} \quad \mathcal{H} \otimes \left(\bigotimes_{j=1}^n \mathcal{H} \right), \quad (\text{A.12})$$

where $B_1 + B_2 := \{\lambda_1 + \lambda_2 \in \mathbb{R} \mid \lambda_j \in B_j, j = 1, 2\}$.

Proof. (i) This follows directly from Lemma A.1.

(ii) Let us note that we can write as

$$T^{(n+1)} = \overline{T \otimes 1 + 1 \otimes T^{(n)}} \quad (\text{A.13})$$

on $\mathcal{H} \otimes \left(\bigotimes_{j=1}^n \mathcal{H} \right)$. The self-adjoint operators $T \otimes I$ and $I \otimes T^{(n)}$ are strongly commuting. Hence using Lemma A.1, the desired result follows. $\square$

Lemma A.3. Let T be a non-negative self-adjoint operator on a Hilbert space $\mathcal{H}$. Then for all $z \in \mathbb{C}$, the following operator equalities hold:

$$\Gamma(e^{izT}) = e^{izd\Gamma(T)}, \quad (\text{A.14})$$

$$\Gamma_b(e^{izT}) = e^{izd\Gamma_b(T)}, \quad (\text{A.15})$$

$$\Gamma_f(e^{izT}) = e^{izd\Gamma_f(T)}. \quad (\text{A.16})$$

Proof. It is well known that for all $z \in \mathbb{C}$ such that $\text{Im} z \geq 0$, $\Gamma(e^{izT})$ and $e^{izd\Gamma(T)}$ are bounded operators, and the operator equality

$$\Gamma(e^{izT}) = e^{izd\Gamma(T)} \quad (\text{A.17})$$

holds (see, e.g., [2, Theorem 3-13]).

Let $z \in \mathbb{C}$ with $\text{Im} z \geq 0$. Set $D_0(T) := \bigcap_{L \geq 0} R(E_T([0, L]))$. Then it follows that

$$\Gamma(e^{-izT})\Gamma(e^{izT})\Psi = \Psi, \quad (\text{A.18})$$

$$\Gamma(e^{izT})\Gamma(e^{-izT})\Psi = \Psi, \quad (\text{A.19})$$

for all $\Psi \in \hat{\oplus}_{n=0}^{\infty} (\hat{\otimes}^n D_0(T))$, where the symbol $\hat{\oplus}_{n=0}^{\infty}$ denotes the algebraic infinite direct sum. By (A.18) and the closedness of $\Gamma(e^{izT})$,

$$\Gamma(e^{-izT})\Gamma(e^{izT}) = I. \quad (\text{A.20})$$

Since $D_0(T)$ is a core of e^{-izT} , we see that $\Psi \in \hat{\oplus}_{n=0}^{\infty} (\hat{\otimes}^n D_0(T))$ is a core of $\Gamma(e^{-izT})$. Hence (A.19) yields that

$$\Gamma(e^{izT})\Gamma(e^{-izT}) \subset I. \quad (\text{A.21})$$

Since e^{izT} is injective, so is $\Gamma(e^{izT})$. Therefore (A.20) and (A.21) imply that

$$\Gamma(e^{-izT}) = \Gamma(e^{izT})^{-1}. \quad (\text{A.22})$$

On the other hand, by the functional calculus, we have

$$e^{-iz\text{d}\Gamma(T)} = (e^{iz\text{d}\Gamma(T)})^{-1}. \quad (\text{A.23})$$

Combining (A.17), (A.22) and (A.23),

$$e^{-iz\text{d}\Gamma(T)} = \Gamma(e^{-izT}), \quad (\text{A.24})$$

which, together with (A.17), yields (A.14).

(A.15) and (A.16) follows directly from (A.14) and the reducibility of $e^{iz\text{d}\Gamma(T)}$ and $\Gamma(e^{-izT})$. $\square$

A.3 Boson creation and annihilation operators

The boson annihilation operator $A(f)$ with $f \in \mathcal{H}$ is defined to be a densely defined closed operator on $\mathcal{F}_b(\mathcal{H})$ whose adjoint is given by

$$(A(f)^*\Psi)^{(0)} := 0, \quad (\text{A.25})$$

$$(A(f)^*\Psi)^{(n)} := \sqrt{n}S_n(f \otimes \Psi^{(n-1)}), \quad \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in D(A(f)^*), \quad n \geq 1. \quad (\text{A.26})$$

We note that $A(f)$ is anti-linear in f and $A(g)^*$ linear in g . The boson creation and annihilation operators leave the finite particle subspace

$$\mathcal{F}_{b,0}(\mathcal{H}) := \left\{ \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_b(\mathcal{H}) \mid \Psi^{(n)} = 0 \text{ for all sufficiently large } n \right\} \quad (\text{A.27})$$

invariant and satisfy the canonical commutation relations:

$$[A(f), A(g)^*] = \langle f, g \rangle_{\mathcal{H}}, \quad [A(f), A(g)] = [A(f)^*, A(g)^*] = 0, \quad f, g \in \mathcal{H}, \quad (\text{A.28})$$

on $\mathcal{F}_{b,0}(\mathcal{H})$.

The following fact is well known.

Lemma A.4. *Let K be an injective, non-negative, self-adjoint operator on $\mathcal{H}$. Then for all $\Psi \in D(\text{d}\Gamma_b(K)^{1/2})$ and $f \in D(K^{-1/2})$,*

$$\|A(f)\Psi\|_{\mathcal{F}_b(\mathcal{H})} \leq \|K^{-1/2}f\|_{\mathcal{H}}\|\text{d}\Gamma_b(K)^{1/2}\Psi\|_{\mathcal{F}_b(\mathcal{H})}, \quad (\text{A.29})$$

$$\|A(f)^*\Psi\|_{\mathcal{F}_b(\mathcal{H})} \leq \|K^{-1/2}f\|_{\mathcal{H}}\|\text{d}\Gamma_b(K)^{1/2}\Psi\|_{\mathcal{F}_b(\mathcal{H})} + \|f\|_{\mathcal{H}}\|\Psi\|_{\mathcal{F}_b(\mathcal{H})}. \quad (\text{A.30})$$

Lemma A.5. *Let T be an injective, non-negative, self-adjoint operator on $\mathcal{H}$. Then for all $f \in D(T^{-1/2}) \cap D(T)$, $A(f)$ and $A(f)^*$ map $D(d\Gamma_b(T)^{3/2})$ into $D(d\Gamma_b(T))$, and satisfy the following commutation relations:*

$$[d\Gamma_b(T), A(f)^*]\Psi = A(Tf)^*\Psi, \quad (\text{A.31})$$

$$[d\Gamma_b(T), A(f)]\Psi = -A(Tf)\Psi, \quad (\text{A.32})$$

for all $\Psi \in D(d\Gamma_b(T)^{3/2})$.

Proof. For a proof, see [2, Theorem 4-27]. $\square$

Lemma A.6. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D} \subset \mathcal{H}$ a dense subspace, Ω_b be the Fock vacuum of $\mathcal{F}_b(\mathcal{H})$. Then it follows that*

$$\bigcap_{f \in \mathcal{D}} \ker A(f) = \mathcal{L}\{\Omega_b\}. \quad (\text{A.33})$$

Lemma A.7. *Let T be a non-negative self-adjoint operator on $\mathcal{H}$, and let $f_n, f \in D(T^{-1/2})$, $\lim_{n \rightarrow \infty} f_n = f$, $\lim_{n \rightarrow \infty} T^{-1/2} f_n = T^{-1/2} f$. Then for all $\Psi \in D(d\Gamma_b(T)^{1/2})$, it follows that*

$$\lim_{n \rightarrow \infty} A(f_n)\Psi = A(f)\Psi, \quad \lim_{n \rightarrow \infty} A(f_n)^*\Psi = A(f)^*\Psi. \quad (\text{A.34})$$

Lemma A.8. *Let T be a non-negative self-adjoint operator on $\mathcal{H}$. Then the following (i) and (ii) hold.*

- (i) *For all $B_1, B_2 \in \mathcal{B}^1$ and $f \in R(E_T(B_1))$, $A(f)^*$ maps $R(E_{d\Gamma_b(T)}(B_2)) \cap D(A(f)^*)$ into $R(E_{d\Gamma_b(T)}(B_1 + B_2))$, where $B_1 + B_2 := \{\lambda_1 + \lambda_2 \in \mathbb{R} \mid \lambda_j \in B_j, j = 1, 2\}$.*
- (ii) *For all $\Lambda \geq 0$ and $f \in \mathcal{H}$, $A(f)$ leave $R(E_{d\Gamma_b(T)}([0, \Lambda]))$ invariant.*

Proof. (i) Let $\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in R(E_{d\Gamma_b(T)}(B_2)) \cap D(A(f)^*)$. By the general theory of direct product operators, it follows that $E_{d\Gamma_b(T)}(B_2) = \oplus_{n=0}^\infty E_{T_b^{(n)}}(B_2)$. Hence $\Psi^{(n)} \in R(E_{T_b^{(n)}}(B_2))$. By the definition of the creation operator $A(f)^*$,

$$(A(f)^*\Psi)^{(n+1)} = \sqrt{n+1} S_{n+1}(f \otimes \Psi^{(n)}), \quad n \geq 0. \quad (\text{A.35})$$

From Lemma A.2, we see that the right-hand side belongs to $R(E_{T_b^{(n)}}(B_1 + B_2))$. Therefore we have $A(f)^*\Psi \in R(E_{d\Gamma_b(T)}(B_1 + B_2))$.

- (ii) Let $f \in \mathcal{H}$ and $\Psi \in R(E_{d\Gamma_b(T)}([0, \Lambda]))$ for some $\Lambda \geq 0$. Since $D(A(f)) \supset D(d\Gamma_b(T)^{1/2})$ from Lemma A.4, we see that $\Psi \in D(A(f))$. To prove the claim, it is sufficient to see that for all $\Phi \in R(E_{d\Gamma_b(T)}([\Lambda, \infty)))$, $\langle \Phi, A(f)\Psi \rangle = 0$.

Now let $\Phi \in R(E_{d\Gamma_b(T)}([\Lambda, \infty)))$ be fixed arbitrarily and set $\Phi_n := E_{d\Gamma_b(T)}([\Lambda, \Lambda + n])\Phi$ ($n \in \mathbb{N}$). Then $\Phi_n \rightarrow \Phi$ ($n \rightarrow \infty$). Moreover, it follow from (i) that $A(f)^*\Phi_n \in R(E_{d\Gamma_b(T)}([\Lambda, \infty)))$ and thus $\langle A(f)^*\Phi_n, \Psi \rangle = 0$ for all $n \in \mathbb{N}$. hence

$$\langle \Phi, A(f)\Psi \rangle = \lim_{n \rightarrow \infty} \langle A(f)^*\Phi_n, \Psi \rangle = 0. \quad (\text{A.36})$$

Therefore the assertion follows. $\square$

Lemma A.9. *Let T be a non-negative self-adjoint operator on $\mathcal{H}$. Then for all $z \in \mathbb{C}$, $f \in \bigcup_{a \geq 0} R(E_T([0, a]))$ and $\Psi \in \bigcup_{\Lambda \geq 0} R(E_{d\Gamma_b(T)}([0, \Lambda]))$, it follows that*

$$e^{izd\Gamma_b(T)} A^*(f) e^{-izd\Gamma_b(T)} \Psi = A^*(e^{izT} f) \Psi, \quad (\text{A.37})$$

$$e^{izd\Gamma_b(T)} A(f) e^{-izd\Gamma_b(T)} \Psi = A(e^{iz^*T} f) \Psi. \quad (\text{A.38})$$

Proof. From Lemmas A.8 and A.3, we see that $\Psi \in D(\Gamma_b(e^{izT})A(f)^*\Gamma_b(e^{-izT}))$. By the definition of creation operator, we have

$$\begin{aligned} (\Gamma_b(e^{izT})A(f)^*\Gamma_b(e^{-izT}))^{(n+1)} &= \sqrt{n+1}S_{n+1}(e^{izT}f \otimes \Psi^{(n)}) \\ &= (A(e^{izT}f)\Psi)^{(n+1)} \end{aligned} \quad (\text{A.39})$$

for all $n \in \mathbb{N}$. Thus we obtain (A.37).

Using (A.37), for all $f \in \bigcup_{a \geq 0} R(E_T([0, a]))$ and $\Psi, \Phi \in \bigcup_{\Lambda \geq 0} R(E_{d\Gamma_b(T)}([0, \Lambda]))$,

$$\begin{aligned} \langle \Phi, e^{izd\Gamma_b(T)}A(f)e^{-izd\Gamma_b(T)}\Psi \rangle &= \langle e^{iz^*d\Gamma_b(T)}A(f)^*e^{-iz^*d\Gamma_b(T)}\Phi, \Psi \rangle \\ &= \langle A(e^{iz^*T}f)^*\Phi, \Psi \rangle \\ &= \langle \Phi, A(e^{iz^*T}f)\Psi \rangle. \end{aligned} \quad (\text{A.40})$$

Since $\Phi \in \bigcup_{\Lambda \geq 0} R(E_{d\Gamma_b(T)}([0, \Lambda]))$ is arbitrary, (A.38) follows. $\square$

A.4 Fermion creation and annihilation operators

The fermion annihilation operator $B(f)$ with $f \in \mathcal{H}$ is defined to be a bounded operator on $\mathcal{F}_f(\mathcal{H})$ whose adjoint is given by

$$(B(f)^*\Psi)^{(0)} := 0, \quad (\text{A.41})$$

$$(B(f)^*\Psi)^{(n)} := \sqrt{n}A_n(f \otimes \Psi^{(n-1)}), \quad \Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_f(\mathcal{H}), \quad (\text{A.42})$$

where A_n denotes the anti-symmetrization operator on $\otimes^n \mathcal{H}$, i.e. $A_n(\otimes^n \mathcal{H}) = \wedge^n \mathcal{H}$. It is well known that, the operator norm of $B(f)^\sharp$ becomes

$$\|B(f)^\sharp\| = \|f\|_{\mathcal{H}}. \quad (\text{A.43})$$

$B(f)$ is anti-linear in f and $B(f)^*$ linear in f . The fermion creation and annihilation operators satisfy the canonical anti-commutation relations:

$$\{B(f), B(g)^*\} = \langle f, g \rangle_{\mathcal{H}}, \quad \{B(f), B(g)\} = \{B(f)^*, B(g)^*\} = 0, \quad f, g \in \mathcal{H}, \quad (\text{A.44})$$

on $\mathcal{F}_f(\mathcal{H})$, where $\{X, Y\} := XY + YX$.

Definition A.1. We define an operator-valued function $\psi(\cdot, \cdot)$ by

$$\psi(f, g) := B(f) + B(g)^*, \quad f, g \in \mathcal{H}. \quad (\text{A.45})$$

Let $\mathcal{E}_f$ be the set consisting of finite linear combinations of finite products of operators $\psi(f, g)$ ($f, g \in \mathcal{H}$). For a product operator $\psi(f_1, g_1) \cdots \psi(f_n, g_n)$ ($f_j, g_j \in \mathcal{H}$, $j = 1, \dots, n$, $n \geq 1$), we define the normal ordering $:\psi(f_1, g_1) \cdots \psi(f_n, g_n):$ by

$$:\psi(f_1, g_1) \cdots \psi(f_n, g_n): = \sum_k' \text{sgn}(\sigma) B(g_{i_1})^* \cdots B(g_{i_k})^* B(f_{j_1}) \cdots B(f_{j_{n-k}}), \quad (\text{A.46})$$

where the symbol $\sum_k'$ denotes the sum over $i_1, \dots, i_k, j_1, \dots, j_{n-k}$ satisfying $i_1 < \dots < i_k$, $j_1 < \dots < j_{n-k}$, $\{i_1, \dots, i_k\} \cap \{j_1, \dots, j_{n-k}\} = \emptyset$, $\{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$, and σ is the permutation $(1, \dots, n) \mapsto (i_1, \dots, i_k, j_1, \dots, j_{n-k})$. We extend it by linearity to $\mathcal{E}_f$.

Lemma A.10. *Let T be a self-adjoint operator in H . Then for all $f \in D(T)$, $B(f)$ and $B(f)^*$ leave $D(d\Gamma_f(T))$ invariant, and satisfy the following commutation relations:*

$$[d\Gamma_f(T), B(f)^*]\Psi = B(Tf)^*\Psi, \quad (\text{A.47})$$

$$[d\Gamma_f(T), B(f)]\Psi = -B(Tf)\Psi, \quad (\text{A.48})$$

for all $\Psi \in D(d\Gamma_f(T))$.

Proof. See [2, Theorem 5-9]. $\square$

Lemma A.11. *Let T be a non-negative self-adjoint operator in $\mathcal{H}$. Then the following (i) and (ii) hold.*

(i) *For all $a, R \geq 0$ and $f \in R(E_T([0, a]))$, $B^*(f)$ maps $R(E_{d\Gamma_f(T)}([0, R]))$ into $R(E_{d\Gamma_f(T)}([0, R+a]))$.*

(ii) *For all $R \geq 0$ and $f \in \mathcal{H}$, $B(f)$ leave $R(E_{d\Gamma_f(T)}([0, R]))$ invariant.*

Proof. Similar to the proof of Lemma A.8. $\square$

Lemma A.12. *Let T be a non-negative self-adjoint operator in $\mathcal{H}$. Then for all $z \in \mathbb{C}$, $f \in \bigcup_{a \geq 0} R(E_T([0, a]))$ and $\Psi \in \bigcup_{\Lambda \geq 0} R(E_{d\Gamma_f(T)}([0, \Lambda]))$, it follows that*

$$e^{izd\Gamma_f(T)} B^*(f) e^{-izd\Gamma_f(T)} \Psi = B^*(e^{izT} f) \Psi, \quad (\text{A.49})$$

$$e^{izd\Gamma_f(T)} B(f) e^{-izd\Gamma_f(T)} \Psi = B(e^{iz^*T} f) \Psi. \quad (\text{A.50})$$

Proof. Similar to the proof of Lemma A.9. $\square$

B A property of tensor product operator

Lemma B.1. *Let K_j ($j = 1, \dots, n$, $n \geq 1$) be non-negative self-adjoint operators, and B_j be densely defined closable operators on Hilbert spaces $\mathcal{H}_j$. Suppose that, for each j , there exists a constant $a_j \geq 0$ such that, for all $L \geq 0$, B_j maps $R(E_{K_j}([0, L]))$ into $R(E_{K_j}([0, L+a_j]))$. Then for a self-adjoint operator*

$$K := \sum_{j=1}^n I \otimes \dots \otimes I \otimes K_j \otimes I \otimes \dots \otimes I \quad (\text{B.1})$$

on $\bigotimes_{j=1}^n \mathcal{H}_j$, the tensor product operator $B_1 \otimes \dots \otimes B_n$ maps $R(E_K([0, L]))$ into $R(E_K([0, L + \sum_j a_j]))$.

Proof. For each $L \geq 0$, set

$$J_L := \{(\lambda_1, \dots, \lambda_n) \in [0, \infty)^n \mid \sum_j \lambda_j \in [0, L]\} \subset \mathbb{R}^n. \quad (\text{B.2})$$

then for all $\varepsilon > 0$ and $L \geq 0$, there exist n dimensional half-closed intervals $I_\varepsilon^{(k)} := I_{\varepsilon,1}^{(k)} \times \dots \times I_{\varepsilon,n}^{(k)} \subset \mathbb{R}^n$ ($I_{\varepsilon,j}^{(k)} = [L_{\varepsilon,j}^{(k)}, \tilde{L}_{\varepsilon,j}^{(k)}) \subset \mathbb{R}$, $j = 1, \dots, n$, $k = 1, \dots, N_\varepsilon$, $1 \leq N_\varepsilon < \infty$) such that $I_\varepsilon^{(k)} \cap I_\varepsilon^{(k')} = \emptyset$ ($k \neq k'$) and

$$J_L \subset \bigcup_{k=1}^{N_\varepsilon} I_\varepsilon^{(k)} \subset J_{L+\varepsilon}. \quad (\text{B.3})$$

Now, we set $\tilde{K}_j := I \otimes \cdots \otimes I \otimes K_j \otimes I \otimes \cdots \otimes I$. Then $\tilde{K}_j$ ($j = 1, \dots, n$) are strongly commuting self-adjoint operators; it follows from Lemma A.1 that

$$R(E_K([0, L])) = R((E_{\tilde{K}_1} \times \cdots \times E_{\tilde{K}_n})(J_L)). \quad (\text{B.4})$$

Using (B.3), we have

$$R((E_{\tilde{K}_1} \times \cdots \times E_{\tilde{K}_n})(J_L)) \subset \bigoplus_{k=1}^{N_\varepsilon} \left(R(E_{\tilde{K}_1}(I_{\varepsilon,1}^{(k)})) \otimes \cdots \otimes R(E_{\tilde{K}_n}(I_{\varepsilon,n}^{(k)})) \right). \quad (\text{B.5})$$

By the present assumption, we see that $B_1 \otimes \cdots \otimes B_n$ maps $R(E_{\tilde{K}_1}(I_{\varepsilon,1}^{(k)})) \otimes \cdots \otimes R(E_{\tilde{K}_n}(I_{\varepsilon,n}^{(k)}))$ into $R(E_{\tilde{K}_1}([0, \tilde{L}_{\varepsilon,1}^{(k)} + a_1])) \otimes \cdots \otimes R(E_{\tilde{K}_n}([0, \tilde{L}_{\varepsilon,n}^{(k)} + a_n]))$ for each $k = 1, \dots, N_\varepsilon$. Combining this with (B.3)-(B.5), we conclude that, for all $\varepsilon > 0$, $B_1 \otimes \cdots \otimes B_n$ maps $R(E_K([0, L]))$ into $R(E_K([0, L + \varepsilon + \sum_j a_j]))$, that is, for all $\Psi \in R(E_K([0, L]))$,

$$E_K([0, L + \varepsilon + \sum_j a_j])(B_1 \otimes \cdots \otimes B_n)\Psi = (B_1 \otimes \cdots \otimes B_n)\Psi, \quad \varepsilon > 0. \quad (\text{B.6})$$

We can take the limit $\varepsilon \downarrow 0$ in (B.6), since the projection-valued function $E_K([0, L])$ ($L \geq 0$) is right-continuous with respect to L . Thus we obtain $E_K([0, L + \sum_j a_j])(B_1 \otimes \cdots \otimes B_n)\Psi = (B_1 \otimes \cdots \otimes B_n)\Psi$, and the desired result follows. $\square$

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