Abnormal geodesics in generalized subriemannian geometry

(一般化されたサブリーマン幾何学における異常測地線)

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Contents

1 Introduction

The control theory is formulated in terms of geometry and analysis. Given an optimal control problem, which is practical in its nature, we need several geometric ideas and methods to understand the problem properly and to solve it precisely. In fact, we apply differential geometric theory of differential systems and vector fields on manifolds to control theory, combined with singularity theory. In this paper, we treat important problem and provide new observations from the view point of geometric control theory.

We generalized in [11] the notion of control systems (see for instance [1]). A *control system* on a finite dimensional manifold is given by the pair of a locally trivial fibration over the manifold and a fiberded mapping over the identity of the manifold. We regarded in [11] the cone structure for distributions of Cartan types having growth $(2, 3, 5)$ as a control system by treating the locally trivial case. As important classes of control systems, two control systems are well-known, which are called the *driftless control-affine system* and the *control-affine system.* For the exact definition, see Example $2.5, 2.6$ of our paper.

Given a control system, a trajectory of the system is defined as an absolutely continuous curve such that its velocity vectors are expressed by a family of vector fields. In particular, if a control system is a driftless control-affine system, then a trajectory of the system is defined as an absolutely continuous curve such that its velocity vectors are linear combinations of the given vector fields. On the other hand, if a control system is a control-affine system, then a trajectory of the system is defined as an absolutely continuous curve such that its velocity vectors are the sum of one vector field called a drift and linear combinations of the other given vector fields.

If we give a control to the system, namely, if we give time-depending coefficients for the family of vector fields, and if we give a point of a finite dimensional manifold, then we have the unique trajectory because we can solve the Cauchy problem of ODE by using the classical Carethéodory theorem (see for instance [3] 2.4.1). For a fixed time interval, a control is called *admissible* if the trajectory exists up to the given time. Given an initial point, we have the endpoint of trajectory for each admissible control. Thus we obtain the well-defined mapping from an appropriate subspace (the space of admissible controls of the space of all controls) to the finite dimensional manifold, which is called the *endpoint mapping.*

The study of singularities of the endpoint mappings is one of important subjects of control theory. A singular point of the endpoint mapping is called a *singular control*, and the corresponding trajectory is called a *singular trajectory.*

Bonnard and Kupka showed in [6] that for a generic control-affine system with one drift and one control, any singular trajectory has always certain good properties, namely, it is "of minimal order" and it is "of corank one". After that, Chitour, Jean and Trélat generalized the result of [6] in the paper [9]. In particular, Chitour, Jean and Trélat also treated the driftless control-affine system case in [8],[9]. They studied, in those papers, the properties of singular controls for generic driftless control-affine systems.

In [8], Chitour, Jean and Trélat introduced and described the notions called "of minimal order" and "of corank one" related to singular controls. For the exact definitions, see Definition 4.2 and Definition 4.3 of our paper. Their results imply that, for a generic linearly-independent driftless control-affine system, any singular bi-extremal with the non-trivial singular trajectory is of minimal order and any singular control with the non-trivial singular trajectory is of corank one.

In the important paper [9], the results in [8] were widely generalized to general driftless control-affine systems including possibly linearly-dependent systems of vector fields. In particular, as a generalization of result in [8], by using a claim (Theorem 2.13 of [9]), they showed in [9] the following result: Let $(X_1, \cdots, X_m), 2 \leqq m \leqq n$ be a generic system of smooth vector fields over an n-dimensional manifold M, regarded as a driftless control-affine system. Then any singular X-bi-extremal with the non-trivial singular X-trajectory is of minimal order. Any singular X-control with the non-trivial singular X-trajectory is of corank one.

However, this assertions in [9] were based on an incorrect theorem(Theorem 2.13 of [9]). In fact, in [18], we pointed out the mistake of Theorem 2.13 in [9] and gave a counterexample(see Introduction of [18]).

Therefore we modified the formulation of theorems in [9]. In fact we adopt the different definition on minimal order property from that in [9] and give the new results in [18] by using the new concept of an "independent" trajectory.

Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields over an n-dimensional manifold M, and Ω be an open subset of $\mathbb{R}^m$. Consider the driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x)$$

with the control parameter $(u_1, \cdots, u_m) \in \Omega$.

To formulate Theorem 1 and Theorem 2, we introduce the new concept of an independent X-trajectory (see Definition 4.6) for a system $X = (X_1, \cdots, X_m)$. An X-trajectory $x : [0, T] \to M$ is called *independent* if the set of $t \in [0, T]$ such that $X_1(x(t)), \cdots, X_m(x(t))$ are linearly dependent over $\mathbb{R}$ has measure zero. An X-trajectory $x : [0, T] \to M$ is called *non-trivial* if for any sub-interval $J \subset [0, T]$, the restriction of x to $J, x|_J : J \to M$ is not constant. We need also the notiton of singular X-controls (see § 3.1) and that of X-bi-extremals (see §3.2).

Let $\mathrm{VF}(M)^m$ denote the set of systems of smooth vector fields $X = (X_1, \cdots, X_m)$ over M. We endow $\mathrm{VF}(M)^m$ with the Whitney smooth topology. Then, the followings hold:

Theorem 1 ([18]) *Suppose $2 \leqq m \leqq n$. Then there exists an open dense subset G_0 of $\mathrm{VF}(M)^m$ such that, if $X \in G_0$, if $z : [0, T] \to T^*M$ is an X-bi-extremal and if the singular trajectory $x = \pi \circ z : [0, T] \to M$ is non-trivial and independent, then the bi-extremal z is of minimal order.*

Here $\pi : T^*M \to M$ is the canonical projection from the cotangent bundle of M.

Theorem 2 ([18]) *Suppose $2 \leqq m \leqq n$. Then there exists an open dense subset G_1 of $\mathrm{VF}(M)^m$ such that, if $X \in G_1$, if $u : [0, T] \to \Omega$ is a singular X-control for a given initial point $x_0 \in M$ and if the corresponding singular trajectory $x : [0, T] \to M, x(0) = x_0$ is non-trivial and independent, then the control u is of corank one.*

Note that the idea of the proofs of Theorem 1 and Theorem 2 are performed basically following the ideas from [8] and Thom's transversality theorem (for instance see [10]) is used in the proofs of Theorem 1 and Theorem 2.

We put a remark on the case $m = 1$ and on the case $m > n$:
Let $m = 1$. A generic vector field on a manifold has isolated singularities. Then the endpoint mapping has singularities everywhere of corank $\geqq n-1$. On the other hand, by definition, no bi-extremal is of minimal order if $m = 1$. Therefore main theorems 1 and 2 never hold in the case.
Let $m > n$. Then the dependent locus turns to be the whole space, and there is no independent trajectory. Therefore in this case main theorems become void.

In general, singular trajectories are very difficult to treat in control problems. Then Theorem 1 and Theorem 2 of guarantee that, for a generic system, the singularities of endpoint mappings are not so bad. Under additional conditions, any singularity is of corank one. Moreover, in the Hamiltonian formalism, if we choose a bi-extremal for a given singular trajectory, then the singular control can be recovered by some differentiations of minimal order (see [8],[9]).

On the other hand, consider singular controls with non-trivial independent trajectory for a generic polynomial driftless control-affine system by using some ideas of Theorem 1 and Theorem 2.

Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ denote an m-tuple of integers, and $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ denote the product space of m-tuple systems of polynomial vector fields over $\mathbb{R}^n$, $Q = (Q_1, \cdots, Q_m)$, such that the degree of Q_i satisfiies $\deg Q_i \leqq d_i$ for every integer $i\,(1 \leqq i \leqq m)$, and we endow $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ with the Euclidean topology.

Let Ω be an open subset of $\mathbb{R}^m$. For $Q = (Q_1, \cdots, Q_m) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$, consider the polynomial driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i\, Q_i(x)$$

with the control parameter $(u_1, \cdots, u_m) \in \Omega$. Then, the followings hold:

Theorem 3 Suppose $2 \leqq m \leqq n$ and suppose that, an m-tuple of even integers $\boldsymbol{D} = (d_1, \cdots, d_m)$ satisfies the inequality : $\min\{d_1, d_2, \cdots, d_m\} \geqq 4n + 2$. Then, there exists an open dense semi-algebraic subset $H_0 \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that, if $Q \in H_0$, if $z : [0, T] \to T^*\mathbb{R}^n$ is a Q-bi-extremal, and if the singular trajectory $x = \pi \circ z : [0, T] \to M$ is non-trivial and independent, then the bi-extremal z is of minimal order.

Here $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection from the cotangent subbundle of $\mathbb{R}^n$.

Theorem 4 Suppose $2 \leqq m \leqq n$ and suppose that, an m-tuple of even integers $\boldsymbol{D} = (d_1, \cdots, d_m)$ satisfies the inequality : $\min\{d_1, d_2, \cdots, d_m\} \geqq 4n + 2$. Then, there exists an open dense semi-algebraic subset $H_1 \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that, if $Q \in H_1$, if $u : [0, T] \to \Omega$ is a singular Q-control for a given initial point $x_0 \in \mathbb{R}^n$, and if the corresponding singular trajectory $x : [0, T] \to \mathbb{R}^n, x(0) = x_0$ is non-trivial and independent, then the control u is of corank one.

Note that the idea of the proofs of Theorem 3 and Theorem 4 are performed basically following the ideas from [8] and Tarski-Seidenberg theorem (for instance see [7]) is used in the proofs of Theorem 3 and Theorem 4.

Now, consider the more geometric meaning of driftless control-affine systems. A *subriemannian structure* on a finite dimensional manifold is given by the pair of a subbundle on the tangent bundle of its manifold and a riemannian metric on the fibers of the subbundle. A singular trajectory for a linearly-independent driftless control-affine system is also called an *abnormal extremal* in subriemannian geometry. A solution of the geodesic equations is associated to the subriemannian metric. By using a rigorous application of the Pontryagin maximal principle of optimal control theory, every length-minimizer associated to a subriemannian structure is either a normal extremal or an abnormal extremal. Note that, abnormal extremals do not depend on the metric, but may be minimizing, and the two possibilities of normal and abnormal extremals are not mutually exclusive. It may happen that an extremal is both normal and abnormal.

Montgomery gave in [14] an example of length minimizer and abnormal extremal, which is not the projection of a normal extremal. Moreover, since different authors had written false proofs of the fact that an abnormal extremal cannot be length-minimizing geodesic associated to a subriemannian structure, Montgomery gave in [14] the list of several false proofs by different authors. These findings gave impetus to wide-ranging researcher (for instance Agrachev and Sarychev [2], Liu and Sussmann [13]).

After that, Bellïaiche widely generalized in [4] a subriemannian structure. The metric in a generalized subriemannian structure is defined by using the system of vector fields on a finite dimensional manifold. Note that, the metric can be defined even if the system of vector fields is not always linearly independent everywhere on a finite dimensional manifold. If it is linearly independent everywhere, then the system generates a subbundle of the tangent bundle of the manifold and the metric in a generalized subriemannian structure is the same as the subriemannian metric in ordinary meaning.

We showed in [12] that a pseudo-product structure in the sense of Tanaka (see [17]) induces locally but naturally a double fibration and cone structures. Since we had been able to regard cone structures as control systems by giving the generalization of control system (see also [11]), we gave in [12] the constrained Hamiltonian systems describing both normal and abnormal extremals for pseudo-product structures with generalized subriemannian metrics and for associated cone structures with metrics.

Bonnard and Heutte showed in the preprint of [5], that for a generic linearly independent driftless control-affine system, any nontrivial abnormal extremal associated to the subriemannian metric in ordinary meaning has always certain good properties, namely, it is "strictly abnormal". After that, Chitour, Jean and Trélat gave a more complete proof in the Appendix in [8].

In our paper, we define the length-minimizer on a generalized subriemannian structure as an analogy of a subriemannian geometry and generalized the result of [8] to general driftless control-affine systems including possibly linearly-dependent systems of vector fields. In particular, we introduce the important concept of a "strictly" abnormal extremal in order to describe Theorem 5.

Let M be an n dimensional manifold. Let Ω be an open subset of $\mathbb{R}^r$. Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields over M. Consider the driftless control-affine system

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x).$$

Moreover, consider the optimal control problem associated to the driftless control-affine system to minimize the energy functional

$$e(u) = \frac{1}{2} \sum_{i=1}^{m} u_i^2.$$

on X-admissible controls with the fixed initial point and the end point.

To formulate Theorem 5, we introduce the important concept of a strictly abnormal X-extremal (see Definition 7.2) for a system $X = (X_1, \cdots, X_m)$. An X-extremal $x : [0, T] \to M$ is called *strictly X-abnormal* if it is not the projection of a normal X-bi-extremal (see §7). We need also the concept, which appeared in Theorem 1 and Theorem 2, that an X-extremal (X-trajectory) is non-trivial and independent.

Let $\mathrm{VF}(M)^m$ denote the set of systems of smooth vector fields $X = (X_1, \cdots, X_m)$ over M. We endow $\mathrm{VF}(M)^m$ with the Whitney smooth topology. Then, the following holds:

Theorem 5 Suppose $2 \leqq m \leqq n$. Then, there exists an open dense subset $G_3 \subset \mathrm{VF}(M)^m$ such that, if $X \in G_3$ and if $x : [0, T] \to M$ is an X-abnormal extremal for the initial point $x_0 \in M$, then x is strictly X-abnormal.

Note that, Theorem 5 is the special case of Proposition 2.19 in [9] and Thom's transversality theorem (for instance see [10]) is used in the proofs of the result. However, since the proof of Proposition 2.19 in [9] is hard to read and in particular, as an important part, the concrete construction of G_3 is not written and the codimension of its complement as a semi-algebraic set is not computed. Therefore, we give the proof of Theorem 5 in this paper, independently from Proposition 2.19 in [9]. Note that the idea of the proofs of Theorem 5 are performed basically following the ideas from [8] and Thom's transversality theorem (for instance see [10]) is used in the proofs of Theorem 5.

Note that a singular trajectory is of corank one if and only if it admits a unique (up to scalar normalization) abnormal extremal lift. It is strictly abnormal and of corank one if and only if it admits a unique extremal lift which is abnormal.

On the other hand, we use some ideas of Theorem 5 and cosider singular controls with non-trivial independent trajectory for a generic polynomial driftless control-affine system.

Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ denote an m-tuple of integers, and $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ denotes the product space of m-tuple systems of polynomial vector fields over $\mathbb{R}^n$, $Q = (Q_1, \cdots, Q_m)$, such that the degree of Q_i satisfiies $\deg Q_i \leqq d_i$ for every integer $i\,(1 \leqq i \leqq m)$, and we endow $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ with the Euclidean topology.

Let Ω be an open subset of $\mathbb{R}^m$. For $Q = (Q_1, \cdots, Q_m) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$, consider the polynomial driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i \, Q_i(x)$$

with the control parameter $(u_1, \cdots, u_m) \in \Omega$. Moreover, consider the optimal control problem associated to the driftless control-affine systems to minimize the energy functional

$$e(u) = \frac{1}{2} \sum_{i=1}^{m} u_i^2.$$

on a Q-admissible control with the fixed initial point and end point. Then, the following holds:

Theorem 6 Suppose $2 \leqq m \leqq n$ and suppose that, an m-tuple of integers $\boldsymbol{D} = (d_1, \cdots, d_m)$ satisfies the inequality : $\min\{d_1, d_2, \cdots, d_m\} \geqq 3n + 2$. Then, there exists an open dense semi-algebraic subset $H_3 \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that, if $Q \in H_3$, if $x : [0, T] \to T^*M$ is a Q-abnormal for the initial point $x_0 \in M$, then x is strictly abnormal.

Note that the ideas of the proof of Theorem 5 are performed basically following the ideas from [8] and Tarski-Seidenberg theorem (for instance see [7]) is used in the proof of Theorem 6.

In §2, we consider the generalized notion of control systems. In §3, we consider driftless control-affine systems and singular controls. In §4, as properties of singular controls, bi-extremals, and trajectories, we define the notion "of corank one", "of minimal order", and "independent". In §5 and §6, we show respectively the main theorem 1 and the main theorem 2.

In §7, we introduce the notion of the optimal controls and define "strictly" abnormal trajectories. In §8, we introduce the notion of generalized subriemannian geometry by Bellïaiche and consider the abnormal extremal in generalized subriemannian geometry. In §9, we show the main theorem 5.

In §10,11,12, we show respectively the main theorem 3, the main theorem 4, and the main theorem 6.

2 Control systems

In this section, we generalized in [11] the notion of control systems. We regarded the cone structure for distributions of Cartan types having growth $(2,3,5)$ as a control system by treating the locally trivial fiber case. As important classes of control systems, two control systems are well-known, we define the driftless control-affine system and the control-affine system (see Example 2.5, 2.6).

Recall the definition of the "generalized" control systems.

Definition 2.1 Let M be an n-dimensional smooth manifold. A *control system* on M is given by the pair $(\pi_{\mathfrak{U}}, F)$ of a locally trivial fibration $\pi_{\mathfrak{U}} : \mathfrak{U} \to M$ over M and a smooth mapping $F : \mathfrak{U} \to TM$ to the tangent bundle TM such that the diagram

$$\begin{tikzcd} \mathfrak{U} \arrow[rr, "F"] \arrow[dr, "\pi_{\mathfrak{U}}"'] & & TM \arrow[dl, "\pi_{TM}"] \\ & M & \end{tikzcd}$$

commute where $\pi_{TM} : TM \to M$ is the projection of tangent bundle. Then the control system is simply written as

$$\mathfrak{U} \xrightarrow{F} TM \xrightarrow{\pi_{TM}} M.$$

$F : \mathfrak{U} \to TM$ is called a *fibred mapping* over the identity of M. In this paper we mainly treat a local case where $\pi_{\mathfrak{U}}$ is trivial.

Example 2.2 Let $D \subset TM$ be a vector subbundle of the tangent bundle TM. A vector subbundle D is also called a *distribution* or a *differential system* on M. Then, $D \subset TM$ is regarded as the control system (π_D, F) of the fibration $\pi_D = \pi_{TM}|_D : D \to M$ and the inclusion $F : D \hookrightarrow TM$:

$$D \overset{F}{\hookrightarrow} TM \xrightarrow{\pi_{TM}} M.$$

Example 2.3 Let $(X_1 \cdots, X_m)$ be a system of smooth vector fields over M. Let $L \subset TM$ be the union of the sets $L_x = \langle X_1(x), \cdots, X_m(x)\rangle_{\mathbb{R}}$ with $x \in M$. Then, the fibration $\pi_L = \pi_{TM}|_L : L \to M$ and $L \subset TM$ is regarded as the control system (π_L, F) of the fibration $\pi_L = \pi_{TM}|_L : L \to M$ and the inclusion $F : L \hookrightarrow TM$:

$$L \overset{F}{\hookrightarrow} TM \xrightarrow{\pi_{TM}} M.$$

If $(X_1 \cdots, X_m)$ is linearly independent everywhere on M, then $L \subset TM$ is a distribution. Therefore, this control system $L \hookrightarrow TM \to M$ is a natural generalization of Example 2.2 in the case where D is trivial.

Example 2.4 Let $\Omega \subset \mathbb{R}^r$ be an open subset. Let $\pi : M \times \Omega \to M$ be the canonical projection. We assume that a smooth mapping $F : M \times \Omega \to TM$ is given. Then, (π, F) is a control system. The control-affine system (π, F) is simply written as

$$M \times \Omega \xrightarrow{F} TM \xrightarrow{\pi_{TM}} M.$$

Note that, a control system (π, F) with the canonical projection $\pi : M \times \Omega \to M$ is given by a family of vector fields $\{X_u\}_{u\in\Omega}$ over M :

$$X_u(x) = F(x, u), \text{ for } u \in \Omega, x \in M.$$

In control theory, a control system $(\pi_{\mathcal{U}}, F)$ with the control parameter $u \in \Omega$ is denoted by

$$\dot{x} = X_u(x).$$

As a typical example of 2.4, a driftless control-affine system and a control-affine system are well-known.

Example 2.5 (Driftless control-affine system) Let $(X_1 \cdots, X_m)$ be a system of smooth vector fields over M. Let $\Omega \subset \mathbb{R}^m$ be an open subset. Then, the *driftless control-affine system* is given by the pair (π, F_D) of the canonical projection $\pi : M \times \Omega \to M$ and the following map $F_D : M \times \Omega \to TM$:

$$F_D(x, u) = \sum_{i=1}^{m} u_i X_i(x), \text{ for } (x, u) \in M \times \Omega.$$

The driftless control-affine system (π, F) is simply written as

$$M \times \Omega \xrightarrow{F_D} TM \xrightarrow{\pi_{TM}} M.$$

In control theory, the driftless control-affine system (π, F_D) with control parameter $u \in \Omega$ is written as

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x).$$

Example 2.6 (Control-affine system) Let $(X_0, X_1 \cdots, X_m)$ be a system of smooth vector fields over M. Let $\Omega \subset \mathbb{R}^{m+1}$ be an open subset. Then, the *control-affine system* is given by the pair (π, F_A) of the canonical projection $\pi : M \times \Omega \to M$ and the following map $F_A : M \times \Omega \to TM$:

$$F_A(x, u) = X_0(x) + \sum_{i=1}^{m} u_i X_i(x), \text{ for } (x, u) \in M \times \Omega.$$

The control-affine system (π, F_A) is simply written as

$$M \times \Omega \xrightarrow{F_A} TM \xrightarrow{\pi_{TM}} M.$$

In control theory, the control-affine system (π, F_A) with control parameter $u \in \Omega$ is written as

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i X_i(x).$$

3 Driftless control-affine systems and singular controls

3.1 Singular controls of driftless control-affine systems

We recall the basic definitions in a driftless control-affine system needed in this paper: Let M be an n-dimensional C^∞ manifold and $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields on M. We consider the control system

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x)$$

defined by X with the control parameters $u_1, \cdots, u_m$, which is called a *driftless control-affine system.*

Let $\Omega \subset \mathbb{R}^m$ be an open set. Let $x_0 \in M$ and $T > 0$. For an L^∞ (i.e. essentially bounded) curve $u : [0,T] \to \Omega$, we consider the *Cauchy problem* $(*)_{x_0,u}$, namely we consider the differential equation with the initial condition:

$$(*)_{x_0,u} \begin{cases} \dot{x}(t) = \sum\limits_{i=1}^{m} u_i(t) X_i(x(t)) \ \text{ for a.e. } t \in [0,T], \\ x(0) = x_0 \end{cases}$$

By using the classical Carathéodory theory ([3] 2.4.1), we have that there exists a locally unique Lipschitz solution. A curve $u : [0,T] \to \Omega$ is called an X-*control* and the solution of $(*)_{x_0,u}$ is called an X-*trajectory.*

In particular, an X-control $u : [0,T] \to \Omega$ is called *admissible* if the global solution $x = x_u$ of $(*)_{x_0,u}$ exists. We call $x(T)$ the endpoint of the trajectory x. We use $\mathcal{U}_{x_0,T}$ to denote the set of admissible X-controls. Then $\mathcal{U}_{x_0,T} \subset L^\infty([0,T],\Omega)$ is a Banach open manifold ([3]).

The endpoint mapping $\mathrm{End}_{x_0}^T : \mathcal{U}_{x_0,T} \to M$ is defined by taking endpoint, namely, $\mathrm{End}_{x_0}^T(u) = x_u(T)$ for the trajectory x_u for $u \in \mathcal{U}_{x_0,T}$.

An X-control $u : [0,T] \to \Omega$ is called a *singular* or an *abnormal* if u is a singular point of $\mathrm{End}_{x_0}^T$, namely if the differential $(\mathrm{End}_{x_0}^T)_* : T_u\mathcal{U}_{x_0,T} \to T_{x_u(T)}M$ is not surjective. When u is a singular X-control, the corresponding trajectory x_u is called a *singular* X-*trajectory* or an *abnormal* X-*extremal.*

Unless otherwise stated, after this we consider only a driftless control-affine case.

3.2 Equivalent condition of singular controls

We explain the equivalent condition of singular X-controls. In general case, it is difficult to study the properties of singular points of a functional from a Banach manifold to a finite dimensional manifold. However, in case of the endpoint mapping, as a part of Pontryagin maximal principle, an admissible X-control of the endpoint mapping is singular if and only if there exists an X-bi-extremal that satisfies the constrained Hamiltonian equations (Proposition 3.1).

Let M be an n-dimensional manifold and $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields on M. Let $\Omega \subset \mathbb{R}^m$ be an open set. Then the local description of the equivalent condition of singular X-controls can be given:

Proposition 3.1 *Let $x_0 \in M$. Let $u : [0,T] \to \Omega$ be an admissible X-control and $x : [0,T] \to M$ be the X-trajectory with $x(0) = x_0$. Then u is a singular X-control if and only if there exists an absolutely continuous curve $z : [0,T] \to T^*M$ such that, $x = \pi \circ z$, and that the following equations hold for any local coordinates $(x, p; u) = (x_1, \cdots, x_n, p_1, \cdots, p_n; u_1, \cdots, u_m)$ of $T^*M \times \Omega$, with a canonical coordinates (x, p) of T^*M :*

$$(\sharp): \begin{cases} (1)\ \dot{x}_i(t) = \frac{\partial H}{\partial p_i}(x(t), p(t); u(t))\,(1 \leqq i \leqq n) & \textit{for a.e. } t \in [0,T] \\ (2)\ \dot{p}_i(t) = -\frac{\partial H}{\partial x_i}(x(t), p(t); u(t))\,(1 \leqq i \leqq n) & \textit{for a.e. } t \in [0,T] \\ (3)\ \frac{\partial H}{\partial u_j}(x(t), p(t); u(t)) = 0\ (1 \leqq j \leqq m) & \textit{for a.e. } t \in [0,T] \\ (4)\ p(t) \neq 0 & \textit{for every } t \in [0,T] \end{cases},$$

where we define $H : T^*M \times \Omega \to \mathbb{R}$ by $H(x, p; u) = \langle p, \sum_{i=1}^m u_i X_i(x) \rangle$.

If a point $x_0 \in M$ is fixed and a singular X-control u is given, then the curve z is called a singular X-bi-extremal or an abnormal X-bi-extremal corresponding to u. The equations $(\sharp)$ is called the *constrained Hamiltonian equations*. Note that for a singular X-control $u : [0,T] \to \Omega$, a corresponding X-bi-extremal z_u is not a unique solution of $(\sharp)$ because the initial condition of (2) is not given.

4 Properties of singular controls, trajectories, and bi-extremals

In this section, we define some concepts of singular controls, trajectories, and bi-extremals. In §4.1, we define the notion of singular bi-extremals of "minimal order". In §4.2, we define the notion of singular controls of "corank one". In §4.3, we define the notion that singular trajectories are non-trivial and independent.

4.1 Singular bi-extremals of minimal order

Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields over an n-dimensional manifold M, and Ω be an open subset of $\mathbb{R}^m$. Consider the driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x)$$

with the control parameter $(u_1, \cdots, u_m) \in \Omega$.

In order to define singular X-controls of minimal order, we prepare the (generalized) Goh matrix ([9], Definition 2.15.).

We describe the (generalized) Goh matrix. For integers i, j $(1 \leqq i, j \leqq m)$, we define the Hamiltonians $H_i : T^*M \to \mathbb{R}$ and $H_{ij} : T^*M \to \mathbb{R}$ by

$$\begin{cases} H_i(z) = \langle z, X_i(x) \rangle, \\ H_{ij}(z) = \langle z, [X_i, X_j](x) \rangle. \end{cases}, z \in T^*M.$$

where $x = \pi(z)$, $\pi : T^*M \to M$ is the canonical projection. Note that

$$\langle z, [X_i, X_j](x) \rangle = \{H_i, H_j\}(x),\ z \in T^*M,$$

where $\{H_i, H_j\}$ is the Poisson bracket of H_i and H_j. Then we define the *Goh matrix* $G : T^*M \to \mathcal{S}_m(\mathbb{R})$ associated to the system of vector fields $X = (X_1, \cdots, X_m)$ on M by :

$$G(z) = (H_{i,j}(z))_{1 \leqq i,j \leqq m},\ z \in T^*M,$$

where $\mathcal{S}_m(\mathbb{R})$ is the set of $m \times m$ skew-symmetric matrices. Note that since $G(z)$ is a skew-symmetric matrix, rank $G(z)$ is even. If m is even, then there exists a polynomial function $P : \mathcal{S}_m(\mathbb{R}) \cong \mathbb{R}^{\frac{m(m-1)}{2}} \to \mathbb{R}$ of degree $\frac{m}{2}$ in the variables $(H_{ij})_{1 \leqq i < j \leqq m}$ such that for any $G \in S_m(\mathbb{R})$, $\det G$ is the square of $P(G)$, which is called the Pfaffian. We define the Hamiltonian $\hat{P} : T^*M \to \mathbb{R}$ associated to the system of vector fields $X = (X_1, \cdots, X_m)$ on M by

$$\hat{P}(z) = P((H_{ij}(z))_{1 \leqq i < j \leqq m}),\ z \in T^*M.$$

Note that the following holds:

$$\det(G(z)) = (\hat{P}(z))^2, z \in T^*M.$$

Then we define the *generalized Goh matrix* $\hat{G} : T^*M \to \mathrm{M}_{m+1,m}(\mathbb{R})$ associated to the system of vector fields $X = (X_1, \cdots, X_m)$ on M by an $(m+1) \times m$ -matrix

$$\hat{G}(z) = \left(\begin{array}{c} (H_{ij}(z))_{1 \leqq i,j \leqq m} \\ \left(\left\{ \hat{P}, H_j \right\} (z) \right)_{1 \leqq j \leqq m} \end{array} \right), z \in T^*M.$$

Now, we show that any X-singular control is obtain as a solution of a homogeneous linear equation by Goh matrices or generalized Goh matrices:
Let $\Omega \subset M$ be an open subset.

Proposition 4.1 *Fix $x_0 \in M$. Let $u : [0,T] \to \Omega$ be a singular X-control and $z : [0,T] \to T^*M$ be a singular X-bi-extremal corresponding to u. Then, the followings hold:*

(1) *u is a solution of the equation $G(z(t))u(t) = 0$ for almost every $t \in [0,T]$.*

(2) *If m is even, then u is a solution of the equation $\hat{G}(z(t))u(t) = 0$ for almost every $t \in [0,T]$.*

Proof: (1) Since u is a singular X-control, by Proposition 3.1, for each integer $i\,(1 \leqq i \leqq m)$ and for every $t \in [0,T]$,

$$H_i(z(t)) = 0.$$

By differentiating both sides, for each integer $i\,(1 \leqq i \leqq m)$ and for almost every $t \in [0,T]$,

$$\sum_{j=1}^{m} H_{ij}(z(t))\, u_j(t) = 0.$$

This is equivalent to the following equation: for almost every $t \in [0,T]$,

$$G(z(t))u(t) = 0.$$

(2) If m is even, then rank $G(z(t)) < m-1$ for $t \in [0,T]$. Therefore for every $t \in [0,T]$,

$$\hat{P}(z(t)) = 0.$$

By differentiating both sides, for almost every $t \in [0,T]$,

$$\sum_{j=1}^{m} \left\{ \hat{P}, H_j \right\} (w(t)) u_j(t) = 0.$$

This is equivalent to the following equation: for almost every $t \in [0,T]$,

$$\hat{G}(z(t))\, u(t) = 0.$$

□

Definition 4.2 A singular X-bi-extremal $z : [0,T] \to T^*M$ is called of *minimal order* if it satisfies the following condition: If m is odd (resp. even), then rank $G(z(t)) = m-1$ (resp. rank $\hat{G}(z(t)) = m-1$) for almost every $t \in [0,T]$.

Remark By Proposition 4.1, if a singular X-bi-extremal $z : [0,T] \to T^*M$ corresponding to singular X-control $u : [0,T] \to \Omega$ is of minimal order, then we can deduce an expression for $u(t)$, up to time reparameterization.

The definition of a singular bi-extremal of minimal order (Definition 4.2 of our paper) is different from Definition 2.16 of [9]. Definition 2.16 of [9] is that, a singular X-bi-extremal $z : [0,T] \to T^*M$ is called of *minimal order* if it satisfies the following conditions:

$$\begin{cases} \text{(i) } \dot{x}(t) = 0, \text{ for almost every } t \in I_{\mathrm{dep}}(x). \\ \text{(ii) If } m \text{ is odd (resp. even), then rank } G(z(t)) = m-1 \text{ (resp. rank } \hat{G}(z(t)) = m-1) \\ \qquad \text{for almost every } t \in [0,T] \setminus I_{\mathrm{dep}}(x). \end{cases}$$

Here

$$I_{\mathrm{dep}}(x) = \{t \in [0,T] \mid X_1(x(t)), \cdots, X_m(x(t)) \text{ are linearly dependent}\}.$$

Theorem 2.13 of [9] claims that any X-trajectory satisfies the above condition (i) for generic X, which is not correct as is mentioned in §1. Therefore our Definition 4.2, we do not suppose the above condition (i). Instead we adopt the condition (ii) as the minimal order condition by replacing $I_{\mathrm{dep}}(x)$ with $[0,T]$, in order to get a natural definition (Definition 4.2).

4.2 Singular controls of corank one

Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields over an n-dimensional manifold M, and Ω be an open subset of $\mathbb{R}^m$. Consider the driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x)$$

with the control parameter $(u_1, \cdots, u_m) \in \Omega$.

Definition 4.3 A singular X-control $u : [0,T] \to \Omega$ is called of *corank one* if the codimension of $(End^T_{x_0})_*$ is one, namely if $\dim M - \dim \mathrm{Im}\left((End^T_{x_0})_*\right) = 1$ at u.

In particular, if the control system is driftless control-affine system, then it is well-known that the following theorem:

Proposition 4.4 ([8]) *A singular X-control $u : [0,T] \to \Omega$ is of corank one if and only if for any corresponding two singular X-bi-extremals $z_1, z_2 : [0,T] \to T^*M$ to u, there exists a non-zero real number λ such that $z_1 = \lambda\, z_2$ on $[0,T]$.*

4.3 Singular non-trivial and independent trajectories

Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields over an n-dimensional manifold M, and Ω be an open subset of $\mathbb{R}^m$. Consider the driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x)$$

with the control parameter $(u_1, \cdots, u_m) \in \Omega$.

Definition 4.5 An singular X-trajectory $x : [0, T] \to M$ is called *non-trivial* if for any sub-interval $J \subset [0, T]$, the restriction of x to J, $x|_J : J \to M$ is not constant.

Definition 4.6 A singular X-trajectory $x : [0, T] \to M$ on a driftless control-affine system $\dot{x} = \sum_{i=1}^{m} u_i X_i$ is called *independent* if the set

$$I_{\text{dep}}(x) = \{t \in [0, T] \mid X_1(x(t)), \cdots, X_m(x(t)) \text{ are linearly dependent over } \mathbb{R}\}$$

has measure zero.

5 Singular bi-extremals with non-trivial independent trajectory of a generic driftless control-affine system

Recall that $\mathrm{VF}(M)$ denotes the space of smooth vector fields over M and $\mathrm{VF}(M)^m$ the m-tuple product set of $\mathrm{VF}(M)$ with the Whitney smooth topology. Then, as is stated in Introduction, the following theorem holds:

Theorem 5.1 (Main theorem 1) *Suppose $2 \leqq m \leqq n$. Then there exists an open dense subset G_0 of $\mathrm{VF}(M)^m$ such that, if $X \in G_0$, if $z : [0,T] \to T^*M$ is an X-bi-extremal and if the singular trajectory $x = \pi \circ z : [0,T] \to M$ is non-trivial and independent, then the bi-extremal z is of minimal order. Here $\pi : T^*M \to M$ is the canonical projection from the cotangent bundle of M.*

The proof goes in parallel with the proof of Theorem 2.4 in [8]. However note that Theorem 2.4 in [8] implies the results only for driftless control-affine systems which are independent everywhere. Since we treat general systems of vector fields which may have non-void locus of dependence, we need appropriate modifications of the proof given in [8].

Outline of proof : Let $d \geqq 1$ be an integer. Put $N = 2d$. We denote the space of all N-jets of vector fields $X \in \mathrm{VF}(M)$ by $J^N(\mathrm{VF}(M))$, and the fibre product over M of m-tuple spaces of $J^N(\mathrm{VF}(M))$, by $J^N(\mathrm{VF}(M))^m$. Then, we will show the main theorem 1 by the following procedures:
[Step1] (See Definition 5.4) Construct the "bad set" with respect to minimal order, $B_{mo}(d) \subset J^N(\mathrm{VF}(M))^m$.
[Step2] (See Lemma 5.9) Show that, if $X \in \mathrm{VF}(M)^m$ satisfies the condition that any $x \in M$, $j_xX \notin B_{mo}(d)$, if $z : [0,T] \to T^*M$ is an X-bi-extremal, and if the singular trajectory $\pi \circ z : [0,T] \to M$ is non-trivial and independent, then z is of minimal order.
[Step3] (See Lemma 5.10) Compute the codimension of $B_{mo}(d)$ in $J^N(\mathrm{VF}(M))^m$.
[Step4] (See the subsection §5.4) For $N > 4n$ ($d > 2n$), let G_0 be the set of $X \in \mathrm{VF}(M)^m$ such that the jet j^N_xX is not included in the closure of $B_{mo}(d)$ in $J^N(\mathrm{VF}(M))^m$. Then, show that, G_0 is an open dense subset of $\mathrm{VF}(M)^m$ in the sense of Whitney smooth topology by Thom transversality theorem (for instance see [10]).

In order to show the main theorem 1, we prepare the subsections: §5.1 to §5.4. In §5.1, after preparing the notations, permutated Hamiltonians, Goh matrices and elementary determinants in Definition 5.2, 5.3, we define the bad set $B_{mo}(d)$ of $J^N(\mathrm{VF}(M))^m$ in Definition 5.4. In §5.2, in Lemma 5.7, 5.8, we describe some properties of rank of Goh matrix for any singular X-bi-extremals if $X \in \mathrm{VF}(M)^m$ satisfies the condition that for any $x \in M$, j^N_xX is included in complement of $B_{mo}(d)$ in $J^N(\mathrm{VF}(M))^m$. In particular, the important Lemma 5.9 prepared for the proof of main theorem 1 can be immediately derived from the Lemmata 5.7, 5.8 : if $X \in \mathrm{VF}(M)^m$ satisfies the condition that , $j^N_xX \notin B_{mo}(d)$ for any $x \in M$, then any singular X-bi-extremal with the non-trivial independent singular X-trajectory is of minimal order. In §5.3, we compute the codimension of $B_{mo}(d)$

in $J^N(\mathrm{VF}(M))^m$ in Lemma 5.10. In §5.4, by using these Lemmata 5.9, 5.10, we show the main theorem 1.

5.1 Construction of bad set with respect to minimal order.

Let $\mathfrak{S}_m$ be the set of permutations with m elements, and $X = (X_1, \cdots, X_m) \in \mathrm{VF}(M)^m$. Then, we prepare some notations, permutated Hamiltonians, Goh matrices and elementary determinants, in order to define the bad set with respect to minimal order, $B_{mo}(d)$ for an integer d :

Definition 5.2 Let $i\,(1 \leqq i \leqq m), j\,(1 \leqq j \leqq m)$ and $r\,(1 \leqq r \leqq m-1)$ be integers. Then, we define *permutated Hamiltonians* $H_i, H_{i,j} : \mathfrak{S}_m \times T^*M \to \mathbb{R}$ and *permutated Goh matrices* $G : \mathfrak{S}_m \times T^*M \to M_m(\mathbb{R})$, $G^r : \mathfrak{S}_m \times T^*M \to M_r(\mathbb{R})$ by

$$\begin{cases} H_i(\sigma, z) = \{z, X_{\sigma(i)}(\pi(z))\}, \quad H_{ij}(\sigma, z) = \{z, [X_{\sigma(i)}, X_{\sigma(j)}](\pi(z))\}, \\ G(\sigma, z) = (H_{i,j}(\sigma, z))_{1 \leqq i,j \leqq m}, \ G^r(\sigma, z) = (H_{i,j}(\sigma, z))_{1 \leqq i,j \leqq r}, \end{cases}$$

where $\pi : T^*M \to M$ is the canonical projection.

Definition 5.3 We inductively define the real valued functions on $\mathfrak{S}_m \times T^*M$, which are called *elementary determinants*: Let $\sigma \in \mathfrak{S}_m, z \in T^*M$. Then,
(I) For an integer $r\,(1 \leqq r \leqq m-1)$,

$$\Delta_0^r(\sigma, z) = \det(G^r(\sigma, z)), \quad \Delta_0^0(\sigma, z) = 1.$$

(II) For integers $r\,(1 \leqq r \leqq m-1)$, $k\,(r+1 \leqq k \leqq m)$, (with the convention that the index $m+1$ stands for $r+1$),

$$\begin{cases} \Delta_{0,s+1}^{r,k}(\sigma, z) = \det\left(\begin{array}{c|c} G^r(\sigma, z) & (H_{ij}(\sigma, z))_{1 \leqq i \leqq r} \\ \hline \multicolumn{2}{c}{\left(\{\Delta_{0,s}^{r,k}, H_j\}(\sigma, z)\right)_{j=1,\cdots,r,k}} \end{array}\right) \quad (s = 0, 1, \cdots), \\ \Delta_{0,0}^{r,k}(\sigma, z) = \det\left(\begin{array}{c|c} G^r(\sigma, z) & (H_{ij}(\sigma, z))_{1 \leqq i \leqq r} \\ \hline (H_{(k+1)j}(\sigma, z))_{1 \leqq j \leqq r} & H_{(k+1)k}(\sigma, z) \end{array}\right). \end{cases}$$

(III) For $r\,(1 \leqq r \leqq m-1)$, $p\,(1 \leqq p \leqq m-r-1)$, and $s_1, \cdots, s_p \geqq 1$,

$$\begin{cases} \Delta_{0,s_1,\cdots,s_p,s+1}^{r,r+1,\cdots,r+p,k}(\sigma, z) = \det\begin{pmatrix} (H_{ij}(\sigma, z))_{\substack{1 \leqq i \leqq r \\ j=1,\cdots,r+p,k}} \\ \hline \left(\{\Delta_{0,s_1-1}^{r,r+1}, H_j\}(\sigma, z)\right)_{j=1,\cdots,r+p,k} \\ \vdots \\ \left(\{\Delta_{0,s_1,\cdots,s_p-1}^{r,r+1,\cdots,r+p}, H_j\}(\sigma, z)\right)_{j=1,\cdots,r+p,k} \\ \left(\{\Delta_{0,s_1,\cdots,s_p,s}^{r,r+1,\cdots,r+p,k}, H_j\}(\sigma, z)\right)_{j=1,\cdots,r+p,k} \end{pmatrix} \quad (s = 0, 1, \cdots), \\ \Delta_{0,s_1,\cdots,s_p,0}^{r,r+1,\cdots,r+p,k}(\sigma, z) = \Delta_{0,s_1,\cdots,s_p-1}^{r,r+1,\cdots,r+p-1,k}(\sigma, z). \end{cases}$$

(IV) Let m be an even integer. We denote the Pfaffian polynomial of G, by $p : \mathbb{R}^{\frac{m(m-1)}{2}} \to \mathbb{R}$. Then, $P(\sigma, z) = p((H_{ij}(\sigma, z))_{1 \leqq i < j \leqq m})$.
(i) for every $k \in \{m-1, m\}$,

$$\begin{cases} \delta^k_{s+1}(\sigma, z) = \det \left(\begin{array}{c|c} G^{m-2}(\sigma, z) & (H_{ik}(\sigma, z))_{1 \leqq i \leqq m-2} \\ \hline (\{\delta^k_s, H_j\}(\sigma, z))_{1 \leqq j \leqq m-2} & \{\delta^k_s, H_k\}(\sigma, z) \end{array} \right) \; (s = 0, 1 \cdots), \\ \delta^k_0(\sigma, z) = P(\sigma, z). \end{cases}$$

(ii) for every integer $s_1 \geqq 1$,

$$\begin{cases} \delta_{s_1, s+1}(\sigma, z) = \left(\begin{array}{c} (H_{ij}(\sigma, z))_{\substack{1 \leqq i \leqq m-2 \\ 1 \leqq j \leqq m}} \\ \hline \left(\{\delta^{m-1}_{s_1 - 1}, H_j\}(\sigma, z)\right)_{1 \leqq j \leqq m} \\ (\{\delta_{s_1, s}, H_j\}(\sigma, z))_{1 \leqq j \leqq m} \end{array} \right) \; (s = 0, 1, \cdots) \\ \delta_{s_1, 0}(\sigma, z) = \delta^m_{s_1}(\sigma, z). \end{cases}$$

By using the elementary determinants, we define the bad set $B_{mo}(d)$ for an integer d :

Definition 5.4 Let d be an integer and $N = 2d$. For an integer p, let $N_{p,d}$ be the set of $(p+1)$-tuples $\overline{s} = (0, s_1, \cdots, s_p)$ in $\{0\} \times (\mathbb{N})^p$ with $s_1 + \cdots + s_p < d + p$. We define the "bad set" with respect to minimal order, $B_{mo}(d) \subset J^N(\mathrm{VF}(M))^m$ by the image of $\hat{B}_{mo}(d)$ by the canonical projection $J^N(\mathrm{VF}(M))^m \times_M T^*M \to J^N(\mathrm{VF}(M))^m$:

$$\hat{B}_{mo}(d) = \{(j^N_x X, z) \,|\, x = \pi(z) \in M, z \in T^*M, \, (j^N_x X, z) \in \hat{B}^0_{mo}(d) \cup \hat{B}^1_{mo}(d)\},$$

Then, $B_{mo}(d)$ is denoted by the following:

$$B_{mo}(d) = \{j^N_x X \,|\, x = \pi(z) \in M, \, (j^N_x X, z) \in \hat{B}^0_{mo}(d) \cup \hat{B}^1_{mo}(d) \,\text{for some}\, z \in T^*M\},$$

where $\hat{B}^0_{mo}(d), \hat{B}^1_{mo}(d) \subset J^N(\mathrm{VF}(M))^m \times_M T^*M$ are written in Definition 10.3, 10.4 respectively:

Definition 5.5 If $m = 2$, then $\hat{B}^0_{mo}(d) = \emptyset$. On the other hand, if $m \geqq 3$, then we define $\hat{B}^0_{mo}(d) \subset J^N(\mathrm{VF}(M))^m \times_M T^*M$ by the union of the sets $\hat{B}^0_{mo}(d, \sigma, r, \overline{s}, z)$ with $\sigma \in \mathfrak{S}_m$, even integers $r \,(0 \leqq r \leqq m-3)$, and $\overline{s} \in N_{p,d}$ with $0 \leqq p < m$. Here the definition of $\hat{B}^0_{mo}(d, \sigma, r, \overline{s}, z)$ is below:
For $z \in T^*M$ with $\pi(z) = x \in M$, $\sigma \in \mathfrak{S}_m$, an even integer $r (0 \leqq r \leqq m-3)$, and $\overline{s} \in N_{p,d}$ with $p \,(0 \leqq p < m)$, let $\hat{B}^0_{mo}(d, \sigma, r, \overline{s}, z)$ be the set of $(j^N_x X, z) \in J^N(\mathrm{VF}(M))^m \times_M T^*M$ such that:
1). $X_1(x), \cdots, X_m(x)$ are linearly independent;
2). $\Delta^r_0(\sigma, z) \neq 0$;
3). for every integer $i \,(0 \leqq i \leqq p)$,
(a) $\Delta^{r, r+1, \cdots, r+i}_{0, s_1, \cdots, s_i}(\sigma, z) \neq 0$;

(b) for every integer $k\,(r+i \leqq k \leqq m)$ and $s\,(1 \leqq s \leqq s_i - 1)$,

$$\Delta^{r,r+1,\cdots,r+i-1,k}_{0,s_1,\cdots,s_{i-1},s}(\sigma, z) = 0;$$

4). for every $k\,(r+p+1 \leqq k \leqq m)$ and $s\,(1 \leqq s \leqq d+p-(s_1+\cdots+s_p)$,

$$\Delta^{r,r+1,\cdots,r+p,k}_{s_1,\cdots,s_p,s}(\sigma, z) = 0.$$

Definition 5.6 If m is an odd integer, then $\hat{B}^1_{mo}(d) = \emptyset$. On the other hand, if $m \geqq 2$ is an even integer, then we define $\hat{B}^1_{mo}(d) \subset J^N(\mathrm{VF}(M))^m \times_M T^*M$ by the union of the sets $\hat{B}^1_{mo}(d, \sigma, s_1, z)$ with $\sigma \in \mathfrak{S}_m$ and integers $s_1\,(1 \leqq s_1 \leqq d)$. Here the definition of $\hat{B}^1_{mo}(d, \sigma, s_1, z)$ is below:
Let $m \geqq 2$ be an even integer. For $\sigma \in \mathfrak{S}_m$ and an integer $s_1 (1 \leqq s_1 \leqq d)$, we define $\hat{B}^1_{mo}(d, \sigma, s_1, z)$ by the set of elements $(j^N_x X, z) \in J^N_x(\mathrm{VF}(M))^m \times_M T^*M$ such that:
1). $X_1(x), \cdots, X_m(x)$ are linearly independent;
2). $\Delta^{m-2}_0(\sigma, z) \neq 0$;
3). (a) if $s_1 < d$, then $\delta^{m-1}_{s_1}(\sigma, z) \neq 0$;
(b) for $k \in \{m-1, m\}$ and $s\,(0 \leqq s \leqq s_1 - 1)$, $\delta^k_s(\sigma, z) = 0$;
4). for $s\,(1 \leqq s \leqq d - s_1)$, $\delta_{s_1,s}(\sigma, z) = 0$.

5.2 The property of singular bi-extremals avoiding bad set with respect to minimal order.

We describe some properties of rank of Goh matrix for any singular X-bi-extremals with the non-trivial and independent singular X-trajectory if $X \in \mathrm{VF}(M)^m$ satisfies the condition that for any $x \in M$, $j^N_x X$ is included in complement of $B_{mo}(d)$ in $J^N(\mathrm{VF}(M))^m$. In particular, we will show the important Lemma 5.9 prepared for the proof of the main theorem 1 by using the Lemmata 5.7, 5.8.

Lemma 5.7 *Suppose that, $2 \leqq m \leqq n$. Let d be a positive integer and $N = 2d$. Let $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j^N_x X \notin B_{mo}(d)$. Then, if $z : [0,T] \to T^*M$ is an X-bi-extremal and if the singular X-trajectory $\pi \circ z : [0,T] \to M$ is non-trivial and independent, then*

$$m - 2 \leqq \mathrm{rank}\ G(z(t)) \leqq m - 1, \ \text{ for } a.e.\, t \in [0,T].$$

Here $\pi : T^*M \to M$ is the canonical projection from cotangent bundle of M.

Proof: By the assumption that $\pi \circ z$ is non-trivial and by Proposition 4.1 (1), we have the inequality rank $G(z(t)) \leqq m-1$ for a.e. $t \in [0,T]$. Therefore it suffices to show the inequality $m - 2 \leqq$ rank $G(z(t))$ for a.e. $t \in [0,T]$.

In $m = 2$ case, Lemma 5.7 clearly holds. We consider $m \geqq 3$. In order to prove by contradiction, assume that, $z : [0,T] \to T^*M$ is an X-bi-extremal and the singular X-trajectory $\pi \circ z : [0,T] \to M$ is non-trivial and independent but there exists a measurable subset of positive measure $K \subset [0,T]$ such that

$$\mathrm{rank}\, G(z(t)) \leqq m - 3 \ \text{ for every } t \in K.$$

Since $\pi \circ z$ is independent, the compliment of $I_{\mathrm{dep}}(\pi \circ z)$, $I_{\mathrm{indep}}(\pi \circ z)$ has positive measure. Then, let $J = I_1 \cap I_{\mathrm{indep}}(\pi \circ z)$. J has positive measure and for every $t \in J$, $X_1((\pi \circ z)(t)), \cdots, X_m((\pi \circ z)(t))$ are linearly independent over $\mathbb{R}$.

After this, in the same way as the proof of Lemma 3.8 in [8], we can prove this Lemma 5.7 :

In fact, for an integer p $(1 \leqq p \leqq m)$, we denote by I_p a subset of $\{1, \cdots, m\}$ with cardinality p. Let r be the maximum of even integers p $(1 \leqq p \leqq m-3)$ such that there exists a subset $I_p \subset \{1, \cdots, m\}$ satisfying $\det(H_{i,j}(z(t)))_{(i,j)\in I_p^2} \not\equiv 0$ on J. Let J_r be the set of $t \in J$ such that there exists $I_r \subset \{1, \cdots, m\}$ satisfying $(H_{i,j}(z(t))_{(i,j)\in I_r^2} \neq 0$. Note J_r has positive measure. Therefore there exists $\sigma \in \mathfrak{S}$ such that

$$\Delta_0^r(\sigma, z(t)) \neq 0 \text{ for every } t \in J_r.$$

On the other hand, for any subset $I_{r+2} \subset \{1, \cdots, m\}$, $\det(H_{i,j}(z(t))_{(i,j)\in I_{r+2}^2} \equiv 0$ on J_r and $\operatorname{rank}(H_{i,j}(z(t))_{(i,j)\in I_{r+2}^2} \leqq r$ on J_r. Therefore for any subset $I_{r+1} \subset I_{r+2}$, $\det(H_{i,j}(z(t))_{(i,j)\in I_{r+1}^2} = 0$ on J_r. In particular, for $k = r+1, \cdots, m$,

$$\Delta_{0,0}^{r,k}(\sigma, z(t)) = 0 \text{ for every } t \in J_r.$$

By differentiating both sides, for $k = r+1, \cdots, m$,

$$\sum_{i=1}^{m} u_i(t) \, \{\Delta_{0,0}^{r,k}, H_i\}(\sigma, z(t)) = 0 \ \text{ for a.e. } t \in J_r.$$

Here, we denote by $G_0(\sigma, z(t))$ the following $m \times m$- matrix:

$$\begin{pmatrix} H_{11}(\sigma, z(t)) & \cdots & H_{1m}(\sigma, z(t)) \\ \vdots & & \vdots \\ H_{r1}(\sigma, z(t)) & \cdots & H_{rm}(\sigma, z(t)) \\ \{\Delta_{0,0}^{r,r+1}, H_1\}(\sigma, z(t)) & \cdots & \{\Delta_{0,0}^{r,r+1}, H_m\}(\sigma, z(t)) \\ \vdots & & \vdots \\ \{\Delta_{0,0}^{r,m}, H_1\}(\sigma, z(t)) & \cdots & \{\Delta_{0,0}^{r,m}, H_m\}(\sigma, z(t)) \end{pmatrix}.$$

Then $G_0(\sigma, z(t)) \not\equiv 0$ on J_r. Note that, the first diagonal minors of order r of $G_0(\sigma, z(t))$ is $\Delta_0^r(\sigma, z(t))$,which never vanishes on J_r. and by definition, the diagonal minors of order $r+1$ containing $\Delta_0^r(\sigma, z(t))$ are $\Delta_{0,1}^{r,k}(\sigma, z(t))$, $k = r+1, \cdots, m$.

Claim. There exists $k_1 \in \{1, \cdots, m\}$, an integer $s_1 (1 \leqq s_1 < d+1)$, and a subset $J_{r+1} \subset J_r$ of positive measure such that

$$\begin{cases} \Delta_{0,\ell}^{r,k}(\sigma, z(t)) \equiv 0 \text{ on } J_{r+1}, \text{ for any integers } k, \ell \ (r+1 \leqq k \leqq m, \, 0 \leqq \ell \leqq s_1 - 1); \\ \Delta_{0,s_1}^{r,k_1}(\sigma, z(t)) \neq 0 \, \text{for every } t \in J_{r+1}. \end{cases}$$

In fact, assume the claim is false. Then for every $k \, (r+1 \leqq k \leqq m)$ $\Delta_{0,1}^{r,k}(\sigma, z(t)) = 0$ on J_r. We consider the matrix $G_1(\sigma, z(t))$ obtained by replacing the last $m-r$ rows of $G_0(\sigma, z(t))$ with rows

$$(\{\Delta_{0,0}^{r,k}, H_\ell\}(\sigma, z(t)))_{1 \leqq \ell \leqq m}, \ \text{ for } k \ (1 \leqq k \leqq m).$$

By construction, $\det G_1(\sigma, z(t)) \equiv 0$ on J_r. The contradiction assumption implies that, for $k\,(r+1 \leqq k \leqq m)$, $\Delta_{0,2}^{r,k}(\sigma, z(t))) \equiv 0$ of J_r. Proceeding similarly, there exists $t \in J_r$ such that $j_{\pi\circ z(t)}X$ belongs to $B(d, \sigma, r, 0, z(t))$. This contradicts the assumption that, for any $x \in M$, $j_x X \in B_{mo}(d)$. Thus the claim is proved.

Up to a permutation, assume $k_1 = r+1$. We define a non-invertible matrix by replacing in G_0 :

$$\begin{cases} \text{the } (r+1)\text{-th line by } (\{\Delta_{0,s_1-1}^{r,r+1}, H_\ell\}(\sigma, z(t)))_{1\leqq \ell \leqq m}; \\ \text{for } j\,(r+2 \leqq j \leqq m), \text{the } j\text{-th line by } (\{\Delta_{0,s_1-1}^{r,j}, H_\ell\}(\sigma, z(t)))_{1\leqq \ell \leqq m}. \end{cases}$$

To this matrix is applied the previous reasoning on G_0. Thus, by a finite number of steps, we obtain that there exists subset $J_{m-1} \subset J$ of positive measure, and $\overline{s} = (0, s_1, \cdots, s_m)$ in $N_{m-1,d}$, such that

$$\begin{cases} \Delta_{0,s_1,\cdots,s_{m-1}}^{r,\cdots,r+m-1}(\sigma, z(t))) \neq 0 \text{ for every } t \in J_{m-1}; \\ \Delta_{0,s_1,\cdots,s_{m-1},\ell}^{r,\cdots,r+m-1,r+m}(\sigma, z(t))) \equiv 0 \text{ on } J_{m-1}, \ell \geqq 0. \end{cases}$$

As a consequence, for every $t \in J_{m-1}$, $j_{\pi\circ z(t)}X$ belongs to $B(d, \sigma, r, \overline{s}, z(t))$. This contradicts the assumption that, for any $x \in M$, $j_x X \in B_{mo}(d)$. $\square$

Lemma 5.8 *Suppose that, m is even and $2 \leqq m \leqq n$. Let d be a positive integer and $N = 2d$. Let $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j_x^N X \notin B_{mo}(d)$. Then, if $z : [0,T] \to T^*M$ is a singular X-bi-extremal and if the singular X-trajectory $\pi \circ z : [0,T] \to M$ is non-trivial and independent, then*

$$\operatorname{rank} \hat{G}(z(t)) = m - 1 \text{ for } a.e.\, t \in [0,T].$$

Proof: By the assumption that $\pi \circ z$ is non-trivial and by Proposition 4.1 (2), we have the inequality $\operatorname{rank} G(z(t)) \leqq m-1$ for a.e. $t \in [0,T]$. Therefore we will show the inequality $\operatorname{rank} G(z(t)) \geqq m-1$ for a.e. $t \in [0,T]$.

In order to prove by contradiction, assume that, $z : [0,T] \to T^*M$ is an X-bi-extremal and the singular X-trajectory $\pi \circ z : [0,T] \to M$ is non-trivial and independent but there exists a measurable subset of positive measure $K \subset [0,T]$ such that

$$\operatorname{rank} \hat{G}(z(t)) \leqq m-2 \text{ for every } t \in K.$$

Since $\pi \circ z$ is independent, $I_{\text{indep}}(\pi \circ z)$ has positive measure, where $I_{\text{indep}}(\pi \circ z)$ is the compliment of $I_{\text{dep}}(\pi \circ z)$ in $[0,T]$. When let $J = I_1 \cap I_{\text{indep}}(\pi \circ z)$, $J \subset [0,T]$ has positive measure and for any $t \in J$, $X_1((\pi \circ z)(t)), \cdots, X_m((\pi \circ z)(t))$ are linearly independent over M. After this, in the same way as the proof of Lemma 3.8 in [8], we can prove this Lemma 5.8:
In fact, By the previous proof, we may assume that there exists $\sigma \in \mathfrak{S}_m$ such that

$$\Delta_0^{m-2}(\sigma, z(t)) \neq 0 \text{ for every } t \in J.$$

In particular, rank $\mathrm{G}(z(t)) = m-2$. Moreover, for $k = m-1$ and $k = m$,

$$\delta_0^k(\sigma, z(t)) = 0 \text{ and } \delta_1^k(\sigma, z(t)) = 0 \text{ for every } t \in J_r.$$

Similarly to the argument of Lemma 3.8, we claim that there exists an integer s_1 ($1 \leqq s_< d$), and $k_1 \in \{m-1, m\}$, such that

$$\delta_{s_1}^{k_1}(\sigma, z(t)) \neq 0 \text{ for every } t \in J.$$

In fact, otherwise, for $s\,(0 \leqq s \leqq d)$ and for $k \in \{m-1, m\}$, $\delta_s^k(\sigma, z(t)) = 0$. In that case, for every $t \in J, j^N_{\pi\circ z(t)}X$ belongs to $B^1(d, \sigma, d, z(t))$. This contradicts for any $x \in M$, $j_x X \in B_{mo}(d)$.

Up to a permutation, we may assume $k_1 = m-1$. Let $J_1 \subset J$ be a subset of positive measure such that

$$\delta_{s_1}^{m-1}(\sigma, z(t)) \neq 0 \text{ for every}\, t \in J_1.$$

Similarly to the argument of Lemma 3.8, for every $s \geqq 0$, we have

$$\delta_{s_1,s}(\sigma, z(t)) = 0 \text{ for } t \in J_r.$$

Then, for every $t \in J_1$, $j^N_{\pi\circ z(t)}X$ belongs to $B^1(d, \sigma, s_1, z(t))$. This contradicts for any $x \in M$, $j_x X \in B_{mo}(d)$. □

Lemma 5.9 *Suppose that, $2 \leqq m \leqq n$. Let d be a positive integer and $N = 2d$. Let $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j_x^N X \notin B_{mo}(d)$. Then, if $z : [0,T] \to T^*M$ is a singular X-bi-extremal and if the singular X-trajectory $\pi \circ z : [0,T] \to M$ is non-trivial and independent, then $z : [0,T] \to T^*M$ is of minimal order.*

Proof: Since rank $G(z(t))$ is even, by Lemma 5.7 if m is odd, then $m-2$ can be replace with $m-1$ in the statement of Lemma 5.7:
If $m\,(2 \leqq m \leqq n)$ is odd, if $z : [0,T] \to T^*M$ is a singular X-bi-extremal and if $\pi \circ z : [0,T] \to M$ is non-trivial and independent, then

$$\text{rank } G(z(t)) = m-1 \text{ for a.e.}\, t \in [0,T].$$

On the other hand, by Lemma 5.8, the following immediately holds:
If $m\,(2 \leqq m \leqq n)$ is even, if $z : [0,T] \to T^*M$ is a singular X-bi-extremal and if $\pi \circ z : [0,T] \to M$ is non-trivial and independent, then

$$\text{rank } \hat{G}(z(t)) = m-1 \text{ for a.e.}\, t \in [0,T].$$

Therefore, these imply that for every integer $m\,(2 \leqq m \leqq n)$, z is of minimal order (see Definition 4.2). □

5.3 Codimension of bad set with respect to minimal order.

Let d be an integer and $N = 2d$. Then

Lemma 5.10 $\operatorname{codim}(\overline{B_{mo}(d)}, J^N(\mathrm{VF}(M))^m) \geqq d - n$.

Proof: We describe only the outline of the proof of Lemma 5.10 because this bad set $B_{mo}(d)$ is the completely same bad set $B_{mo}(d)$ defined as 3.1.2 in [8]: Let $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ be the m-tuple product space of polynomial vector fields of degree $\leqq N$ over $\mathbb{R}^n$.

Step 1: Construct the typical fiber $G(d)$ of $B_{mo}(d)$: The typical fiber $G(d) \subset \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ is constructed by the union of $G^0(d)$ and $G^1(d)$. Note that $G^0(d), G^1(d)$ are semi-algebraic sets for d. Therefore $G(d) = G^0(d) \cup G^1(d)$ and its closure $\overline{G(d)}$ are semi-algebraic also. In particular dimensions of $G^0(d), G^1(d), G(d), \overline{G(d)}$ are well-defined and we have that

$$\dim(\overline{G(d)}) = \dim(G(d)) = \max\{\dim(G^0(d)), \dim(G^1(d))\}.$$

Moreover we have that the dimensions of $B_{mo}(d)$ and $\overline{B_{mo}(d)}$ are well-defined and that they are equal.

The definition of $G^0(d), G^1(d) \subset \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ is below:

(1) Construction of the trivial fiber $G^0(d)$ of $\hat{B}^0_{mo}(d)$: If $m = 2$, then $G^0(d) = \emptyset$. If $m \geqq 3$, then $G^0(d)$ is the canonical projection of $G^0(d; T^*_0\mathbb{R}^n)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. $G^0(d; T^*_0\mathbb{R}^n)$ is defined by the set of $(Q, p) = (Q_1, \cdots, Q_m, p) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^m$ such that there exist $\sigma \in \mathfrak{S}_m$, an even integer $r\,(r \leqq m-3)$, and $\overline{s} \in N_{q,d}$ with $q(0 \leqq q < m)$ such that (Q, p) satisfies the conditions 1) to 4):

1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;
2). $\Delta^r_0(\sigma, z_0) \neq 0$;
3). for every integer $i\,(0 \leqq i \leqq q)$, $\Delta^{r,r+1,\cdots,r+i}_{0,s_1,\cdots,s_i}(\sigma, z_0) \neq 0$.
4). (a) for every integers $i\,(0 \leqq i \leqq q)$, $k\,(r + i \leqq k \leqq m)$, and $s\,(1 \leqq s \leqq s_i - 1)$,

$$\Delta^{r,r+1,\cdots,r+i-1,k}_{0,s_1,\cdots,s_i-1,s}(\sigma, z_0) = 0;$$

(b) for every integer $k\,(r + p + 1 \leqq k \leqq m)$ and $s\,(1 \leqq s \leqq d + q - (s_1 + \cdots + s_p))$,

$$\Delta^{r,r+1,\cdots,r+q,k}_{0,s_1,\cdots,s_q,s}(\sigma, z_0) = 0,$$

where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0, p)$.

(2) Construction of the trivial fiber $G^1(d)$ of $\hat{B}^1_{mo}(d)$: In m is an odd integer, then $G^1(d) = \emptyset$. If m is an even integer, then $G^1(d)$ is the canonical projection of $G^1(d; T^*_0\mathbb{R}^n)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. $G^1(d; T^*_0\mathbb{R}^n)$ is defined by the set of $(Q, p) = (Q_1, \cdots, Q_m, p) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^m$ such that there exists a positive integer $s_1\,(1 \leqq s_1 \leqq d)$ and $\sigma \in \mathfrak{S}_m$ such that (Q, p) satisfies the conditions 1) to 4):

1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;
2). $\Delta^{m-2}_0(\sigma, z_0) \neq 0$;
3). if $s_1 < d$, then $\delta^{m-1}_{s_1}(\sigma, z_0) \neq 0$;

4). (a) for every integer $k \in \{m-1.m\}$ and $s\,(0 \leqq s \leqq s_1 - 1)$, $\delta_s^k(\sigma, z_0) = 0$;
(b) for every integer $s\,(1 \leqq s \leqq d - s_1)$, $\delta_{s_1,s}(\sigma, z_0) = 0$,
where z_0 is the elements of $T_0^*\mathbb{R}^n$ given in coordinates by $(0,p)$.

Step 2: Construct the two mappings $\phi^0(\sigma, r, \overline{s}), \phi^1(\sigma, s_1) : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n \to \mathbb{R}^d$:
(1) Construction of $\phi^0(\sigma, r, \overline{s})$: Let $\sigma \in \mathfrak{S}_m$, $r\,(0 \leqq r \leqq m-3)$ be an even integer, and $\overline{s} = (0, s_1, \cdots, s_q) \in N_{q,d}$ with $0 \leqq q < m$. Then, we define the mapping $\phi^0(\sigma, r, \overline{s})$: $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q,p) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$,

$$\phi^0(\sigma, r, \overline{s})(Q,p) = \begin{cases} \Delta^{r,r+1,\cdots,r+i-1,r+i}_{0,s_1,\cdots,s_{i-1},\overline{s}}(Q)(\sigma, z_0) & i = 1, \cdots, q, \text{and } \overline{s} = 1, \cdots, s_i - 1, \\ \Delta^{r,r+1,\cdots,r+q,r+q+1}_{0,s_1,\cdots,s_q,\overline{s}}(Q)(\sigma, z_0) & \overline{s} = 1, 2, \cdots, d + q - (s_1 + \cdots + s_q), \end{cases}$$

where z_0 is the elements of $T_0^*\mathbb{R}^n$ given in coordinates by $(0,p)$, and $\Delta^{r,r+1,\cdots,r+i-1,r+i}_{0,s_1,\cdots,s_{i-1},\overline{s}}(Q)$, $\Delta^{r,r+1,\cdots,r+q,r+q+1}_{0,s_1,\cdots,s_q,\overline{s}}(Q)$ are the elementary determinants associated to Q.
(2) Construction of $\phi^1(\sigma, s_1)$: Let $\sigma \in \mathfrak{S}_m$ and $s_1\,(1 \leqq s_1 \leqq d)$. Then, we define the mapping $\phi^1(\sigma, s_1) : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q,p) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$,

$$\phi^1(\sigma, s_1)(Q,p) = \begin{cases} \delta_s^m(Q)(\sigma, z_0), & s = 0, 1 \cdots, s_1 - 1; \\ \delta_{s_1,s}(Q)(\sigma, z_0), & s = 1, 2 \cdots, d - s_1, \end{cases}$$

where z_0 is the elements of $T_0^*\mathbb{R}^n$ given in coordinates by $(0,p)$, and $\delta_s^m(Q), \delta_{s_1,s}(Q)$ are the elementary determinants associated to Q.

Step 3: Construct the two open sets $T^0_{\sigma,r,\overline{s}}, T^1_{\sigma,s_1} \subset \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$:
(1) Construction of $T^0_{\sigma,r,\overline{s}}$: Let $\sigma \in \mathfrak{S}_m$, $r\,(0 \leqq r \leqq m-3)$ be an even integer, and $\overline{s} = (0, s_1, \cdots, s_q) \in N_{q,d}$ with $0 \leqq q < m$. Then, $T^0_{\sigma,r,\overline{s}}$ is defined by the set of $(Q,p) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$ such that (Q,p) satisfies the conditions 1) and 2):
1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;
2). for every integer $i\,(0 \leqq i \leqq p)$, $\Delta^{r,r+1,\cdots,r+i}_{0,s_1,\cdots,s_i}(\sigma, z_0) \neq 0$,
where z_0 is the elements of $T_0^*\mathbb{R}^n$ given in coordinates by $(0,p)$.

(2) Construction of T^1_{σ,s_1} : Let $\sigma \in \mathfrak{S}_m$ and $s_1\,(1 \leqq s_1 \leqq d)$. Then, T^1_{σ,s_1} is defined by the set of $(Q,p) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$ such that (Q,p) satisfies the conditions 1) to 3):
1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;
2). $\Delta_0^{m-2}(\sigma, z_0) \neq 0$;
3). if s_1 is $s_1 < d$, then $\delta_{s_1}^{m-1}(\sigma, z_0) \neq 0$,
where z_0 is the elements of $T_0^*\mathbb{R}^n$ given in coordinates by $(0,p)$.

Then, $T^0_{\sigma,r,\overline{s}}$ and T^1_{σ,s_1} are open subsets of $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$.

Step 4: The followings immediately hold:

(1) If $m \geqq 3$, then $G_0(d; T_0^*\mathbb{R}^n)$ is the union of the kernels of the restriction to $T^0_{\sigma,r,\overline{s}}$ of the mapping $\phi^0(\sigma, r, \overline{s})$ with $\sigma \in \mathfrak{S}_m$, even integers $r\,(0 \leqq r \leqq m-3)$, and $\overline{s} \in N_{q,d}$ with $0 \leqq q < m$.

(2) If $m \geqq 2$ is even, then $G^1(d; T_0^*\mathbb{R}^n)$ is the union of the kernels of the restriction to T^1_{σ,s_1} of the mapping $\phi^1(\sigma, s_1)$ with $\sigma \in \mathfrak{S}_m$ and $s_1(1 \leqq s_1 \leqq d)$.

Step 5: Let Ω_0 be the set of $Q \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that $Q_1(0), \cdots, Q_m(0)$ are linearly independent. Then, the followings hold (see Lemma 3.5, 3.6 in [8]):

(1) If $m \geqq 3, \sigma \in \mathfrak{S}_m$, and $r\,(0 \leqq r \leqq m-3)$ is an even integer, then the restriction to the intersection $T^0_{\sigma,r,\overline{s}} \cap \hat{V}$ of the mapping $\phi^0(\sigma, r, \overline{s})$ is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0 \times \mathbb{R}^n$.

(2) If $m \geqq 2$ is even, $\sigma \in \mathfrak{S}_m$, and an integer s_1 satisfies $1 \leqq s_1 \leqq d$, then the restriction to the intersection $T^1_{\sigma,s_1} \cap \hat{V}$ of the mapping $\phi^1(\sigma, s_1)$ is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0 \times \mathbb{R}^n$.

Step 6: $\mathrm{codim}(B_{mo}(d), J^N(\mathrm{VF}(M))^m) \geqq d - n$:
Let $k \in \{0, 1\}$. By step 4, 5, $\mathrm{codim}(G^k(d; T^*_0\mathbb{R}^n), \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^m) = d$. On the other hand, $G^k(d)$ is the canonical projection of $G^k(d; T^*_0\mathbb{R}^n)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. Therefore, $\mathrm{codim}(G^k(d), \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \geqq d - n$. Since $G(d) = G^0(d) \cup G^1(d)$ is the typical fiber of $B_{mo}(d)$,

$$\mathrm{codim}(B_{mo}(d), J^N(\mathrm{VF}(M))^m) = \mathrm{codim}(G(d), \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \geqq d - n.$$

Since the dimensions of $B_{mo}(d)$ and $\overline{B_{mo}(d)}$ are equal,

$$\mathrm{codim}(\overline{B_{mo}(d)}, J^N(\mathrm{VF}(M))^m) \geqq d - n.$$

□

5.4 Proof of main theorem 1

Proof of Theorem 5.1 (Main theorem 1) :
Let $d > 2n$ be a positive integer. Let $N = 2d (> 4n)$. Let G_0 be the set of $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j^N_x X$ is not included in the closure of $B_{mo}(d)$ in $J^N(\mathrm{VF}(M))^m$:

$$G_0 = \left\{ X \in \mathrm{VF}(M))^m \;\middle|\; j^N_x X \notin \overline{B_{mo}(d)} \text{ for any } x \in M. \right\}.$$

By Lemma 5.10,

$$\mathrm{codim}(\overline{B_{mo}(d)}, J^N(\mathrm{VF}(M))^m) \geqq d - n > n.$$

Then G_0 is an open dense subset of $\mathrm{VF}(M)^m$ by using the transversality theorem (see [10]).

Let $X = (X_1, \cdots, X_m) \in G_0$. Then, for any $x \in M$, $j^N_x X \notin B_{mo}(d)$. Therefore, by using Lemma 5.9, if $z : [0, T] \to T^*M$ is a singular X-bi-extremal and if the singular X-trajectory $x = \pi \circ z : [0, T] \to M$ is non-trivial and independent, then $z : [0, T] \to T^*M$ is of minimal order. □

6 Singular controls with non-trivial independent trajectory of generic driftless control-affine system

Recall that $\mathrm{VF}(M)$ denotes the space of smooth vector fields over M and $\mathrm{VF}(M)^m$ the m-tuple product set of $\mathrm{VF}(M)$ with the Whitney smooth topology. Then, as is stated in Introduction, the following theorem holds:

Theorem 6.1 (Main theorem 2) *Suppose $2 \leqq m \leqq n$. Then there exists an open dense subset G_1 of $\mathrm{VF}(M)^m$ such that, if $X \in G_1$, if $u : [0,T] \to \Omega$ is a singular X-control for a given initial point $x_0 \in M$ and if the corresponding singular trajectory $x : [0,T] \to M, x(0) = x_0$ is non-trivial and independent, then the control u is of corank one.*

Outline of proof : Let $d' \geqq 1$ be an integer. We set $d = 2d' - 1$ and $N = d + 1 = 2d'$.
[Step1] Construct the "bad set" with respect to corank one, $B_C(d) \subset J^N(\mathrm{VF}(M))^m$.
[Step2] (See Lemma 6.7) Show that, if $X \in \mathrm{VF}(M)^m$ satisfies for any $x \in M, j^N_x X \notin B_{mo}(d') \cup B_C(d)$, if $u : [0,T] \to \Omega$ is a singular X-control for a given initial point $x_0 \in M$ and if the corresponding X-trajectory $x : [0,T] \to M, x(0) = x_0$ is non-trivial and independent, then the control u is of corank one.
[Step3] (See Lemma 6.8) Compute the codimension of $B_C(d)$ in $J^N(\mathrm{VF}(M))^m$.
[Step4] (See the subsection §6.4) Let $d' > 2n$. Then $N = 2d' > 4n$ and $d = 2d' - 1 > 3n$. Let G_1 be the set of $X \in \mathrm{VF}(M)^m$ such that, for any $x \in M$, the jets $j^N_x X$ is not included in the closure of $B_{mo}(d') \cup B_C(d)$ in $J^N(\mathrm{VF}(M))^m$. Then, show that G_1 is an open dense subset of $\mathrm{VF}(M)^m$ in the sense of Whitney smooth topology by Thom transversality theorem (for instance see [10]).

In order to show the main theorem 2, we prepare the subsections: §6.1 to §6.4. In §6.1, after preparing some notations, namely, permutated Hamiltonians, extended Lie derivatives and elementary determinants in Definition 6.2, 6.3, 6.4, we define the bad set $B_C(d)$ in Definition 5.4. In §6.2, we show the important Lemma 6.7 prepared for the proof of main theorem 2: if $X \in \mathrm{VF}(M)^m$ satisfies the condition that, for any $x \in M$, $j^N_x X \notin B_{mo}(d)$, then any singular X-control with the non-trivial independent singular X-trajectory is of corank one. In §6.3, we compute the codimension of $B_C(d)$ in $J^N(\mathrm{VF}(M))^m$ in Lemma 6.8. In §6.4, by using these Lemmata 6.7, 6.8, we show the main theorem 2.

6.1 Construction of bad set with respect to corank one.

Let $\mathfrak{S}_m$ be the set of permutations with m elements, and $X = (X_1, \cdots, X_m) \in \mathrm{VF}(M)^m$. Then, we prepare some notations, permutated Hamiltonians, extended Lie derivatives and elementary determinants, in order to define the bad set $B_C(d) \subset J^N(\mathrm{VF}(M))^m$ for an integer d.

Definition 6.2 For $k \in \{1,2\}$, we define the real valued functions $H^{[k]}_{ij}, \Delta^{[k],r}_0, P^{[k]}, \delta^{[k],i}_s$ on $\mathfrak{S}_m \times T^*M \times_M T^*M$, which are called *permutated Hamiltonians*:

Let $\sigma \in \mathfrak{S}_m$ and $(z^{[1]}, z^{[2]}) \in T^*M \times_M T^*M$. Then,

$$
\begin{aligned}
&H^{[k]}_{ij}(\sigma, z^{[1]}, z^{[2]}) = H_{ij}(\sigma, z^{[k]}) \text{ for } i, j \ (1 \leqq i, j \leqq m)\\
&\Delta^{[k],r}_0(\sigma, z^{[1]}, z^{[2]}) = \Delta^r_0(\sigma, z^{[k]}) \text{ for every integer } r \ (0 \leqq r \leqq m-1)\\
&P^{[k]}(\sigma, z^{[1]}, z^{[2]}) = P(z^{[k]}), \text{and } P^{[k],m-2}(\sigma, z^{[1]}, z^{[2]}) = P^{m-2}(z^{[k]})\\
&\delta^{[k],i}_s(\sigma, z^{[1]}, z^{[2]}) = \delta^i_s(\sigma, z), \text{ for every integer } s \geqq 0 \text{ and } i \in \{m-1, m\},
\end{aligned}
$$

where P (resp. P^{m-2}) is defined by

$$
P(\sigma, z) = P\left((H_{ij}(\sigma, z))_{1 \leqq i<j \leqq m}\right) \ \left(\text{resp. } P^{m-2}(\sigma, z) = P\left((H_{ij}(\sigma, z))_{1 \leqq i<j \leqq m-2}\right)\right),
$$

for $\sigma \in \mathfrak{S}, z \in T^*M$ by using the Pfaffian polynomial of G (resp. G^{m-2}) .

Definition 6.3 Let $\mathcal{F}_\bullet$ be the set of $F \in C^\infty(T^*M \times_M T^*M)$ such that there exists $F_1, F_2 \in C^\infty(T^*M)$ such that $F(z^{[1]}, z^{[2})= F_1(z^{[1]})F_2(z^{[2]})$ for $(z^{[1]}, z^{[2})\in T^*M \times_M T^*M$. Let $\mathcal{F}$ be the set of $F \in C^\infty(T^*M \times_M T^*M)$ such that F is a linear combination of a finite number of elements of $\mathcal{F}_\bullet$ on $\mathbb{R}$. Then, for the Hamiltonian vector field $\overrightarrow{H}$ of $X \in \mathrm{VF}(M)$, we define *an extended Lie derivative* $\mathcal{L}_{\overrightarrow{H_j}} : \mathcal{F} \to \mathcal{F}$ by the following: For $F \in \mathcal{F}_\bullet$ and $(z^{[1]}, z^{[2})\in T^*M \times_M T^*M$,

$$
\mathcal{L}_{\overrightarrow{H}}(F)(z^{[1]}, z^{[2]}) = \mathcal{L}_{\overrightarrow{H}}(F_1)(z^{[1]})\, F_2(z^{[2]}) + F_1(z^{[1]})\, \mathcal{L}_{\overrightarrow{H}}(F_2)(z^{[2]}),
$$

and extend it by linearly to $\mathcal{F}$. Here, $\mathcal{L}_{\overrightarrow{H}}(F_k)$ is the Lie derivative of F_k with respect to $\overrightarrow{H}$ for $k \in \{1, 2\}$.

Definition 6.4 We inductively define the real valued functions on $\mathfrak{S}_m \times T^*M \times_M T^*M$, which are called elementary determinants: Let $\sigma \in \mathfrak{S}_m$ and $(z^{[1]}, z^{[2]}) \in T^*M \times_M T^*M$. Then

$$
\begin{cases}
\Theta_{s+1}(\sigma, z^{[1]}, z^{[2]}) = \det \left(\dfrac{\left(H^{[1]}_{ij}(\sigma, z^{[1]}, z^{[2]})\right)_{\substack{1 \leqq i \leqq m-1,\\ 1 \leqq j \leqq m}}}{\left(\mathcal{L}_{\overrightarrow{H_j}} \Theta_s(\sigma, z^{[1]}, z^{[2]})\right)_{1 \leqq j \leqq m}} \right) \ (s = 0, 1, \cdots),\\[4ex]
\Theta_0(\sigma, z^{[1]}, z^{[2]}) = \det \left(\dfrac{\left(H^{[1]}_{ij}(\sigma, z^{[1]}, z^{[2]})\right)_{\substack{1 \leqq i \leqq m-1,\\ 1 \leqq j \leqq m}}}{\left(H^{[2]}_{ij}(\sigma, z^{[1]}, z^{[2]})\right)_{1 \leqq j \leqq m}} \right),
\end{cases}
$$

and

$$\begin{cases} \theta_s(\sigma, z^{[1]}, z^{[2]}) = \det \left(\begin{array}{c} \left(H^{[1]}_{ij}(\sigma, z^{[1]}, z^{[2]}) \right)_{\substack{1 \leqq i \leqq m-1, \\ 1 \leqq j \leqq m}} \\ \hline \left(\{P^{[1]}, H^{[1]}_j\}(\sigma, z^{[1]}, z^{[2]}) \right)_{1 \leqq j \leqq m} \\ \hline \left(\mathcal{L}_{\overrightarrow{H_j}} \theta_s(\sigma, z^{[1]}, z^{[2]}) \right)_{1 \leqq j \leqq m} \end{array} \right) \quad (s = 0, 1, \cdots) \\ \theta_0(\sigma, z^{[1]}, z^{[2]}) = \det \left(\begin{array}{c} \left(H^{[1]}_{ij}(\sigma, z^{[1]}, z^{[2]}) \right)_{\substack{1 \leqq i \leqq m-1, \\ 1 \leqq j \leqq m}} \\ \hline \left(\{P^{[1]}, H^{[1]}_j\}(\sigma, z^{[1]}, z^{[2]}) \right)_{1 \leqq j \leqq m} \\ \hline \left(\{P^{[12]}, H^{[2]}_j\}(\sigma, z^{[1]}, z^{[2]}) \right)_{1 \leqq j \leqq m} \end{array} \right), \end{cases}$$

Definition 6.5 Let $N = d + 1$. We define the "bad set" with respect to corank one, $B_C(d)$ by the canonical projection of $\hat{B}_C(d)$ by $J^N(\mathrm{VF}(M))^m \times_M T^*M \times_M T^*M \to J^N(\mathrm{VF}(M))^m$. The definition of $\hat{B}_C(d)$ is written in Definition 11.6:

Definition 6.6 We define $\hat{B}_C(d) \subset J^N(\mathrm{VF}(M))^m \times_M T^*M \times_M T^*M$ by the union of the sets $B_C(d, \sigma)$ with $\sigma \in \mathfrak{S}_m$. Here the definition of $B_C(d, \sigma)$ is below:
For $\sigma \in \mathfrak{S}_m$, let $B_C(d, \sigma)$ be the subset of $J^N(\mathrm{VF}(M))^m \times_M T^*M \times_M T^*M$ of all triples $(j^N_x X, z^{[1]}, z^{[2]})$ such that $q = \pi(z^{[1]}) = \pi(z^{[2]})$:

1). $X_1(x), \cdots, X_m(x)$ are linearly independent;

2). $z^{[1]}, z^{[2]}$ are linearly independent;

3). $\begin{cases} \text{if } m \text{ is odd, then } \Delta^{m-1}_0(\sigma, z^{[1]} \neq 0, \\ \text{if } m \text{ is even, then } \delta^{m-1}_1(\sigma, z^{[1]}) P^{m-2}(\sigma, z^{[2]}) \neq 0; \end{cases}$

4). for $s \; (0 \leqq s \leqq d - 1)$, $\begin{cases} \text{if } m \text{ is odd, then } \Theta_s(\sigma, z^{[1]}, z^{[2]}) = 0, \\ \text{if } m \text{ is even, then } \theta_s(\sigma, z^{[1]}, z^{[2]}) = 0. \end{cases}$

6.2 Property of singular controls avoiding the bad set with respect to corank one.

We show the important Lemma 6.7 prepared for the proof of the main theorem 2.

Lemma 6.7 *Suppose that $2 \leqq m \leqq n$. Let $d' \geqq 1$ be an integer. We set $d = 2d' - 1$ and $N = d + 1 = 2d'$. Let $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j^N_x X \notin B_{mo}(d') \cup B_C(d)$. Then, if $u : [0, T] \to \Omega$ is a singular X-control for a given initial point $x_0 \in M$ and if the corresponding X-trajectory $x : [0, T] \to M, x(0) = x_0$ is non-trivial and independent, then the control u is of corank one.*

Proof: In order to prove by contradiction, assume that, $u : [0, T] \to \Omega$ is a singular X-control for a given initial point $x_0 \in M$ and $x : [0, T] \to M, x(0) = x_0$ is non-trivial and independent but the control u is not of corank one. Then by Proposition 4.4, there exists two bi-extremals $z^{[1]}, z^{[2]} : [0,T] \to T^*M$ with $z^{[1]} \circ \pi = z^{[2]} \circ \pi = x$ such that $z^{[1]}(t)$ and $z^{[2]}(t)$ are linearly independent for every $t \in [0,1]$.

Case 1 We consider the case m is odd. For $k = 1,2$, we denote by $G^{[k]}$ the Goh matrix $G(z^{[k]}) = (H_{ij}(z^{[k]}))_{1 \leqq i,j \leqq m}$. Since $X = (X_1, \cdots, X_m) \in \mathrm{VF}(M)^m$ is for any $x \in M$, $j_x^N X \notin B_{mo}(d')$, by the proof of Lemma 5.7, there exists $I_{m-1} \subset \{1, \cdots, m\}$ with cardinality $m-1$ such that $\det \left(H_{ij}(z^{[1]}(t))\right)_{(i,j) \in I_{m-1}^2} \not\equiv 0$ on $[0,T]$. Then, up to a permutation, we may assume that there exist an open subinterval $K \subset [0,T]$ and $\sigma \in \mathfrak{S}_m$ such that

$$\Delta_0^{[1],m-1}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \text{ for every } t \in K.$$

On the other hand, since x is independent, $I_{\mathrm{indep}}(x)$ has positive measure, where $I_{\mathrm{indep}}(x)$ is the compliment of $I_{\mathrm{dep}}(x)$ in $[0,T]$. Then Let $J = K \cap I_{\mathrm{indep}}(x)$.

Now, since for $k = 1,2$, $H_i(\sigma, z^{[k]}(t)) = 0$ on $[0,T]$, by differentiating both sides, for $i\,(1 \leqq i \leqq m)$,

$$\sum_{j=1}^{m} u_j(t)\, H_{ij}(\sigma, z^{[k]}(t)) = 0 \text{ a.e. } t \in [0,T].$$

Hence, the matrix

$$\begin{pmatrix} H_{11}(\sigma, z^{[1]}(t)) & \cdots & H_{1m}(\sigma, z^{[1]}(t)) \\ \vdots & & \vdots \\ H_{(m-1)1}(\sigma, z^{[1]}(t)) & \cdots & H_{(m-1)m}(\sigma, z^{[1]}(t)) \\ H_{11}(\sigma, z^{[2]}(t)) & \cdots & H_{1m}(\sigma, z^{[2]}(t)) \end{pmatrix}$$

is not invertible. Therefore $\Theta_0(\sigma, z^{[1]}(t), z^{[2]}(t)) \equiv 0$ on $[0,T]$. By differentiating both sides,

$$\sum_{j=1}^{m} u_j(t)\, \mathcal{L}_{\overrightarrow{H_j}} \Theta_0(\sigma, z^{[1]}(t), z^{[2]}(t)) = 0 \text{ a.e.}\, t \in [0,T].$$

This implies $\Theta_1 \equiv 0$. By proceeding similarly, for $k(0 \leqq k \leqq d-1)$.

$$\begin{cases} \Delta_0^{[1],m-1}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \\ \Theta_k(\sigma, z^{[1]}(t), z^{[2]}(t)) \equiv 0 \end{cases} \text{for every } t \in J.$$

This contradicts the assumption that, for any $x \in M$, $j_x^N X \in B_C(d)$.

Case 2 We consider the case m is even. By proofs of Lemma 5.7, 5.8, up to permutation, there exists an subinterval $K \subset [0,T]$ such that

$$\begin{cases} P^{[1],m-2}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \\ \delta_1^{[1],m-1}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \end{cases} \text{for every } t \in K.$$

Then Let $J = K \cap I_{\mathrm{indep}}(x)$.

We show that $P^{[2],m-2}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0$ for every $t \in J$. For $\alpha \in [0,T]$, consider $z^\alpha = (1-\alpha) z^{[1]} + \alpha z^{[2]}$. Since $P^{m-2}(z^{[\alpha]}(t))$ depends continuously on α, for α small enough, $P^{m-2}(z^{[\alpha]}(t)) = 0$ for every $t \in J$. Moreover, the set of singular X-bi-extremals of singular X-trajectory x is a vector space, $z^{[\alpha]}$ is singular X-bi-extremals of x, and for $\alpha > 0$, $z^{[\alpha]}(t)$ and $z^{[1]}(t)$ are linearly independent for every $t \in J$. Then, it suffices to replace $z^{[2]}$ by z^{α}, for some $\alpha > 0$ small enough.

Similar to the case m odd, for $k\,(0 \leqq k \leqq d-1)$,

$$\begin{cases} \delta_1^{[1],m-1}(\sigma, z^{[1]}(t), z^{[2]}(t))\, P^{[2],m-2}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \\ \theta_k(\sigma, z^{[1]}(t), z^{[2]}(t)) \equiv 0 \end{cases} \quad \text{for every } t \in J.$$

This contradicts the assumption that, for any $x \in M$, $j_x^N X \in B_C(d)$.

□

6.3 Codimension of bad set with respect to corank one.

Let d be a positive integer and $N = d+1$.

Lemma 6.8 $\operatorname{codim}(\overline{B_C(d)}, J^N(\mathrm{VF}(M))^m) \geqq d - 2n$.

Proof: We describe only the outline of the proof of Lemma 6.8 because this bad set $B_C(d)$ is the completely same bad set $B_C(d)$ defined as 3.1.2 in [8]: Let $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ be the m-tuple product space of polynomial vector fields of degree $\leqq d_i$ over $\mathbb{R}^n$.

Step 1: Construct the typical fiber $G_C(d)$ of $B_C(d)$:

Typical fiber $G_C(d)$ of $B_C(d)$ is the canonical projection of $G_C(d; T_0^*\mathbb{R}^n \times T_0^*\mathbb{R}^n)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. $G_C(d; T_0^*\mathbb{R}^n \times T_0^*\mathbb{R}^n)$ is defined by the set of $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that there exists $\sigma \in \mathfrak{S}_k$ such that $(Q, p^{[1]}, p^{[2]})$ satisfies the conditions 1) to 4):

1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;
2). $p^{[1]}, p^{[2]}$ are linearly independent;
3). (a) if m is odd, then $\Delta_0^{m-1}(\sigma, z_0^{[1]}) \neq 0$,
(b) if m is even, then $\delta_1^{m-1}(\sigma, z_0^{[1]}) P^{m-2}(\sigma, z_0^{[2]})$;
4). (a) if m is odd, then $\Theta_s(\sigma, z_0^{[1]}, z_0^{[2]})$ for every integer $s\,(0 \leqq s \leqq d-1)$,
(b) if m is even, then $\theta_s(\sigma, z_0^{[1]}, z_0^{[2]})$ for every integer $s\,(0 \leqq s \leqq d-1)$,

where $z_0^{[1]}, z_1^{[2]}$ are the elements of $T_0^*\mathbb{R}^n$ given in coordinates by $(0, p_1), (0, p_2)$.

Step 2: Construct the mapping $\phi_\sigma : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$:

Let $\sigma \in \mathfrak{S}_k$. Then construct the mapping $\phi_{\sigma,V}$ by dividing two cases of m being odd and even:

Case1 If m be an odd integer, then we define $\phi_\sigma : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\phi_{\sigma,V}(Q, p^{[1]}, p^{[2]}) = \Big(\Theta_s(Q)(\sigma, z_0^{[1]}, z_0^{[2]})\Big)_{0 \leqq s \leqq d-1},$$

where $z_0^{[1]}, z_0^{[2]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p^{[1]}, p^{[2]})$ and $\Theta_s(Q)$ is the elementary determinants associated to Q.

Case2 If m be an even integer, then we define $\phi_\sigma : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\phi_{\sigma,V}(Q, p^{[1]}, p^{[2]}) = \Big(\theta_s(\sigma, z_0^{[1]}, z_0^{[2]})\Big)_{0 \leqq s \leqq d-1},$$

where $z_0^{[1]}, z_0^{[2]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p^{[1]}, p^{[2]})$ and $\theta_s(Q)$ is the elementary determinants associated to Q.

Step 3: Construct the open subset $V_\sigma \subset \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$:

Let $\sigma \in \mathfrak{S}$. Then, construct V_σ by dividing two cases of m being odd and even:

Case1 If m is an odd integer, then V_σ is defined by the set of $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that $(Q, p^{[1]}, p^{[2]})$ satisfies the conditions 1) to 2):

1). $p^{[1]}, p^{[2]}$ are linearly independent;

2). $\Delta_0^{m-1}(\sigma, z_0^{[1]}) \neq 0$.

Case2 If m is an even integer, then V_σ is defined by the set of $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that

1). $p^{[1]}, p^{[2]}$ are linearly independent;

2). $\delta_1^{m-1}(\sigma, z_0^{[1]}) P^{m-2}(\sigma, z_0^{[2]}) \neq 0$,

where $z_0^{[1]}, z_0^{[2]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p^{[1]}, p^{[2]})$.

Then, V_σ is an open subset of $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$.

Step 4: $G_C(d; T_0^*\mathbb{R}^n \times T_0^*\mathbb{R}^n)$ is the union of kernels of restriction to V_σ of the mapping ϕ_σ with $\sigma \in \mathfrak{S}_m$.

Step 5: Let Ω_0 be the set of $Q \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that $Q_1(0), \cdots, Q_m(0)$ are linearly independent. It is well-known that the local coordinate systems on Ω_0 can be constructed (see Coordinate systems in [6],[8]).Then, if $\sigma \in \mathfrak{S}_m$, then the restriction to the intersection $V_\sigma \cap \hat{V}$ of the mapping ϕ_σ is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0 \times \mathbb{R}^n \times \mathbb{R}^n$. (The proof is Lemma 4.2. in [8].)

Step 6: $\mathrm{codim}(B_C(d), J^N(\mathrm{VF}(M))^m) \geqq d - 2n$:

By Step 4,5, $\mathrm{codim}(G_C(d; T_0^*\mathbb{R}^n \times T_0^*\mathbb{R}^n), \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n) = d$. On the other hand, $G_C(d)$ is the canonical projection of $G_C(d; T_0^*\mathbb{R}^n \times T_0^*\mathbb{R}^n)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. Therefore, $\mathrm{codim}(G_C(d), \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \geqq d - 2n$. Since $G_C(d)$ is the typical fiber of $B_C(d)$,

$$\mathrm{codim}(B_C(d), J^N(\mathrm{VF}(M)) = \mathrm{codim}(G_C(d), \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \geqq d - 2n.$$

Since the dimensions of $B_C(d)$ and $\overline{B_C(d)}$ are equal,

$$\mathrm{codim}(\overline{B_C(d)}, J^N(\mathrm{VF}(M)) \geqq d - 2n.$$

□

6.4 Proof of main theorem 2

Proof of Theorem 6.1 (Main theorem 2):

Let $d' > 2n$ be an integer. Let $d = 2d' - 1$ and $N = d + 1 = 2d'(> 4n)$.

Then, let G_1 be the set of $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j^N_x X$ is not included in the closure of $B_{mo}(d') \cup B_C(d)$ in $J^N(\mathrm{VF}(M))^m$:

$$G_1 = \left\{ X \in \mathrm{VF}(M))^m \;\middle|\; j^N_x X \notin \overline{B_{mo}(d') \cup B_C(d)} \text{ for any } x \in M. \right\}.$$

By Lemmata 5.10, 6.8 and $n \geqq 2$,

$$\mathrm{codim}(\overline{B_{mo}(d') \cup B_C(d)}, J^N(\mathrm{VF}(M))^m) \geqq \min\{d' - n, d - 2n\} > n.$$

Then G_1 is an open dense subset of $\mathrm{VF}(M)^m$ by using transversality theorem (see [10]).

Let $X = (X_1, \cdots, X_m) \in G_1$. Then, for any $x \in M$, $j^N_x X \notin B_{mo}(d') \cup B_C(d)$. Therefore, by using Lemma 6.7, if $u : [0, T] \to \Omega$ is a singular X-control for a given initial point $x_0 \in M$ and if the corresponding X-trajectory $x : [0, T] \to M, x(0) = x_0$ is non-trivial and independent, then the control u is of corank one. $\square$

7 Strictly abnormal extremals with respect to an optimal control problem

In this section, we describe the optimal control problem with respect to the driftless control system. In §8.2, as a typical example, we consider the geodesic on a generalized subriemannian manifold.

Let M be an n dimensional manifold. Let $x_0 \in M$ and $T > 0$. Let Ω be an open subset of $\mathbb{R}^m$. Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields over M. Consider the driftless control systems

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x).$$

We denote by $\mathfrak{U}_{x_0,x_1,T}$ the set of admissible X-controls from $[0,T]$ to Ω such that the corresponding trajectory to u has a fixed initial point x_0 and a fixed point x_1. Given a smooth function $e : \Omega \to \mathbb{R}$, consider an (X,e)-*optimal control problem* to minimize a cost functional C_e. Here, a *cost* functional $C_e : \mathfrak{U}_{x_0,x_1,T} \to \mathbb{R}$ is defined by

$$C_e(u) = \int_{[0,T]} e(u(t))dt$$

for $u \in \mathfrak{U}_{x_0,T}$. An admissible control $u_* \in \mathfrak{U}_{x_0,T}$ is called an (X,e)-*optimal control*, an (X,e)-*minimal control* or an (X,e)-*minimizer* if $C(u_*) \leqq C(u)$ for any $u \in \mathfrak{U}_{x_0,T}$. Then, $e : \Omega \to \mathbb{R}$ is called a *cost fuction* of the (X,e)-optimal control problem.

The *Hamiltonian function* $H = H_{(X,e)} : (T^*M \times \Omega) \times \mathbb{R} \to \mathbb{R}$ of the optimal control problem (X,e) is defined by for $(x,p,u\,;\,p_0) \in (T^*M \times \Omega) \times \mathbb{R}$,

$$H_{(X,e)}(x,p,u\,;\,p_0) = H_X(x,p,u) + p_0\, e(u),$$

where $(x,p,u) = (x_1, \cdots, x_n, p_1, \cdots, p_n, u_1, \cdots, u_m)$ is the local coordinate of $T^*M \times \Omega$ with a canonical coordinate of (x,p) of T^*M and we define $H_X : T^*M \to \mathbb{R}$ by $H_X(x,p,u) = \langle p, \sum_{i=1}^m u_i X_i(x) \rangle$.

By the Pontryagin maximum principle (see [16],[3]), the following property holds:

Proposition 7.1 *Let $x_0 \in M$. If $u : [0,T] \to \Omega$ is an X-optimal control and $x : [0,T] \to \Omega$ be the corresponding X-trajectory with $x(0) = x_0$, then there exists a pair (z, p_0) of an absolute continuous curve $z : [0,T] \to T^*M$ and a real number $p_0 \leqq 0$ such that, $x = \pi \circ z$, and that the following equations hold: for any local canonical coordinates $(x,p,u) = (x_1, \cdots, x_n, p_1, \cdots, p_n, u_1, \cdots, u_m)$ of $T^*M \times \Omega$ with a canonical coordinate of (x,p) of T^*M :*

$$(\star):\begin{cases} (1)\ \dot{x}_i(t) = \frac{\partial H}{\partial p_i}(x(t),p(t),u(t)\,;\,p_0)\ (1 \leqq i \leqq n) & \textit{for a.e. } t \in [0,T] \\ (2)\ \dot{p}_i(t) = -\frac{\partial H}{\partial x_i}(x(t),p(t);u(t)\,;\,p_0)\ (1 \leqq i \leqq n) & \textit{for a.e. } t \in [0,T] \\ (3)\ \frac{\partial H}{\partial u_j}(x(t),p(t);u(t)\,;\,p_0) = 0\ (1 \leqq j \leqq m) & \textit{for a.e. } t \in [0,T] \\ (4)\ (p(t),p_0) \neq 0. & \textit{for every } t \in [0,T] \end{cases},$$

A curve $z : [0,T] \to T^*M$ is called a *normal* X*-bi-extremal* (resp. an *abnormal* X*-bi-extremal*) if $p_0 < 0$ (resp. $p_0 = 0$). A curve $x : [0,T] \to T^*M$ is called a *normal* X*-extremal* (resp. an *abnormal* X*-extremal*) if it possesses an X-normal bi-extremal lift (resp. an X-abnormal bi-extremal lift). Note that, if $x : [0,T] \to T^*M$ is an abnormal X-extremal, then x is a singular X-trajectory, namely the corresponding singular X-trajectory to the singular X-control, which is the singular point of the end-point mapping. Moreover, note that, if $z : [0,T] \to T^*M$ is an abnormal X-bi-extremal, then z is a singular X-bi-extremal.

Definition 7.2 *An* X*-extremal* $x : [0,T] \to M$ *is called strictly* X*-abnormal if it is not the projection of a normal* X*-bi-extremal.*

Note that it may happen that an extremal $x : [0,T] \to M$ is both normal and abnormal.

8 Generalized subriemannian geometry and length minimizers

8.1 Generalized subriemanian geometry

We recall the "generalized" subriemannian geometry by Bellïche (see [4]):

Let $X = (X_1, \cdots, X_m)$ be a system of vector fields over an n-dimensional smooth manifold M. Given a point $x \in M$, let $L_x \subset T_x M$ be the vector space over $\mathbb{R}$ generated by $X_1(x), \cdots, X_m(x)$, namely $L_x = \langle X_1(x), \cdots, X_m(x) \rangle_{\mathbb{R}}$. Let $L \subset TM$ be the union of the sets L_x with $x \in M$. In particular, if the system of vector fields $X = (X_1, \cdots, X_m)$ is linearly independent, then $L \subset TM$ is a subbundle of TM, and $L \subset TM$ is called a *distribution* of TM.

Definition 8.1 Let $X = (X_1, \cdots, X_m)$ be a system of vector fields over an n-dimensional smooth manifold M. Let $L \subset TM$ be the union of the sets $L_x = \langle X_1(x), \cdots, X_m(x) \rangle_{\mathbb{R}}$ with $x \in M$. Then $g : L \longrightarrow \mathbb{R}$ is called a *generalized subriemannian metric* or a *generalized sub-Riemannain metric* if for $w = (x, v) \in L$,

$$g(w) = g(x, v) = \inf\{(u_1)^2 + \cdots + (u_m)^2 \mid u_1 X_1(x) + \cdots + u_r X_m(x) = v\},$$

where $w = (x, v)$ is canonical coordinates of $L \subset TM$, namely, $x \in M, v \in L_x$.

Note that, if $X = (X_1, \cdots, X_m)$ is linearly independent everywhere on M, then the system generates a distribution and the metric g in a generalized subriemannian structure is the same as the subriemannian metric in ordinary meaning.

Let $x : [0, T] \to M$ be an absolutely continuous curve. Then, $x : [0, T] \longrightarrow M$ is called X-*admissible* (or L-*admissible*) if for a.e. $t \in [0, T]$,

$$\dot{x}(t) \in L_{x(t)} = \langle X_1(x(t)), \cdots, X_m(x(t)) \rangle_{\mathbb{R}}.$$

Then, *generalized Carnot-Carathéodry distance* $d_{CC} : M \times M \longrightarrow \mathbb{R} \cup \{\infty\}$ is defined by the following:
for $p, q \in M$,

$$d_{CC}(p, q) = \inf \left\{ \int_{[0,T]} \sqrt{g(x(t), \dot{x}(t))}dt \;\middle|\; \begin{array}{l} x : [0, T] \longrightarrow M : X\text{-admissible} \\ x(0) = p, \ x(T) = q \end{array} \right\}.$$

Definition 8.2 An X-admissible curve $x : [0, T] \to M$ is called a *length-minimizer* if the length of x is equal to $d_{CC}(x(0), x(T))$:

$$d_{CC}(x(0), x(T)) = \int_0^T \sqrt{g(x(t), \dot{x}(t))}dt.$$

We denote by (M, X, g) the triplet of an n dimensional manifold M, a system of smooth vector fields over M, $X = (X_1, \cdots, X_m)$, and the subriemannian metric g on L, which is the union of the sets $L_x = \langle X_1(x), \cdots, X_m(x) \rangle_{\mathbb{R}}$ with $x \in M$.

8.2 Abnormal controls in generalized subriemannian geometry

In previous section §7, we considered the optimal control problem with respect to the driftless control system. In this section, as a typical example of this optimal control problem, consider the geodesic on generalized subriemannian manifold.

Let X be an n dimensional manifold. Let $x_0 \in M$ and $T > 0$. Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields over M. Consider the driftless control systems

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x).$$

with the control parameter $u \in \mathbb{R}^m$. We denote by $\mathfrak{U}_{x_0,x_1,T}$ the set of admissible X-controls from $[0, T]$ to $\mathbb{R}^m$ such that the corresponding trajectory to u has a fixed initial point x_0 and fixed end point x_1.

We define an *energy* function $e : \mathbb{R}^m \to \mathbb{R}$ by

$$e(u) = \frac{1}{2} \sum_{i=1}^{m} u_i^2, \text{ for } u \in \mathbb{R}^m.$$

Consider the optimal control problem to minimize the *energy* functional $C_e : \mathfrak{U}_{x_0,x_1,T} \to \mathbb{R}$

$$C_e(u) = \int_{[0,T]} e(u(t))dt = \int_{[0,T]} \frac{1}{2} \sum_{i=1}^{m} u_i(t)^2 dt, \text{ for } u \in \mathfrak{U}_{x_0,T}.$$

It is known that the problem is equivalent to minimizing the *length* :

$$\ell(u) = \int_{[0,T]} \sqrt{\sum_{i=1}^{m} u_i(t)^2 dt}, \text{ for } u \in \mathfrak{U}_{x_0,T}.$$

If $X_1, \cdots, X_m$ are linearly independent everywhere, then the optimal problem (X, e) is exactly to minimize the Carnot-Carathéodory distances in subriemannian geometry (see [15]).

The Hamiltonian function $H = H_{(X,e)} : (T^*M \times \mathbb{R}^m) \times \mathbb{R} \to \mathbb{R}$ of the optimal control problem (X, e) is given by

$$H(x, p, u\,;\, p_0) = \sum_{i=1}^{m} \langle p, u_i X_i(x) \rangle + \frac{1}{2} p_0 (\sum_{i=1}^{m} u_i^2).$$

where $(x, p, u) = (x_1, \cdots, x_n, p_1, \cdots, p_n, u_1, \cdots, u_m)$ is the local coordinate of $T^*M \times \mathbb{R}^m$ with a canonical coordinate of (x, p) of T^*M. Then the constraint $\frac{\partial H}{\partial u} = 0$ is equivalent to the following

$$p_0\, u_j = -\langle p, X_j(x) \rangle, (1 \leqq j \leqq m).$$

For an X-normal extremal, we have $p_0 < 0$. Then we have

$$u_j = -\frac{1}{p_0} \langle p, X_j(x) \rangle, (1 \leqq j \leqq m).$$

Then

$$H = -\frac{1}{2p_0}\sum_{i=1}^{m}\langle p, X_i(x)\rangle^2.$$

From the linearity of Hamiltonian function on (p, p_0), we can normalize p_0, so that

$$H = \frac{1}{2}\sum_{i=1}^{m}\langle p, X_i(x)\rangle^2.$$

Thus our formulation coincides with the ordinary normal Hamiltonian formalism in sub-riemannian geometry.

Therefore, for an X-abnormal extremal $u : [0,T] \to \mathbb{R}^m$, the equation reduces to

$$\begin{cases} (1)\ \dot{x}_i(t) = \frac{\partial H}{\partial p_i}\,(x(t), p(t), u(t))\,(1 \leqq i \leqq n) & \text{for a.e. } t \in [0,T] \\ (2)\ \dot{p}_i(t) = -\frac{\partial H}{\partial x_i}\,(x(t), p(t); u(t))\,(1 \leqq i \leqq n) & \text{for a.e. } t \in [0,T] \\ (3)\ \langle p(t), X_j(x(t))\rangle = 0\ (1 \leqq j \leqq m),\ \ p(t) \neq 0 & \text{for every } t \in [0,T] \end{cases},$$

with $H(x,p,u) = H_X(x,p,u) = \langle p, \sum_{i=1}^m u_i\, X_i(x)\rangle$.

9 Abnormal extremals of generic driftless control-affine system in generalized subriemannian geometry

In this section, we prove Theorem 3 (Theorem 9.2). In order to formulate the Theorem 3, we recall the strictly abnormal extremal: Let X be an n dimensional manifold M. Let Ω be an open subset of $\mathbb{R}^r$. Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields over M. Consider the driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x).$$

Moreover, consider the optimal control problem associated to the driftless control-affine systems to minimize the energy functional

$$e(u) = \frac{1}{2} \sum_{i=1}^{m} u_i^2.$$

on an X-admissible control with the initial point $x_0 \in M$. The Hamiltonian function $H = H_{(X,e)} : (T^*M \times \mathbb{R}^m) \times \mathbb{R} \to \mathbb{R}$ of the optimal control problem (X, e) is given by

$$H(x, p, u\,;\, p_0) = \sum_{i=1}^{m} \langle p, u_i X_i(x) \rangle + \frac{1}{2}\, p_0 (\sum_{i=1}^{m} u_i^2).$$

where $(x, p, u) = (x_1, \cdots, x_n, p_1, \cdots, p_n, u_1, \cdots, u_m)$ is the local coordinate of $T^*M \times \mathbb{R}^m$ with a canonical coordinate of (x, p) of T^*M.

Definition 9.1 An X-abnormal extremal $x : [0, T] \to M$ is called *strictly* if it is not the projection of a normal X-bi-extremal.

Let $\mathrm{VF}(M)^m$ denote the set of systems of smooth vector fields $X = (X_1, \cdots, X_m)$ over M. We endow $\mathrm{VF}(M)^m$ with the Whitney smooth topology. Then, the following Theorem 9.2 holds:

Theorem 9.2 *Suppose $2 \leqq m \leqq n$. Then, there exists an open dense subset $G_3 \subset \mathrm{VF}(M)^m$ such that, if $X \in G_3$ and if $x : [0, T] \to M$ is an X-abnormal extremal for the initial point $x_0 \in M$, then x is strictly X-abnormal.*

This Theorem 9.2 is the special case of Proposition 2.19 of [9]. However the proof of Proposition 2.19 is hard to read , because the construction of G_3 is not written. We will improve the proof of Proposition 2.19 clearly.

Outline of proof Let $d \geqq 1$ be an integer. Put $N = d + 1$. We denote the space of all N-jets of vector fields $X \in \mathrm{VF}(M)$ by $J^N(\mathrm{VF}(M))$, and the fibre product over M of m-tuple spaces of $J^N(\mathrm{VF}(M))$, by $J^N(\mathrm{VF}(M))^m$. Then, we will show the main theorem 3 by the following procedures:

[Step1] Construct the "bad set" with respect to minimal order, $B_{sa}(d) \subset J^N(\mathrm{VF}(M))^m$.
[Step2] Show that, if $X \in \mathrm{VF}(M)^m$ satisfies the condition that any $x \in M$, $j_x X \notin B_{sa}(d)$ and if $x : [0,T] \to M$ is an abnormal X-bi-extremal, then x is of strictly abnormal.
[Step3] Compute the codimension of $B_{sa}(d)$ in $J^N(\mathrm{VF}(M))^m$.
[Step4] For $N > 3n+1$ $(d > 3n)$, let G_3 be the set of $X \in \mathrm{VF}(M)^m$ such that the jet $j_x^N X$ is not included in the closure of $B_{sa}(d)$ in $J^N(\mathrm{VF}(M))^m$. Then, show that, G_3 is an open dense subset of $\mathrm{VF}(M)^m$ in the sense of Whitney smooth topology by Thom transversality theorem (for instance see [10]).

9.1 Construction of bad set with respect to strictly abnormal

Let $(z^{[n]}, z^{[a]}) \in T^*M \times_M T^*M$ and $x = \pi(z^{[n]}) = \pi(z^{[a]})$. For every multi index I of $\{1, \cdots, m\}$, set

$$H_I^{[n]}(z^{[n]}, z^{[a]}) = H_I(z^{[n]}) \,\text{and}\, H_I^{[a]}(z^{[n]}, z^{[a]}) = H_I(z^{[a]}).$$

and define inductively the following functions in $\mathcal{F}$, depending on $(z^{[n]}, z^{[a]})$

$$\begin{cases} \beta_{i,0} = H_i^{[a]} \\ \beta_{i,s+1} = \sum_{j=1}^{m} H_j^{[n]} \mathcal{L}_{\overrightarrow{H_j}} \beta_{i,s}, \ (s = 1, 2, \cdots), \end{cases}$$

where $\mathcal{F}$ and $\mathcal{L}_{\overrightarrow{H_j}}$ are defined in before section.

Definition 9.3 Let d be a positive integer. Let $N = d+1$. For every integer $i\,(1 \leqq i \leqq m)$ and $(z^{[n]}, z^{[a]}) \in T^*M \times_M T^*M$, we define $\hat{B}(d, i, z^{[n]}, z^{[a]})$ by the set of $j_x^N X \in J^N(\mathrm{VF}(M))^m$ such that the following conditions hold:

1)$X_i(x) \neq 0$;

2)$H_i^{[n]}(z^{[n]}, z^{[a]}) \neq 0$;

3)$\beta_{i,s}(z^{[n]}, z^{[a]}) = 0$ for every integer $s\,(0 \leqq s \leqq d-1)$.

$\hat{B}((d, z^{[n]}, z^{[a]}) \subset J^N(\mathrm{VF}(M))^m$ is the union of $\hat{B}(d, i, z^{[n]}, z^{[a]})$ with $i\,(1 \leqq i \leqq m)$.

Definition 9.4 Let d be a positive integer. Let $N = d+1$. we define $\hat{B}_{sa}(d) \subset J^N(\mathrm{VF}(M))^m \times_M T^*M \times_M T^*M$ by

$$\hat{B}_{sa}(d) = \{(j_x^N X, z^{[n]}, z^{[a]}) | j_x^N X \in \hat{B}((d, z^{[n]}, z^{[a]})\}.$$

Definition 9.5 Let d be a positive integer. Let $N = d+1$. we define the bad set with respect to strictly abnormal $B_{sa}(d)$ by the canonical projection of $\hat{B}_{sa}(d)$ on $J^N(\mathrm{VF}(M))^m$.

9.2 The property of singular bi-extremals avoiding bad set with respect to strictly abnormal

Lemma 9.6 *Suppose that, $2 \leqq m \leqq n$. Let d be a positive integer and $N = d+1$. Let $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j^N_x X \notin B_{sa}(d)$. Then, if $x : [0,T] \to M$ is an abnormal X-bi-extremal, then x is of strictly abnormal.*

Proof: By contradiction, assume that there exists a nontrivial singular X-trajectory $x : [0,T] \to M$ with a singular X-control $u : [0,T] \to \mathbb{R}^m$ such that $x = \pi \circ z^{[n]} = \pi \circ z^{[a]}$, where $z^{[n]}$ is a normal X-bi-extremal lift of x, and $z^{[a]}$ is an abnormal X-bi-extremal lift of x.

For every multi-index $I \subset \{0, \cdots, m\}$ and $t \in [0,T]$, set

$$H_I(z^{[n]}(t)) = \langle z^{[n]}(t), X_I(x(t)) \rangle, H_I(z^{[a]}(t)) = \langle z^{[a]}(t), X_I(x(t)) \rangle.$$

After time differentiation, we have on $[0,T]$,

$$\begin{cases} \frac{d}{dt} H_I(z^{[n]}(t)) = \sum\limits_{i=1}^{m} u_i(t) H_{Ii}(z^{[n]}(t)), \\ \frac{d}{dt} H_I(z^{[a]}(t)) = \sum\limits_{i=1}^{m} u_i(t) H_{Ii}(z^{[a]}(t)). \end{cases}$$

By Pontryagin maximum principle,

$$\begin{cases} u_i(t) = H_i(z^{[n]}(t)) = H^{[n]}_i(z^{[n]}(t), z^{[a]}(t)), \\ H^{[a]}_i(z^{[n]}(t), z^{[a]}(t)) = H_i(z^{[a]}(t)) = 0 \cdots (\star) \end{cases} \quad \text{for every integer } i \, (1 \leqq i \leqq m), t \in [0,T]$$

Since $x : [0,T] \to \mathbb{R}^n$ is nontrivial, there exists an open subset $J \subset [0,T]$ and an integer $i_0 \, (1 \leqq i_0 \leqq m)$ such that $u_{i_0}(t) X_{i_0}(x(t)) \neq 0$ on J. Therefore, $u_{i_0}(t) \neq 0$ and $X_{i_0}(x(t)) \neq 0$ on J. Since $u_{i_0}(t) = H_{i_0}(z^{[n]}(t))$,

$$H_{i_0}(z^{[n]}(t)) \neq 0.$$

on the other hand, by differentiating $(\star)$ with respect to $t \in [0,T]$,

$$\begin{aligned} 0 &= \frac{d}{dt} H^{[a]}_{i_0+1}(z^{[n]}(t), z^{[a]}(t)) \\ &= \sum_{j=1}^{m} u_j(t) H^{[a]}_{(i_0+1)j}(z^{[n]}(t), z^{[a]}(t)) \\ &= \sum_{j=1}^{m} H^{[n]}_j(z^{[n]}(t), z^{[a]}(t)) \, H^{[a]}_{(i_0+1)j}(z^{[n]}(t), z^{[a]}(t)) \\ &= \beta_{i_0,1}(z^{[n]}(t), z^{[a]}(t)) \end{aligned}$$

For every $t \in [0,T]$, by induction,

$$\beta_{i_0,s}(z^{[n]}(t), z^{[a]}(t)) = 0.$$

for every $s \, (0 \leqq s \leqq d-1)$ and $t \in J$. Hence, $j^N_x X \in \hat{B}(d, i_0, z^{[n]}, z^{[a]})$ for $t \in J$, which contradicts the hypothesis. □

9.3 Codimension of bad set with respect to strictly abnormal

Lemma 9.7 $\operatorname{codim}(\overline{B_{sa}(d)}; J^N(VF(M))^m) \geqq d - 2n$.

Proof: We describe only the outline of the proof of Lemma 12.6. Let $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$ be the m-tuple product space of polynomial vector fields of degree $\leqq N$ over $\mathbb{R}^n$.

Step1: Construct the typical fiber $G_{sa}(d)$ of $B_{sa}(d)$.

Typical fiber $G_{sa}(d)$ of $B_{sa}(d)$ is the canonical projection of $G_{sa}(d; T_0^*\mathbb{R}^m \times T_0^*\mathbb{R}^m)$ by $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$. $G_{sa}(d; T_0^*M \times_M T_0^*M)$ is defined by the set of $(Q, p^{[n]}, p^{[a]}) \in \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that there exists $i\,(1 \leqq i \leqq m)$ such that $(Q, p^{[n]}, p^{[a]})$ satisfies the following conditions 1) to 4):

1)$Q_i(0)$ are linearly independet;

2)$H_i^{[n]}(z_0^{[n]}, z_0^{[a]}) \neq 0$;

3)$\beta_{i,s}(z_0^{[n]}, z_0^{[a]}) = 0$ for every integer $s\,(0 \leqq s \leqq d-1)$.

where $z_0^{[n]}, z_0^{[a]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p_1), (0, p_2)$.

Step2: Construct the mapping $\phi_i : \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$:

Let $i(1 \leqq i \leqq m)$ be a positive integer. Then we define the mapping $\phi_i : \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q, p_1, p_2) \in \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\phi_i(Q, p_1, p_2) = \beta_{i,s}(z_0^{[n]}, z_0^{[a]}),$$

where $z_0^{[n]}, z_0^{[a]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p_1), (0, p_2)$.

Step3: Construct the open subset $V_i \subset \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$

Let $i(1 \leqq i \leqq m)$ be a positive integer. Then V_i is the defined by the set of $(Q, p_1, p_2) \in \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that (Q, p_1, p_2) satisfies the following condition:

$$H_i^{[n]}(z_0^{[n]}, z_0^{[a]}) \neq 0,$$

where $z_0^{[n]}, z_0^{[a]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p_1), (0, p_2)$. Then, V_i is an open subset of $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$.

Step4: $G_{sa}(d; T_0^*\mathbb{R}^m \times T_0^*\mathbb{R}^m)$ is the union of the kernel of restriction to V_i of the mapping ϕ_i with $i\,(1 \leqq i \leqq m)$.

Step5: Let Ω_0^i be the set of $Q \in \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$ such that $Q_i \neq 0$. It is well-known that the local coordinate systems on Ω_0^i can be constructed (see Coordinate systems in [6],[8]) Then, for every integer $i\,(1 \leqq i \leqq m)$, the restriction to the intersection $V_i \cap \hat{V}$ of the mapping ϕ_i is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0 \times \mathbb{R}^n$.

Step6: $\operatorname{codim}(\overline{B_{sa}(d)}; J^N(\mathrm{VF}(M))^m) \geqq d - 2n$.

By step 4,5, $\operatorname{codim}(G_{sa}(d; T_0^*\mathbb{R}^n \times T_0^*\mathbb{R}^n); \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n) = d$. On the other hand, $G_{sa}(d)$ of $B_{sa}(d)$ is the canonical projection of $G_{sa}(d; T_0^*\mathbb{R}^m \times T_0^*\mathbb{R}^m)$ by $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$. Therefore, $\operatorname{codim}(G_{sa}(d); \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)) \geqq d - 2n$. Since $G_{sa}(d)$ is the tyupical fiber of $B_{sa}(d)$,

$$\operatorname{codim}(B_{sa}(d); J^N(\mathrm{VF}(M))^m) = \operatorname{codim}(G_{sa}(d); \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)) \geqq d - 2n.$$

Since the dimensions of $B_{sa}(d)$ and $\overline{B_{sa}(d)}$ are equal,

$$\operatorname{codim}(\overline{B_{sa}(d)}; J^N(\mathrm{VF}(M))^m) \geqq d - 2n$$

$\square$

9.4 Proof of main Theorem 3

Let $d > 2n$ be a positive integer. Let $N = 2d(> 4n)$. Let G_3 be the set of $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j^N_x X$ is not included in the closure of $B_{sa}(d)$ in $J^N(\mathrm{VF}(M))^m$:

$$G_3 = \left\{ X \in \mathrm{VF}(M))^m \,\middle|\, j^N_x X \notin \overline{B_{sa}(d)} \text{ for any } x \in M. \right\}.$$

By Lemma 9.7,

$$\operatorname{codim}(\overline{B_{mo}(d)}, J^N(\mathrm{VF}(M))^m) \geqq d - n > n.$$

Then G_3 is an open dense subset of $\mathrm{VF}(M)^m$ by using the transversality theorem (see [10]).

Let $X = (X_1, \cdots, X_m) \in G_3$. Then, for any $x \in M$, $j^N_x X \notin B_{sa}(d)$. Therefore, by using Lemma 9.6, if $x : [0,T] \to M$ is abnormal, then $x : [0,T] \to T^*M$ is strictly abnormal. $\square$

10 Singular bi-extremals of generic polynomial driftless control-affine systems

Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ denote an m-tuple of integers, and $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ denote the product space of m-tuple systems of polynomial vector fields over $\mathbb{R}^n$, $Q = (Q_1, \cdots, Q_m)$, such that the degree of Q_i satisfiies $\deg Q_i \leqq d_i$ for every integer $i\,(1 \leqq i \leqq m)$, and we endow $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ with the Euclidean topology.

Let Ω be an open subset of $\mathbb{R}^m$. For $Q = (Q_1, \cdots, Q_m) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$, consider the polynomial driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i\, Q_i(x)$$

with the control parameter $(u_1, \cdots, u_m) \in \Omega$. Then, the following holds:

Theorem 10.1 *Suppose $2 \leqq m \leqq n$ and suppose that, an m-tuple of even integers $\boldsymbol{D} = (d_1, \cdots, d_m)$ satisfies the inequality : $\min\{d_1, d_2, \cdots, d_m\} \geqq 4n + 2$. Then, there exists an open dense semi-algebraic subset $H_0 \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that, if $Q \in H_0$, if $z : [0, T] \to T^*\mathbb{R}^n$ is a Q-bi-extremal, and if the singular trajectory $x = \pi \circ z : [0, T] \to \mathbb{R}^n$ is non-trivial and independent, then the bi-extremal z is of minimal order.*

Here $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection from the cotangent subbundle of $\mathbb{R}^n$.

Theorem 10.1 is the polynomial version of Theorem 5.1. However, the method of proofs of these theorems is different. In the proofs of Theorem 5.1, Thom's transversality theorem (for instance [10]) is used. On the other hand, Tarski-Seidenberg theorem (for instance [7]) is used.

Outline of proof : Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be an m-tuple of even integers. Then, we will show the main theorem 4 by the following procedures:
[Step1] (See Definition 10.2) Construct the "bad set" with respect to minimal order, $B_{mo}(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$. Note that $B_{mo}(\boldsymbol{D})$ is semi-algebraic and in particular, dimension of $B_{mo}(\boldsymbol{D})$ is well-defined and in pariticular, its closure the dimension of $\overline{\pi(B_{mo}(\boldsymbol{D})}$ is well-defined, where $\pi : \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ is the canonical projection.
[Step2] (See Lemma 10.7) Show that, if $Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ satisfies the condition that any $x \in \mathbb{R}^n$, $(Q, x) \notin B_{mo}(\boldsymbol{D})$, if $z : [0, T] \to T^*\mathbb{R}^n$ is an Q-bi-extremal, and if the singular trajectory $\pi \circ z : [0, T] \to \mathbb{R}^n$ is non-trivial and independent, then z is of minimal order.
[Step3] (See Lemma 10.8) By using Tarski-Seidenberg theorem (for instance see [10]), Compute the codimension of $\overline{\pi(B_{mo}(\boldsymbol{D}))}$ in $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ by the canonical projection $\pi : \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$
[Step4] (See the subsection §10.4) For $\min\{d_1, d_2, \cdots, d_m\} > 4n$, namely $\min\{d_1, d_2, \cdots, d_m\} \geqq 4n+2$, let H_0 be the set of $Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that Q is not included in the closure of the projection of $B_{mo}(\boldsymbol{D})$ in $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ by the canonical projection $\pi : \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$. Then, show that, H_0 is an open dense subset of $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ in the sense of Euclidean topology.

In order to show the main theorem 3, we prepare the subsections: §10.1 to §10.4. In §10.1, we define the bad set $B_{mo}(\boldsymbol{D})$ of $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ in Definition 10.2. In §10.2, in Lemma 10.5, 10.6, we describe some properties of rank of Goh matrix for any singular Q-bi-extremals if $Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ satisfies the condition that for any $x \in \mathbb{R}^n$, (Q, x) is included in complement of $B_{mo}(\boldsymbol{D})$ in $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$. In particular, the important Lemma 10.7 prepared for the proof of main theorem 1 can be immediately derived from the Lemmata 10.5, 10.6 : if $Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ satisfies the condition that , $(Q, x) \notin B_{mo}(\boldsymbol{D})$ for any $x \in \mathbb{R}^n$, then any singular X-bi-extremal with the non-trivial independent singular X-trajectory is of minimal order. In §10.3, we compute the codimension of $B_{mo}(\boldsymbol{D})$ in $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ in Lemma 10.8. In §10.4, by using these Lemmata 10.7, 10.8, we show the main theorem 1.

10.1 Construction of bad set

In this section, we construct the semi-algebraic set $B_{mo}(\boldsymbol{D})$, which is called the bad set with respect to minimal order for an m-tuple of even integers $(d_1, \cdots, d_m)$:
By using the elementary determinants, we define the bad set $B_{mo}(\boldsymbol{D})$ for an m-tuple of even integers $(d_1, \cdots, d_m)$:

Definition 10.2 Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be an m-tuple of even integers For an integer p, let $N_{p,\boldsymbol{D}}$ be the set of $(p+1)$-tuples $\overline{s} = (0, s_1, \cdots, s_p)$ in $\{0\} \times (\mathbb{N})^p$ such that there exists an integer $i\,(1 \leqq i \leqq m)$ such that $s_1 + \cdots + s_p < d_i + p$. We define the "bad set" with respect to minimal order, $B_{mo}(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$ by the image of $\hat{B}_{mo}(\boldsymbol{D})$ by the canonical projection $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times T^*\mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$:

$$\hat{B}_{mo}(\boldsymbol{D}) = \{((Q,x),z) \,|\, x = \pi(z) \in \mathbb{R}^n,\, z \in T^*\mathbb{R}^n,\, ((Q,x),z) \in \hat{B}^0_{mo}(\boldsymbol{D}) \cup \hat{B}^1_{mo}(\boldsymbol{D})\},$$

Then, $B_{mo}(\boldsymbol{D})$ is defined by the following:

$$B_{mo}(\boldsymbol{D}) = \{(Q,x) \,|\, x = \pi(z) \in \mathbb{R}^n,\, ((Q,x),z) \in \hat{B}^0_{mo}(\boldsymbol{D}) \cup \hat{B}^1_{mo}(\boldsymbol{D}) \text{ for some } z \in T^*\mathbb{R}^n\},$$

where $\hat{B}^0_{mo}(\boldsymbol{D}), \hat{B}^1_{mo}(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times T^*\mathbb{R}^n$ are written in Definition 10.3, 10.4 respectively:

Definition 10.3 Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be an m-tuple of even integers. If $m = 2$, then $\hat{B}^0_{mo}(\boldsymbol{D}) = \emptyset$. On the other hand, if $m \geqq 3$, then we define $\hat{B}^0_{mo}(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times T^*\mathbb{R}^n$ by the union of the sets $\hat{B}^0_{mo}(i, \sigma, r, \overline{s}, z)$ with integers $i(1 \leqq i \leqq m)$, $\sigma \in \mathfrak{S}_m$, even integers $r\,(0 \leqq r \leqq m-3)$, and $\overline{s} \in N_{p,d_i}$ with $0 \leqq p < m$. Here the definition of $\hat{B}^0_{mo}(i, \sigma, r, \overline{s}, z)$ is below:

For $z \in T^*\mathbb{R}^n$ with $\pi(z) = x \in \mathbb{R}^n$, $\sigma \in \mathfrak{S}_m$, an integer $i(1 \leqq i \leqq m)$, an even integer $r(0 \leqq r \leqq m-3)$, and $\overline{s} \in N_{p,d_i}$ and $p\,(0 \leqq p < m)$, let $\hat{B}^0_{mo}(i, \sigma, r, \overline{s}, z)$ be the set of $((Q,x),z) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times T^*\mathbb{R}^n$ such that:

1). $Q_1(x), \cdots, Q_m(x)$ are linearly independent;
2). $\Delta^r_0(\sigma, z) \neq 0$;
3). for every integer $j\,(0 \leqq j \leqq p)$,

(a) $\Delta^{r,r+1,\cdots,r+j}_{0,s_1,\cdots,s_j}(\sigma,z) \neq 0$;
(b) for every integer $k\,(r+j \leqq k \leqq m)$ and $s\,(1 \leqq s \leqq s_j - 1)$,

$$\Delta^{r,r+1,\cdots,r+j-1,k}_{0,s_1,\cdots,s_{j-1},s}(\sigma,z) = 0;$$

4). for every $k\,(r+p+1 \leqq k \leqq m)$ and $s\,(1 \leqq s \leqq d_i + p - (s_1 + \cdots + s_p)$,

$$\Delta^{r,r+1,\cdots,r+p,k}_{s_1,\cdots,s_p,s}(\sigma,z) = 0.$$

Definition 10.4 Let $\boldsymbol{D} = (d_1,\cdots,d_m)$ be an m-tuple of even integers. If m is an odd integer, then $\hat{B}^1_{mo}(\boldsymbol{D}) = \emptyset$. On the other hand, if $m \geqq 2$ is an even integer, then we define $\hat{B}^1_{mo}(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times T^*\mathbb{R}^n$ by the union of the sets $\hat{B}^1_{mo}(i,\sigma,s_1,z)$ with integers $i\,(1 \leqq i \leqq m)$, $\sigma \in \mathfrak{S}_m$, and integers $s_1\,(1 \leqq s_1 \leqq d)$. Here the definition of $\hat{B}^1_{mo}(i,\sigma,s_1,z)$ is below:
Let $m \geqq 2$ be an even integer. For an integer $i\,(1 \leqq i \leqq m)$, $\sigma \in \mathfrak{S}_m$ and an integer $s_1(1 \leqq s_1 \leqq d)$, we define $\hat{B}^1_{mo}(i,\sigma,s_1,z)$ by the set of elements $((Q,x),z) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times T^*\mathbb{R}^n$ such that:

1). $Q_1(x),\cdots,Q_m(x)$ are linearly independent;
2). $\Delta^{m-2}_0(\sigma,z) \neq 0$;
3). (a) if $s_1 < d_i$, then $\delta^{m-1}_{s_1}(\sigma,z) \neq 0$;
(b) for $k \in \{m-1,m\}$ and $s\,(0 \leqq s \leqq s_1 - 1), \delta^k_s(\sigma,z) = 0$;
4). for $s\,(1 \leqq s \leqq d_i - s_1), \delta_{s_1,s}(\sigma,z) = 0$.

10.2 The property of singular bi-extremals avoiding bad set

We describe some properties of rank of Goh matrix for any singular Q-bi-extremals with the non-trivial and independent singular Q-trajectory if $Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ satisfies the condition that for any $x \in \mathbb{R}^n$, (Q,x) is included in complement of $B_{mo}(\boldsymbol{D})$ in $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$. In particular, we will show the important Lemma 10.7 prepare10-d for the proof of the main theorem 1 by using the Lemmata 5.7, 10.6.

Lemma 10.5 *Suppose that, $2 \leqq m \leqq n$. Let $\boldsymbol{D} = (d_1,\cdots,d_m)$ be an m-tuple of even integers. Let $Q \in \mathrm{VF}(M)^m$ such that for any $x \in \mathbb{R}^n$, $(Q,x) \notin B_{mo}(\boldsymbol{D})$. Then, if $z : [0,T] \to T^*\mathbb{R}^n$ is an Q-bi-extremal and if the singular Q-trajectory $\pi \circ z : [0,T] \to \mathbb{R}^n$ is non-trivial and independent, then*

$$m - 2 \leqq \mathrm{rank}\ G(z(t)) \leqq m - 1, \ \text{ for } a.e.\, t \in [0,T].$$

Here $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection from cotangent bundle of $\mathbb{R}^n$.

Proof: Since $(d_1,\cdots,d_m)$ is an m-tuple of even integers, there exists an integer $i\,(1 \leqq i \leqq m)$ such that $2d_i = \min\{d_1,\cdots,d_m\}$.

By the assumption that $\pi \circ z$ is non-trivial and by Proposition 4.1 (1), we have the inequality rank $G(z(t)) \leqq m - 1$ for a.e. $t \in [0,T]$. Therefore it suffices to show the inequality $m - 2 \leqq$ rank $G(z(t))$ for a.e. $t \in [0,T]$.

In $m = 2$ case, Lemma 10.5 clearly holds. We consider $m \geqq 3$. In order to prove by contradiction, assume that, $z : [0, T] \to T^*\mathbb{R}^n$ is an Q-bi-extremal and the singular Q-trajectory $\pi \circ z : [0, T] \to \mathbb{R}^n$ is non-trivial and independent but there exists a measurable subset of positive measure $K \subset [0, T]$ such that

$$\operatorname{rank} G(z(t)) \leqq m - 3 \text{ for every } t \in K.$$

Since $\pi \circ z$ is independent, the compliment of $I_{\text{dep}}(\pi \circ z)$, $I_{\text{indep}}(\pi \circ z)$ has positive measure. Then, let $J = I_1 \cap I_{\text{indep}}(\pi \circ z)$. J has positive measure and for every $t \in J$, $Q_1((\pi \circ z)(t)), \cdots, Q_m((\pi \circ z)(t))$ are linearly independent over $\mathbb{R}$.

After this, in the same way as the proof of Lemma 3.8 in [8], we can prove this Lemma 10.5 :

In fact, for an integer p $(1 \leqq p \leqq m)$, we denote by I_p a subset of $\{1, \cdots, m\}$ with cardinality p. Let r be the maximum of even integers p $(1 \leqq p \leqq m - 3)$ such that there exists a subset $I_p \subset \{1, \cdots, m\}$ satisfying $\det (H_{i,j}(z(t)))_{(i,j) \in I_p^2} \not\equiv 0$ on J. Let J_r be the set of $t \in J$ such that there exists $I_r \subset \{1, \cdots, m\}$ satisfying $(H_{i,j}(z(t))_{(i,j) \in I_r^2} \neq 0$. Note J_r has positive measure. Therefore there exists $\sigma \in \mathfrak{S}$ such that

$$\Delta_0^r(\sigma, z(t)) \neq 0 \text{ for every } t \in J_r.$$

On the other hand, for any subset $I_{r+2} \subset \{1, \cdots, m\}$, $\det (H_{i,j}(z(t))_{(i,j) \in I_{r+2}^2} \equiv 0$ on J_r and $\operatorname{rank} (H_{i,j}(z(t))_{(i,j) \in I_{r+2}^2} \leqq r$ on J_r. Therefore for any subset $I_{r+1} \subset I_{r+2}$, $\det (H_{i,j}(z(t))_{(i,j) \in I_{r+1}^2} = 0$ on J_r. In particular, for $k = r + 1, \cdots, m$,

$$\Delta_{0,0}^{r,k}(\sigma, z(t)) = 0 \text{ for every } t \in J_r.$$

By differentiating both sides, for $k = r + 1, \cdots, m$,

$$\sum_{i=1}^{m} u_i(t) \{\Delta_{0,0}^{r,k}, H_i\}(\sigma, z(t)) = 0 \text{ for a.e. } t \in J_r.$$

Here, we denote by $G_0(\sigma, z(t))$ the following $m \times m$- matrix:

$$\begin{pmatrix} H_{11}(\sigma, z(t)) & \cdots & H_{1m}(\sigma, z(t)) \\ \vdots & & \vdots \\ H_{r1}(\sigma, z(t)) & \cdots & H_{rm}(\sigma, z(t)) \\ \{\Delta_{0,0}^{r,r+1}, H_1\}(\sigma, z(t)) & \cdots & \{\Delta_{0,0}^{r,r+1}, H_m\}(\sigma, z(t)) \\ \vdots & & \vdots \\ \{\Delta_{0,0}^{r,m}, H_1\}(\sigma, z(t)) & \cdots & \{\Delta_{0,0}^{r,m}, H_m\}(\sigma, z(t)) \end{pmatrix}.$$

Then $G_0(\sigma, z(t)) \not\equiv 0$ on J_r. Note that, the first diagonal minors of order r of $G_0(\sigma, z(t))$ is $\Delta_0^r(\sigma, z(t))$,which never vanishes on J_r. and by definition, the diagonal minors of order $r + 1$ containing $\Delta_0^r(\sigma, z(t))$ are $\Delta_{0,1}^{r,k}(\sigma, z(t))$, $k = r + 1, \cdots, m$.

Claim. There exists $k_1 \in \{1, \cdots, m\}$, an integer $s_1 (1 \leqq s_1 < d+1)$, and a subset $J_{r+1} \subset J_r$ of positive measure such that

$$\begin{cases} \Delta^{r,k}_{0,\ell}(\sigma, z(t)) \equiv 0 \text{ on } J_{r+1}, \text{ for any integers } k, \ell \ (r+1 \leqq k \leqq m,\ 0 \leqq \ell \leqq s_1 - 1); \\ \Delta^{r,k_1}_{0,s_1}(\sigma, z(t)) \neq 0 \text{ for every } t \in J_{r+1}. \end{cases}$$

In fact, assume the claim is false. Then for every $k\,(r+1 \leqq k \leqq m)$ $\Delta^{r,k}_{0,1}(\sigma, z(t)) = 0$ on J_r. We consider the matrix $G_1(\sigma, z(t))$ obtained by replacing the last $m - r$ rows of $G_0(\sigma, z(t))$ with rows

$$(\{\Delta^{r,k}_{0,0}, H_\ell\}(\sigma, z(t)))_{1 \leqq \ell \leqq m}, \text{ for } k \ (1 \leqq k \leqq m).$$

By construction, $\det G_1(\sigma, z(t)) \equiv 0$ on J_r. The contradiction assumption implies that, for $k\,(r+1 \leqq k \leqq m)$, $\Delta^{r,k}_{0,2}(\sigma, z(t))) \equiv 0$ of J_r. Proceeding similarly, there exists $t \in J_r$ such that $(Q, (\pi \circ z)(t)$ belongs to $B(i, \sigma, r, 0, z(t))$ for every integer $i\,(1 \leqq i \leqq m)$. This contradicts the assumption that, for any $x \in \mathbb{R}^n$, $(Q, x) \in B_{mo}(\boldsymbol{D})$. Thus the claim is proved.

Up to a permutation, assume $k_1 = r + 1$. We define a non-invertible matrix by replacing in G_0 :

$$\begin{cases} \text{the } (r+1)\text{-th line by } (\{\Delta^{r,r+1}_{0,s_1-1}, H_\ell\}(\sigma, z(t)))_{1 \leqq \ell \leqq m}; \\ \text{for } j\,(r+2 \leqq j \leqq m), \text{the } j\text{-th line by } (\{\Delta^{r,j}_{0,s_1-1}, H_\ell\}(\sigma, z(t)))_{1 \leqq \ell \leqq m}. \end{cases}$$

To this matrix is applied the previous reasoning on G_0. Thus, by a finite number of steps, we obtain that there exists subset $J_{m-1} \subset J$ of positive measure, and $\overline{s} = (0, s_1, \cdots, s_m)$ in N_{m-1,d_i}, such that

$$\begin{cases} \Delta^{r,\cdots,r+m-1}_{0,s_1,\cdots,s_{m-1}}(\sigma, z(t))) \neq 0 \text{ for every } t \in J_{m-1}; \\ \Delta^{r,\cdots,r+m-1,r+m}_{0,s_1,\cdots,s_{m-1},\ell}(\sigma, z(t))) \equiv 0 \text{ on } J_{m-1}, \ell \geqq 0. \end{cases}$$

As a consequence, for every $t \in J_{m-1}, (Q, (\pi \circ z)(t)$ belongs to $B(i, \sigma, r, \overline{s}, z(t))$. This contradicts the assumption that, for any $x \in \mathbb{R}^n$, $(Q, x) \in B_{mo}(\boldsymbol{D})$. □

Lemma 10.6 *Suppose that, m is even and $2 \leqq m \leqq n$. Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be a pair of even integers. Let $Q \in \mathrm{VF}(\mathbb{R}^n)^m$ such that for any $x \in \mathbb{R}^n$, $(Q, x) \notin B_{mo}(\boldsymbol{D})$. Then, if $z : [0, T] \to T^*\mathbb{R}^n$ is a singular X-bi-extremal and if the singular X-trajectory $\pi \circ z : [0, T] \to \mathbb{R}^n$ is non-trivial and independent, then*

$$\operatorname{rank} \hat{G}(z(t)) = m - 1 \text{ for } a.e.\, t \in [0, T].$$

Proof: Since $(d_1, \cdots, d_m)$ is an m-tuple of even integers, there exists an integer $i\,(1 \leqq i \leqq m)$ such that $2d_i = \min\{d_1, \cdots, d_m\}$.

By the assumption that $\pi \circ z$ is non-trivial and by Proposition 4.1 (2), we have the inequality rank $G(z(t)) \leqq m-1$ for a.e. $t \in [0,T]$. Therefore we will show the inequality rank $G(z(t)) \geqq m-1$ for a.e. $t \in [0,T]$.

In order to prove by contradiction, assume that, $z : [0,T] \to T^*\mathbb{R}^n$ is an Q-bi-extremal and the singular Q-trajectory $\pi \circ z : [0,T] \to \mathbb{R}^n$ is non-trivial and independent but there exists a measurable subset of positive measure $K \subset [0,T]$ such that

$$\operatorname{rank} \hat{G}(z(t)) \leqq m-2 \text{ for every } t \in K.$$

Since $\pi \circ z$ is independent, $I_{\text{indep}}(\pi \circ z)$ has positive measure, where $I_{\text{indep}}(\pi \circ z)$ is the compliment of $I_{\text{dep}}(\pi \circ z)$ in $[0,T]$. When let $J = I_1 \cap I_{\text{indep}}(\pi \circ z)$, $J \subset [0,T]$ has positive measure and for any $t \in J$, $Q_1((\pi \circ z)(t)), \cdots, Q_m((\pi \circ z)(t))$ are linearly independent over $\mathbb{R}^n$. After this, in the same way as the proof of Lemma 3.8 in [8], we can prove this Lemma 10.6:
In fact, By the previous proof, we may assume that there exists $\sigma \in \mathfrak{S}_m$ such that

$$\Delta_0^{m-2}(\sigma, z(t)) \neq 0 \text{ for every } t \in J.$$

In particular, rank $\mathrm{G}(z(t)) = m-2$. Moreover, for $k = m-1$ and $k = m$,

$$\delta_0^k(\sigma, z(t)) = 0 \text{ and } \delta_1^k(\sigma, z(t)) = 0 \text{ for every } t \in J_r.$$

Similarly to the argument of Lemma 3.8, we claim that there exists an integer s_1 ($1 \leqq s_1 < d_i$), and $k_1 \in \{m-1, m\}$, such that

$$\delta_{s_1}^{k_1}(\sigma, z(t)) \neq 0 \text{ for every } t \in J.$$

In fact, otherwise, for s ($0 \leqq s \leqq d_i$) and for $k \in \{m-1, m\}$, $\delta_s^k(\sigma, z(t)) = 0$. In that case, for every $t \in J, (Q, \pi \circ z(t))$ belongs to $B^1(i, \sigma, d, z(t))$. This contradicts for any $x \in \mathbb{R}^n$, $(Q, x) \in B_{mo}(\boldsymbol{D})$.

Up to a permutation, we may assume $k_1 = m-1$. Let $J_1 \subset J$ be a subset of positive measure such that

$$\delta_{s_1}^{m-1}(\sigma, z(t)) \neq 0 \text{ for every } t \in J_1.$$

Similarly to the argument of Lemma 3.8, for every $s \geqq 0$, we have

$$\delta_{s_1,s}(\sigma, z(t)) = 0 \text{ for } t \in J_r.$$

Then, for every $t \in J_1$, $j^N_{\pi \circ z(t)} X$ belongs to $B^1(i, \sigma, s_1, z(t))$. This contradicts for any $x \in \mathbb{R}^n$, $(Q, x) \in B_{mo}(\boldsymbol{D})$. □

Lemma 10.7 *Suppose that, $2 \leqq m \leqq n$. Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be a pair of even integers. Let $Q \in \mathrm{VF}(\mathbb{R}^n)^m$ such that for any $x \in \mathbb{R}^n$, $(Q, x) \notin B_{mo}(\boldsymbol{D})$. Then, if $z : [0,T] \to T^*\mathbb{R}^n$ is a singular Q-bi-extremal and if the singular Q-trajectory $\pi \circ z : [0,T] \to \mathbb{R}^n$ is non-trivial and independent, then $z : [0,T] \to T^*\mathbb{R}^n$ is of minimal order.*

Proof: Since rank $G(z(t))$ is even, by Lemma 10.5 if m is odd, then $m-2$ can be replace with $m-1$ in the statement of Lemma 10.5:
If $m\,(2 \leqq m \leqq n)$ is odd, if $z : [0,T] \to T^*\mathbb{R}^n$ is a singular Q-bi-extremal and if $\pi \circ z : [0,T] \to \mathbb{R}^n$ is non-trivial and independent, then

$$\text{rank } G(z(t)) = m-1 \text{ for a.e.}\, t \in [0,T].$$

On the other hand, by Lemma 10.6, the following immediately holds:
If $m\,(2 \leqq m \leqq n)$ is even, if $z : [0,T] \to T^*\mathbb{R}^n$ is a singular Q-bi-extremal and if $\pi \circ z : [0,T] \to \mathbb{R}^n$ is non-trivial and independent, then

$$\text{rank } \hat{G}(z(t)) = m-1 \text{ for a.e.}\, t \in [0,T].$$

Therefore, these imply that for every integer $m\,(2 \leqq m \leqq n)$, z is of minimal order (see Definition 4.2). $\square$

10.3 Codimension of bad set

Lemma 10.8 *Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be a pair of even integers. Let d be an integer such that $2d = \min\{d_1, \cdots, d_m\}$. Then,*

$$\operatorname{codim}(\overline{\pi(B_{mo}(\boldsymbol{D}))}, \mathrm{VF}^{\boldsymbol{D}}_{\text{poly}}(\mathbb{R}^n)) \geqq d-n.$$

where, $\pi : \mathrm{VF}^{\boldsymbol{D}}_{\text{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\text{poly}}(\mathbb{R}^n)$ is the canonical projection.

Step1: Construct the typical fiber $G(\boldsymbol{D})$ of $\pi(B_{mo}(\boldsymbol{D}))$: The typical fiber $G(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{N}}_{\text{poly}}(\mathbb{R}^n)$ is constructed by the union of $G^0(\boldsymbol{D})$ and $G^1(\boldsymbol{D})$. Note that $G^0(\boldsymbol{D}), G^1(\boldsymbol{D})$ are semi-algebraic sets for $\boldsymbol{D}$. Therefore $G(\boldsymbol{D}) = G^0(\boldsymbol{D}) \cup G^1(\boldsymbol{D})$ and its closure $\overline{G(\boldsymbol{D})}$ are semi-algebraic also. In particular dimensions of $G^0(\boldsymbol{D}), G^1(\boldsymbol{D}), G(\boldsymbol{D}), \overline{G(\boldsymbol{D})}$ are well-defined and we have that

$$\dim(\overline{G(\boldsymbol{D})}) = \dim(G(\boldsymbol{D})) = \max\{\dim(G^0(\boldsymbol{D})), \dim(G^1(\boldsymbol{D}))\}.$$

Moreover we have that the dimension of $\overline{\pi(B_{mo}(\boldsymbol{D}))}$ is well-defined and that they are equal.

The definition of $G^0(\boldsymbol{D}), G^1(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\text{poly}}(\mathbb{R}^n)$ is below:
(1) Construction of the trivial fiber $G^0(\boldsymbol{D})$ of $\pi(B^0_{mo}(\boldsymbol{D}))$: If $m = 2$, then $G^0(\boldsymbol{D}) = \emptyset$. If $m \geqq 3$, then $G^0(\boldsymbol{D})$ is the union of $\pi(G^0(d_i; T^*_0\mathbb{R}^n))$ with integers $i\,(1 \leqq i \leqq m)$ by $\mathrm{VF}^{\boldsymbol{D}}_{\text{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\text{poly}}(\mathbb{R}^n)$.

$G^0(d_i; T^*_0\mathbb{R}^n)$ is defined by the set of $(Q,p) = (Q_1, \cdots, Q_m, p) \in \mathrm{VF}^{\boldsymbol{D}}_{\text{poly}}(\mathbb{R}^n) \times \mathbb{R}^m$ such that there exist $\sigma \in \mathfrak{S}_m$, an even integer $r\,(0 \leqq r \leqq m-3)$, and $\overline{s} \in N_{q,d}$ with $q(0 \leqq q < m)$ such that (Q,p) satisfies the conditions 1) to 4):
1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;
2). $\Delta^r_0(\sigma, z_0) \neq 0$;
3). for every integer $j\,(0 \leqq j \leqq q)$, $\Delta^{r,r+1,\cdots,r+j}_{0,s_1,\cdots,s_j}(\sigma, z_0) \neq 0$.

4). (a) for every integers $j\,(0 \leqq j \leqq q)$, $k\,(r+j \leqq k \leqq m)$, and $s\,(1 \leqq s \leqq s_j - 1)$,

$$\Delta^{r,r+1,\cdots,r+j-1,k}_{0,s_1,\cdots,s_j-1,s}(\sigma, z_0) = 0;$$

(b) for every integer $k\,(r+p+1 \leqq k \leqq m)$ and $s\,(1 \leqq s \leqq d_i + q - (s_1 + \cdots + s_p))$,

$$\Delta^{r,r+1,\cdots,r+q,k}_{0,s_1,\cdots,s_q,s}(\sigma, z_0) = 0,$$

where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0, p)$.

(2) Construction of the trivial fiber $G^1(\boldsymbol{D})$ of $\pi(B^1_{mo}(\boldsymbol{D}))$: In m is an odd integer, then $G^1(\boldsymbol{D}) = \emptyset$. If m is an even integer, then $G^1(\boldsymbol{D})$ is the union of $\pi(G^1(d_i; T^*_0\mathbb{R}^n))$ with integers $i\,(1 \leqq i \leqq m)$ by $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$.

$G^1(d_i; T^*_0\mathbb{R}^n)$ is defined by the set of $(Q, p) = (Q_1, \cdots, Q_m, p) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^m$ such that there exist a positive integer $s_1\,(1 \leqq s_1 \leqq d_i)$ and $\sigma \in \mathfrak{S}_m$ such that (Q, p) satisfies the conditions 1) to 4):

1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;
2). $\Delta^{m-2}_0(\sigma, z_0) \neq 0$;
3). if $s_1 < d_i$, then $\delta^{m-1}_{s_1}(\sigma, z_0) \neq 0$;
4). (a) for every integer $k \in \{m-1, m\}$ and $s\,(0 \leqq s \leqq s_1 - 1)$, $\delta^k_s(\sigma, z_0) = 0$;
(b) for every integer $s\,(1 \leqq s \leqq d_i - s_1), \delta_{s_1,s}(\sigma, z_0) = 0$,

where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0, p)$.

Step 2: Construct the two mappings $\phi^0_i(\sigma, r, \overline{s})$ and $\phi^1_i(\sigma, s_1)$

(1) Construction of $\phi^0_i(\sigma, r, \overline{s})$: Let $i\,(1 \leqq i \leqq m)$ be a positive integer, $\sigma \in \mathfrak{S}_m$, $r\,(0 \leqq r \leqq m-3)$ be an even integer, and $\overline{s} = (0, s_1, \cdots, s_q) \in N_{q,d_i}$ with $0 \leqq q < m$. Then, we define the mapping $\phi^0_i(\sigma, r, \overline{s}) : \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n \to \mathbb{R}^d_i$ by for $(Q, p) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$,

$$\phi^0_i(\sigma, r, \overline{s})(Q, p) = \begin{cases} \Delta^{r,r+1,\cdots,r+j-1,r+j}_{0,s_1,\cdots,s_{j-1},\overline{s}}(Q)(\sigma, z_0) & j = 1, \cdots, q, \text{and } \overline{s} = 1, \cdots, s_j - 1, \\ \Delta^{r,r+1,\cdots,r+q,r+q+1}_{0,s_1,\cdots,s_q,\overline{s}}(Q)(\sigma, z_0) & \overline{s} = 1, 2, \cdots, d_i + q - (s_1 + \cdots + s_q), \end{cases}$$

where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0, p)$, and $\Delta^{r,r+1,\cdots,r+i-1,r+i}_{0,s_1,\cdots,s_{i-1},\overline{s}}(Q)$, $\Delta^{r,r+1,\cdots,r+q,r+q+1}_{0,s_1,\cdots,s_q,\overline{s}}(Q)$ are the elementary determinants associated to Q.

(2) Construction of $\phi^1_i(\sigma, s_1)$: Let $i\,(1 \leqq i \leqq m)$ be a positive integer, $\sigma \in \mathfrak{S}_m$ and $s_1\,(1 \leqq s_1 \leqq d_i)$. Then, we define the mapping $\phi^1_i(\sigma, s_1) : \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n \to \mathbb{R}^d_i$ by for $(Q, p) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$,

$$\phi^1_i(\sigma, s_1)(Q, p) = \begin{cases} \delta^m_s(Q)(\sigma, z_0), & s = 0, 1 \cdots, s_1 - 1; \\ \delta_{s_1,s}(Q)(\sigma, z_0), & s = 1, 2 \cdots, d_i - s_1, \end{cases}$$

where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0, p)$, and $\delta^m_s(Q), \delta_{s_1,s}(Q)$ are the elementary determinants associated to Q.

Step 3: Construct the two open sets $T^0_{i,\sigma,r,\overline{s}}, T^1_{i,\sigma,s_1} \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$:

(1) Construction of $T^0_{i,\sigma,r,\overline{s}}$: Let $i\,(1 \leqq i \leqq m)$ be a positive integer, $\sigma \in \mathfrak{S}_m$, $r\,(0 \leqq r \leqq m-3)$ be an even integer, and $\overline{s} = (0, s_1, \cdots, s_q) \in N_{q,d_i}$ with $0 \leqq q < m$.

Then, $T^0_{\sigma,r,\overline{s}}$ is defined by the set of $(Q,p) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$ such that (Q,p) satisfies the conditions 1) and 2):
1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;
2). for every integer $i\,(0 \leqq i \leqq p)$, $\Delta^{r,r+1,\cdots,r+i}_{0,s_1,\cdots,s_i}(\sigma, z_0) \neq 0$,
where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0,p)$.

(2) Construction of T^1_{i,σ,s_1} : Let $i\,(1 \leqq i \leqq m)$ be a positive integer, $\sigma \in \mathfrak{S}_m$ and $s_1\,(1 \leqq s_1 \leqq d_i)$. Then, T^1_{σ,s_1} is defined by the set of $(Q,p) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$ such that (Q,p) satisfies the conditions 1) to 3):

1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;

2). $\Delta^{m-2}_0(\sigma, z_0) \neq 0$;

3). if s_1 is $s_1 < d_i$, then $\delta^{m-1}_{s_1}(\sigma, z_0) \neq 0$,

where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0,p)$.

Then, $T^0_{i,\sigma,r,\overline{s}}$ and T^1_{i,σ,s_1} are open subsets of $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$.

Step 4: The followings immediately hold:

(1) If $m \geqq 3$ and $i\,(1 \leqq i \leqq m)$ is an integer, then $G^0(d_i; T^*_0\mathbb{R}^n)$ concides with the union of the kernels of the restriction to $T^0_{i,\sigma,r,\overline{s}}$ of the mapping $\phi^0_i(\sigma, r, \overline{s})$ with $\sigma \in \mathfrak{S}_m$, even integers $r\,(0 \leqq r \leqq m-3)$, and $\overline{s} \in N_{q,d}$ with $0 \leqq q < m$.

(2) If $m \geqq 2$ is an even integer and $i\,(1 \leqq i \leqq m)$ is an integer, then $G^1(d_i; T^*_0\mathbb{R}^n)$ concides with the union of the kernels of the restriction to T^1_{i,σ,s_1} of the mapping $\phi^1(\sigma, s_1)$ with $\sigma \in \mathfrak{S}_m$ and $s_1(1 \leqq s_1 \leqq d)$.

Step 5: Let Ω_0 be the set of $Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that $Q_1(0), \cdots, Q_m(0)$ are linearly independent. Let $i\,(1 \leqq i \leqq m)$ be an integer such that $2d_i = \min\{d_1, \cdots, d_m\}$. Then, the followings hold (see Lemma 3.5, 3.6 in [8]):

(1) If $m \geqq 3, \sigma \in \mathfrak{S}_m$, and $r\,(0 \leqq r \leqq m-3)$ is an even integer, then the restriction to the intersection $T^0_{i,\sigma,r,\overline{s}} \cap \hat{V}$ of the mapping $\phi^0_i(\sigma, r, \overline{s})$ is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0 \times \mathbb{R}^n$.

(2) If $m \geqq 2$ is even, $\sigma \in \mathfrak{S}_m$, and an integer s_1 satisfies $1 \leqq s_1 \leqq d$, then the restriction to the intersection $T^1_{i,\sigma,s_1} \cap \hat{V}$ of the mapping $\phi^1_i(\sigma, s_1)$ is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0 \times \mathbb{R}^n$.

Step 6: Let d be an integer $2d = \min\{d_1, \cdots, d_m\}$.Then,

$$\mathrm{codim}(\overline{\pi(B_{mo}(\boldsymbol{D}))}, \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)) \geqq d - n.$$

where, $\pi : \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$ is the canonical projection.
Let $k \in \{0, 1\}$. By step 4, 5, if j $(1 \leqq j \leqq m)$ is a positive integer, then

$$\mathrm{codim}(G^k(d; T_0^*\mathbb{R}^n), \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n) \times \mathbb{R}^n) = d.$$

By Tarski-Seidenberg theorem for $\pi : \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$,

$$\mathrm{codim}(G^k(\boldsymbol{D}), \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)) = \mathrm{codim}(\pi(G^k(d; T_0^*\mathbb{R}^n), \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)) \geqq d - n.$$

Since $G(\boldsymbol{D}) = G^0(\boldsymbol{D}) \cup G^1(\boldsymbol{D})$,

$$\mathrm{codim}(G(\boldsymbol{D}), \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)) = \min\{\mathrm{codim}(G^k(\boldsymbol{D}), \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n) \times \mathbb{R}^n) | k = 0, 1\}$$

Therefore,since $\mathrm{codim}(G(\boldsymbol{D}), \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)) \geqq d - n$,

$$\begin{aligned} \mathrm{codim}(\pi(B_{mo}(\boldsymbol{D}), \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)) &\geqq \mathrm{codim}(B_{mo}(\boldsymbol{D}), \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n) \times \mathbb{R}^n) - n \\ &= \mathrm{codim}(G(\boldsymbol{D}), \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)) - n \\ &\geqq d - 2n. \end{aligned}$$

Therefore, $\mathrm{codim}(\overline{\pi(B_{mo}(\boldsymbol{D})}, \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)) \geqq d - 2n$. □

10.4 Proof of main theorem 4

It is well-known that if $B \subset \mathbb{R}^K$ is a semi-algebraic, then the complement of B in $\mathbb{R}^K$ is dense if and only if $\dim(\mathbb{R}^K, B) > 0$. In particular, the complement of the clousure of B in $\mathbb{R}^K$, $\mathbb{R}^K \setminus \overline{B}$ is open dense subset of $\mathbb{R}^K$.

Let $d > 2n$ be an integer such that $2d = \min\{D_1, D_2, \cdots, D_m\} (> 4n)$. Let H_0 be the set of $Q \in \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$ such that Q is not included in the closure of the projection of $B_{mo}(\boldsymbol{D})$ in $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$ by the canonical projection $\pi : \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$, namely,

$$H_0 = \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n) \setminus \overline{\pi(B_{mo}(\boldsymbol{D}))}.$$

Since $\pi(B_{mo}(\boldsymbol{D})) \subset \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$ is semi-algebraic and

$$\mathrm{codim}(\overline{\pi(B_{mo}(\boldsymbol{D}))}, \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)) \geqq d - 2n > 0,$$

Therefore, H_0 is an open dense subset of $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$ in the sense of Euclidean topology.

11 Singular controls of generic polynomial driftless control-affine systems

Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ denote an m-tuple of integers, and $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ denote the product space of m-tuple systems of polynomial vector fields over $\mathbb{R}^n$, $Q = (Q_1, \cdots, Q_m)$, such that the degree of Q_i satisfiies $\deg Q_i \leqq d_i$ for every integer $i\,(1 \leqq i \leqq m)$, and we endow $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ with the Euclidean topology.

Let Ω be an open subset of $\mathbb{R}^m$. For $Q = (Q_1, \cdots, Q_m) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$, consider the polynomial driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i\, Q_i(x)$$

with the control parameter $(u_1, \cdots, u_m) \in \Omega$. Then, the following holds:

Theorem 11.1 *Suppose $2 \leqq m \leqq n$ and suppose that, an m-tuple of even integers $\boldsymbol{D} = (d_1, \cdots, d_m)$ satisfies the inequality : $\min\{d_1, d_2, \cdots, d_m\} \geqq 4n+2$. Then, there exists an open dense semi-algebraic subset $H_2 \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that, if $Q \in H_2$, if $u : [0,T] \to \Omega$ is a singular Q-control for a given initial point $x_0 \in \mathbb{R}^n$, and if the corresponding singular trajectory $x : [0,T] \to \mathbb{R}^n, x(0) = x_0$ is non-trivial and independent, then the control u is of corank one.*

Theorem 11.1 is the polynomial version of Theorem 6.1. However, the method of proofs of these theorems is different. In the proofs of Theorem 5.1, Thom's transversality theorem (for instance [10]) is used. On the other hand, Tarski-Seidenberg theorem (for instance [7]) is used.

Outline of proof : Let $k' \leqq 1$ be an integer. We set such that $k' = 2k - 1$. Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be an m-tuple of even integers $2k = k' + 1$. Then, we will show the main theorem 4 by the following procedures:

[Step1] (See Definition 11.5) Construct the "bad set" with respect to corank one, $B_C(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$. Note that $B_C(\boldsymbol{D})$ is semi-algebraic and in particular, by Tarski-Seidenberg theorem, $\pi(B_C(\boldsymbol{D}))$ is semi-algebraic by $\pi : \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$. Dimensions of $\pi(B_C(\boldsymbol{D}))$ and its closure $\overline{\pi(B_C(\boldsymbol{D}))}$ are well-defined.

[Step2] (See Lemma 11.7) Show that, if $X \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ satisfies the condition that any $x \in \mathbb{R}^n, (Q,x) \notin B_C(\boldsymbol{D})$, if $u : [0,T] \to \Omega$ is a singular Q-control for a given initial point $x_0 \in \mathbb{R}^n$, and if the corresponding singular trajectory $x : [0,T] \to \mathbb{R}^n, x(0) = x_0$ is non-trivial and independent, then the control u is of corank one.

[Step3] (See Lemma 11.8) By using Tarski-Seidenberg theorem (for instance see [10]), Compute the codimension of $\overline{B_C(\boldsymbol{D})}$ in $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$.

[Step4] (See the subsection §11.4) For $\min\{d_1, d_2, \cdots, d_m\} > 4n$, namely $\min\{d_1, d_2, \cdots, d_m\} \geqq 4n+2$, let H_0 be the set of $Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that (Q,x) is not included in the closure of $B_C(\boldsymbol{D})$ in $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$. Then, show that, H_0 is an open dense subset of $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ in the sense of Euclidean topology.

In order to show the main theorem 3, we prepare the subsections: §11.1 to §11.4. In §11.1, after preparing the notations, permutated Hamiltonians, Goh matrices and elementary determinants in Definition 5.2, 5.3, we define the bad set $B_C(d)$ of $J^N(\mathrm{VF}(M))^m$ in Definition 5.4. In §11.2, in Lemma 5.7, 5.8, we describe some properties of rank of Goh matrix for any singular X-bi-extremals if $X \in \mathrm{VF}(M)^m$ satisfies the condition that for any $x \in M$, $j_x^N X$ is included in complement of $B_C(d)$ in $J^N(\mathrm{VF}(M))^m$. In particular, the important Lemma 5.9 prepared for the proof of main theorem 1 can be immediately derived from the Lemmata 5.7, 5.8 : if $X \in \mathrm{VF}(M)^m$ satisfies the condition that , $j_x^N X \notin B_C(d)$ for any $x \in M$, then any singular X-bi-extremal with the non-trivial independent singular X-trajectory is of minimal order. In §11.3, we compute the codimension of $B_C(d)$ in $J^N(\mathrm{VF}(M))^m$ in Lemma 5.10. In §11.4, by using these Lemmata 5.9, 5.10, we show the main theorem 1.

11.1 Construction of bad set

Let $\mathfrak{S}_m$ be the set of permutations with m elements, and $X = (X_1, \cdots, X_m) \in \mathrm{VF}(M)^m$. Then, we prepare some notations, permutated Hamiltonians, extended Lie derivatives and elementary determinants, in order to define the bad set $B_C(d) \subset J^N(\mathrm{VF}(M))^m$ for an integer d.

Definition 11.2 For $k \in \{1, 2\}$, we define the real valued functions $H_{ij}^{[k]}, \Delta_0^{[k],r}, P^{[k]}, \delta_s^{[k],i}$ on $\mathfrak{S}_m \times T^*M \times_M T^*M$, which are called *permutated Hamiltonians*:
Let $\sigma \in \mathfrak{S}_m$ and $(z^{[1]}, z^{[2]}) \in T^*M \times_M T^*M$. Then,

$$
\begin{aligned}
&H_{ij}^{[k]}(\sigma, z^{[1]}, z^{[2]}) = H_{ij}(\sigma, z^{[k]}) \text{ for } i, j \; (1 \leqq i, j \leqq m)\\
&\Delta_0^{[k],r}(\sigma, z^{[1]}, z^{[2]}) = \Delta_0^r(\sigma, z^{[k]}) \text{ for every integer } r \; (0 \leqq r \leqq m-1)\\
&P^{[k]}(\sigma, z^{[1]}, z^{[2]}) = P(z^{[k]}), \text{and } P^{[k],m-2}(\sigma, z^{[1]}, z^{[2]}) = P^{m-2}(z^{[k]})\\
&\delta_s^{[k],i}(\sigma, z^{[1]}, z^{[2]}) = \delta_s^i(\sigma, z), \text{ for every integer } s \geqq 0 \text{ and } i \in \{m-1, m\},
\end{aligned}
$$

where P (resp. P^{m-2}) is defined by

$$
P(\sigma, z) = P\left((H_{ij}(\sigma, z))_{1 \leqq i<j \leqq m}\right) \; \left(\text{resp. } P^{m-2}(\sigma, z) = P\left((H_{ij}(\sigma, z))_{1 \leqq i<j \leqq m-2}\right)\right),
$$

for $\sigma \in \mathfrak{S}, z \in T^*M$ by using the Pfaffian polynomial of G (resp. G^{m-2}) .

Definition 11.3 Let $\mathcal{F}_\bullet$ be the set of $F \in C^\infty(T^*M \times_M T^*M)$ such that there exists $F_1, F_2 \in C^\infty(T^*M)$ such that $F(z^{[1]}, z^{[2]}) = F_1(z^{[1]})F_2(z^{[2]})$ for $(z^{[1]}, z^{[2]}) \in T^*M \times_M T^*M$. Let $\mathcal{F}$ be the set of $F \in C^\infty(T^*M \times_M T^*M)$ such that F is a linear combination of a finite number of elements of $\mathcal{F}_\bullet$ on $\mathbb{R}$. Then, for the Hamiltonian vector field $\overrightarrow{H}$ of $X \in \mathrm{VF}(M)$, we define *an extended Lie derivative* $\mathcal{L}_{\overrightarrow{H_j}} : \mathcal{F} \to \mathcal{F}$ by the following: For $F \in \mathcal{F}_\bullet$ and $(z^{[1]}, z^{[2]}) \in T^*M \times_M T^*M$,

$$
\mathcal{L}_{\overrightarrow{H}}(F)(z^{[1]}, z^{[2]}) = \mathcal{L}_{\overrightarrow{H}}(F_1)(z^{[1]})\, F_2(z^{[2]}) + F_1(z^{[1]})\, \mathcal{L}_{\overrightarrow{H}}(F_2)(z^{[2]}),
$$

and extend it by linearly to $\mathcal{F}$. Here, $\mathcal{L}_{\overrightarrow{H}}(F_k)$ is the Lie derivative of F_k with respect to $\overrightarrow{H}$ for $k \in \{1, 2\}$.

Definition 11.4 We inductively define the real valued functions on $\mathfrak{S}_m \times T^*M \times_M T^*M$, which are called elementary determinants: Let $\sigma \in \mathfrak{S}_m$ and $(z^{[1]}, z^{[2]}) \in T^*M \times_M T^*M$. Then

$$\begin{cases} \Theta_{s+1}(\sigma, z^{[1]}, z^{[2]}) = \det \left(\begin{array}{c} \left(H^{[1]}_{ij}(\sigma, z^{[1]}, z^{[2]}) \right)_{\substack{1 \leqq i \leqq m-1, \\ 1 \leqq j \leqq m}} \\ \hline \left(\mathcal{L}_{\overrightarrow{H_j}} \Theta_s(\sigma, z^{[1]}, z^{[2]}) \right)_{1 \leqq j \leqq m} \end{array} \right) \quad (s = 0, 1, \cdots), \\ \Theta_0(\sigma, z^{[1]}, z^{[2]}) = \det \left(\begin{array}{c} \left(H^{[1]}_{ij}(\sigma, z^{[1]}, z^{[2]}) \right)_{\substack{1 \leqq i \leqq m-1, \\ 1 \leqq j \leqq m}} \\ \hline \left(H^{[2]}_{ij}(\sigma, z^{[1]}, z^{[2]}) \right)_{1 \leqq j \leqq m} \end{array} \right), \end{cases}$$

and

$$\begin{cases} \theta_s(\sigma, z^{[1]}, z^{[2]}) = \det \left(\begin{array}{c} \left(H^{[1]}_{ij}(\sigma, z^{[1]}, z^{[2]}) \right)_{\substack{1 \leqq i \leqq m-1, \\ 1 \leqq j \leqq m}} \\ \hline \left(\{P^{[1]}, H^{[1]}_j\}(\sigma, z^{[1]}, z^{[2]}) \right)_{1 \leqq j \leqq m} \\ \hline \left(\mathcal{L}_{\overrightarrow{H_j}} \theta_s(\sigma, z^{[1]}, z^{[2]}) \right)_{1 \leqq j \leqq m} \end{array} \right) \quad (s = 0, 1, \cdots) \\ \theta_0(\sigma, z^{[1]}, z^{[2]}) = \det \left(\begin{array}{c} \left(H^{[1]}_{ij}(\sigma, z^{[1]}, z^{[2]}) \right)_{\substack{1 \leqq i \leqq m-1, \\ 1 \leqq j \leqq m}} \\ \hline \left(\{P^{[1]}, H^{[1]}_j\}(\sigma, z^{[1]}, z^{[2]}) \right)_{1 \leqq j \leqq m} \\ \hline \left(\{P^{[12]}, H^{[2]}_j\}(\sigma, z^{[1]}, z^{[2]}) \right)_{1 \leqq j \leqq m} \end{array} \right), \end{cases}$$

Now, we construct the semi-algebraic set $B_C(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$, which is called the bad set with respect to minimal order for an m-tuple of integers $\boldsymbol{D} = (d_1, \cdots, d_m)$:

Definition 11.5 Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be an m-tuple of integers such that $d_i \geqq 2$ for every integer $i\,(1 \leqq i \leqq m)$. We define the "bad set" with respect to corank one, $B_C(\boldsymbol{D})$ by the canonical projection of $\hat{B}_C(\boldsymbol{D})$ by $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times T^*\mathbb{R}^n \times_{\mathbb{R}^n} T^*\mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$. The definition of $\hat{B}_C(\boldsymbol{D})$ is written in Definition 11.6:

Definition 11.6 Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be an m-tuple of integers such that $d_i \geqq 2$ for every integer $i\,(1 \leqq i \leqq m)$. Let $d = \min\{d_1, \cdots, d_m\} - 1$. We define $\hat{B}_C(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times T^*\mathbb{R}^n \times_{\mathbb{R}^n} T^*\mathbb{R}^n$ by the union of the sets $B_C(d, \sigma)$ with $\sigma \in \mathfrak{S}_m$. Here the definition of $B_C(d, \sigma)$ is below:
For $\sigma \in \mathfrak{S}_m$, let $B_C(d, \sigma)$ be the subset of $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times_M T^*\mathbb{R}^n \times_M T^*\mathbb{R}^n$ of all triples $((Q, x), z^{[1]}, z^{[2]})$ such that $q = \pi(z^{[1]}) = \pi(z^{[2]})$:

1). $X_1(x), \cdots, X_m(x)$ are linearly independent;

2). $z^{[1]}, z^{[2]}$ are linearly independent;

3). $\begin{cases} \text{if } m \text{ is odd, then } \Delta_0^{m-1}(\sigma, z^{[1]} \neq 0, \\ \text{if } m \text{ is even, then } \delta_1^{m-1}(\sigma, z^{[1]}) P^{m-2}(\sigma, z^{[2]}) \neq 0; \end{cases}$

4). for $s\,(0 \leqq s \leqq d-1)$, $\begin{cases} \text{if } m \text{ is odd, then } \Theta_s(\sigma, z^{[1]}, z^{[2]}) = 0, \\ \text{if } m \text{ is even, then } \theta_s(\sigma, z^{[1]}, z^{[2]}) = 0. \end{cases}$

11.2 Property of singular controls avoiding the bad set with respect to corank one.

We show the important Lemma 11.7 prepared for the proof of the main theorem 4.

Lemma 11.7 *Suppose that $2 \leqq m \leqq n$. Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be an m-tuple of even integers such that $d_i \geqq 2$ for every integer $i\,(1 \leqq i \leqq m)$. Let $Q \in VF^{\boldsymbol{D}}_{poly}(\mathbb{R}^n)$ such that for any $x \in \mathbb{R}^n$, $(Q, x) \notin B_{mo}(\boldsymbol{D}) \cup B_C(\boldsymbol{D})$. Then, if $u : [0,T] \to \Omega$ is a singular Q-control for a given initial point $x_0 \in \mathbb{R}^n$ and if the corresponding Q-trajectory $x : [0,T] \to \mathbb{R}^n, x(0) = x_0$ is non-trivial and independent, then the control u is of corank one.*

Proof: Since $(d_1, \cdots, d_m)$ is an m-tuple of even integers, there exists an integer $i\,(1 \leqq i \leqq m)$ such that $2d_i = \min\{d_1, \cdots, d_m\}$. Let $d' = 2d_i - 1$.

In order to prove by contradiction, assume that, $u : [0,T] \to \Omega$ is a singular X-control for a given initial point $x_0 \in \mathbb{R}^n$ and $x : [0,T] \to \mathbb{R}^n, x(0) = x_0$ is non-trivial and independent but the control u is not of corank one. Then by Proposition 4.4, there exists two bi-extremals $z^{[1]}, z^{[2]} : [0,T] \to T^*\mathbb{R}^n$ with $z^{[1]} \circ \pi = z^{[2]} \circ \pi = x$ such that $z^{[1]}(t)$ and $z^{[2]}(t)$ are linearly independent for every $t \in [0,1]$.

Case 1 We consider the case m is odd. For $k = 1, 2$, we denote by $G^{[k]}$ the Goh matrix $G(z^{[k]}) = (H_{ij}(z^{[k]}))_{1 \leqq i,j \leqq m}$. Since $Q = (Q_1, \cdots, Q_m) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ is for any $x \in Q$, $j^N_x X \notin B_{mo}(d')$, by the proof of Lemma 5.7, there exists $I_{m-1} \subset \{1, \cdots, m\}$ with cardinality $m-1$ such that $\det\left(H_{ij}(z^{[1]}(t))\right)_{(i,j) \in I^2_{m-1}} \not\equiv 0$ on $[0,T]$. Then, up to a permutation, we may assume that there exist an open subinterval $K \subset [0,T]$ and $\sigma \in \mathfrak{S}_m$ such that

$$\Delta_0^{[1],m-1}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \text{ for every } t \in K.$$

On the other hand, since x is independent, $I_{\mathrm{indep}}(x)$ has positive measure, where $I_{\mathrm{indep}}(x)$ is the compliment of $I_{\mathrm{dep}}(x)$ in $[0,T]$. Then Let $J = K \cap I_{\mathrm{indep}}(x)$.

Now, since for $k = 1, 2$, $H_i(\sigma, z^{[k]}(t)) = 0$ on $[0,T]$, by differentiating both sides, for $i\,(1 \leqq i \leqq m)$,

$$\sum_{j=1}^{m} u_j(t)\, H_{ij}(\sigma, z^{[k]}(t)) = 0 \text{ a.e. } t \in [0,T].$$

Hence, the matrix

$$\begin{pmatrix} H_{11}(\sigma, z^{[1]}(t)) & \cdots & H_{1m}(\sigma, z^{[1]}(t)) \\ \vdots & & \vdots \\ H_{(m-1)1}(\sigma, z^{[1]}(t)) & \cdots & H_{(m-1)m}(\sigma, z^{[1]}(t)) \\ H_{11}(\sigma, z^{[2]}(t)) & \cdots & H_{1m}(\sigma, z^{[2]}(t)) \end{pmatrix}$$

is not invertible. Therefore $\Theta_0(\sigma, z^{[1]}(t), z^{[2]}(t)) \equiv 0$ on $[0, T]$. By differentiating both sides,

$$\sum_{j=1}^{m} u_j(t)\, \mathcal{L}_{\overrightarrow{H_j}} \Theta_0(\sigma, z^{[1]}(t), z^{[2]}(t)) = 0 \text{ a.e. } t \in [0, T].$$

This implies $\Theta_1 \equiv 0$. By proceeding similarly, for $k (0 \leqq k \leqq d-1)$.

$$\begin{cases} \Delta_0^{[1],m-1}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \\ \Theta_k(\sigma, z^{[1]}(t), z^{[2]}(t)) \equiv 0 \end{cases} \text{for every } t \in J.$$

This contradicts the assumption that, for any $x \in \mathbb{R}^n$, $(Q, x) \in B_C(d)$.

Case 2 We consider the case m is even. By proofs of Lemma 5.7, 5.8, up to permutation, there exists an subinterval $K \subset [0, T]$ such that

$$\begin{cases} P^{[1],m-2}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \\ \delta_1^{[1],m-1}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \end{cases} \text{for every } t \in K.$$

Then Let $J = K \cap I_{\text{indep}}(x)$.

We show that $P^{[2],m-2}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0$ for every $t \in J$. For $\alpha \in [0, T]$, consider $z^{\alpha} = (1-\alpha)z^{[1]} + \alpha z^{[2]}$. Since $P^{m-2}(z^{[\alpha]}(t))$ depends continuously on α, for α small enough, $P^{m-2}(z^{[\alpha]}(t)) = 0$ for every $t \in J$. Moreover, the set of singular X-bi-extremals of singular X-trajectory x is a vector space, $z^{[\alpha]}$ is singular X-bi-extremals of x, and for $\alpha > 0$, $z^{[\alpha]}(t)$ and $z^{[1]}(t)$ are linearly independent for every $t \in J$. Then, it suffices to replace $z^{[2]}$ by z^{α}, for some $\alpha > 0$ small enough.

Similar to the case m odd, for $k\,(0 \leqq k \leqq d-1)$,

$$\begin{cases} \delta_1^{[1],m-1}(\sigma, z^{[1]}(t), z^{[2]}(t))\, P^{[2],m-2}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \\ \theta_k(\sigma, z^{[1]}(t), z^{[2]}(t)) \equiv 0 \end{cases} \text{for every } t \in J.$$

This contradicts the assumption that, for any $x \in \mathbb{R}^n$, $(Q, x) \in B_C(\boldsymbol{D})$.

□

11.3 Codimension of bad set with respect to corank one.

Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be a pair of positive integers $d_i \geqq 2$ for every integer $i\,(1 \leqq i \leqq m)$. Let $d = \min\{d_1, \cdots, d_m\} - 1$. Then,

Lemma 11.8 $\mathrm{codim}(\overline{\pi(B_C(\boldsymbol{D}))}, J^N(\mathrm{VF}(M))^m) \geqq d - 2n$.

Proof: We describe only the outline of the proof of Lemma 6.8 because this bad set $B_C(\boldsymbol{D})$ is the completely same bad set $B_C(\boldsymbol{D})$ defined as 3.1.2 in [8]: Let $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ be the m-tuple product space of polynomial vector fields of degree $\leqq d_i$ over $\mathbb{R}^n$.

Step 1: Construct the typical fiber $G_C(\boldsymbol{D})$ of $\pi(B_C(\boldsymbol{D}))$:

Typical fiber $G_C(\boldsymbol{D})$ of $\pi(B_C(\boldsymbol{D}))$ is the canonical projection of $G_C(d; T^*_0\mathbb{R}^n \times T^*_0\mathbb{R}^n)$ by $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$. $G_C(d; T^*_0\mathbb{R}^n \times T^*_0\mathbb{R}^n)$ is defined by the set of $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that there exists $\sigma \in \mathfrak{S}_k$ such that $(Q, p^{[1]}, p^{[2]})$ satisfies the conditions 1) to 4):

1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;

2). $p^{[1]}, p^{[2]}$ are linearly independent;

3). (a) if m is odd, then $\Delta^{m-1}_0(\sigma, z^{[1]}_0) \neq 0$,

(b) if m is even, then $\delta^{m-1}_1(\sigma, z^{[1]}_0)P^{m-2}(\sigma, z^{[2]}_0)$;

4). (a) if m is odd, then $\Theta_s(\sigma, z^{[1]}_0, z^{[2]}_0)$ for every integer $s\,(0 \leqq s \leqq d-1)$,

(b) if m is even, then $\theta_s(\sigma, z^{[1]}_0, z^{[2]}_0)$ for every integer $s\,(0 \leqq s \leqq d-1)$,

where $z^{[1]}_0, z^{[2]}_1$ are the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0, p_1), (0, p_2)$.

Step 2: Construct the mapping $\phi_\sigma : \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$:

Let $\sigma \in \mathfrak{S}_k$. Then construct the mapping $\phi_{\sigma,V}$ by dividing two cases of m being odd and even:

Case1 If m be an odd integer, then we define $\phi_\sigma : \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\phi_{\sigma,V}(Q, p^{[1]}, p^{[2]}) = \Big(\Theta_s(Q)(\sigma, z^{[1]}_0, z^{[2]}_0)\Big)_{0 \leqq s \leqq d-1},$$

where $z^{[1]}_0, z^{[2]}_0$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p^{[1]}, p^{[2]})$ and $\Theta_s(Q)$ is the elementary determinants associated to Q.

Case2 If m be an even integer, then we define $\phi_\sigma : \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\phi_{\sigma,V}(Q, p^{[1]}, p^{[2]}) = \Big(\theta_s(\sigma, z^{[1]}_0, z^{[2]}_0)\Big)_{0 \leqq s \leqq d-1},$$

where $z^{[1]}_0, z^{[2]}_0$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p^{[1]}, p^{[2]})$ and $\theta_s(Q)$ is the elementary determinants associated to Q.

Step 3: Construct the open subset $V_\sigma \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$:

Let $\sigma \in \mathfrak{S}$. Then, construct V_σ by dividing two cases of m being odd and even:

Case1 If m is an odd integer, then V_σ is defined by the set of $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that $(Q, p^{[1]}, p^{[2]})$ satisfies the conditions 1) to 2):

1). $p^{[1]}, p^{[2]}$ are linearly independent;

2). $\Delta^{m-1}_0(\sigma, z^{[1]}_0) \neq 0$.

Case2 If m is an even integer, then V_σ is defined by the set of $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that

1). $p^{[1]}, p^{[2]}$ are linearly independent;
2). $\delta_1^{m-1}(\sigma, z_0^{[1]})P^{m-2}(\sigma, z_0^{[2]}) \neq 0$,
where $z_0^{[1]}, z_0^{[2]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p^{[1]}, p^{[2]})$.

Then, V_σ is an open subset of $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$.

Step 4: $G_C(d;T_0^*\mathbb{R}^n\times T_0^*\mathbb{R}^n)$ is the union of kernels of restriction to V_σ of the mapping ϕ_σ with $\sigma\in\mathfrak{S}_m$.

Step 5: Let Ω_0 be the set of $Q\in \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$ such that $Q_1(0),\cdots,Q_m(0)$ are linearly independent. It is well-known that the local coordinate systems on Ω_0 can be constructed (see Coordinate systems in [6],[8]).Then, if $\sigma\in\mathfrak{S}_m$, then the restriction to the intersection $V_\sigma\cap\hat{V}$ of the mapping ϕ_σ is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0\times\mathbb{R}^n\times\mathbb{R}^n$. (The proof is Lemma 4.2. in [8].)

Step 6: $\mathrm{codim}(\overline{\pi(B_C(\boldsymbol{D}))},\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n))\geqq d-3n$:

By Step 4,5, $\mathrm{codim}(G_C(d;T_0^*\mathbb{R}^n\times T_0^*\mathbb{R}^n),\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n)=d$. On the other hand, $G_C(d)$ is the canonical projection of $G_C(d;T_0^*\mathbb{R}^n\times T_0^*\mathbb{R}^n)$ by $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n\to\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$. Therefore, by Tarski-Seidenberg theorem,

$$\mathrm{codim}(G_C(d),\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)\geqq d-2n$$

. Therefore,by Tarski-Seidenberg theorem,

$$\begin{aligned}\mathrm{codim}(\overline{\pi(B_C(\boldsymbol{D}))},\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)) &\geqq \mathrm{codim}(\pi(B_C(\boldsymbol{D}),\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)\times\mathbb{R}^n)-n\\ &= \mathrm{codim}(G_C(d),\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)-n\\ &= d-3n.\end{aligned}$$

□

11.4 Proof of main theorem 5

It is well-known that if $B\subset\mathbb{R}^K$ is a semi-algebraic, then the complement of B in $\mathbb{R}^K$ is dense if and only if $\dim(\mathbb{R}^K,B)>0$. In particular, the complement of the clousure of B in $\mathbb{R}^K$, $\mathbb{R}^K\backslash\overline{B}$ is open dense subset of $\mathbb{R}^K$.

Since $(d_1,\cdots,d_m)$ is an m-tuple of even integers, there exists an integer d' such that $d'=\min\{d_1,\cdots,d_m\}$. Let $d'>2n$ be an integer. Let $d=2d'-1$ and $\min\{d_1,\cdots,d_m\}=d+1=2d'(>4n)$. Then, $\min\{d_1,\cdots,d_m\}$

Then, let H_1 be the set of $X\in\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$ such that for any $x\in M$, j_x^NX is not included in the closure of $\pi(B_{mo}(\boldsymbol{D}))\cup\pi(B_C(\boldsymbol{D}))$ in $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$:

$$H_1=\left\{X\in\mathrm{VF}(M))^m\;\middle|\;Q\notin\overline{\pi(B_{mo}(\boldsymbol{D}))\cup\pi(B_C(\boldsymbol{D})).}\right\}.$$

By Lemmata 11.7, 11.8 and $n\geqq 2$,

$$\mathrm{codim}(\overline{\pi(B_{mo}(\boldsymbol{D}))\cup\pi(B_C(\boldsymbol{D}))},\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n))\geqq\min\{d'-2n,d-3n\}>0.$$

Then H_1 is an open dense subset of $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$ by using Tarski-Seidenberg theorem.

Let $Q\in G_1$. Then, by using Lemma 6.7, if $u:[0,T]\to\Omega$ is a singular Q-control for a given initial point $x_0\in\mathbb{R}^n$ and if the corresponding Q-trajectory $x:[0,T]\to Q,x(0)=x_0$ is non-trivial and independent, then the control u is of corank one.

12 Properties of abnormal extremals on a generic polynomial system in generalized subriemannian geometry

Recall that, $\boldsymbol{D} = (d_1, \cdots, d_m)$ denotes an m-tuple of integers, and $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ denotes the product space of m-tuples of polynomial vector fields over $\mathbb{R}^n$: $(Q_1, \cdots, Q_m)$, such that the degree of Q_i satisfiies $\deg Q_i \leqq d_i$ for every integer $i\,(1 \leqq i \leqq m)$, and we endow $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ with the Euclidean topology.

Let Ω be an open subset of $\mathbb{R}^m$. For $Q = (Q_1, \cdots, Q_m) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$, consider the polynomial driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i\, Q_i(x)$$

with the control parameter $(u_1, \cdots, u_m) \in \Omega$. Moreover, consider the optimal control problem associated to the driftless control-affine systems to minimize the energy functional

$$e(u) = \frac{1}{2} \sum_{i=1}^{m} u_i^2.$$

on Q-admissible controls with the fixed initial point $x_0 \in M$ and the fixed end point x_1 Then, the following holds:

Theorem 12.1 *Suppose* $2 \leqq m \leqq n$ *and suppose that,* $\boldsymbol{D} = (d_1, \cdots, d_m)$ *satisfies the inequality* : $\min\{d_1, d_2, \cdots, d_m\} \geqq 3n+2$. *Then, there exists an open dense semi-algebraic subset* $H_3 \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ *such that, if* $Q \in H_3$, *if* $x : [0, T] \to T^*M$ *is a* Q*-abnormal bi-extremal, then* x *is strictly abnormal.*

Outline of proof Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be an m-tuple Let $d = \min\{d_1, \cdots, d_m\}$. Then, we will show the main theorem 3 by the following procedures:
[Step1] Construct the “bad set” with respect to minimal order, $B_{sa}(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$.
[Step2] Show that, if $Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ satisfies the condition that any $x \in Q$, $(Q, x) \notin B_{sa}(\boldsymbol{D})$ and if $x : [0, T] \to \mathbb{R}^n$ is an abnormal Q-bi-extremal, then x is of strictly abnormal.
[Step3] Compute the codimension of $\pi(B_{sa}(d))$ in $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ by $\pi : \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$.
[Step4] For $d > 3n - 1$, let H_3 be the set of $Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that (Q, x) is not included in the closure of $\pi(B_{sa}(\boldsymbol{D}))$ in $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$. Then, show that, H_3 is an open dense subset of $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ in the sense of Whitney smooth topology.

12.1 Construction of bad set

Let $(z^{[n]}, z^{[a]}) \in T^*M \times_M T^*M$ and $x = \pi(z^{[n]}) = \pi(z^{[a]})$. For every multi index I of $\{1, \cdots, m\}$, set

$$H_I^{[n]}(z^{[n]}, z^{[a]}) = H_I(z^{[n]}) \,\text{and}\, H_I^{[a]}(z^{[n]}, z^{[a]}) = H_I(z^{[a]}).$$

and define inductively the following functions in $\mathcal{F}$, depending on $(z^{[n]}, z^{[a]})$

$$\begin{cases} \beta_{i,0} = H_i^{[a]} \\ \beta_{i,s+1} = \sum_{j=1}^{m} H_j^{[n]} \mathcal{L}_{\overrightarrow{H_j}} \beta_{i,s}, \;\; (s = 1, 2, \cdots), \end{cases}$$

where $\mathcal{F}$ and $\mathcal{L}_{\overrightarrow{H_j}}$ are defined in before section.

Definition 12.2 Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be a pair of positive integers such that $d_i \geqq 2$ for every integer $i\,(1 \leqq i \leqq m)$. Let $d = \min\{d_1, \cdots, d_m\} - 1$. For every integer $i\,(1 \leqq i \leqq m)$ and $(z^{[n]}, z^{[a]}) \in T^*M \times_M T^*M$, we define $\hat{B}(\boldsymbol{D}, i, z^{[n]}, z^{[a]})$ by the set of $(Q, x) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$ such that the following conditions hold:

1)$X_i(x) \neq 0$;

2)$H_i^{[n]}(z^{[n]}, z^{[a]}) \neq 0$;

3)$\beta_{i,s}(z^{[n]}, z^{[a]}) = 0$ for every integer $s\,(0 \leqq s \leqq d-1)$.

$\hat{B}((\boldsymbol{D}, z^{[n]}, z^{[a]}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$ is the union of $\hat{B}(\boldsymbol{D}, i, z^{[n]}, z^{[a]})$ with $i\,(1 \leqq i \leqq m)$.

Definition 12.3 Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be a pair of positive integers such that $d_i geqq 2$ for every integer $i\,(1 \leqq i \leqq m)$. Let $d = \min\{d_1, \cdots, d_m\} - 1$. we define $\hat{B}_{sa}(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times_M T^*M \times_M T^*M$ by

$$\hat{B}_{sa}(\boldsymbol{D}) = \{((Q, x), z^{[n]}, z^{[a]}) | (Q, x) \in \hat{B}((\boldsymbol{D}, z^{[n]}, z^{[a]})\}.$$

Definition 12.4 Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be a pair of positive integers such that $d_i \geqq 2$ for every integer $i\,(1 \leqq i \leqq m)$. we define the bad set with respect to strictly abnormal $B_{sa}(\boldsymbol{D})$ by the canonical projection of $\hat{B}_{sa}(d)$ on $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$.

12.2 The property of singular bi-extremals avoiding bad set with respect to strictly abnormal

Lemma 12.5 *Suppose that, $2 \leqq m \leqq n$. Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be a pair of positive integers such that $d_i geqq 2$ for every integer $i\,(1 \leqq i \leqq m)$. Let $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $(Q, x) \notin B_{sa}(\boldsymbol{D})$. Then, if $x : [0, T] \to \mathbb{R}^n$ is an abnormal Q-bi-extremal, then x is of strictly abnormal.*

Proof: By contradiction, assume that there exists a nontrivial singular X-trajectory $x:[0,T]\to M$ with a singular X-control $u:[0,T]\to\mathbb{R}^m$ such that $x=\pi\circ z^{[n]}=\pi\circ z^{[a]}$, where $z^{[n]}$ is a normal X-bi-extremal lift of x, and $z^{[a]}$ is an abnormal X-bi-extremal lift of x.

For every multi-index $I\subset\{0,\cdots,m\}$ and $t\in[0,T]$, set

$$H_I(z^{[n]}(t))=\langle z^{[n]}(t),X_I(x(t))\rangle,\, H_I(z^{[a]}(t))=\langle z^{[a]}(t),X_I(x(t))\rangle.$$

After time differentiation, we have on $[0,T]$,

$$\begin{cases} \frac{d}{dt}H_I(z^{[n]}(t))=\sum_{i=1}^m u_i(t)H_{Ii}(z^{[n]}(t)),\\ \frac{d}{dt}H_I(z^{[a]}(t))=\sum_{i=1}^m u_i(t)H_{Ii}(z^{[a]}(t)).\end{cases}$$

By Pontryagin maximum principle,

$$\begin{cases} u_i(t)=H_i(z^{[n]}(t))=H_i^{[n]}(z^{[n]}(t),z^{[a]}(t)),\\ H_i^{[a]}(z^{[n]}(t),z^{[a]}(t))=H_i(z^{[a]}(t))=0\cdots(\star)\end{cases}\quad \text{for every integer } i\,(1\leqq i\leqq m), t\in[0,T]$$

Since $x:[0,T]\to\mathbb{R}^n$ is nontrivial, there exists an open subset $J\subset[0,T]$ and an integer $i_0\,(1\leqq i_0\leqq m)$ such that $u_{i_0}(t)X_{i_0}(x(t))\neq 0$ on J. Therefore, $u_{i_0}(t)\neq 0$ and $X_{i_0}(x(t))\neq 0$ on J. Since $u_{i_0}(t)=H_{i_0}(z^{[n]}(t))$,

$$H_{i_0}(z^{[n]}(t))\neq 0.$$

on the other hand, by differentiating $(\star)$ with respect to $t\in[0,T]$,

$$\begin{aligned} 0 &= \frac{d}{dt}H^{[a]}_{i_0+1}(z^{[n]}(t),z^{[a]}(t))\\ &= \sum_{j=1}^m u_j(t)H^{[a]}_{(i_0+1)j}(z^{[n]}(t),z^{[a]}(t))\\ &= \sum_{j=1}^m H_j^{[n]}(z^{[n]}(t),z^{[a]}(t))\,H^{[a]}_{(i_0+1)j}(z^{[n]}(t),z^{[a]}(t))\\ &= \beta_{i_0,1}(z^{[n]}(t),z^{[a]}(t))\end{aligned}$$

For every $t\in[0,T]$, by induction,

$$\beta_{i_0,s}(z^{[n]}(t),z^{[a]}(t))=0.$$

for every $s\,(0\leqq s\leqq d-1)$ and $t\in J$. Hence, $j_x^N X\in\hat{B}(d,i_0,z^{[n]},z^{[a]})$ for $t\in J$, which contradicts the hypothesis. $\square$

12.3 Codimension of bad set with respect to strictly abnormal

Lemma 12.6 $\operatorname{codim}(\overline{\pi(B_{sa}(\boldsymbol{D}))};\, VF^{\boldsymbol{D}}_{poly}(\mathbb{R}^n))\geqq d-3n.$

Proof: We describe only the outline of the proof of Lemma 12.6. Let $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ be the m-tuple product space of polynomial vector fields of degree $\leqq N$ over $\mathbb{R}^n$.

Step1: Construct the typical fiber $G_{sa}(d)$ of $B_{sa}(d)$.

Typical fiber $G_{sa}(d)$ of $B_{sa}(d)$ is the canonical projection of $G_{sa}(d; T^*_0\mathbb{R}^m \times T^*_0\mathbb{R}^m)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. $G_{sa}(d; T^*_0 M \times_M T^*_0 M)$ is defined by the set of $(Q, p^{[n]}, p^{[a]}) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that there exists $i\,(1 \leqq i \leqq m)$ such that $(Q, p^{[n]}, p^{[a]})$ satisfies the following conditions 1) to 4):

1)$Q_i(0)$ are linearly independet;

2)$H^{[n]}_i(z^{[n]}_0, z^{[a]}_0) \neq 0$;

3)$\beta_{i,s}(z^{[n]}_0, z^{[a]}_0) = 0$ for every integer $s\,(0 \leqq s \leqq d-1)$.

where $z^{[n]}_0, z^{[a]}_0$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p_1), (0, p_2)$.

Step2: Construct the mapping $\phi_i : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$:

Let $i(1 \leqq i \leqq m)$ be a positive integer. Then we define the mapping $\phi_i : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q, p_1, p_2) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\phi_i(Q, p_1, p_2) = \beta_{i,s}(z^{[n]}_0, z^{[a]}_0),$$

where $z^{[n]}_0, z^{[a]}_0$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p_1), (0, p_2)$.

Step3: Construct the open subset $V_i \subset \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$

Let $i(1 \leqq i \leqq m)$ be a positive integer. Then V_i is the defined by the set of $(Q, p_1, p_2) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that (Q, p_1, p_2) satisfies the following condition:

$$H^{[n]}_i(z^{[n]}_0, z^{[a]}_0) \neq 0,$$

where $z^{[n]}_0, z^{[a]}_0$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p_1), (0, p_2)$. Then, V_i is an open subset of $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$.

Step4: $G_{sa}(d; T^*_0\mathbb{R}^m \times T^*_0\mathbb{R}^m)$ is the union of the kernel of restriction to V_i of the mapping ϕ_i with $i\,(1 \leqq i \leqq m)$.

Step5: Let Ω^i_0 be the set of $Q \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that $Q_i \neq 0$. It is well-known that the local coordinate systems on Ω^i_0 can be constructed (see Cooordinate systems in [6],[8]) Then, for every integer $i\,(1 \leqq i \leqq m)$, the restriction to the intersection $V_i \cap \hat{V}$ of the mapping ϕ_i is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0 \times \mathbb{R}^n$.

Step6: $\mathrm{codim}(\overline{B_{sa}(d)}; J^N(\mathrm{VF}(M))^m) \geqq d - 2n$.

By step 4,5, $\mathrm{codim}(G_{sa}(d; T^*_0\mathbb{R}^n \times T^*_0\mathbb{R}^n); \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n) = d$. On the other hand, $G_{sa}(d)$ of $B_{sa}(d)$ is the canonical projection of $G_{sa}(d; T^*_0\mathbb{R}^m \times T^*_0\mathbb{R}^m)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. Therefore, $\mathrm{codim}(G_{sa}(d); \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \geqq d - 2n$. Since $G_{sa}(d)$ is the tyupical fiber of $B_{sa}(d)$,

$$\mathrm{codim}(B_{sa}(d); J^N(\mathrm{VF}(M))^m) = \mathrm{codim}(G_{sa}(d); \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \geqq d - 2n.$$

Since the dimensions of $B_{sa}(d)$ and $\overline{B_{sa}(d)}$ are equal,

$$\begin{aligned}
\mathrm{codim}(\overline{\pi(B_{sa}(d))}; \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)) &= \mathrm{codim}(\overline{\pi(B_{sa}(d))}; \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)) \\
&\geqq \mathrm{codim}(\overline{B_{sa}(d)}; \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n) - n \\
&= \mathrm{codim}(\overline{G_{sa}(d)}; \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)) - n \\
&\geqq d - 3n
\end{aligned}$$

□

12.4 Proof of main theorem 6

It is well-known that if $B \subset \mathbb{R}^K$ is a semi-algebraic, then the complement of B in $\mathbb{R}^K$ is dense if and only if $\dim(\mathbb{R}^K, B) > 0$. In particular, the complement of the clousure of B in $\mathbb{R}^K$, $\mathbb{R}^K \backslash \overline{B}$ is open dense subset of $\mathbb{R}^K$.

Let $d > 3n$ be an integer such that $\min\{D_1, D_2, \cdots, D_m\} = d + 1 (> 3n + 1)$. Let H_3 be the set of $Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that for any (Q, x) is not included in the closure of $\pi(B_{sa}(\boldsymbol{D}))$ by $\pi : \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$:

$$H_3 = \left\{ Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \;\middle|\; (Q, x) \notin \overline{\pi(B_{sa}(\boldsymbol{D}))} \text{ for any } x \in M. \right\}.$$

By Lemma 12.6,

$$\mathrm{codim}(\overline{B_{mo}(d)}, \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)) \geqq d - 3n > 0.$$

Then G_3 is an open dense subset of $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$.

Let $Q = (Q_1, \cdots, Q_m) \in H_3$. Then, for any $x \in M$, $(Q, x) \notin B_{sa}(\boldsymbol{D})$. Therefore, by using Lemma 12.5, if $x : [0, T] \to \mathbb{R}^n$ is abnormal then $x : [0, T] \to T^*\mathbb{R}^n$ is strictly abnormal. □

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