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**The lifespan of solutions to initial  
value problem for nonlinear  
wave equations**

(非線形波動方程式に対する初期値問題の解の最大存在時間)

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# Contents

|          |                                                                                                            |           |
|----------|------------------------------------------------------------------------------------------------------------|-----------|
| <b>1</b> | <b>Introduction</b>                                                                                        | <b>2</b>  |
| <b>2</b> | <b>Almost global solutions of semilinear wave equations with the critical exponent in high dimensions</b>  | <b>4</b>  |
| 2.1      | Historical Background . . . . .                                                                            | 4         |
| 2.2      | Semilinear Case . . . . .                                                                                  | 7         |
| 2.3      | Main theorems . . . . .                                                                                    | 9         |
| 2.4      | Weighted $L^\infty$ space . . . . .                                                                        | 13        |
| 2.5      | A priori estimates . . . . .                                                                               | 15        |
| 2.6      | Lower bound in odd space dimensions . . . . .                                                              | 28        |
| 2.7      | Lower bound in even space dimensions . . . . .                                                             | 34        |
| 2.8      | Upper bound of the lifespan for the critical case in odd dimensions . . . . .                              | 46        |
| 2.9      | Upper bound of the lifespan for the subcritical case in odd dimensions . . . . .                           | 54        |
| 2.10     | Upper bound of the lifespan for the critical case in even dimensions . . . . .                             | 56        |
| 2.11     | Upper bound of the lifespan for the subcritical case in even dimensions . . . . .                          | 66        |
| <b>3</b> | <b>The lifespan of classical solutions to one dimensional wave equations with weighted nonlinear terms</b> | <b>69</b> |
| 3.1      | Introduction . . . . .                                                                                     | 69        |
| 3.2      | Notations . . . . .                                                                                        | 71        |
| 3.3      | Upper bound of the lifespan . . . . .                                                                      | 72        |
| 3.4      | Lower bound of the lifespan . . . . .                                                                      | 81        |
|          | <b>Appendix</b>                                                                                            | <b>85</b> |
|          | <b>Acknowledgment</b>                                                                                      | <b>99</b> |

# Chapter 1

## Introduction

In this thesis, we are concerned with the initial value problem for nonlinear wave equations,

$$\begin{cases} u_{tt} - \Delta u = F(u) & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & \text{for } x \in \mathbf{R}^n \end{cases} \quad (1.0.1)$$

where  $n \geq 1$ ,  $f, g \in C_0^\infty(\mathbf{R}^n)$ ,  $F(u) = |u|^p$  or  $F(u) = |u|^{p-1}u$  with  $p > 1$ , and  $\varepsilon > 0$  is a parameter.

For this initial value problem, Strauss [30] conjectured the following. Let  $n \geq 2$  and  $p_0(n)$  is the positive root of the quadratic equation  $(n-1)p^2 - (n+1)p - 2 = 0$ . Then the solution of (1.0.1) exists globally in time for small  $\varepsilon$  if  $p > p_0(n)$ , and the solution blows up in finite time if  $1 < p \leq p_0(n)$ . This is first proved by F. John [13] in the case of  $n = 3$ , except for  $p = p_0(3)$ , and after that, many contributions for this conjecture has been done. (See the section 1 in Chapter 2 for details.)

The first topic of this thesis is due to Takamura & Wakasa [33]. We shall briefly introduce the result.

We consider an integral equation related to (1.0.1):

$$u(x, t) = u^0(x, t) + L(F(u))(x, t) \quad \text{for } (x, t) \in \mathbf{R}^n \times [0, \infty), \quad (1.0.2)$$

where  $u^0$  is a solution of the homogeneous wave equations, that is the solution of (1.0.1) with  $H \equiv 0$ , and

$$L(F(u))(x, t) = \frac{1}{2m-1} \int_0^t (t-\tau) M(F(u(\cdot, \tau)))(x, t-\tau) d\tau,$$

and

$$M(\phi|x, r) = \begin{cases} \frac{1}{\omega_n} \int_{|\omega|=1} \phi(x + r\omega) dS_\omega & \text{for } n = 2m+1, \\ \frac{2}{\omega_{n+1}} \int_{|\xi| \leq 1} \frac{\phi(x + r\xi)}{\sqrt{1-|\xi|^2}} d\xi & \text{for } n = 2m, \end{cases}$$

for  $\phi \in C(\mathbf{R}^n)$  and  $m \in \mathbf{N}$ , where  $r = |x|$  and  $\omega_n$  is the area of the unit sphere in  $\mathbf{R}^n$ .

When  $n \geq 4$  and  $F(u) = |u|^p$  in (1.0.2), Agemi & Kubota & Takamura [1] showed that the solution of (1.0.2) exists globally in time if  $p > p_0(n)$ . In [33], we proved that the solution of (1.0.2) blows up in finite time if  $1 < p \leq p_0(n)$ , and we have obtained some estimates for the lifespan of solution  $u \in C^1(\mathbf{R}^n \times [0, T])$  to (1.0.2) which is the maximal existence time of solutions. The estimate of the lifespan is the same order as in the one for (1.0.1) with  $F(u) = |u|^p$  and  $1 < p \leq p_0(n)$ , with respect to  $\varepsilon$ . We note that if we assume that  $u$  is the  $C^2$  solution, and impose appropriate regularity on  $f$ ,  $g$ , and  $F$ , then  $u$  satisfies some nonlinear wave equations. See Remark 2.3.1 below.

In the second topic of this thesis is due to Wakasa [34]. We consider the following initial value problem for nonlinear wave equations:

$$\begin{cases} u_{tt} - u_{xx} = \frac{|u|^{p-1}u}{(1+x^2)^{(1+a)/2}}, & (x, t) \in \mathbf{R} \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbf{R}, \end{cases} \quad (1.0.3)$$

where  $a \geq -1$ ,  $\varepsilon > 0$  and  $p > 1$ . Suzuki [31] showed that (1.0.3) has a global in-time solution for  $p > (1 + \sqrt{5})/2$  and  $pa > 1$  if  $f$  and  $g$  are odd functions and  $\varepsilon$  is small enough, and Kubo & Osaka & Yazici [20] have obtained the same conclusion for any  $p > 1$  satisfying  $pa > 1$ . On the other hand, they showed that the solution of (1.0.3) blows up in finite time if  $f$  and  $g$  are not odd functions. In addition, [20] obtained an upper bound of the lifespan of solution to (1.0.3). In [34], we have improved the upper bound of the lifespan and derive its lower bound which shows the optimality of our new upper bound.

In Appendix, we derive a representation formula for the following homogeneous wave equation:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = 0, \quad u_t(x, 0) = g(x), & x \in \mathbf{R}^n. \end{cases}$$

The proof of the formula divided into two steps. First, we show the representation formula by using Fourier transform in odd space dimensions. Next, we use Hadamard's method of descent to get the representation formula in even space dimensions from the case of odd space dimensions. The argument is based on Courant & Hilbert [4] and Yajima [35].

## Chapter 2

# Almost global solutions of semilinear wave equations with the critical exponent in high dimensions

In this chapter, we introduce the result of Takamura & Wakasa [33].

### 2.1 Historical Background

In this section, we review known results for the following initial value problem for fully nonlinear wave equations:

$$\begin{cases} u_{tt} - \Delta u = H(u, Du, D_x Du) & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), \end{cases} \quad (2.1.1)$$

where  $u = u(x, t)$  is a scalar unknown function of space-time variables,

$$\begin{aligned} Du &= (u_{x_0}, u_{x_1}, \dots, u_{x_n}), \quad x_0 = t, \\ D_x Du &= (u_{x_i x_j}, \quad i, j = 0, 1, \dots, n, \quad i + j \geq 1), \end{aligned}$$

$f, g \in C_0^\infty(\mathbf{R}^n)$  and  $\varepsilon > 0$  is a “small” parameter.

Let

$$\widehat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \dots, n; (\lambda_{ij}), i, j = 0, 1, \dots, n, \quad i + j \geq 1).$$

Suppose that the nonlinear term  $H = H(\widehat{\lambda})$  is a sufficiently smooth function with

$$H(\widehat{\lambda}) = O(|\widehat{\lambda}|^{1+\alpha})$$

in a neighborhood of  $\widehat{\lambda} = 0$ , where  $\alpha \geq 1$  is an integer.

Let us define the lifespan  $\widetilde{T}_\varepsilon$  of classical solutions of (2.1.1) by

$$\widetilde{T}_\varepsilon \equiv \widetilde{T}_\varepsilon(f, g) := \sup\{T \in [0, \infty) : \text{There exists a unique solution } u \in C^2(\mathbf{R} \times [0, T)) \text{ of (2.1.1)}\}$$

with arbitrarily fixed  $(f, g)$ . When  $\widetilde{T}_\varepsilon = \infty$ , the problem (2.1.1) admits a global-in-time solution, while we only have a local-in-time solution on  $[0, \widetilde{T}_\varepsilon)$  when  $\widetilde{T}_\varepsilon < \infty$ . From now on, we omit “-in-time” and simply use “global” and “local”.

We are interested in “small” initial data, because there exist different cases in which we have  $\widetilde{T}_\varepsilon < \infty$ , even if  $\varepsilon$  is “small enough”. We note that when the initial data is “large”, there are blow-up results (For example, see Glassey [7], Levine [21], or Sideris [29]). The meaning of blow-up is that

$$\sup_{x \in \mathbf{R}^n} |u(x, t)| \rightarrow \infty \quad \text{if } t \rightarrow \widetilde{T}_\varepsilon - 0$$

holds for the classical solution  $u$  of (2.1.1).

Therefore, in what follows, we assume that  $\varepsilon$  is “small” enough. As is shown in Chapter 2 of Li and Chen [22], we have long histories on the estimate for  $\widetilde{T}_\varepsilon$ . Various lower bounds of  $\widetilde{T}_\varepsilon$  are summarized in the following table, where  $a = a(\varepsilon)$  satisfies

$$a^2 \varepsilon^2 \log(a + 1) = 1 \tag{2.1.2}$$

and  $c$  stands for a positive constant independent of  $\varepsilon$ .

| $\tilde{T}_\varepsilon \geq$ | $\alpha = 1$                                                                                                                                                                                                                                         | $\alpha = 2$                                                                                                                                                              | $\alpha \geq 3$                   |
|------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------|
| $n = 1$                      | $\varepsilon^{-\alpha/2}$<br>in general,<br>$c\varepsilon^{-\alpha(1+\alpha)/(2+\alpha)}$<br>if $\int_{\mathbf{R}} g(x)dx = 0$ ,<br>$c\varepsilon^{-\alpha}$<br>if $\partial_u^\beta H(0) = 0$<br>for $1 + \alpha \leq \forall \beta \leq 2\alpha$ . | same as<br>$(n, \alpha) = (1, 1)$                                                                                                                                         | same as<br>$(n, \alpha) = (1, 1)$ |
| $n = 2$                      | $ca(\varepsilon)$<br>in general ,<br>$c\varepsilon^{-1}$<br>if $\int_{\mathbf{R}^2} g(x)dx = 0$ ,<br>$c\varepsilon^{-2}$<br>if $\partial_u^2 H(0) = 0$                                                                                               | $c\varepsilon^{-6}$<br>in general,<br>$c\varepsilon^{-18}$<br>if $\partial_u^3 H(0) = 0$ ,<br>$\exp(c\varepsilon^{-2})$<br>if $\partial_u^3 H(0) = \partial_u^4 H(0) = 0$ | $\infty$                          |
| $n = 3$                      | $c\varepsilon^{-2}$<br>in general,<br>$\exp(c\varepsilon^{-1})$<br>if $\partial_u^2 H(0) = 0$                                                                                                                                                        | $\infty$                                                                                                                                                                  | $\infty$                          |
| $n = 4$                      | $\exp(c\varepsilon^{-2})$<br>in general,<br>$\infty$<br>if $\partial_u^2 H(0) = 0$                                                                                                                                                                   | $\infty$                                                                                                                                                                  | $\infty$                          |
| $n \geq 5$                   | $\infty$                                                                                                                                                                                                                                             | $\infty$                                                                                                                                                                  | $\infty$                          |

We would not like to present references for each results. But we note that two parts in this table are different from the original one in Li and Chen [22]. One is the general case in  $(n, \alpha) = (4, 1)$ . In this part, the lower bound of  $\tilde{T}_\varepsilon$  is  $\exp(c\varepsilon^{-1})$  in Li and Chen [22]. But later, it has been improved by Li and Zhou [23]. The other is the case for  $\partial_u^3 H(0) = 0$  in  $(n, \alpha) = (2, 2)$ . This part is due to Katayama [16]. But this result was missed in Li and Chen [22].

Sharpness of these lower bounds is established if one could prove that there is no possibility to improve the lower bound of  $\tilde{T}_\varepsilon$  in sense of order of  $\varepsilon$  by showing a blow-up result for some special equations and special data. We note that the sharpness for the general case in  $(n, \alpha) = (4, 1)$  has been open more than 20 years. The answer was obtained by our previous work, Takamura and Wakasa [32], by studying  $H = u^2$  with some special data. It is interesting to notice that this equation is in the critical level concerning the Strauss conjecture, since  $p_0(4) = 2$ . We mention whole histories for the semilinear equation in the next section. We also note that Zhou and Han



[41] have obtained the sharpness in  $(n, \alpha) = (2, 2)$  by studying  $H = u_t^2 u + u^4$ . (note that  $\partial_u^3 H(0) = 0$  but  $\partial_u^4 H(0) \neq 0$ ).

**Remark 2.1.1** *It is remarkable that when  $(n, \alpha) = (3, 1)$ , Klainerman [18] showed the global existence by using a special structure on  $H = H(Du, D_x Du)$ . It is so-called “null condition”. (Christodoulou [3] also showed the global existence by using the conformal mapping method.) When  $(n, \alpha) = (2, 2)$ , the null condition has been also established independently by Godin [10] for  $H = H(Du)$  and Katayama [15] for  $H = H(Du, D_x Du)$ . Later, Professor R. Agemi conjectured global existence result under “non-positive condition” in this case which is weaker than the null condition with  $(n, \alpha) = (2, 2)$ . This conjecture has been verified by Hoshiga [11] and Kubo [19] independently.*

## 2.2 Semilinear Case

We consider the following semilinear wave equations:

$$\begin{cases} u_{tt} - \Delta u = |u|^p, & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), \end{cases} \quad (2.2.1)$$

assuming that  $p > 1$  and  $\varepsilon > 0$  is “small”. Let us define the lifespan  $\widehat{T}_\varepsilon$  of a solution of (2.2.1) by

$$\widehat{T}_\varepsilon \equiv \widehat{T}_\varepsilon(f, g) := \sup\{T \in [0, \infty) : \text{There exists a unique weak solution } u(x, t) \text{ of (2.2.1)}\}$$

with arbitrarily fixed  $(f, g)$ .

**Remark 2.2.1** *In the definition of the lifespan, we consider weak solutions which is a  $C^1$ -solution of associated integral equations to (2.2.1). However, when  $p \geq 2$ , we can obtain the classical solution.*

When  $n = 1$ , we have  $\widehat{T}_\varepsilon < \infty$  for any power  $p > 1$  by Kato [17]. When  $n \geq 2$ , the following conjecture was proposed by Strauss [30]. Let  $p_0(n)$  be the positive root of the quadratic equation,

$$\gamma(p, n) = 2 + (n + 1)p - (n - 1)p^2 = 0. \quad (2.2.2)$$

That is,

$$p_0(n) = \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)}. \quad (2.2.3)$$

Then we have

$$\begin{aligned}\widehat{T}_\varepsilon &= \infty & \text{if } p > p_0(n) \text{ and } \varepsilon \text{ is "small"} & \quad (\text{global existence}), \\ \widehat{T}_\varepsilon &< \infty & \text{if } 1 < p \leq p_0(n) & \quad (\text{blow-up in finite time}).\end{aligned}$$

We note that  $p_0(n)$  is monotonously decreasing in  $n$ , and  $p_0(4) = 2$ . This conjecture had been verified by many authors in different cases. All the references on the final result in each part can be summarized in the following table.

|            | $1 < p < p_0(n)$ | $p = p_0(n)$                               | $p > p_0(n)$                       |
|------------|------------------|--------------------------------------------|------------------------------------|
| $n = 2$    | Glassey [8]      | Schaeffer [27]                             | Glassey [9]                        |
| $n = 3$    | John [13]        | Schaeffer [27]                             | John [13]                          |
| $n \geq 4$ | Sideris [28]     | Yordanov & Zhang [36]<br>Zhou [40], indep. | Georgiev & Lindblad<br>& Sogge [5] |

We are interested in the estimate of the lifespan  $\widehat{T}_\varepsilon$  in the blow-up case. From now on,  $c$  and  $C$  stand for positive constants independent of  $\varepsilon$ . When  $n = 1$ , we have the following estimate of the lifespan  $\widehat{T}_\varepsilon$  for any  $p > 1$ :

$$\left\{ \begin{array}{ll} c\varepsilon^{-(p-1)/2} \leq \widehat{T}_\varepsilon \leq C\varepsilon^{-(p-1)/2} & \text{if } \int_{\mathbf{R}} g(x)dx \neq 0, \\ c\varepsilon^{-p(p-1)/(p+1)} \leq \widehat{T}_\varepsilon \leq C\varepsilon^{-p(p-1)/(p+1)} & \text{if } \int_{\mathbf{R}} g(x)dx = 0. \end{array} \right. \quad (2.2.4)$$

This result has been obtained by Zhou [37]. Moreover, Lindblad [24] has obtained a more precise result for  $p = 2$ ,

$$\left\{ \begin{array}{ll} \exists \lim_{\varepsilon \rightarrow +0} \varepsilon^{1/2} \widehat{T}_\varepsilon > 0 & \text{for } \int_{\mathbf{R}} g(x)dx \neq 0, \\ \exists \lim_{\varepsilon \rightarrow +0} \varepsilon^{2/3} \widehat{T}_\varepsilon > 0 & \text{for } \int_{\mathbf{R}} g(x)dx = 0. \end{array} \right. \quad (2.2.5)$$

Similarly to this, Lindblad [24] has also obtained the following result for  $(n, p) = (2, 2)$ .

$$\left\{ \begin{array}{ll} \exists \lim_{\varepsilon \rightarrow +0} a(\varepsilon)^{-1} \widehat{T}_\varepsilon > 0 & \text{for } \int_{\mathbf{R}^2} g(x)dx \neq 0, \\ \exists \lim_{\varepsilon \rightarrow +0} \varepsilon \widehat{T}_\varepsilon > 0 & \text{for } \int_{\mathbf{R}^2} g(x)dx = 0, \end{array} \right. \quad (2.2.6)$$

where  $a(\varepsilon)$  is the one in (2.1.2).

When  $1 < p < p_0(n)$  ( $n \geq 3$ ) or  $2 < p < p_0(2)$  ( $n = 2$ ), we have

$$c\varepsilon^{-2p(p-1)/\gamma(p,n)} \leq \widehat{T}_\varepsilon \leq C\varepsilon^{-2p(p-1)/\gamma(p,n)}, \quad (2.2.7)$$

where  $\gamma(p, n)$  is defined by (2.2.2), as is summarized in the following table.

|            | lower bound of $\widehat{T}_\varepsilon$                         | upper bound of $\widehat{T}_\varepsilon$ |
|------------|------------------------------------------------------------------|------------------------------------------|
| $n = 2$    | Zhou [39]                                                        | Zhou [39]                                |
| $n = 3$    | Lindblad [24]                                                    | Lindblad [24]                            |
| $n \geq 4$ | Lindblad & Sogge [25]<br>: $n \leq 8$ or radially symmetric sol. | Georgiev & Takamura<br>& Zhou [6]        |

On the other hand, when  $p = p_0(n)$ , we have

$$\exp(c\varepsilon^{-p(p-1)}) \leq \widehat{T}_\varepsilon \leq \exp(C\varepsilon^{-p(p-1)}), \quad (2.2.8)$$

as is summarized in the following table.

|            | lower bound of $\widehat{T}_\varepsilon$                     | upper bound of $\widehat{T}_\varepsilon$ |
|------------|--------------------------------------------------------------|------------------------------------------|
| $n = 2$    | Zhou [39]                                                    | Zhou [39]                                |
| $n = 3$    | Zhou [38]                                                    | Zhou [38]                                |
| $n \geq 4$ | Lindblad & Sogge [25]<br>: $n \leq 8$ or radially symm. sol. | Takamura & Wakasa [32]                   |

## 2.3 Main theorems

In this section, we introduce the main results of [33]. From now on, we assume that  $n \geq 2$  and write  $n = 2m, 2m + 1$  ( $m = 1, 2, 3, \dots$ ). For a function  $\phi \in C(\mathbf{R})$ , we define

$$M(\phi|x, r) = \begin{cases} \frac{1}{\omega_n} \int_{|\omega|=1} \phi(x + r\omega) dS_\omega & \text{for } n = 2m + 1, \\ \frac{2}{\omega_{n+1}} \int_{|\xi| \leq 1} \frac{\phi(x + r\xi)}{\sqrt{1 - |\xi|^2}} d\xi & \text{for } n = 2m, \end{cases} \quad (2.3.1)$$

where  $\omega_n$  is the area of the unit sphere in  $\mathbf{R}^n$ . We consider the following integral equation,

$$u(x, t) = \varepsilon u^0(x, t) + L(F(u))(x, t) \quad \text{for } (x, t) \in \mathbf{R}^n \times [0, \infty), \quad (2.3.2)$$

where  $F$  is a given function,

$$L(F(u))(x, t) = \frac{1}{2m - 1} \int_0^t (t - \tau) M(F(u(\cdot, \tau))|x, t - \tau) d\tau, \quad (2.3.3)$$

and

$$u^0(x, t) = \partial_t R(f|x, t) + R(g|x, t). \quad (2.3.4)$$

Here

$$R(\phi|x, t) = \frac{1}{(2m-1)!!} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{m-1} \{ t^{2m-1} M(\phi|x, t) \}. \quad (2.3.5)$$

Note that (2.3.4) is a representation formula of the solution to the homogeneous wave equation. This fact is proved in Appendix below.

**Remark 2.3.1** Assume that  $u$  is a  $C^2$  solution of (2.3.2) with  $f \in C^{m+3}(\mathbf{R}^n)$ ,  $g \in C^{m+2}(\mathbf{R}^n)$  and  $F \in C^2(\mathbf{R})$ , then  $u$  satisfies

$$\begin{cases} u_{tt} - \Delta u = F(u) - G(u) \\ \quad - \frac{2(m-1)}{2m-1} M(F(\varepsilon f)|x, t) & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbf{R}^n \end{cases} \quad (2.3.6)$$

where

$$G(u) = \frac{2(m-1)}{2m-1} \int_0^t M(F'(u(\cdot, \tau)) u_t(\cdot, \tau) |x, t - \tau) d\tau. \quad (2.3.7)$$

We note that  $G \equiv 0$  when  $n = 2, 3$  ( $m = 1$ ). Moreover, assuming  $M(F(f)) \in C^{m+1}(\mathbf{R}^n \times [0, \infty))$ , and replacing  $\varepsilon u^0(x, t)$  by a classical solution  $v = v(x, t)$  of

$$\begin{cases} v_{tt} - \Delta v = \frac{2(m-1)}{2m-1} M(F(\varepsilon f)|x, t) & \text{in } \mathbf{R}^n \times [0, \infty), \\ v(x, 0) = \varepsilon f(x), \quad v_t(x, 0) = \varepsilon g(x) & x \in \mathbf{R}^n \end{cases} \quad (2.3.8)$$

in (2.3.2), we have that a  $C^2$  solution  $u$  of

$$u(x, t) = v(x, t) + L(F(u))(x, t) \quad \text{for } (x, t) \in \mathbf{R}^n \times [0, \infty) \quad (2.3.9)$$

satisfies

$$\begin{cases} u_{tt} - \Delta u = F(u) - G(u) & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbf{R}^n. \end{cases} \quad (2.3.10)$$

We shall explain some relation between (2.3.6) and the following initial value problem:

$$\begin{cases} u_{tt} - \Delta u = F(u) & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), \end{cases} \quad (2.3.11)$$

where  $f \in C^{m+3}(\mathbf{R}^n)$ ,  $g \in C^{m+2}(\mathbf{R}^n)$  and  $F \in C^{m+1}(\mathbf{R})$ . The solution of (2.3.11) is written as

$$u(x, t) = \varepsilon u^0(x, t) + \int_0^t R(F(u(\cdot, \tau)) |x, t - \tau) d\tau, \quad (2.3.12)$$

where  $u^0$  and  $R$  are defined in (2.3.4) and (2.3.5), respectively. Making use of the result of Proposition A.2 and (2.3.5), we see that the second term of the right hand side of (2.3.12) is equal to

$$\begin{aligned} & \int_0^t R(F(u(\cdot, \tau))|x, t - \tau) d\tau \\ &= \int_0^t (t - \tau) M(F(u(\cdot, \tau))|x, t - \tau) d\tau \\ & \quad + \int_0^t \sum_{k=1}^{m-1} \alpha_{m-1,k} (t - \tau)^{k+1} \frac{d^k(t - \tau)}{d(t - \tau)^k} \{M(F(u(\cdot, \tau))|x, t - \tau)\} d\tau, \end{aligned}$$

where  $\alpha_{m-1,k}$  are suitable constants. If we put the coefficient,  $(2m - 1)^{-1}$  in the first term, and remove the second term of the right hand side of above, then (2.3.12) coincides with (2.3.2). By the argument in 254-255pp. in Agemi, Kubota and Takamura [1], we see that if  $u$  is the  $C^2$  solution of (2.3.2), then  $u$  satisfies (2.3.6).

From now on, let  $F(u) = |u|^p$  and let us define a lifespan  $\bar{T}_\varepsilon$  by

$$\bar{T}_\varepsilon \equiv \bar{T}_\varepsilon(f, g) := \sup\{T \in [0, \infty) : \text{There exists a unique weak solution } u \text{ of (2.3.2)}\}$$

with arbitrarily fixed  $(f, g)$ . Agemi, Kubota and Takamura [1] have obtained that  $\bar{T}_\varepsilon = \infty$  for  $p > p_0(n)$  if  $\varepsilon$  is small enough, where  $p_0(n)$  is Strauss' exponent. Their theorem is written for (2.3.9) only, but it is available also for (2.3.2).

Our purpose is to establish the same estimates for  $\bar{T}_\varepsilon$  as in (2.2.7) and (2.2.8) when  $n \geq 4$  and  $1 < p \leq p_0(n)$ . They are divided into two theorems below. We note that one can expect to get a  $C^2$  solution only for  $n = 4$  and  $F(u) = u^2$ .

We shall state our main result. We assume on the data that

$$\begin{cases} \text{both } f \in C_0^{m+3}(\mathbf{R}^n) \text{ and } g \in C_0^{m+2}(\mathbf{R}^n) \text{ have} \\ \text{compact support contained in } \{x \in \mathbf{R}^n : |x| \leq k\} \\ \text{with some constant } k > 0. \end{cases} \quad (2.3.13)$$

Then, we have the following existence theorem for large time interval.

**Theorem 2.3.1** *Let  $n \geq 4$  and  $F(u) = |u|^p$  with  $1 < p \leq p_0(n)$ . Suppose that (2.3.13) is fulfilled. Then there exists a positive constant  $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$  such that the lifespan  $\bar{T}_\varepsilon$  satisfies*

$$\begin{aligned} \bar{T}_\varepsilon &\geq c\varepsilon^{-2p(p-1)/\gamma(p,n)} & \text{if } 1 < p < p_0(n), \\ \bar{T}_\varepsilon &\geq \exp(c\varepsilon^{-p(p-1)}) & \text{if } p = p_0(n) \end{aligned} \quad (2.3.14)$$

for any  $\varepsilon$  with  $0 < \varepsilon \leq \varepsilon_0$ , where  $c$  is a positive constant independent of  $\varepsilon$ .

In the proof of Theorem 2.3.1, we shall employ a similar weighted  $L^\infty$  iteration method to Agemi, Kubota and Takamura [1]. Such an argument had been firstly introduced by John [13] in the case of  $n = 3$ .

For the counter part, the following assumptions on the data are required.

$$\left\{ \begin{array}{l} \text{Let } f \equiv 0, g(x) = g(|x|) \text{ and } g \in C_0^1([0, \infty)) \text{ satisfy that there} \\ \text{exist positive constants } k_0 \text{ and } k_1 \text{ with } 0 < k_0 < k_1 < k \\ \text{such that the following three conditions hold.} \\ \text{(i) } \text{supp } g \subset \{x \in \mathbf{R}^n : |x| \leq k\} \\ \text{(ii) } g(|x|) \geq 0 \text{ for } k_0 < |x| < k \text{ and } \int_{(k_1+k)/2}^k \lambda^{[n/2]} g(\lambda) d\lambda > 0, \\ \text{(iii) } k_0 \text{ is sufficiently close to } k \text{ so that} \\ \quad P_m(z) > \frac{1}{2} \text{ and } T_m(z) > \frac{1}{2} \text{ for all } z > \frac{k_0}{k}, \\ \quad \text{where } P_m \text{ or } T_m \text{ denote Legendre or Tschebyscheff} \\ \quad \text{polynomials of degree } m \text{ respectively.} \end{array} \right. \quad (2.3.15)$$

Then, we have the following blow-up theorem.

**Theorem 2.3.2** *Let  $n \geq 4$  and  $F(u) = |u|^p$  with  $1 < p \leq p_0(n)$ . Assume that (2.3.15) holds. Then there exist a positive constant  $\varepsilon_1 = \varepsilon_1(g, n, p, k)$  such that the lifespan  $\overline{T}_\varepsilon$  satisfies*

$$\begin{aligned} \overline{T}_\varepsilon &\leq C\varepsilon^{-2p(p-1)/\gamma(p,n)} & \text{if } 1 < p < p_0(n), \\ \overline{T}_\varepsilon &\leq \exp(C\varepsilon^{-p(p-1)}) & \text{if } p = p_0(n) \end{aligned} \quad (2.3.16)$$

for any  $\varepsilon$  with  $0 < \varepsilon \leq \varepsilon_1$ , where  $C$  is a positive constant independent of  $\varepsilon$ .

**Remark 2.3.2** *The uniqueness of the solution of (2.3.6) is an open problem. Although in Agemi, Kubota and Takamura [1] the uniqueness is claimed in the below of (1.8) at 242p, it seems difficult to apply the uniqueness theorem in Appendix 1 in John [14] because (99a) in [14] fails.*

The proof of Theorem 2.3.2 is done by an iteration argument concerning point-wise estimates from below which is basically introduced by John [13] for  $n = 3$ . But in the critical case, we have to reduce the proof to the argument of Zhou [38] in which the solution is compared with a blowing-up solution of a nonlinear ordinary differential equation of the second order. We also have to employ the slicing method of the blow-up set which is introduced by Agemi, Kurokawa and Takamura [2] due to technical difficulties in high dimensions. See the sections 2.8, 2.9, 2.10 and 2.11 below.

## 2.4 Weighted $L^\infty$ space

First, we shall state the following two lemmas for  $v$  of (2.3.8) which play key roles in proofs of Theorem 2.3.1 and Theorem 2.3.2. The first one is Huygens' principle in odd space dimensions.

**Lemma 2.4.1 (Agemi, Kubota and Takamura [1])** *Let  $n = 5, 7, 9, \dots$ . Under the same assumption as in Theorem 2.3.1, there exists a classical solution  $v$  of (2.3.8) which satisfies*

$$\text{supp } v \subset \{x \in \mathbf{R}^n : t - k \leq |x| \leq t + k\}, \quad (2.4.1)$$

where  $k$  is the one in (2.3.13).

See 253p. in [1] for the proof of this lemma.

Next, we shall introduce the decay estimate for  $v$ . First, we write  $v$  in the form

$$v = v_0 + v_1. \quad (2.4.2)$$

Here,  $v_0 = \varepsilon u^0$  is in (2.3.4) and  $v_1$  is a solution to the inhomogeneous wave equation

$$\begin{cases} (v_1)_{tt} - \Delta v_1 = \frac{2(m-1)}{2m-1} M(F(\varepsilon f)|x, t) & \text{in } \mathbf{R}^n \times [0, \infty), \\ v_1(x, 0) = (v_1)_t(x, 0) = 0, & x \in \mathbf{R}^n \end{cases} \quad (2.4.3)$$

where  $M$  is defined in (2.3.1). Then we have the following lemma.

**Lemma 2.4.2 (Agemi, Kubota and Takamura [1])** *Under the same assumption as in Theorem 2.3.1, there exists positive constants  $C_{n,k,f,g}$  and  $C_{n,k,f^p}$  such that  $v_0$  and  $v_1$  satisfies*

$$\sum_{|\alpha| \leq 1} |\nabla_x^\alpha v_0(x, t)| \leq \frac{C_{n,k,f,g} \varepsilon}{(t + |x| + 2k)^{(n-1)/2}} \quad (2.4.4)$$

and

$$\sum_{|\alpha| \leq 1} |\nabla_x^\alpha v_1(x, t)| \leq \frac{C_{n,k,f^p} \varepsilon^p}{(t + |x| + 2k)^{(n-1)/2}} \quad (2.4.5)$$

for  $n = 5, 7, 9, \dots$ , and

$$\sum_{|\alpha| \leq 2} |\nabla_x^\alpha v_0(x, t)| \leq \frac{C_{n,k,f,g} \varepsilon}{(t + |x| + 2k)^{(n-1)/2} (t - |x| + 2k)^{(n-1)/2}} \quad (2.4.6)$$

and

$$\sum_{|\alpha| \leq 2} |\nabla_x^\alpha v_1(x, t)| \leq \frac{C_{n,k,f^p} \varepsilon^p}{(t + |x| + 2k)^{(n-1)/2} (t - |x| + 2k)^{(n-3)/2}} \quad (2.4.7)$$

for  $n = 4, 6, 8, \dots$ .

This lemma directly follows from Lemma 3.2 and Lemma 3.4 in [1]. We omit the proof here.

It is obvious that  $v$  is a global classical solution of (2.3.8). Therefore our unknown function shall be  $U = u - v$ . Then, (2.3.9) can be rewritten into

$$U = L(F(U + v)). \quad (2.4.8)$$

In what follows, we shall construct a solution of (2.4.8) in a weighted  $L^\infty$  space. For this purpose, define a sequence of functions  $\{U_l\}_{l \in \mathbf{N}}$  by

$$U_l = L(F(U_{l-1} + U_0)), \quad U_0 = v, \quad (2.4.9)$$

where  $L$  is the one in (2.3.3).

Taking into account of these Lemma 2.4.2, we shall introduce a weighted  $L^\infty$  space as follows. We denote a weighted  $L^\infty$  norm of  $U$  by

$$\|U\| = \sup_{(x,t) \in \mathbf{R}^n \times [0,T]} \{w(|x|, t)|U(x, t)|\} \quad (2.4.10)$$

with a weighted function

$$w(r, t) = \begin{cases} \tau_+(r, t)^{(n-1)/2} \tau_-(r, t)^q & \text{if } p > \frac{n+1}{n-1}, \\ \tau_+(r, t)^{(n-1)/2} \left( \log 4 \frac{\tau_+(r, t)}{\tau_-(r, t)} \right)^{-1} & \text{if } p = \frac{n+1}{n-1}, \\ \tau_+(r, t)^{(n-1)/2+q} & \text{if } 1 < p < \frac{n+1}{n-1}, \end{cases} \quad (2.4.11)$$

where we set

$$\tau_+(r, t) = \frac{t + r + 2k}{k}, \quad \tau_-(r, t) = \frac{t - r + 2k}{k}$$

and  $q$  is defined by

$$q = \frac{n-1}{2}p - \frac{n+1}{2}. \quad (2.4.12)$$

In order to get a  $C^1$  solution of (2.4.8), we shall show the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  in a function space  $X$  defined by

$$X = \{U \in C^1(\mathbf{R}^n \times [0, T]) : \|U\|_X < \infty, \text{ supp } U(x, t) \subset \{|x| \leq t + k\}\}$$

which equips a norm

$$\|U\|_X = \sum_{|\alpha| \leq 1} \|\nabla_x^\alpha U\|.$$



In view of (2.3.9), we note that  $\partial U/\partial t$  can be expressed in terms of  $\nabla_x U$ . Hence we consider spatial derivatives of  $U$  only. Moreover, we see that  $X$  is a Banach space for any fixed  $T > 0$ . It follows from the definition of the norm (2.4.10) that there exists a positive constant  $C_T$  depending on  $T$  such that

$$\|U\| \geq C_T |U(x, t)|, \quad t \in [0, T].$$

Later, we shall make use of Hölder's inequality

$$\| |U_1|^a |U_2|^b \| \leq \|U_1\|^a \|U_2\|^b, \quad a + b = 1, \quad a, b \in [0, 1]. \quad (2.4.13)$$

Furthermore, for  $v_0$  and  $v_1$  in (2.4.2), we define

$$U_{00} = v_0 \quad \text{and} \quad U_{01} = v_1.$$

We also denote  $\partial/\partial x_i$  by  $\partial_i$  for  $i = 1, 2, \dots, n$ , and set

$$\begin{aligned} \partial W_l &= \max_{|\alpha| \leq 1} \{ |\nabla_x^\alpha U_l|, |\nabla_x^\alpha U_{l-1}| \}, \quad W_l = \max \{ |U_l|, |U_{l-1}| \}, \\ \partial W_0 &= \max_{|\alpha| \leq 1} \{ |\nabla_x^\alpha U_0| \}, \quad \partial W_{0a} = \max_{|\alpha| \leq 1} \{ |\nabla_x^\alpha U_{0a}| \} \quad (a = 0, 1). \end{aligned}$$

## 2.5 A priori estimates

In this section, we show *a priori* estimates which plays a key role in the contraction mapping argument. The following lemmas are the essential estimates.

**Lemma 2.5.1** *Let  $L$  be the linear integral operator defined by (2.3.3). Assume that  $U \in C^0(\mathbf{R}^n \times [0, T])$  with  $\text{supp } U \subset \{(x, t) \in \mathbf{R}^n \times [0, T] : |x| \leq t + k\}$  and  $\|U\| < \infty$ . Then, there exists a positive constant  $C$  independent of  $k$  and  $T$  such that*

$$\|L(|U|^p)\| \leq C k^2 \|U\|^p D(T), \quad (2.5.1)$$

where  $D(T)$  is defined by

$$D(T) = \begin{cases} 1 & \text{if } p > p_0(n), \\ \log \frac{2T + 3k}{k} & \text{if } p = p_0(n), \\ \left( \frac{2T + 3k}{k} \right)^{\gamma(p, n)/2} & \text{if } 1 < p < p_0(n) \end{cases} \quad (2.5.2)$$

and  $\gamma(p, n)$  is the one in (2.2.2).

**Lemma 2.5.2** *Let  $n = 5, 7, 9, \dots$  and  $L$  be the linear integral operator defined by (2.3.3). Assume that  $U, U_0 \in C^0(\mathbf{R}^n \times [0, T])$  with  $\text{supp } U \subset \{(x, t) \in \mathbf{R}^n \times [0, T] : |x| \leq t + k\}$ ,  $\text{supp } U_0 \subset \{(x, t) \in \mathbf{R}^n \times [0, T] : t - k \leq |x| \leq t + k\}$  and  $\|U\|, \|\tau_+^{(n-1)/2} U_0 w^{-1}\| < \infty$ . Then, there exists a positive constant  $C_{n,\nu,p}$  depending on  $n, \nu$  and  $p$  such that*

$$\|L(|U_0|^{p-\nu}|U|^\nu)\| \leq C_{n,\nu,p} k^2 \left\| \frac{\tau_+^{(n-1)/2}}{w} U_0 \right\|^{p-\nu} \|U\|^\nu E_\nu(T), \quad (2.5.3)$$

where  $0 \leq \nu \leq p$ .  $E_\nu(T)$  is defined by

$$E_\nu(T) = \begin{cases} 1 & \text{if } p > \frac{n+1}{n-1}, \\ \left(\frac{2T+3k}{k}\right)^{\nu\delta} & \text{if } p = \frac{n+1}{n-1}, \\ \left(\frac{2T+3k}{k}\right)^{-\nu q} & \text{if } p < \frac{n+1}{n-1} \end{cases} \quad \text{for } 0 \leq \nu < p, \quad (2.5.4)$$

where  $q$  is the one in (2.4.12) and  $\delta$  is a small positive constant. When  $\nu = p$ , (2.5.3) coincides with (2.5.1) as  $E_p(T) = D(T)$  and  $C_{n,p,p} = C$ .

**Lemma 2.5.3** *Let  $n = 4, 6, 8, \dots$  and  $L$  be the linear integral operator defined by (2.3.3). Assume that  $U, U_{00}, U_{01} \in C^0(\mathbf{R}^n \times [0, T])$  with  $\text{supp}(U, U_{00}, U_{01}) \subset \{(x, t) \in \mathbf{R}^n \times [0, T] : |x| \leq t + k\}$  and  $\|U\|, \|(\tau_+ \tau_-)^{(n-1)/2} U_{0a} (w \tau_-^a)^{-1}\| < \infty$  ( $a = 0, 1$ ). Then, there exists a positive constant  $C_{n,\nu,p}$  depending on  $n, \nu$ , and  $p$  such that*

$$\|L(|U_{0a}|^{p-\nu}|U|^\nu)\| \leq C_{n,\nu,p} k^2 \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{0a}}{w \tau_-^a} \right\|^{p-\nu} \|U\|^\nu E_{\nu,a}(T), \quad (2.5.5)$$

where  $0 \leq \nu \leq p$  and  $a = 0, 1$ . When  $0 \leq \nu < p$ ,  $E_{\nu,a}(T)$  is defined by

$$E_{\nu,a}(T) = \begin{cases} 1 & \text{if } \mu < -1, \\ \log \frac{2T+3k}{k} & \text{if } \mu = -1, \\ \left(\frac{2T+3k}{k}\right)^{1+\mu} & \text{if } \mu > -1 \end{cases} \quad \text{for } p > \frac{n+1}{n-1}, \quad (2.5.6)$$

where  $\mu = (p - \nu) \left( a - \frac{n-1}{2} \right) - \nu q$  and  $q$  is the one in (2.4.12), and

$$E_{\nu,a}(T) = \begin{cases} \log \frac{2T+3k}{k} & \text{if } \sigma = -1, \nu = 0, \\ \left( \frac{2T+3k}{k} \right)^{1+\sigma} & \text{if } \sigma > -1, \\ \left( \frac{2T+3k}{k} \right)^{\nu\delta} & \text{otherwise} \end{cases} \quad \text{for } p = \frac{n+1}{n-1}, \quad (2.5.7)$$

where  $\sigma = (p - \nu) \left( a - \frac{n-1}{2} \right)$  and  $\delta$  stands for any positive constant, and

$$E_{\nu,a}(T) = \begin{cases} \left( \frac{2T+3k}{k} \right)^{-\nu q} & \text{if } \sigma < -1, \\ \log \frac{2T+3k}{k} \left( \frac{2T+3k}{k} \right)^{-\nu q} & \text{if } \sigma = -1, \text{ for } p < \frac{n+1}{n-1}. \\ \left( \frac{2T+3k}{k} \right)^{1+\mu} & \text{if } \sigma > -1 \end{cases} \quad (2.5.8)$$

When  $\nu = p$ , (2.5.5) coincides with (2.5.1) as  $E_{p,a}(T) = D(T)$  for  $a = 0, 1$  and  $C_{n,p,p} = C$ .

**Lemma 2.5.4** Suppose that the same assumption as in Lemma 2.5.3 is fulfilled. Then, there exists a positive constant  $C_{n,\nu,p}$  depending on  $n$ ,  $\nu$ , and  $p$  such that

$$\begin{aligned} & \|L(|U_{00}|^{p-\nu}|U_{01}|^\nu)\| \\ & \leq C_{n,\nu,p} k^2 \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{00}}{w} \right\|^{p-\nu} \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{01}}{w\tau_-} \right\|^\nu F_\nu(T), \end{aligned} \quad (2.5.9)$$

where  $0 \leq \nu \leq p$ . When  $0 < \nu < p$ ,  $F_\nu(T)$  is defined by

$$F_\nu(T) = \begin{cases} 1 & \text{if } \kappa < -1, \\ \log \frac{2T+3k}{k} & \text{if } \kappa = -1, \\ \left( \frac{2T+3k}{k} \right)^{1+\kappa} & \text{if } \kappa > -1, \end{cases} \quad (2.5.10)$$

where  $\kappa = \nu - \frac{n-1}{2}p$ . When  $\nu = 0$  or  $\nu = p$ , (2.5.9) coincides with (2.5.5) as  $F_0(T) = E_{0,0}(T)$  for  $a = \nu = 0$  and  $F_p(T) = E_{0,1}(T)$  for  $a = 1$ ,  $\nu = p$ .

Four lemmas above follow from the following basic estimate.

**Lemma 2.5.5 (Basic estimate)** *Let  $L$  be the linear integral operator defined by (2.3.3) and  $a_1 \geq 0$ ,  $a_2 \in \mathbf{R}$  and  $a_3 \geq 0$ . Then, there exists a positive constant  $C_{n,p,a_1,a_2,a_3}$  such that*

$$\begin{aligned} & L \left\{ \tau_+^{-(n-1)p/2+a_1} \tau_-^{a_2} (\log(4\tau_+/\tau_-))^{a_3} \right\} (x, t) \\ & \leq C_{n,p,a_1,a_2,a_3} k^2 w(r, t)^{-1} \left( \frac{2T+3k}{k} \right)^{a_1} E_{a_1,a_2,a_3}(T) \end{aligned} \quad (2.5.11)$$

for  $|x| \leq t + k$ ,  $t \in [0, T]$ , where  $E_{a_1,a_2,a_3}(T)$  is defined by

$$E_{a_1,a_2,a_3}(T) = \begin{cases} 1 & \text{if } a_2 < -1 \text{ and } a_3 = 0, \\ \log \frac{2T+3k}{k} & \text{if } a_2 = -1 \text{ and } a_3 = 0, \\ \left( \frac{2T+3k}{k} \right)^{\delta a_3} & \text{if } a_2 \leq -1 \text{ and } a_3 > 0, \\ \left( \frac{2T+3k}{k} \right)^{1+a_2} & \text{if } a_2 > -1, \end{cases} \quad (2.5.12)$$

where  $\delta$  stands for any positive constant.

**Proof.** First we introduce the following fundamental identity for spherical means.

**Lemma 2.5.6 (John [12])** *Let  $b \in C([0, \infty))$ . Then, the identity*

$$\int_{|\omega|=1} b(|x + \rho\omega|) dS_\omega = 2^{3-n} \omega_{n-1} (r\rho)^{2-n} \int_{|\rho-r|}^{\rho+r} \lambda h(\lambda, \rho, r) b(\lambda) d\lambda \quad (2.5.13)$$

holds for  $x \in \mathbf{R}^n$ ,  $r = |x|$  and  $\rho > 0$ , where

$$h(\lambda, \rho, r) = \{\lambda^2 - (\rho - r)^2\}^{(n-3)/2} \{(\rho + r)^2 - \lambda^2\}^{(n-3)/2}. \quad (2.5.14)$$

See [12] for the proof of this lemma.

In order to continue the proof of Lemma 2.5.5, we need radially symmetric versions of  $L$  which follows from Lemma 2.5.6. From now on, a positive constant  $C$  depending only on  $n$  and  $p$  may change from line to line.

**Lemma 2.5.7** *Let  $L$  be the linear integral operator defined by (2.3.3) and  $\Psi = \Psi(|x|, t) \in C([0, \infty)^2)$ ,  $x \in \mathbf{R}^n$ . Then,*

$$L(\Psi)(x, t) = L_{\text{odd}}(\Psi)(r, t), \quad r = |x| \quad (2.5.15)$$

holds for  $n = 5, 7, 9 \dots$  and

$$L(\Psi)(x, t) = L_{\text{even},1}(\Psi)(r, t) + L_{\text{even},2}(\Psi)(r, t), \quad r = |x| \quad (2.5.16)$$

holds for  $n = 4, 6, 8 \dots$ , where  $L_{\text{odd}}(\Psi)$  is defined by

$$\begin{aligned} & L_{\text{odd}}(\Psi)(r, t) \\ &= Cr^{2-n} \int_0^t (t-\tau)^{3-n} d\tau \int_{|t-\tau-r|}^{t-\tau+r} \lambda h(\lambda, t-\tau, r) \Psi(\lambda, \tau) d\lambda \end{aligned} \quad (2.5.17)$$

and each  $L_{\text{even},i}(\Psi)$  ( $i = 1, 2$ ) is defined by

$$\begin{aligned} L_{\text{even},1}(\Psi)(r, t) = & Cr^{2-n} \int_0^t (t-\tau)^{2-n} d\tau \int_{|t-r-\tau|}^{t+r-\tau} \lambda \Psi(\lambda, \tau) d\lambda \times \\ & \times \int_{|\lambda-r|}^{t-\tau} \frac{\rho h(\lambda, \rho, r)}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho, \end{aligned} \quad (2.5.18)$$

$$\begin{aligned} L_{\text{even},2}(\Psi)(r, t) = & Cr^{2-n} \int_0^{(t-r)_+} (t-\tau)^{2-n} d\tau \int_0^{t-r-\tau} \lambda \Psi(\lambda, \tau) d\lambda \times \\ & \times \int_{|\lambda-r|}^{\lambda_0+r} \frac{\rho h(\lambda, \rho, r)}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho. \end{aligned} \quad (2.5.19)$$

Here the usual notation  $a_+ = \max\{a, 0\}$  is used.

**Proof.** (2.5.15) immediately follows from Lemma 2.5.6. For (2.5.16), we make use of changing variables by  $y - x = (t - \tau)\xi$  in (2.3.3). Then, we obtain

$$L(\Psi)(x, t) = C \int_0^t (t-\tau)^{2-n} d\tau \int_{|y-x| \leq t-\tau} \frac{\Psi(|y|, \tau)}{\sqrt{(t-\tau)^2 - |y-x|^2}} dy.$$

Introducing polar coordinates, we have

$$\begin{aligned} L(\Psi)(x, t) = & C \int_0^t (t-\tau)^{2-n} d\tau \int_0^{t-\tau} \frac{\rho^{n-1} d\rho}{\sqrt{(t-\tau)^2 - \rho^2}} \times \\ & \times \int_{|\omega|=1} \Psi(|x + \rho\omega|, \tau) dS_\omega. \end{aligned}$$

Thus Lemma 2.5.6 yields

$$\begin{aligned} L(\Psi)(x, t) = & Cr^{2-n} \int_0^t (t-\tau)^{2-n} d\tau \int_0^{t-\tau} \frac{\rho d\rho}{\sqrt{(t-\tau)^2 - \rho^2}} \times \\ & \times \int_{|\rho-r|}^{\rho+r} \lambda \Psi(\lambda, \tau) h(\lambda, \rho, r) d\lambda. \end{aligned} \quad (2.5.20)$$

Therefore, (2.5.16) follows from inverting the order of  $(\rho, \lambda)$ -integral in (2.5.20).

□

In order to estimate the kernel  $h(\lambda, \rho, r)$ , we need the following lemma.

**Lemma 2.5.8 (Agemi, Kubota and Takamura [1])** *Let  $h(\lambda, \rho, r)$  be the one in (2.5.14). Suppose that  $|\rho - r| \leq \lambda \leq \rho + r$  and  $\rho \geq 0$ . Then, the inequality*

$$|\lambda - r| \leq \rho \leq \lambda + r \quad (2.5.21)$$

*holds. Moreover, the following three estimates are available.*

$$h(\lambda, \rho, r) \leq Cr^{n-3}\lambda^{n-3}, \quad (2.5.22)$$

$$h(\lambda, \rho, r) \leq C\rho^{n-3}r^{(n-3)/2}\lambda^{(n-3)/2}, \quad (2.5.23)$$

$$h(\lambda, \rho, r) \leq Cr^{n-3}\rho^{n-3}. \quad (2.5.24)$$

See 257-258pp. in [1] for the proof of this lemma.

Let us continue the proof of Lemma 2.5.5. For simplicity, we set

$$\begin{aligned} I_{odd}(r, t) &= L_{odd} \left\{ \tau_+^{-(n-1)p/2+a_1} \tau_-^{a_2} (\log(4\tau_+/\tau_-))^{a_3} \right\} (r, t), \\ I_{even,i}(r, t) &= L_{even,i} \left\{ \tau_+^{-(n-1)p/2+a_1} \tau_-^{a_2} (\log(4\tau_+/\tau_-))^{a_3} \right\} (r, t) \quad (i = 1, 2). \end{aligned}$$

**Estimates for  $I_{odd}$  and  $I_{even,1}$ .** We shall estimate  $I_{odd}$  and  $I_{even,1}$  on the following three domains.

$$\begin{aligned} D_1 &= \{(r, t) \mid r \geq t - r > -k \text{ and } r \geq 2k\}, \\ D_2 &= \{(r, t) \mid r \geq t - r > -k \text{ and } r \leq 2k\}, \\ D_3 &= \{(r, t) \mid t - r \geq r\}. \end{aligned}$$

(i) Estimate in  $D_1$ ,

Making use of (2.5.23), we get

$$\begin{aligned} I_{odd}(r, t) &\leq Cr^{-(n-1)/2} \int_0^t d\tau \int_{|t-\tau-r|}^{t+r-\tau} \lambda^{(n-1)/2} \tau_-^{a_2} \times \\ &\quad \times \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda \end{aligned} \quad (2.5.25)$$

and

$$\begin{aligned} I_{even,1}(r, t) &\leq Cr^{-(n-1)/2} \int_0^t (t-\tau)^{2-n} d\tau \int_{|t-\tau-r|}^{t+r-\tau} \lambda^{(n-1)/2} \times \\ &\quad \times \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \tau_-^{a_2} \times \\ &\quad \times \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda \int_{|\lambda-r|}^{t-\tau} \frac{\rho^{n-2}}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho. \end{aligned} \quad (2.5.26)$$

If one apply the simple inequality

$$\int_{|\lambda-r|}^{t-\tau} \frac{\rho^{n-2}}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho \leq (t-\tau)^{n-2} \quad \text{for } 0 \leq \tau \leq t \quad (2.5.27)$$

to the right-hand side of (2.5.26), the same quantity as the right-hand side of (2.5.25) appears. Hence, we shall estimate for  $I_{odd}$  only from now on.

Changing variables in (2.5.25) by

$$\alpha = \tau + \lambda, \quad \beta = \tau - \lambda, \quad (2.5.28)$$

we get

$$\begin{aligned} I_{odd}(r, t) &\leq C r^{-(n-1)/2} \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{a_2} d\beta \int_{|t-r|}^{t+r} (\alpha - \beta)^{(n-1)/2} \times \\ &\quad \times \left( \frac{\alpha + 2k}{k} \right)^{-(n-1)p/2+a_1} \left( \log 4 \frac{\alpha + 2k}{\beta + 2k} \right)^{a_3} d\alpha. \end{aligned}$$

It follows from

$$\frac{r}{k} = \frac{2r + r + r}{4k} \geq \frac{\tau_+(r, t)}{4}$$

that

$$\begin{aligned} I_{odd}(r, t) &\leq C \tau_+(r, t)^{-(n-1)/2} \left( \frac{t + r + 2k}{k} \right)^{a_1} \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{a_2} d\beta \times \\ &\quad \times \int_{t-r}^{t+r} \left( \frac{\alpha + 2k}{k} \right)^{-1-q} \left( \log 4 \frac{\alpha + 2k}{\beta + 2k} \right)^{a_3} d\alpha. \end{aligned} \quad (2.5.29)$$

When  $a_3 = 0$ ,  $\alpha$ -integral in (2.5.29) is dominated by

$$\begin{cases} Ck\tau_-^{-q} & \text{if } p > \frac{n+1}{n-1}, \\ k \log \frac{\tau_+}{\tau_-} & \text{if } p = \frac{n+1}{n-1}, \\ Ck\tau_+^{-q} & \text{if } p < \frac{n+1}{n-1} \end{cases}$$

and  $\beta$ -integral in (2.5.29) is dominated by

$$\begin{cases} \frac{k}{-(1+a_2)} & \text{if } a_2 < -1, \\ k \log \frac{t-r+2k}{k} & \text{if } a_2 = -1, \\ \frac{k}{(1+a_2)} \left( \frac{t-r+2k}{k} \right)^{1+a_2} & \text{if } a_2 > -1. \end{cases}$$

(2.5.11) is now established for  $a_3 = 0$ .

When  $a_3 > 0$ , we employ the following simple lemma.

**Lemma 2.5.9** *Let  $\delta > 0$  be any given constant. Then, we have*

$$\log X \leq \frac{X^\delta}{\delta} \text{ for } X \geq 1. \quad (2.5.30)$$

The proof of this lemma follows from elementary computation. We shall omit it. Then, it follows from Lemma 2.5.9 that

$$\begin{aligned} I_{odd}(r, t) &\leq C(4\delta^{-1})^{a_3} \tau_+(r, t)^{-(n-1)/2} \left( \frac{t+r+2k}{k} \right)^{a_1+\delta a_3} \times \\ &\quad \times \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2-\delta a_3} d\beta \int_{t-r}^{t+r} \left( \frac{\alpha+2k}{k} \right)^{-1-q} d\alpha. \end{aligned}$$

The  $\alpha$ -integral above can be estimated by the same manner in the case of  $a_3 = 0$ . The  $\beta$ -integral is dominated by

$$\begin{cases} \frac{-k}{1+a_2-\delta a_3} & \text{if } a_2 \leq -1, \\ \frac{k}{1+a_2-\delta a_3} \left( \frac{t-r+2k}{k} \right)^{1+a_2-\delta a_3} & \text{if } a_2 > -1 \end{cases} \quad (2.5.31)$$

with  $\delta > 0$  satisfying  $1+a_2-\delta a_3 > 0$ . Therefore  $I_{odd}$  and  $I_{even,1}$  are bounded in  $D_1$  by the quantity in the right-hand side of (2.5.11) as desired.

(ii) Estimate in  $D_2$ .

By making use of (2.5.24), we have

$$\begin{aligned} I_{odd}(r, t) &\leq Cr^{-1} \int_0^t d\tau \int_{|t-\tau-r|}^{t+r-\tau} \lambda \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \times \\ &\quad \times \tau_-(\lambda, \tau)^{a_2} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda \end{aligned} \quad (2.5.32)$$

and

$$\begin{aligned} I_{even,1}(r, t) &\leq Cr^{-1} \int_0^t (t-\tau)^{-1} d\tau \int_{|t-\tau-r|}^{t+r-\tau} \lambda \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \times \\ &\quad \times \tau_-(\lambda, \tau)^{a_2} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda \int_{|\lambda-r|}^{t-\tau} \frac{\rho}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho. \end{aligned}$$

Similarly to the estimate in  $D_1$ , the simple inequality

$$\int_{|\lambda-r|}^{t-\tau} \frac{\rho}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho \leq t-\tau \quad \text{for } 0 \leq \tau \leq t$$



helps us to estimate  $I_{odd}$  only. Changing variables by (2.5.28), we get

$$I_{odd}(r, t) \leq Ckr^{-1} \left( \frac{t+r+2k}{k} \right)^{a_1} \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2} d\beta \times \int_{t-r}^{t+r} \left( \frac{\alpha+2k}{k} \right)^{1-(n-1)p/2} \left( \log 4 \frac{\alpha+2k}{\beta+2k} \right)^{a_3} d\alpha. \quad (2.5.33)$$

Note that both  $\tau_+$  and  $\tau_-$  are numerical constants in this domain, and that the integrand of both  $\alpha$ -integral and  $\beta$ -integral in (2.5.33) is numerical constant  $C_{a_1, a_2, a_3}$  depending on  $a_1$ ,  $a_2$  and  $a_3$ . Hence we have

$$I_{odd}(r, t) \leq CC_{a_1, a_2, a_3} kr^{-1} \int_{-k}^{t-r} d\beta \int_{t-r}^{t+r} d\alpha \leq CC_{a_1, a_2, a_3} k^2. \quad (2.5.34)$$

This is the desired estimate in  $D_2$ .

(iii) Estimate in  $D_3$ .

By the same reason, we have to estimate  $I_{odd}$  in (2.5.33) only. In  $D_3$ , since  $1 - (n-1)p/2 < 0$  is trivial, we get

$$\left( \frac{\alpha+2k}{k} \right)^{1-(n-1)p/2} \leq \left( \frac{t-r+2k}{k} \right)^{1-(n-1)p/2} \leq Cw(r, t)^{-1}$$

because  $t-r \geq r$  is equivalent to  $3(t-r) \geq t+r$ . Hence, when  $a_3 = 0$ , we obtain

$$I_{odd}(r, t) \leq Ckw(r, t)^{-1} \left( \frac{t+r+2k}{k} \right)^{a_1} \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2} d\beta.$$

When  $a_3 > 0$ , due to Lemma 2.5.9, we have

$$I_{odd}(r, t) \leq C(4\delta^{-1})^{a_3} kw(r, t)^{-1} \times \left( \frac{t+r+2k}{k} \right)^{a_1+\delta a_3} \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2-\delta a_3} d\beta.$$

Therefore, in view of (2.5.31),  $I_{odd}$  and  $I_{even,1}$  are bounded in  $D_3$  by the quantity in the right-hand side of (2.5.11).

**Estimates for  $I_{even,2}$ .** We shall estimate  $I_{even,2}$  on the following three domains.

$$\begin{aligned} D_4 &= \{(r, t) \mid 0 < t-r \leq k \text{ and } t \leq 2k\}, \\ D_5 &= \{(r, t) \mid 0 < t-r \leq k \text{ and } t \geq 2k\}, \\ D_6 &= \{(r, t) \mid t-r \geq k\}. \end{aligned}$$

(iv) Estimate in  $D_4$ ,

Note that  $\tau_+$  and  $\tau_-$  are numerical constants in this case. By virtue of (2.5.21) and (2.5.22), we get

$$\begin{aligned} I_{even,2}(r, t) &\leq CC_{a_1, a_2, a_3} r^{-1} \int_0^{t-r} (t-\tau)^{2-n} d\tau \int_0^{t-r-\tau} \lambda^{n-2} d\lambda \\ &\quad \times \int_{|\lambda-r|}^{\lambda+r} \frac{\rho}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho. \end{aligned}$$

It follows from

$$\int_{|\lambda-r|}^{\lambda+r} \frac{\rho d\rho}{\sqrt{(t-\tau)^2 - \rho^2}} \leq \frac{2r\lambda}{\sqrt{t-\tau+\lambda+r}\sqrt{t-\tau-\lambda-r}} \quad (2.5.35)$$

that

$$\begin{aligned} I_{even,2}(r, t) &\leq CC_{a_1, a_2, a_3} \int_0^{t-r} (t-\tau)^{3/2-n} d\tau \int_0^{t-r-\tau} \frac{\lambda^{n-1}}{\sqrt{t-\tau-\lambda-r}} d\lambda. \end{aligned}$$

Noticing that

$$\lambda \leq t-r-\tau \leq t-\tau \quad \text{for } \tau \geq 0,$$

we obtain

$$I_{even,2}(r, t) = CC_{a_1, a_2, a_3} \int_0^{t-r} (t-\tau)^{1/2} d\tau \int_0^{t-r-\tau} \frac{d\lambda}{\sqrt{t-\tau-\lambda-r}}.$$

Making use of (2.5.28), we have

$$\begin{aligned} I_{even,2}(r, t) &\leq CC_{a_1, a_2, a_3} k^{1/2} \int_{-k}^{t-r} d\beta \int_{\beta}^{t-r} \frac{d\alpha}{\sqrt{t-r-\alpha}} \\ &\leq CC_{a_1, a_2, a_3} k^2. \end{aligned}$$

This is the desired estimate in  $D_4$ .

(v) Estimate in  $D_5$ .

In this domain, (2.5.11) follows from

$$I_{even,2}(r, t) \leq Ck^2 C_{a_1, 12, a_3} \tau_+(r, t)^{-(n-1)/2}.$$

To see this, we shall employ (2.5.21) and (2.5.23). Then we have

$$\begin{aligned} I_{even,2}(r, t) &\leq Cr^{-(n-1)/2} \int_0^{t-r} (t-\tau)^{2-n} d\tau \int_0^{t-r-\tau} \lambda^{(n-1)/2} \times \\ &\quad \times \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \tau_-(\lambda, \tau)^{a_2} \times \\ &\quad \times \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda \int_{|\lambda-r|}^{\lambda+r} \frac{\rho^{n-2}}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho. \end{aligned}$$

Similarly to (2.5.34), it follows from (2.5.27) that

$$I_{even,2}(r, t) \leq CC_{a_1, a_2, a_3} r^{-(n-1)/2} k^2.$$

In  $D_5$ , we have  $r \geq k$  which implies  $r/k \geq C\tau_+(r, t)$ . Hence, we obtain the desired estimate.

(vi) Estimate in  $D_6$ .

By virtue of (2.5.22) and (2.5.35), we get

$$\begin{aligned} & I_{even,2}(r, t) \\ & \leq C \int_0^{t-r} (t-\tau)^{3/2-n} d\tau \int_0^{t-r-\tau} \frac{\lambda^{n-1}}{\sqrt{t-\tau-\lambda-r}} \times \\ & \quad \times \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \tau_-(\lambda, \tau)^{a_2} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda. \end{aligned}$$

Then, we divide the integral of the right-hand side above as

$$I_{even,3}(r, t) + I_{even,4}(r, t),$$

where

$$\begin{aligned} I_{even,3}(r, t) &= C \int_0^{(t-r)/2} (t-\tau)^{3/2-n} d\tau \int_0^{t-r-\tau} \frac{\lambda^{n-1}}{\sqrt{t-\tau-\lambda-r}} \times \\ & \quad \times \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \tau_-(\lambda, \tau)^{a_2} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda \end{aligned}$$

and

$$\begin{aligned} I_{even,4}(r, t) &= C \int_{(t-r)/2}^{t-r} (t-\tau)^{3/2-n} d\tau \int_0^{t-r-\tau} \frac{\lambda^{n-1}}{\sqrt{t-\tau-\lambda-r}} \times \\ & \quad \times \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \tau_-(\lambda, \tau)^{a_2} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda. \end{aligned}$$

First, we shall estimate  $I_{even,3}$ . It follows from (2.5.28) that

$$\begin{aligned} & I_{even,3}(r, t) \\ & \leq Ck^{1/2} \left( \frac{t+r+2k}{k} \right)^{3/2-n} \left( \frac{t-r+2k}{k} \right)^{n-1-(n-1)p/2+a_1} \times \\ & \quad \times \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2} d\beta \int_{\beta}^{t-r} \left( \log 4 \frac{\alpha+2k}{\beta+2k} \right)^{a_3} \frac{d\alpha}{\sqrt{t-r-\alpha}} \end{aligned}$$

because of  $n-1-(n-1)p/2 \geq 0$  for  $p \leq 2$ . When  $a_3 = 0$ , we get

$$\begin{aligned} I_{even,3}(r, t) &\leq Ck \left( \frac{t+r+2k}{k} \right)^{3/2-n} \times \\ & \quad \times \left( \frac{t-r+2k}{k} \right)^{-q+(n-2)/2+a_1} \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2} d\beta. \end{aligned}$$

Hence the desired estimate follows from

$$\frac{\tau_-(r, t)^{-q+(n-2)/2}}{\tau_+(r, t)^{n-3/2}} \leq Cw(r, t)^{-1} \quad (2.5.36)$$

in this case. When  $a_3 > 0$ , Lemma 2.5.9 yields that

$$I_{even,3}(r, t) \leq Ck(4\delta^{-1})^{a_3} \left( \frac{t+r+2k}{k} \right)^{3/2-n} \times \left( \frac{t-r+2k}{k} \right)^{-q+(n-2)/2+a_1+\delta a_3} \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2-\delta a_3} d\beta.$$

Hence the desired estimate follows from (2.5.31) and (2.5.36).

Next, we shall estimate  $I_{even,4}$ . If  $r \geq t-r \geq k$ , (2.5.28) yields that

$$I_{even,4}(r, t) \leq \frac{Ck^{n-1}}{r^{n-3/2}} \left( \frac{t-r+2k}{k} \right)^{n-1-(n-1)p/2+a_1} \times \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2} d\beta \int_{\beta}^{t-r} \frac{d\alpha}{\sqrt{t-r-\alpha}} \left( \log 4 \frac{\alpha+2k}{\beta+2k} \right)^{a_3}.$$

In this case, we have  $r/k \geq C\tau_+(r, t)$ , so that (2.5.11) follows from the same argument as for  $I_{even,3}$ . On the other hand, if  $t-r \geq r$  and  $t-r \geq k$ , we have

$$\tau + \lambda + 2k \geq \frac{t-r}{2} + 2k \geq \frac{t+r+2k}{6} \text{ for } \tau \geq \frac{t-r}{2}, \lambda \geq 0.$$

Hence (2.5.35) yields that

$$I_{even,4}(r, t) \leq Ck^{1/2} \left( \frac{t+r+2k}{k} \right)^{1/2-(n-1)p/2+a_1} \times \int_0^{t-r} d\tau \int_0^{t-r-\tau} \frac{\tau_-(\lambda, \tau)^{a_2}}{\sqrt{t-\tau-\lambda-r}} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda.$$

Changing variables by (2.5.28), we have

$$I_{even,4}(r, t) \leq Ck^{1/2} \left( \frac{t+r+2k}{k} \right)^{1/2-(n-1)p/2+a_1} \times \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2} d\beta \int_{\beta}^{t-r} \left( \log 4 \frac{\alpha+2k}{\beta+2k} \right)^{a_3} \frac{d\alpha}{\sqrt{t-r-\alpha}}.$$

Therefore, applying the simple inequality

$$\tau_+(r, t)^{1-(n-1)p/2} \leq Cw(r, t)^{-1},$$

we obtain the desired estimates by the same argument as for  $I_{even,3}$ . The proof of Lemma 2.5.5 is now completed.  $\square$

**Proof of Lemma 2.5.1.** Since (2.3.3) and (2.4.10) yield that

$$L(|U|^p)(x, t) \leq \|U\|^p L(w^{-p})(x, t),$$

it is enough to show the inequality

$$w(r, t)L(w^{-p})(x, t) \leq Ck^2 D(T).$$

This is established by (2.5.11) with setting

$$\begin{cases} a_1 = a_3 = 0, & a_2 = -pq & \text{if } p > \frac{n+1}{n-1}, \\ a_1 = a_2 = 0, & a_3 = p & \text{if } p = \frac{n+1}{n-1}, \\ a_1 = -pq, & a_2 = a_3 = 0 & \text{if } p < \frac{n+1}{n-1}. \end{cases}$$

$\square$

**Proof of Lemma 2.5.2.** Due to Huygens' principle in Lemma 2.4.1, one can replace  $\tau_-$  by  $\tau_- \chi_{\{-k \leq t-r \leq k\}}$  in (2.5.11). Then, the integral with respect to the variable of  $\beta = \tau - \lambda$  is bounded. In order to establish (2.5.3), it is enough to show the inequality

$$w(r, t)L\left(\tau_+^{-(n-1)(p-\nu)/2} w^{-\nu} \chi_{\{-k \leq t-r \leq k\}}\right) \leq C_{n,\nu,p} k^2 E_\nu(T).$$

This is established by (2.5.11) with setting

$$\begin{cases} a_1 = a_3 = 0, & a_2 = -\nu q & \text{if } p > \frac{n+1}{n-1}, \\ a_1 = a_2 = 0, & a_3 = \nu & \text{if } p = \frac{n+1}{n-1}, \\ a_1 = -\nu q, & a_2 = a_3 = 0 & \text{if } p < \frac{n+1}{n-1}. \end{cases}$$

$\square$

**Proof of Lemma 2.5.3 and Lemma 2.5.4.** In order to prove (2.5.5) and (2.5.9), it is enough to show inequalities

$$w(r, t)L\left(\tau_+^{-(n-1)(p-\nu)/2} \tau_-^\sigma w^{-\nu}\right) \leq C_{n,\nu,p} k^2 E_{\nu,a}(T)$$

and

$$w(r, t)L\left(\tau_+^{-(n-1)p/2} \tau_-^\kappa\right) \leq C_{n,\nu,p} k^2 F_\nu(T).$$

If we set

$$\begin{cases} a_1 = a_3 = 0, a_2 = \mu & \text{if } p > \frac{n+1}{n-1}, \\ a_1 = 0, a_2 = \sigma, a_3 = \nu & \text{if } p = \frac{n+1}{n-1}, \\ a_1 = -\nu q, a_2 = \sigma, a_3 = 0 & \text{if } p < \frac{n+1}{n-1} \end{cases}$$

in (2.5.11), we have (2.5.5). If we set  $a_1 = a_3 = 0, a_2 = \kappa$  in (2.5.11), we have (2.5.9).  $\square$

## 2.6 Lower bound in odd space dimensions

In this section, we prove Theorem 2.3.1 in odd space dimensions. It is obviously enough for this to show the following proposition.

**Proposition 2.6.1** *Let  $n = 5, 7, 9, \dots$ . Assume (3.1.5). Then, there exists a positive constant  $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$  such that each of (2.3.2) and (2.3.9) admits a unique solution  $u \in C^1(\mathbf{R}^n \times [0, T])$  as far as  $T$  satisfies*

$$T \leq \begin{cases} c\varepsilon^{-2p(p-1)/\gamma(p,n)} & \text{if } 1 < p < p_0(n) \\ \exp(c\varepsilon^{-p(p-1)}) & \text{if } p = p_0(n) \end{cases} \quad (2.6.1)$$

for  $0 < \varepsilon \leq \varepsilon_0$ , where  $c$  is a positive constant independent of  $\varepsilon$ .

Our purpose is to construct a solution of the integral equation (2.4.8) as a limit of  $\{U_l\}_{l \in \mathbf{N}}$  in  $X$ . To end this, we define a closed subspace  $Y_0$  in  $X$  by

$$Y_0 = \{U \in X : \|\nabla_x^\alpha U\| \leq 2M_0\varepsilon^p (|\alpha| \leq 1)\},$$

where we set

$$M_0 = 2^p p C_{n,0,p} k^2 C_0^p > 0.$$

Recall that  $C_{n,0,p}$  is the one in (2.5.3). We note that there exists a positive constant  $C_0$  independent of  $\varepsilon$  which satisfies that

$$\left\| \frac{\tau_+^{(n-1)/2}}{w} \partial W_0 \right\| \leq C_0 \varepsilon \quad \text{for } 0 < \varepsilon \leq 1. \quad (2.6.2)$$

It is easy to check that this fact follows from the definition of the weight function in (2.4.11) and the decay estimate for  $U_0 = v$  in (2.4.4) and (2.4.5).

**Proof of Proposition 2.6.1.** First of all, we assume that

$$0 < \varepsilon \leq 1 \quad (2.6.3)$$

to make use of (2.6.2). We shall show the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ . The boundedness of  $\{U_l\}_{l \in \mathbf{N}}$ ,

$$\|U_l\| \leq 2M_0\varepsilon^p \quad (l \in \mathbf{N}), \quad (2.6.4)$$

can be obtained by induction with respect to  $l$  as follows. Recall that  $L$  is a positive and linear operator by its definition, (2.3.3). It follows from (2.4.9), (2.5.3) with  $\nu = 0$  and (2.6.2) that

$$\begin{aligned} \|U_1\| &= 2^p \|L(|U_0|^p)\| \\ &\leq 2^p C_{n,0,p} k^2 \|\tau_+^{(n-1)/2} w^{-1} U_0\|^p E_0(T) \leq M_0 \varepsilon^p, \end{aligned} \quad (2.6.5)$$

where we have used  $E_0(T) = 1$  for  $p > 1$ . Assume that  $\|U_{l-1}\| \leq 2M_0\varepsilon^p$  ( $l \geq 2$ ). It follows from the simple estimate

$$\begin{aligned} |U_l| &= L(|U_{l-1} + U_0|^p) \\ &\leq 2^p \{L(|U_{l-1}|^p) + L(|U_0|^p)\}, \end{aligned} \quad (2.6.6)$$

(2.5.1) and (2.5.3) with  $\nu = 0$  that

$$\|U_l\| \leq 2^p k^2 \{C \|U_{l-1}\|^p D(T) + C_{n,0,p} \|\tau_+^{(n-1)/2} w^{-1} U_0\|^p E_0(T)\}.$$

Hence (2.6.2) and the assumption of the induction yield that

$$\|U_l\| \leq 2^p C k^2 (2M_0\varepsilon^p)^p D(T) + M_0\varepsilon^p.$$

This inequality shows (2.6.4) provided

$$2^p C k^2 (2M_0)^p \varepsilon^{p^2} D(T) \leq M_0 \varepsilon^p. \quad (2.6.7)$$

Next we shall estimate the differences of  $\{U_l\}_{l \in \mathbf{N}}$  under the conditions, (2.6.3) and (2.6.7) ensuring the boundedness (2.6.4). Then there exists a  $\theta \in (0, 1)$  such that

$$\begin{aligned} |U_{l+1} - U_l| &= |L\{F'(U_{l-1} + U_0 + \theta(U_l - U_{l-1}))(U_l - U_{l-1})\}| \\ &\leq pL\{|U_{l-1} + U_0 + \theta(U_l - U_{l-1})|^{p-1} |U_l - U_{l-1}|\}. \end{aligned} \quad (2.6.8)$$

Hence we have

$$|U_{l+1} - U_l| \leq 2^{p-1} pL\{(|3W_l|^{p-1} + |U_0|^{p-1})|U_l - U_{l-1}|\}. \quad (2.6.9)$$

Hölder's inequality (2.4.13) and *a priori* estimate (2.5.1) yield that

$$\begin{aligned} &\|L(|3W_l|^{p-1} |U_l - U_{l-1}|)\| \\ &= \|L\{(|3W_l|^{(p-1)/p} |U_l - U_{l-1}|^{1/p})^p\}\| \\ &\leq C k^2 \| |3W_l|^{(p-1)/p} |U_l - U_{l-1}|^{1/p} \|^p D(T) \\ &\leq C k^2 \|3W_l\|^{p-1} D(T) \|U_l - U_{l-1}\|. \end{aligned} \quad (2.6.10)$$

We note that (2.6.4) implies  $\|W_l\| \leq 2M_0\varepsilon^p$  ( $l \in \mathbf{N}$ ). Moreover, (2.5.3) with  $\nu = 1$  implies that

$$\begin{aligned} & \|L(|U_0|^{p-1}|U_l - U_{l-1}|)\| \\ & \leq C_{n,1,p}k^2\|\tau_+^{(n-1)/2}w^{-1}U_0\|^{p-1}\|U_l - U_{l-1}\|E_1(T). \end{aligned} \quad (2.6.11)$$

Since (2.6.4) implies that  $\|W_l\| \leq 2M_0\varepsilon^p$  for  $l \geq 2$ , the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  follows from

$$\|U_{l+1} - U_l\| \leq \frac{1}{2}\|U_l - U_{l-1}\| \quad \text{for } l \geq 2$$

provided

$$2^{p-1}pk^2\{C(6M_0\varepsilon^p)^{p-1}D(T) + C_{n,1,p}(C_0\varepsilon)^{p-1}E_1(T)\} \leq \frac{1}{2}, \quad (2.6.12)$$

In fact, we obtain

$$\|U_{l+1} - U_l\| \leq \frac{1}{2^{l-1}}\|U_2 - U_1\| \quad \text{for } l \geq 2 \quad (2.6.13)$$

which implies the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ .

Now we shall show the convergence of  $\{\partial_i U_l\}_{l \in \mathbf{N}}$  for  $i = 1, 2, \dots, n$  under the conditions, (2.6.3), (2.6.7) and (2.6.12) which ensure the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ . As before, the boundedness of  $\{\partial_i U_l\}_{l \in \mathbf{N}}$  for  $i = 1, 2, \dots, n$ ,

$$\|\partial_i U_l\| \leq 2M_0\varepsilon^p \quad (l \in \mathbf{N}, i = 1, 2, \dots, n), \quad (2.6.14)$$

can be obtained by induction as follows. Similarly to (2.6.5), we have

$$\begin{aligned} \|\partial_i U_1\| &= 2^p p \|L(|U_0|^{p-1}|\partial_i U_0|)\| \\ &\leq 2^p p \|L(|\partial W_0|^p)\| \leq 2^p p C_{n,0,p}k^2\|\tau_+^{(n-1)/2}w^{-1}\partial W_0\|^p E_0(T). \end{aligned}$$

Hence (2.6.2) and  $E_0(T) = 1$  for  $p > 1$  implies  $\|\partial_i U_1\| \leq M_0\varepsilon^p$ . Assume that  $\|\partial_i U_{l-1}\| \leq 2M_0\varepsilon^p$  ( $l \geq 2$ ). We note that this means  $\|\partial W_{l-1}\| \leq 2M_0\varepsilon^p$  ( $l \geq 2$ ). It follows from (2.4.9) that

$$\begin{aligned} |\partial_i U_l| &= p|L(|U_{l-1} + U_0|^{p-1}|\partial_i(U_{l-1} + U_0)|)| \\ &\leq 2^{p-1}pL\{|U_{l-1}|^{p-1} + |U_0|^{p-1}\}(|\partial_i U_{l-1}| + |\partial_i U_0|). \end{aligned}$$

Similarly to (2.6.10) and (2.6.11), we obtain that

$$\begin{aligned} & \|L(|U_{l-1}|^{p-1}|\partial_i U_{l-1}|)\| \leq Ck^2\|U_{l-1}\|^{p-1}\|\partial_i U_{l-1}\|D(T), \\ & \|L(|U_{l-1}|^{p-1}|\partial_i U_0|)\| \leq C_{n,p-1,p}k^2\|U_{l-1}\|^{p-1}\|\tau_+^{(n-1)/2}w^{-1}\partial_i U_0\|E_{p-1}(T), \\ & \|L(|U_0|^{p-1}|\partial_i U_{l-1}|)\| \leq C_{n,1,p}k^2\|\tau_+^{(n-1)/2}U_0w^{-1}\|^{p-1}\|\partial_i U_{l-1}\|E_1(T), \\ & \|L(|U_0|^{p-1}|\partial_i U_0|)\| \leq C_{n,0,p}k^2\|\tau_+^{(n-1)/2}\partial W_0w^{-1}\|^p E_0(T). \end{aligned}$$



Hence, we get

$$\begin{aligned}
\|\partial_i U_l\| &\leq 2^{p-1}pk^2\{C\|\partial W_{l-1}\|^p D(T) \\
&\quad + C_{n,p-1,p}\|\partial W_{l-1}\|^{p-1}\|\tau_+^{(n-1)/2}w^{-1}\partial W_0\|E_{p-1}(T) \\
&\quad + C_{n,1,p}\|\partial W_{l-1}\|\|\tau_+^{(n-1)/2}w^{-1}\partial W_0\|^{p-1}E_1(T) \\
&\quad + C_{n,0,p}\|\tau_+^{(n-1)/2}w^{-1}\partial W_0\|^p\} \\
&\leq 2^{p-1}pk^2\{C(2M_0\varepsilon^p)^p D(T) + C_{n,p-1,p}(2M_0\varepsilon^p)^{p-1}C_0\varepsilon E_{p-1}(T) \\
&\quad + C_{n,1,p}(2M_0\varepsilon^p)(C_0\varepsilon)^{p-1}E_1(T) + C_{n,0,p}(C_0\varepsilon)^p\}.
\end{aligned}$$

This inequality shows (2.6.14) provided

$$\begin{aligned}
(3/2)M_0\varepsilon^p &\geq 2^{p-1}pk^2\{C(2M_0)^p\varepsilon^{p^2}D(T) \\
&\quad + C_{n,p-1,p}(2M_0)^{p-1}C_0\varepsilon^{p^2-p+1}E_{p-1}(T) \\
&\quad + C_{n,1,p}2M_0C_0^{p-1}\varepsilon^{2p-1}E_1(T)\}.
\end{aligned} \tag{2.6.15}$$

Next we shall estimate the differences of  $\{\partial_i U_l\}_{l \in \mathbf{N}}$  for  $i = 1, 2, \dots, n$ , under the conditions, (2.6.3), (2.6.7), (2.6.12) and (2.6.15) which ensure the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  and the boundedness of  $\{\partial W_l\}_{l \in \mathbf{N}}$ . (2.4.9) implies that

$$\begin{aligned}
&|\partial_i U_{l+1} - \partial_i U_l| \\
&= p|L\{|U_l + U_0|^{p-1}(\partial_i U_l + \partial_i U_0) - |U_{l-1} + U_0|^{p-1}(\partial_i U_{l-1} + \partial_i U_0)\}| \\
&\leq pL\{|U_l + U_0|^{p-1}|\partial_i U_l - \partial_i U_{l-1}|\} \\
&\quad + pL\{|U_l - U_{l-1}|^{p-1}|\partial_i U_{l-1} + \partial_i U_0|\} \\
&\leq 2^{p-1}pL\{(|U_l|^{p-1} + |U_0|^{p-1})|\partial_i U_l - \partial_i U_{l-1}|\} \\
&\quad + pL\{|U_l - U_{l-1}|^{p-1}(|\partial_i U_{l-1}| + |\partial_i U_0|)\}.
\end{aligned}$$

Similarly to the proof of the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ , we obtain that

$$\begin{aligned}
&\|L\{|U_l|^{p-1}|\partial_i U_l - \partial_i U_{l-1}|\}\| \leq Ck^2\|U_l\|^{p-1}\|\partial_i U_l - \partial_i U_{l-1}\|D(T), \\
&\|L\{|U_0|^{p-1}|\partial_i U_l - \partial_i U_{l-1}|\}\| \\
&\quad \leq C_{n,1,p}k^2\|\tau_+^{(n-1)/2}w^{-1}U_0\|^{p-1}\|\partial_i U_l - \partial_i U_{l-1}\|E_1(T), \\
&\|L\{|U_l - U_{l-1}|^{p-1}|\partial_i U_{l-1}|\}\| \leq Ck^2\|U_l - U_{l-1}\|^{p-1}\|\partial_i U_{l-1}\|D(T), \\
&\|L\{|U_l - U_{l-1}|^{p-1}|\partial_i U_0|\}\| \\
&\quad \leq C_{n,p-1,p}k^2\|U_l - U_{l-1}\|^{p-1}\|\tau_+^{(n-1)/2}w^{-1}\partial_i U_0\|E_{p-1}(T).
\end{aligned}$$

Therefore, due to (2.6.13), all the assumptions imply that

$$\begin{aligned}
&\|\partial_i U_{l+1} - \partial_i U_l\| \\
&\leq 2^{p-1}pk^2\{C\|W_l\|^{p-1}D(T) + C_{n,1,p}\|\tau_+^{(n-1)/2}w^{-1}U_0\|^{p-1}E_1(T)\} \times \\
&\quad \times \|\partial_i U_l - \partial_i U_{l-1}\| \\
&\quad + pk^2\{C\|\partial W_{l-1}\|D(T) + C_{n,p-1,p}\|\tau_+^{(n-1)/2}w^{-1}\partial W_0\|E_{p-1}(T)\} \times \\
&\quad \times \|U_l - U_{l-1}\|^{p-1} \\
&\leq 2^{p-1}pk^2\{C(2M_0\varepsilon^p)^{p-1}D(T) + C_{n,1,p}(C_0\varepsilon)^{p-1}E_1(T)\}\|\partial_i U_l - \partial_i U_{l-1}\| \\
&\quad + pk^2\{C(2M_0\varepsilon^p)D(T) + C_{n,p-1,p}C_0\varepsilon E_{p-1}(T)\}(\|U_2 - U_1\|2^{-(l-1)})^{p-1}.
\end{aligned}$$

This inequality yields

$$\|\partial_i U_{l+1} - \partial_i U_l\| \leq \frac{1}{2} \|\partial_i U_l - \partial_i U_{l-1}\| + \frac{N_0(\varepsilon, T)}{2^{(l-1)(p-1)}}, \quad (2.6.16)$$

where we set

$$N_0(\varepsilon, T) = pk^2 \|U_2 - U_1\|^{p-1} \{C(2M_0\varepsilon^p)D(T) + C_{n,p-1,p}C_0\varepsilon E_{p-1}(T)\}$$

provided

$$2^{p-1}pk^2 \{C(2M_0)^{p-1}\varepsilon^{p(p-1)}D(T) + C_{n,1,p}(C_0)^{p-1}\varepsilon^{p-1}E_1(T)\} \leq \frac{1}{2}. \quad (2.6.17)$$

We note that (2.6.16) implies

$$\|\partial_i U_{l+1} - \partial_i U_l\| \leq \frac{1}{2^{l-1}} \|\partial_i U_2 - \partial_i U_1\| + \frac{N_0(\varepsilon, T)}{2^{(l-1)(p-1)}} \sum_{\nu=0}^{l-2} \frac{1}{2^\nu} \quad \text{for } l \geq 2.$$

The convergence of  $\{\partial_i U_l\}_{l \in \mathbf{N}}$  ( $i = 1, 2, 3, \dots, n$ ) follows from this estimate.

In this way, the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  in  $Y_0 \subset X$  can be established if all the five conditions, (2.6.3), (2.6.7), (2.6.12), (2.6.15), (2.6.17), are satisfied. In order to complete the proof of Proposition 2.6.1, we shall fix  $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$  and  $c$  in the statement of Proposition 2.6.1. First, we propose a sufficient condition to (2.6.3), (3.4.7) as well as related factors in (2.6.12), (2.6.15), (2.6.17) to  $D(T)$  by

$$2^{2p}3^{p-1}pCk^2M_0^{p-1}\varepsilon^{p(p-1)}D(T) \leq 1. \quad (2.6.18)$$

Next, we propose sufficient conditions to related factors in (2.6.12), (2.6.15), (2.6.17) to  $E_1(T)$  and  $E_{p-1}(T)$  according to  $p$  by the following.

In the case of  $1 < p < \frac{n+1}{n-1}$ , such conditions are

$$2^{p+1}pk^2C_{n,1,p}C_0^{p-1}\varepsilon^{p-1} \left( \frac{2T+3k}{k} \right)^{-q} \leq 1 \quad (2.6.19)$$

and

$$1 \geq 2^{2p-1}pk^2C_{n,p-1,p}M_0^{p-2}C_0\varepsilon^{(p-1)^2} \left( \frac{2T+3k}{k} \right)^{-(p-1)q}. \quad (2.6.20)$$

If we put

$$c = \min \left\{ (2^{2p}3^{p-1}pCk^2(M_0)^{p-1})^{-1}, (2^{p+1}pk^2C_{n,1,p}C_0^{p-1})^{-p}, (2^{2p-1}pk^2C_{n,p-1,p}M_0^{p-2}C_0)^{-p/(p-1)} \right\} > 0,$$

the inequality  $\varepsilon^{p(p-1)}D(T) \leq c$  implies (2.6.18), (2.6.19) and (2.6.20), because of  $-pq < \gamma(n, p)/2$ . Furthermore, one can readily check that  $\varepsilon_0 = 1$  by (2.6.3).

In the case of  $p = \frac{n+1}{n-1}$ , let us fix  $\delta$  with

$$0 < \delta < \frac{1}{p}. \quad (2.6.21)$$

Then, similarly to the above, our conditions are

$$2^{p+1}pC_{n,1,p}k^2C_0^{p-1}\varepsilon^{p-1}\left(\frac{2T+3k}{k}\right)^\delta \leq 1 \quad (2.6.22)$$

and

$$1 \geq 2^{2p-1}pk^2C_{n,p-1,p}M_0^{p-2}C_0\varepsilon^{(p-1)^2}\left(\frac{2T+3k}{k}\right)^{\delta(p-1)}. \quad (2.6.23)$$

If we put

$$c = \min \left\{ (2^{2p}3^{p-1}pCk^2(M_0)^{p-1})^{-1}, (2^{p+1}pk^2C_{n,1,p}C_0^{p-1})^{-p}, (2^{2p-1}pk^2C_{n,p-1,p}M_0^{p-2}C_0)^{-p/(p-1)} \right\} > 0,$$

the inequality  $\varepsilon^{p(p-1)}D(T) \leq c$  implies (2.6.18), (2.6.22) and (2.6.23), because of (2.6.21). Furthermore, one can readily check that  $\varepsilon_0 = 1$  by (2.6.3).

Finally, in the case of  $p > \frac{n+1}{n-1}$ , our conditions are

$$2^{p+1}pk^2C_{n,1,p}C_0^{p-1}\varepsilon^{p-1} \leq 1 \quad (2.6.24)$$

and

$$2^{2p-1}pk^2C_{n,p-1,p}M_0^{p-2}C_0\varepsilon^{(p-1)^2} \leq 1. \quad (2.6.25)$$

If we put

$$c = (2^{2p}3^{p-1}pCk^2(M_0)^{p-1})^{-1},$$

the inequality  $\varepsilon^{p(p-1)}D(T) \leq c$  implies (2.6.18). Furthermore, one can find that

$$\varepsilon_0 = \min \left\{ 1, (2^{p+1}pk^2C_{n,1,p})^{-1/(p-1)}C_0^{-1}, (2^{2p-1}pk^2M_0^{p-2}C_{n,p-1,p}C_0)^{-1/(p-1)^2} \right\} > 0,$$

by (2.6.3), (2.6.24) and (2.6.25). Therefore, the proof of proposition 2.6.1 is completed.  $\square$

## 2.7 Lower bound in even space dimensions

Similarly to the previous section, we investigate the lower bound of the lifespan in the even dimensional case. Our purpose is to show the following proposition.

**Proposition 2.7.1** *Let  $n = 4, 6, 8, \dots$ . Suppose that the same assumption in Theorem 2.3.1, (3.1.5) is fulfilled. Then, there exists a positive constant  $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$  such that each of (2.3.6) and (2.3.10) admits a unique classical solution  $u \in C^2(\mathbf{R}^4 \times [0, T])$  if  $n = 4$  and  $p = p_0(4) = 2$ , or each of (2.3.2) and (2.3.9) admits a unique solution  $u \in C^1(\mathbf{R}^n \times [0, T])$  otherwise, as far as  $T$  satisfies*

$$T \leq \begin{cases} c\varepsilon^{-2p(p-1)/\gamma(p,n)} & \text{if } 1 < p < p_0(n) \\ \exp(c\varepsilon^{-p(p-1)}) & \text{if } p = p_0(n) \end{cases} \quad (2.7.1)$$

for  $0 < \varepsilon \leq \varepsilon_0$  and  $c$  is a positive constant independent of  $\varepsilon$ .

Employing the similar argument to odd dimensions, we shall construct a  $C^1$  solution of the integral equation (2.4.8) as a limit of  $\{U_l\}_{l \in \mathbf{N}}$  in  $X$ . We also remark that it is possible to construct a  $C^2$  solution if and only if  $(n, p) = (4, 2)$  in our problem. However, its construction is almost the same as for  $C^1$  solution. Therefore we shall omit it. Now, define a closed subspace  $Y_1$  in  $X$  by

$$Y_1 = \{U \in X : \|\nabla_x^\alpha U\| \leq M_1 \varepsilon^p \text{ } (|\alpha| \leq 1)\},$$

where  $M_1$  is defined by

$$M_1 = 2^{2p} p k^2 (C_{n,0,p} C_1^p + C_{n,1,p} C_1^{p-1} C_2 + C_{n,p-1,p} C_1 C_2^{p-1} + C_{n,0,p} C_2^p) > 0.$$

Recall that  $C_{n,0,p}$ ,  $C_{n,1,p}$  and  $C_{n,p-1,p}$  are the one in (2.5.5) and (2.5.9).  $C_1$  and  $C_2$  are positive constants independent of  $\varepsilon$  which satisfy that

$$\left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{\partial W_{00}}{w} \right\| \leq C_1 \varepsilon, \quad \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{\partial W_{01}}{w \tau_-} \right\| \leq C_2 \varepsilon^p. \quad (2.7.2)$$

The existence of  $C_1$  and  $C_2$  is trivial by the definition of the weight function in (2.4.11) and the decay estimate for  $U_{00} = v_0$  and  $U_{01} = v_1$  in (2.4.6) and (2.4.7).

**Proof of the Proposition 2.7.1.** First of all, we shall show the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ . The boundedness of  $\{U_l\}_{l \in \mathbf{N}}$  in  $Y_1$ ,

$$\|U_l\| \leq 2M_1 \varepsilon^p \text{ } (l \in \mathbf{N}), \quad (2.7.3)$$

can be obtained by induction with respect to  $l$  as follows. Similarly to (2.6.5), it follows from (2.4.9) and (2.5.5) of Lemma 2.5.3 with  $\nu = 0$  and  $a = 0, 1$  that

$$\begin{aligned} \|U_1\| \leq & 2^{2p} k^2 C_{n,0,p} \left( \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{00}}{w} \right\|^p E_{0,0}(T) + \right. \\ & \left. + \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{01}}{w \tau_-} \right\|^p E_{0,1}(T) \right). \end{aligned} \quad (2.7.4)$$

By the definition of (2.5.6), (2.5.7) and (2.5.8) with  $\nu = a = 0$ , we get

$$E_{0,0}(T) = 1. \quad (2.7.5)$$

It follows from (2.7.2) that

$$\|U_1\| \leq 2^{2p} k^2 C_{n,0,p} \varepsilon^p (C_1^p + C_2^p \varepsilon^{p(p-1)} E_{0,1}(T)).$$

This inequality show  $\|U_1\| \leq M_1 \varepsilon^p$  provided

$$\varepsilon^{p(p-1)} E_{0,1}(T) \leq 1. \quad (2.7.6)$$

Assume that  $\|U_{l-1}\| \leq 2M_1 \varepsilon^p$  ( $l \geq 2$ ). Making use of (2.6.6), the assumption of the induction and (2.5.1) yield that

$$\begin{aligned} \|U_l\| & \leq 2^p k^2 \left\{ C \|U_{l-1}\|^p D(T) + C_{n,0,p} \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{00}}{w} \right\|^p E_{0,0}(T) \right. \\ & \quad \left. + C_{n,0,p} \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{01}}{w \tau_-} \right\|^p E_{0,1}(T) \right\} \\ & \leq 2^p C k^2 (2M_1 \varepsilon^p)^p D(T) + M_1 \varepsilon^p. \end{aligned}$$

This inequality shows (2.7.3) provided

$$2^p C k^2 (2M_1)^p \varepsilon^{p^2} D(T) \leq M_1 \varepsilon^p. \quad (2.7.7)$$

Next we shall estimate the differences under (2.7.6) and (2.7.7) ensuring the boundedness (2.7.3). We note that (2.7.3) implies  $\|W_l\| \leq 2M_1 \varepsilon^p$  ( $l \in \mathbf{N}$ ). Making use of the inequalities (2.6.8) and (2.6.9), we have

$$|U_{l+1} - U_l| \leq 2^{p-1} p L \{ \{3W_l\}^{p-1} + 2^{p-1} (|U_{00}|^{p-1} + |U_{01}|^{p-1}) \} |U_l - U_{l-1}| \}.$$

Applying (2.5.5) of Lemma 2.5.3 with  $\nu = 1$  and  $a = 0, 1$ , we get

$$\begin{aligned} & \|L(|U_{0a}|^{p-1} |U_l - U_{l-1}|)\| \\ & \leq k^2 C_{n,1,p} E_{1,a}(T) \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{U_{0a}}{w \tau_-^a} \right\|^{p-1} \|U_l - U_{l-1}\| \end{aligned} \quad (2.7.8)$$

for  $a = 0, 1$ . Since (2.7.3) implies that  $\|W_l\| \leq 2M_1\varepsilon^p$  for  $l \geq 2$ , the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  follows from

$$\|U_{l+1} - U_l\| \leq \frac{1}{2}\|U_l - U_{l-1}\| \quad \text{for } l \geq 2$$

provided

$$\begin{aligned} 2^{p-1}pk^2\{C(6M_1\varepsilon^p)^{p-1}D(T) + 2^{p-1}C_{n,1,p}(C_1\varepsilon)^{p-1}E_{1,0}(T) \\ + 2^{p-1}C_{n,1,p}C_2^{p-1}E_{1,1}(T)\varepsilon^{p(p-1)}\} \leq \frac{1}{2}, \end{aligned} \quad (2.7.9)$$

Thus, we obtain

$$\|U_{l+1} - U_l\| \leq \frac{1}{2^{l-1}}\|U_2 - U_1\| \quad \text{for } l \geq 2 \quad (2.7.10)$$

which implies the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ .

Now we shall show the convergence of  $\{\partial_i U_l\}_{l \in \mathbf{N}}$  for  $i = 1, 2, \dots, n$  under the conditions, (2.7.6), (2.7.7), (2.7.9) which ensure the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ . As before, the boundedness

$$\|\partial_i U_l\| \leq 2M_1\varepsilon^p \quad (l \in \mathbf{N}, i = 1, 2, \dots, n) \quad (2.7.11)$$

can be obtained by induction as follows. Similarly to (2.7.4), we have

$$\begin{aligned} \|\partial_i U_1\| &= 2^p p \|L(|U_0|^{p-1} |\partial_i U_0|)\| \\ &\leq 2^{2p-1} p \|L((|U_{00}|^{p-1} + |U_{01}|^{p-1})(|\partial_i U_{00}| + |\partial_i U_{01}|))\| \\ &\leq 2^{2p-1} p \|L(\partial W_{00}^p + U_{00}^{p-1} \partial W_{01} + U_{01}^{p-1} \partial W_{00} + \partial W_{01}^p)\|. \end{aligned}$$

Applying (2.5.9) of Lemma 2.5.3 with  $\kappa = 1$  and  $\kappa = p - 1$ , we get

$$\begin{aligned} &\|L(U_{00}^{p-1} \partial W_{01})\| \\ &\leq C_{n,1,p} k^2 \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{U_{00}}{w} \right\|^{p-1} \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{\partial W_{01}}{w \tau_-} \right\| F_1(T) \end{aligned}$$

and

$$\begin{aligned} &\|L(U_{01}^{p-1} \partial W_{00})\| \\ &\leq C_{n,p-1,p} k^2 \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{U_{01}}{w \tau_-} \right\|^{p-1} \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{\partial W_{00}}{w} \right\| F_{p-1}(T). \end{aligned}$$

By virtue of (2.7.2) and (2.7.5), we obtain

$$\begin{aligned} \|\partial_i U_1\| &\leq 2^{2p-1} p k^2 \varepsilon^p \{C_{n,0,p} C_1^p + C_{n,1,p} C_1^{p-1} C_2 \varepsilon^{p-1} F_1(T) \\ &\quad + C_{n,p-1,p} C_2^{p-1} C_1 \varepsilon^{(p-1)^2} F_{p-1}(T) \\ &\quad + C_{n,0,p} C_2^p \varepsilon^{p(p-1)} E_{0,1}(T)\}. \end{aligned}$$

This inequality shows  $\|\partial_i U_1\| \leq M_1 \varepsilon^p$  provided (2.7.6),

$$\varepsilon^{p-1} F_1(T) \leq 1 \quad (2.7.12)$$

and

$$\varepsilon^{(p-1)^2} F_{p-1}(T) \leq 1 \quad (2.7.13)$$

hold.

Assume that  $\|\partial_i U_{l-1}\| \leq 2M_1 \varepsilon^p$  ( $l \geq 2$ ). Then we get

$$\begin{aligned} |\partial_i U_l| &= p|L(|U_{l-1} + U_0|^{p-1}|\partial_i(U_{l-1} + U_0)|)| \\ &\leq 2^{p-1}pL\{(|U_{l-1}|^{p-1} + |U_0|^{p-1})(|\partial_i U_{l-1}| + |\partial_i U_0|)\} \\ &\leq 2^{p-1}pL\{(|U_{l-1}|^{p-1} + 2^{p-1}(|U_{00}|^{p-1} + |U_{01}|^{p-1})) \\ &\quad \times (|\partial_i U_{l-1}| + |\partial_i U_{00}| + |\partial_i U_{01}|)\}. \end{aligned}$$

Similarly to (2.7.8), we obtain that

$$\begin{aligned} &\|L(|U_{l-1}|^{p-1}|\partial_i U_{0a}|)\| \\ &\leq C_{n,p-1,p} k^2 \|U_{l-1}\|^{p-1} \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{\partial_i U_{0a}}{w \tau_-^a} \right\| E_{p-1,a}(T), \\ &\|L(|U_{0a}|^{p-1}|\partial_i U_{l-1}|)\| \\ &\leq C_{n,1,p} k^2 \|\partial_i U_{l-1}\| \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{U_{0a}}{w \tau_-^a} \right\|^{p-1} E_{1,a}(T) \end{aligned}$$

for  $a = 0, 1$ . Making use of (2.7.2) and (2.7.5), we get

$$\begin{aligned} &\|\partial_i U_l\| \\ &\leq 2^{p-1}pk^2[C(2M_1\varepsilon^p)^p D(T) + C_{n,p-1,p}(2M_1\varepsilon^p)^{p-1}C_1\varepsilon E_{p-1,0}(T) \\ &\quad + C_{n,p-1,p}(2M_1\varepsilon^p)^{p-1}C_2\varepsilon^p E_{p-1,1}(T) \\ &\quad + 2^{p-1}\{C_{n,1,p}(C_1\varepsilon)^{p-1}(2M_1\varepsilon^p)E_{1,0}(T) + C_{n,0,p}C_1^p\varepsilon^p \\ &\quad + C_{n,1,p}(C_1\varepsilon)^{p-1}C_2\varepsilon^p F_1(T) + C_{n,1,p}(C_2\varepsilon^p)^{p-1}(2M_1\varepsilon^p)E_{1,1}(T) \\ &\quad + C_{n,p-1,p}(C_2\varepsilon^p)^{p-1}C_1\varepsilon F_{p-1}(T) + C_{n,0,p}(C_2\varepsilon^p)^p E_{0,1}(T)\}]. \end{aligned}$$

Under the assumptions (2.7.6), (2.7.12) and (2.7.13), this inequality shows (2.7.11) provided

$$\begin{aligned} &(7/4)M_1\varepsilon^p \\ &\geq 2^{p-1}pk^2[C(2M_1)^p\varepsilon^{p^2} D(T) \\ &\quad + C_{n,p-1,p}(2M_1)^{p-1}C_1\varepsilon^{p^2-p+1}E_{p-1,0}(T) \\ &\quad + C_{n,p-1,p}(2M_1)^{p-1}C_2\varepsilon^{p^2}E_{p-1,1}(T) + 2^{p-1}\{C_{n,1,p}C_1^{p-1}\varepsilon^{2p-1} \\ &\quad \times 2M_1E_{1,0}(T) + C_{n,1,p}C_2^{p-1}2M_1\varepsilon^{p^2}E_{1,1}(T)\}]. \end{aligned} \quad (2.7.14)$$

Next we shall estimate the differences under the conditions, (2.7.6), (2.7.7), (2.7.9), (2.7.12), (2.7.13), (2.7.14) which ensure the convergence of  $\{U_l\}_{l \in \mathbb{N}}$

and the boundedness of  $\{\partial U_l\}_{l \in \mathbf{N}}$ . Similarly to odd dimensions, we have

$$\begin{aligned} & |\partial_i U_{l+1} - \partial_i U_l| \\ & \leq 2^{p-1} p L \{ \{ |U_l|^{p-1} + 2^{p-1} \{ |U_{00}|^{p-1} + |U_{01}|^{p-1} \} \} |\partial_i U_l - \partial_i U_{l-1}| \} \\ & \quad + p L \{ |U_l - U_{l-1}|^{p-1} (|\partial_i U_{l-1}| + |\partial_i U_{00}| + |\partial_i U_{01}|) \}. \end{aligned}$$

Applying (2.5.5) of Lemma 2.5.3 with  $\nu = 1$ ,  $\nu = p - 1$  and  $a = 0, 1$ , we get

$$\begin{aligned} & \|L\{|U_{0a}|^{p-1}|\partial_i U_l - \partial_i U_{l-1}|\}\| \\ & \leq C_{n,1,p} k^2 \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{U_{0a}}{w \tau_-^a} \right\|^{p-1} \|\partial_i U_l - \partial_i U_{l-1}\| E_{1,a}(T), \\ & \|L\{|U_l - U_{l-1}|^{p-1}|\partial_i U_{0a}|\}\| \\ & \leq C_{n,p-1,p} k^2 \|U_l - U_{l-1}\|^{p-1} \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{\partial_i U_{0a}}{w \tau_-^a} \right\| E_{p-1,a}(T) \end{aligned}$$

for  $a = 0, 1$ . Then, all the assumptions imply that

$$\begin{aligned} & \|\partial_i U_{l+1} - \partial_i U_l\| \\ & \leq 2^{p-1} p k^2 \{ C(2M_1 \varepsilon^p)^{p-1} D(T) + 2^{p-1} \{ C_{n,1,p} (C_1 \varepsilon)^{p-1} E_{1,0}(T) \\ & \quad + C_{n,1,p} (C_2 \varepsilon^p)^{p-1} E_{1,1}(T) \} \|\partial_i U_l - \partial_i U_{l-1}\| \\ & \quad + p k^2 \{ C(2M_1 \varepsilon^p) D(T) + C_{n,p-1,p} C_1 \varepsilon E_{p-1,0}(T) \\ & \quad + C_{n,p-1,p} C_2 \varepsilon^p E_{p-1,1}(T) \} (\|U_2 - U_1\| 2^{-(l-1)})^{p-1} \}. \end{aligned}$$

This inequality yields

$$\|\partial_i U_{l+1} - \partial_i U_l\| \leq \frac{1}{2} \|\partial_i U_l - \partial_i U_{l-1}\| + \frac{N_1(\varepsilon, T)}{2^{(l-1)(p-1)}}, \quad (2.7.15)$$

where we set

$$\begin{aligned} N_1(\varepsilon, T) &= p k^2 \{ C(2M_1 \varepsilon^p) D(T) + C_{n,p-1,p} C_1 \varepsilon E_{p-1,0}(T) \\ & \quad + C_{n,p-1,p} C_2 \varepsilon^p E_{p-1,1}(T) \} \end{aligned}$$

provided

$$\begin{aligned} & 2^{p-1} p k^2 \{ C(2M_1)^{p-1} \varepsilon^{p(p-1)} D(T) \\ & \quad + 2^{p-1} \{ C_{n,1,p} (C_1)^{p-1} \varepsilon^{p-1} E_{1,0}(T) \\ & \quad + C_{n,1,p} C_2^{p-1} \varepsilon^{p(p-1)} E_{1,1}(T) \} \} \leq \frac{1}{2}. \end{aligned} \quad (2.7.16)$$

Hence, the convergence of  $\{\partial_i U_l\}_{l \in \mathbf{N}}$  ( $i = 1, 2, 3, \dots, n$ ) follows from this estimate.

In this way, the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  in  $Y_0 \subset X$  can be established if all the seven conditions, (2.7.6), (2.7.7), (2.7.9), (2.7.12), (2.7.13), (2.7.14), (2.7.16) are satisfied. In order to complete the proof of Proposition 2.7.1, we shall fix  $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$  and  $c$  in the statement of Proposition 2.7.1.



First, we propose a sufficient condition to (2.7.7) as well as related factors in (2.7.9), (2.7.14), (2.7.16) to  $D(T)$  as

$$2^{2p-1}3^p p C k^2 M_1^{p-1} \varepsilon^{p(p-1)} D(T) \leq 1. \quad (2.7.17)$$

Next we propose sufficient conditions to related factors in (2.7.6), (2.7.9), (2.7.12), (2.7.13), (2.7.14) and (2.7.16) to

$$E_{0,1}(T), E_{1,0}(T), E_{1,1}(T), F_1(T), F_{p-1}(T), E_{p-1,0}(T), E_{p-1,1}(T)$$

up to  $p$ .

**Conditions in the case of  $1 < p < \frac{n+1}{n-1}$ .**

(i) Conditions from  $E_{0,1}(T)$ .

In (2.5.8), setting  $\nu = 0$  and  $a = 1$ , we have that  $\sigma = -(n-3)p/2 < -1$  when  $n \geq 6$  and  $\sigma = \mu = -p/2 > -1$  when  $n = 4$ , which imply

$$E_{0,1}(T) = \begin{cases} 1 & \text{if } n \geq 6, \\ \left(\frac{2T+3k}{k}\right)^{1-p/2} & \text{if } n = 4. \end{cases} \quad (2.7.18)$$

Since  $E_{0,1}(T)$  appears in (2.7.6), the conditions are

$$\varepsilon^{p(p-1)} \leq 1 \quad \text{if } n \geq 6, \quad (2.7.19)$$

$$\left(\frac{2T+3k}{k}\right)^{1-p/2} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } n = 4. \quad (2.7.20)$$

(ii) Condition from  $E_{1,0}(T)$ .

In (2.5.8), setting  $\nu = 1$  and  $a = 0$ , we have that  $\sigma = -(n-1)(p-1)/2 > -1$  and  $\mu = -2q - 1$ , which imply

$$E_{1,0}(T) = \left(\frac{2T+3k}{k}\right)^{-2q}.$$

Since  $E_{1,0}(T)$  appears in (2.7.9), (2.7.14) and (2.7.16), the condition is

$$2^{2p-1}3^p k^2 C_{n,1,p} C_1^{p-1} \varepsilon^{p-1} \left(\frac{2T+3k}{k}\right)^{-2q} \leq 1. \quad (2.7.21)$$

(iii) Condition from  $E_{1,1}(T)$ .

In (2.5.8), setting  $\nu = a = 1$ , we have that  $\sigma = -(n-3)(p-1)/2 > -1$  and  $\mu - 1 - (n-2)p$ , which imply

$$E_{1,1}(T) = \left(\frac{2T+3k}{k}\right)^{n-(n-2)p}.$$

Since  $E_{1,1}(T)$  appears in (2.7.9), (2.7.14) and (2.7.16), the condition is

$$2^{2p-1}3pk^2C_{n,1,p}C_2^{p-1}\varepsilon^{p(p-1)}\left(\frac{2T+3k}{k}\right)^{n-(n-2)p} \leq 1. \quad (2.7.22)$$

(iv) Conditions from  $F_1(T)$ .

In (2.5.10), setting  $\nu = 1$ , we have that

$$\begin{aligned} \kappa &= 1 - \frac{n-1}{2}p < -1 && \text{if } n \geq 6, \text{ or } n = 4 \text{ and } p > \frac{4}{3}, \\ \kappa &= -1 && \text{if } n = 4 \text{ and } p = \frac{4}{3}, \\ \kappa &> -1 && \text{if } n = 4 \text{ and } 1 < p < \frac{4}{3}, \end{aligned}$$

which imply

$$F_1(T) = \begin{cases} 1 & \text{if } n \geq 6, \text{ or } n = 4 \text{ and } p > \frac{4}{3}, \\ \log \frac{2T+3k}{k} & \text{if } n = 4 \text{ and } p = \frac{4}{3}, \\ \left(\frac{2T+3k}{k}\right)^{2-3p/2} & \text{if } n = 4 \text{ and } 1 < p < \frac{4}{3}. \end{cases} \quad (2.7.23)$$

Since  $F_1(T)$  appears in (2.7.12), the conditions are

$$\varepsilon^{p(p-1)} \leq 1 \quad \text{if } n \geq 6, \text{ or } n = 4 \text{ and } p > \frac{4}{3}, \quad (2.7.24)$$

$$\log \frac{2T+3k}{k} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } n = 4 \text{ and } p = \frac{4}{3}, \quad (2.7.25)$$

$$\left(\frac{2T+3k}{k}\right)^{2-3p/2} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } n = 4 \text{ and } 1 < p < \frac{4}{3}. \quad (2.7.26)$$

(v) Condition from  $F_{p-1}(T)$ .

In (2.5.10), setting  $\nu = p-1$ , we have  $\kappa = -(n-3)p/2 - 1 < -1$ , which implies  $F_{p-1}(T) = 1$ . Since  $F_{p-1}(T)$  appears in (2.7.13), the condition is

$$\varepsilon^{p(p-1)} \leq 1. \quad (2.7.27)$$

(vi) Condition from  $E_{p-1,0}(T)$ .

In (2.5.8), setting  $\nu = p-1$  and  $a = 0$ , we have  $\sigma = -(n-1)/2 < -1$ , which implies

$$E_{p-1,0}(T) = \left(\frac{2T+3k}{k}\right)^{-(p-1)q}.$$

Since  $E_{p-1,0}(T)$  appears in (2.7.14), the condition is

$$2^{2p}5 \cdot 7^{-1}pk^2C_{n,p-1,p}C_1M_1^{p-2}\varepsilon^{(p-1)^2}\left(\frac{2T+3k}{k}\right)^{-(p-1)q} \leq 1. \quad (2.7.28)$$

(vii) Conditions from  $E_{p-1,1}(T)$ .

In (2.5.8), setting  $\nu = p-1$  and  $a = 1$ , we have that  $\sigma = -(n-3)/2 < -1$  when  $n \geq 6$  and  $\sigma > -1, \mu = -1/2 - (p-1)q$  when  $n = 4$ , which imply

$$E_{p-1,1}(T) = \begin{cases} \left(\frac{2T+3k}{k}\right)^{-(p-1)q} & \text{if } n \geq 6, \\ \left(\frac{2T+3k}{k}\right)^{1/2-(p-1)q} & \text{if } n = 4. \end{cases} \quad (2.7.29)$$

Since  $E_{p-1,1}(T)$  appear in (2.7.14), the conditions are

$$\begin{aligned} & 2^{2p}5 \cdot 7^{-1}pk^2C_{n,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)} \\ & \times \left(\frac{2T+3k}{k}\right)^{-(p-1)q} \leq 1 \quad \text{if } n \geq 6, \end{aligned} \quad (2.7.30)$$

$$\begin{aligned} & 2^{2p}5 \cdot 7^{-1}pk^2C_{4,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)} \\ & \times \left(\frac{2T+3k}{k}\right)^{1/2-(p-1)q} \leq 1 \quad \text{if } n = 4. \end{aligned} \quad (2.7.31)$$

Now, we are in a position to summarize all the conditions in (i)-(vii) above. Set

$$\varepsilon_0 = 1 \quad (2.7.32)$$

Then, (2.7.32) implies (2.7.19), (2.7.24) and (2.7.27). In order to make that (2.7.17) includes (2.7.25), we employ Lemma 2.5.9 with  $\delta = \gamma(p, 4)/2 > 0$  and  $X = (2T+3k)/k > 1$ . Then, if we set

$$\begin{aligned} c = \min \{ & (2^{2p-1}3^p p C k^2 (M_1)^{p-1})^{-1}, 1, (2^{2p-1}3^p k^2 C_{n,1,p} C_1^{p-1})^{-p}, \\ & (2^{2p-1}3^p k^2 C_{n,1,p} C_2^{p-1})^{-1}, \gamma(4/3, 4)/2, \\ & (2^{2p}5 \cdot 7^{-1}pk^2C_{n,p-1,p}C_1M_1^{p-2})^{-p/(p-1)}, \\ & (2^{2p}5 \cdot 7^{-1}pk^2C_{n,p-1,p}C_2M_1^{p-2})^{-1} \} > 0, \end{aligned}$$

the inequality  $\varepsilon^{p(p-1)}D(T) \leq c$  implies (2.7.17), (2.7.20), (2.7.21), (2.7.22), (2.7.25), (2.7.26), (2.7.28), (2.7.30) and (2.7.31) because of  $1-p/2 \leq \gamma(p, 4)/2$  in (2.7.20),  $-2pq \leq \gamma(p, n)/2$  in (2.7.21),  $n-(n-2)p \leq \gamma(p, n)/2$  in (2.7.22),  $2-3p/2 \leq \gamma(p, 4)/2$  in (2.7.26),  $-pq \leq \gamma(p, n)/2$  in (2.7.28),  $-(p-1)q \leq \gamma(p, n)/2$  in (2.7.30) and  $1/2-(p-1)q \leq \gamma(p, 4)/2$  in (2.7.31).

**Conditions in the case of  $p = \frac{n+1}{n-1}$ .**

(i) Conditions from  $E_{0,1}(T)$ .

In (2.5.7), setting  $\nu = 0$  and  $a = 1$ , we have that  $\sigma = -(n-3)(n+1)/2(n-1) < -1$  when  $n \geq 6$  and  $\sigma > -1$  when  $n = 4$ , which imply

$$E_{0,1}(T) = \begin{cases} 1 & \text{if } n \geq 6, \\ \left(\frac{2T+3k}{k}\right)^{1-p/2} & \text{if } n = 4. \end{cases} \quad (2.7.33)$$

Since  $E_{0,1}(T)$  appears in (2.7.6), the conditions are

$$\varepsilon^{p(p-1)} \leq 1 \text{ if } n \geq 6, \quad (2.7.34)$$

$$\left(\frac{2T+3k}{k}\right)^{1-p/2} \varepsilon^{p(p-1)} \leq 1 \text{ if } n = 4. \quad (2.7.35)$$

(ii) Condition from  $E_{1,0}(T)$ .

In (2.5.7), setting  $\nu = 1$  and  $a = 0$ , we have  $\sigma = -(n-1)(p-1)/2 = -1$ , which implies

$$E_{1,0}(T) = \left(\frac{2T+3k}{k}\right)^\delta.$$

Since  $E_{1,0}(T)$  appears in (2.7.9), (2.7.14) and (2.7.16), the conditions is

$$2^{2p-1} 3pk^2 C_{n,1,p} C_1^{p-1} \varepsilon^{p-1} \left(\frac{2T+3k}{k}\right)^\delta \leq 1. \quad (2.7.36)$$

(iii) Condition from  $E_{1,1}(T)$ .

In (2.5.7), setting  $\nu = a = 1$ , we have  $\sigma = -(n-3)/(n-1) > -1$ , which implies

$$E_{1,1}(T) = \left(\frac{2T+3k}{k}\right)^{2/(n-1)}.$$

Since  $E_{1,1}(T)$  appears in (2.7.9) (2.7.14) and (2.7.16), the condition is

$$2^{2p-1} 3pk^2 C_{n,1,p} C_2^{p-1} \varepsilon^{p(p-1)} \left(\frac{2T+3k}{k}\right)^{2/(n-1)} \leq 1. \quad (2.7.37)$$

(iv) Condition from  $F_1(T)$ .

In (2.5.10), setting  $\nu = 1$ , we have  $\kappa < -1$ , which implies  $F_1(T) = 1$ . Since,  $F_1(T)$  appears in (2.7.12), the condition is

$$\varepsilon^{p(p-1)} \leq 1. \quad (2.7.38)$$

(v) Condition from  $F_{p-1,0}(T)$ .

In (2.5.10), setting  $\nu = p-1$ , we have  $\kappa < -1$ , which implies  $F_{p-1}(T) = 1$ . Since,  $F_{p-1}(T)$  appears in (2.7.13), the condition is (2.7.38).

(vi) Condition from  $E_{p-1,0}(T)$ .

In (2.5.7), setting  $\nu = p-1$  and  $a = 0$ , we have  $\sigma = -(n-1)/2 < -1$ , which implies

$$E_{p-1,0}(T) = \left( \frac{2T+3k}{k} \right)^{(p-1)\delta}.$$

Since  $E_{p-1,0}(T)$  appears in (2.7.14), the condition is

$$2^{2p}5 \cdot 7^{-1}pk^2C_{n,p-1,p}C_1M_1^{p-2}\varepsilon^{(p-1)^2} \left( \frac{2T+3k}{k} \right)^{(p-1)\delta} \leq 1. \quad (2.7.39)$$

(vii) Conditions from  $E_{p-1,1}(T)$ .

In (2.5.7), setting  $\nu = p-1$  and  $a = 1$ , we have that  $\sigma = -(n-3)/2 < -1$  when  $n \geq 6$  and  $\sigma > -1$  when  $n = 4$ , which imply

$$E_{p-1,1}(T) = \begin{cases} \left( \frac{2T+3k}{k} \right)^{(p-1)\delta} & \text{if } n \geq 6, \\ \left( \frac{2T+3k}{k} \right)^{1/2} & \text{if } n = 4. \end{cases} \quad (2.7.40)$$

Since  $E_{p-1,1}(T)$  appears in (2.7.16), the conditions are

$$\begin{aligned} & 2^{2p}5 \cdot 7^{-1}pk^2C_{n,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)} \\ & \times \left( \frac{2T+3k}{k} \right)^{(p-1)\delta} \leq 1 \quad \text{if } n \geq 6, \end{aligned} \quad (2.7.41)$$

$$\begin{aligned} & 2^{2p}5 \cdot 7^{-1}pk^2C_{4,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)} \\ & \times \left( \frac{2T+3k}{k} \right)^{1/2} \leq 1. \quad \text{if } n = 4. \end{aligned} \quad (2.7.42)$$

Now, we are in a position to summarize all the conditions in (i)-(vii) above. First we note that (2.7.32) implies (2.7.34) and (2.7.38). Then, if we assume (2.6.21) and set

$$c = \min \{ (2^{2p-1}3^p p C k^2 (M_1)^{p-1})^{-1}, 1, (2^{2p-1}3^p k^2 C_{n,1,p} C_1^{p-1})^{-p}, \\ (2^{2p-1}3^p k^2 C_{n,1,p} C_2^{p-1})^{-1}, \\ (2^{2p}5 \cdot 7^{-1}pk^2C_{n,p-1,p}C_1M_1^{p-2})^{-p/(p-1)}, \\ (2^{2p}5 \cdot 7^{-1}pk^2C_{n,p-1,p}C_2M_1^{p-2})^{-1} \} > 0,$$

the inequality  $\varepsilon^{p(p-1)}D(T) \leq c$  implies (2.7.17), (2.7.35), (2.7.36), (2.7.37), (2.7.39), (2.7.41) and (2.7.42) because of  $1 - p/2 \leq 1$  in (2.7.35),  $p\delta < 1$  in (2.7.36) and (2.7.39),  $2/(n-1) < 1$  in (2.7.37) and  $(p-1)\delta < 1$  in (2.7.41).

**Conditions in the case of  $p > \frac{n+1}{n-1}$ .**

(i) Conditions from  $E_{0,1}(T)$ .

In (2.5.6), setting  $\nu = 0$  and  $a = 1$ , we have that  $\mu = -(n-3)p/2 < -1$  when  $n \geq 6$  and  $\mu = -p/2 \geq -1$  when  $n = 4$ , which imply

$$E_{0,1}(T) = \begin{cases} 1 & \text{if } n \geq 6, \\ \left(\frac{2T+3k}{k}\right)^{1-p/2} & \text{if } n = 4, p < 2, \\ \log \frac{2T+3k}{k} & \text{if } n = 4, p = 2. \end{cases} \quad (2.7.43)$$

Since  $E_{0,1}(T)$  appear (2.7.6), the conditions are

$$\varepsilon^{p(p-1)} \leq 1 \quad \text{if } n \geq 6, \quad (2.7.44)$$

$$\left(\frac{2T+3k}{k}\right)^{1-p/2} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } n = 4, p < 2, \quad (2.7.45)$$

$$\log \frac{2T+3k}{k} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } n = 4, p = 2. \quad (2.7.46)$$

(ii) Condition from  $E_{1,0}(T)$ .

In (2.5.6), setting  $\nu = 1$  and  $a = 0$ , we have  $\mu = -(n-1)(p-1)/2 > -1$ , which implies  $E_{1,0}(T) = 1$ . Since  $E_{1,0}(T)$  appears in (2.7.9), (2.7.14) and (2.7.16), the condition is

$$2^{2p-1}3pk^2C_{n,1,p}C_1^{p-1}\varepsilon^{p-1} \leq 1. \quad (2.7.47)$$

(iii) Conditions from  $E_{1,1}(T)$ .

In (2.5.6), setting  $\nu = a = 1$ , we have  $\mu = -(n-3)(p-1)/2 - q - 1 - (n-2)p$ , which implies

$$E_{1,1}(T) = \begin{cases} 1 & \text{if } p > \frac{n}{n-2}, \\ \log \frac{2T+3k}{k} & \text{if } p = \frac{n}{n-2}, \\ \left(\frac{2T+3k}{k}\right)^{n-(n-2)p} & \text{if } p < \frac{n}{n-2}. \end{cases} \quad (2.7.48)$$

Since  $E_{1,1}(T)$  appears in (2.7.9), (2.7.14) and (2.7.16), the conditions are

$$\left. \begin{aligned} & 2^{2p-1} 3pk^2 C_{n,1,p} C_2^{p-1} \varepsilon^{p(p-1)} \times \\ & \times \left( \frac{2T+3k}{k} \right)^{n-(n-2)p} \end{aligned} \right\} \leq 1 \quad \text{if } p < \frac{n}{n-2}, \quad (2.7.49)$$

$$2^{2p-1} 3pk^2 C_{n,1,p} C_2^{p-1} \varepsilon^{p(p-1)} \log \frac{2T+3k}{k} \leq 1 \quad \text{if } p = \frac{n}{n-2}, \quad (2.7.50)$$

$$2^{2p-1} 3pk^2 C_{n,1,p} C_2^{p-1} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } p > \frac{n}{n-2}. \quad (2.7.51)$$

(iv) Condition from  $F_1(T)$  and  $F_{p-1}(T)$ .

In (2.5.10), setting  $\nu = 1$  and  $\nu = p-1$ , we have  $\kappa < -1$ , which implies  $F_1(T) = F_{p-1}(T) = 1$ . Since  $F_1(T)$  or  $F_{p-1}(T)$  appears in (2.7.12) or (2.7.13) respectively, the condition is

$$\varepsilon^{p(p-1)} \leq 1. \quad (2.7.52)$$

(v) Condition from  $E_{p-1,0}(T)$ .

In (2.5.6), setting  $\nu = p-1$  and  $a = 0$ , we have  $\mu = -(n-1)/2 - q(p-1) < -1$ , which implies  $E_{p-1,0}(T) = 1$ . Since  $E_{p-1,0}(T)$  appears in (2.7.14), the condition is

$$2^{2p} 5 \cdot 7^{-1} pk^2 C_{n,p-1,p} C_1 M_1^{p-2} \varepsilon^{(p-1)^2} \leq 1. \quad (2.7.53)$$

(vi) Conditions from  $E_{p-1,1}(T)$ .

In (2.5.6), setting  $\nu = p-1$  and  $a = 1$ , we have that  $\mu = -(n-3)/2 - (p-1)q < -1$  when  $n \geq 6$  and  $\mu > -1$  when  $n = 4$ , which imply

$$E_{p-1,1}(T) = \begin{cases} 1 & \text{if } n \geq 6, \\ \left( \frac{2T+3k}{k} \right)^{1/2-(p-1)q} & \text{if } n = 4, p < 2, \\ \log \frac{2T+3k}{k} & \text{if } n = 4, p = 2. \end{cases} \quad (2.7.54)$$

Since  $E_{p-1,1}(T)$  appears in (2.7.16), the condition are

$$2^{2p} 5 \cdot 7^{-1} pk^2 C_{n,p-1,p} C_2 M_1^{p-2} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } n \geq 6, \quad (2.7.55)$$

$$\left. \begin{aligned} & 2^{2p} 5 \cdot 7^{-1} pk^2 C_{4,p-1,p} C_2 M_1^{p-2} \varepsilon^{p(p-1)} \times \\ & \times \left( \frac{2T+3k}{k} \right)^{1/2-(p-1)q} \end{aligned} \right\} \leq 1 \quad \text{if } n = 4, p < 2 \quad (2.7.56)$$

$$\left. \begin{aligned} & 2^{2p} 5 \cdot 7^{-1} pk^2 C_{4,p-1,p} C_2 M_1^{p-2} \varepsilon^{p(p-1)} \times \\ & \times \log \frac{2T+3k}{k} \end{aligned} \right\} \leq 1 \quad \text{if } n = 4, p = 2 \quad (2.7.57)$$

Now, we are in a position to summarize all the conditions in (i)-(vi) above. Set

$$\begin{aligned} \varepsilon_0 = \min \quad & \{1, (2^{2p-1}3pk^2C_{n,1,p}C_1^{p-1})^{-1/(p-1)}, \\ & \{2^{2p-1}3pk^2C_{n,1,p}C_2^{p-1}\}^{-1/p(p-1)}, \\ & \{2^{2p}5 \cdot 7^{-1}pk^2C_{n,p-1,p}C_1M_1^{p-2}\}^{-1/(p-1)^2}, \\ & \{2^{2p}5 \cdot 7^{-1}pk^2C_{n,p-1,p}C_2M_1^{p-2}\}^{-1/p(p-1)}\} > 0. \end{aligned} \quad (2.7.58)$$

Then, (2.7.58) implies (2.7.44), (2.7.47), (2.7.51), (2.7.52), (2.7.53) and (2.7.55). In order to make that (2.7.50) includes (2.7.17) when  $n \geq 6$ , we employ the lemma 2.5.9 with  $\delta = \gamma(p, n)/2 > 0$  and  $X = (2T + 3k)/k > 1$ . Then, if we set

$$\begin{aligned} c = \min \quad & \{(2^{2p-1}3pCk^2(M_1)^{p-1})^{-1}, 1, (2^{2p-1}3pk^2C_{n,1,p}C_2^{p-1})^{-1}, \\ & (2^{2p-1}3pk^2C_{n,1,p}C_2^{p-1})^{-1}\gamma(n/(n-2), n)/2, \\ & (2^{2p}5 \cdot 7^{-1}pk^2C_{4,p-1,p}C_2M_1^{p-2})^{-1}\} > 0, \end{aligned}$$

the inequality  $\varepsilon^{p(p-1)}D(T) \leq c$  implies (2.7.17), (2.7.45), (2.7.46), (2.7.49), (2.7.50) and (2.7.56) and (2.7.57), because of  $1 - p/2 \leq \gamma(p, 4)/2$  in (2.7.45),  $n - (n-2)p \leq \gamma(p, n)/2$  in (2.7.49), and  $1/2 - (p-1)q \leq \gamma(p, 4)/2$  in (2.7.56). Therefore the proof of proposition 2.7.1 is now completed.  $\square$

## 2.8 Upper bound of the lifespan for the critical case in odd dimensions

In this section, we prove Theorem 2.3.2 for the critical case in odd space dimensions. The proof is divided into three steps. In the first step, we get a point-wise estimate of the linear term  $u^0$  from below by means of the representation formula due to Rammaha [26]. In the second step, we employ the comparison argument between the solution of integral equations (2.3.9) and the blowing-up solution of ODE, basically introduced by Zhou [38], in order to overcome the difficulty in the critical case. In the last step, we also employ the slicing method introduced by Agemi, Kurokawa and Takamura [2].

**Proposition 2.8.1** *Suppose that the assumptions of Theorem 2.3.2 are fulfilled. Let  $u$  be a  $C^0$ -solution of (2.3.9) in  $\mathbf{R}^n \times [0, T]$ . Then, there exists a positive constant  $\varepsilon_0 = \varepsilon_0(g, n, p, k)$  such that  $T$  cannot be taken as*

$$T > \exp(c\varepsilon^{-p(p-1)}) \quad \text{if } p = p_0(n) \quad (2.8.1)$$

for  $0 < \varepsilon \leq \varepsilon_0$ , where  $c$  is a positive constant independent of  $\varepsilon$ .



**Proof.** First we note that one may assume that the solution of (2.3.9) is radially symmetric without loss of the generality. To see this, we employ the spherical mean which is defined by

$$\tilde{v}(r, t) = \frac{1}{\omega_n} \int_{|\omega|=1} v(r\omega, t) dS_\omega \quad (r > 0),$$

for  $v \in C(\mathbf{R}^n \times [0, \infty))$ . If we take the spherical mean of (2.3.9), we get

$$\tilde{u} = \varepsilon \tilde{u}^0 + L(\widetilde{|u|^p}).$$

Thanks to the fundamental identity for iterated spherical means, we have

$$L(\widetilde{|u|^p}) = L_{odd}(\widetilde{|u|^p}),$$

where  $L_{odd}$  is the one in (2.5.17). See 78-81pp. of John [12] for details. Thus, it follows from Jensen's inequality  $\widetilde{|u|^p} \geq |\tilde{u}|^p$  ( $p > 1$ ) and the positivity of  $L_{odd}$  that

$$\tilde{u} \geq \varepsilon \tilde{u}^0 + L_{odd}(|\tilde{u}|^p).$$

We estimate  $\tilde{u}$  from below all the time in this section, so that we may assume that the equality holds here.

Let  $u = u(r, t)$  be a  $C^0$ -solution of

$$u = \varepsilon u^0 + L_{odd}(|u|^p) \quad \text{in} \quad (0, \infty) \times [0, T] \quad (2.8.2)$$

which is associated by (2.3.9). Note that  $u^0 = u^0(r, t)$  is a solution of

$$\begin{cases} u_{tt}^0 - \frac{n-1}{r} u_r^0 - u_{rr}^0 = 0 & \text{in } (0, \infty) \times [0, \infty) \\ u^0(r, 0) = 0, \quad u_t^0(r, 0) = g(r), \quad r \in (0, \infty). \end{cases} \quad (2.8.3)$$

### [The 1st step] Estimate of $u^0$ .

We have the following representation of  $u^0$ .

**Lemma 2.8.1 (Rammaha [26])** *Let  $n = 5, 7, 9, \dots$  and  $u^0$  be a solution of (2.8.3). Then,  $u^0$  is represented by*

$$u^0(r, t) = \frac{1}{2r^{(n-1)/2}} \int_{|r-t|}^{r+t} \lambda^{(n-1)/2} g(\lambda) P_{(n-3)/2} \left( \frac{\lambda^2 + r^2 - t^2}{2r\lambda} \right) d\lambda,$$

where  $P_k$  is Legendre polynomial of degree  $k$  defined by

$$P_k(z) = \frac{1}{2^k k!} \frac{d^k}{dz^k} (z^2 - 1)^k.$$

See (6a) on 681p. in [26] for the proof. This lemma implies the following estimate.

**Lemma 2.8.2 (Rammaha [26])** *Let  $n = 5, 7, 9, \dots$ . Assume (3.1.3). Then there exists a positive constant  $C_g$  such that for  $t + k_0 < r < t + k_1$  and  $t \geq k_2$ ,*

$$u^0(r, t) \geq \frac{C_g}{r^{(n-1)/2}}, \quad (2.8.4)$$

where  $k_2 = k - k_0$ .

See Lemma 2 on 682p. in [26] for the proof.

In order to prove the blow up result, we employ the iteration argument originally introduced by John [13]. Our frame in the argument is obtained by the following lemma.

**Lemma 2.8.3** *Let  $u$  be a  $C^0$ -solution of (2.8.2). Assume (3.1.3). Then  $u$  in  $\Sigma_0 = \{(r, t) : 2k \leq t - r \leq r\}$  satisfies*

$$\begin{aligned} u(r, t) &\geq \frac{C2^{(n-3)/2}(t-r)^{(n-1)/2}}{r^{(3n-7)/2}} \times \\ &\times \iint_{R(r,t)} \{(t-r-\tau+\lambda)(t+r-\tau-\lambda)\}^{(n-3)/2} |u(\lambda, \tau)|^p d\lambda d\tau + \\ &+ \frac{E_1(t-r)^{(3n-5)/2-(n-1)p/2}}{r^{(3n-7)/2}} \varepsilon^p, \end{aligned} \quad (2.8.5)$$

where  $C$  is the one in (2.5.17),

$$E_1 = \frac{CC_g^p(k_1 - k_0)}{(n-1)2^{(n-1)p/2-(3n-11)/2}}$$

and

$$R(r, t) = \{(\lambda, \tau) : t - r \leq \lambda, \tau + \lambda \leq t + r, 2k \leq \tau - \lambda \leq t - r\}.$$

**Proof.** By virtue of Huygens' principle on  $u^0$  and (2.5.17), we have

$$u \geq I_1 + I_2 \quad \text{in } \Sigma_0,$$

where we set

$$\begin{aligned} I_1(r, t) &= Cr^{2-n} \int_{R(r,t)} (t-\tau)^{3-n} h(\lambda, t-\tau, r) \lambda |u(\lambda, \tau)|^p d\lambda d\tau, \\ I_2(r, t) &= Cr^{2-n} \int_{S(r,t)} (t-\tau)^{3-n} h(\lambda, t-\tau, r) \lambda |u(\lambda, \tau)|^p d\lambda d\tau, \\ S(r, t) &= \{(\lambda, \tau) : t - r \leq \lambda, \tau + \lambda \leq t + r, -k_1 \leq \tau - \lambda \leq -k_0\}. \end{aligned}$$

Changing variable by (2.5.28) in  $I_1$ , we have

$$\begin{aligned} I_1(r, t) \geq & \frac{Cr^{2-n}}{4} \int_{2k}^{t-r} \{(t-r-\beta)(t+r-\beta)\}^{(n-3)/2} d\beta \times \\ & \times \int_{2(t-r)+\beta}^{t+r} \{(\alpha-(t-r))(t+r-\alpha)\}^{(n-3)/2} \times \\ & \times (t-(\alpha+\beta)/2)^{3-n} (\alpha-\beta) |u(\lambda, \tau)|^p d\alpha \end{aligned}$$

in  $\Sigma_0$ . Noticing that

$$\begin{aligned} t+r-\beta & \geq 2r, \quad t - \frac{\alpha+\beta}{2} \leq r, \\ \alpha-\beta & \geq 2(t-r) \text{ and } \alpha-(t-r) \geq t-r \end{aligned}$$

hold in the domain of the integral above, we have

$$\begin{aligned} I_1(r, t) \geq & \frac{C2^{(n-5)/2}(t-r)^{(n-1)/2}}{r^{(3n-7)/2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-3)/2} d\beta \times \\ & \times \int_{2(t-r)+\beta}^{t+r} (t+r-\alpha)^{(n-3)/2} |u(\lambda, \tau)|^p d\alpha \end{aligned}$$

in  $\Sigma_0$ . Hence, we have the first term of the right-hand side of (2.8.5).

Next, we shall show the second term of (2.8.5). Similarly to the above, we have

$$\begin{aligned} I_2(r, t) \geq & \frac{Cr^{2-n}}{4} \int_{-k_1}^{-k_0} \{(t-r-\beta)(t+r-\beta)\}^{(n-3)/2} d\beta \times \\ & \times \int_{2(t-r)+\beta}^{t+r} \{(\alpha-(t-r))(t+r-\alpha)\}^{(n-3)/2} \times \\ & \times (t-(\alpha+\beta)/2)^{3-n} (\alpha-\beta) |u(\lambda, \tau)|^p d\alpha \end{aligned}$$

in  $\Sigma_0$ . Note that

$$\begin{aligned} t+r-\beta & \geq r, \quad t - \frac{\alpha+\beta}{2} \leq r-\beta \leq 2r, \\ \alpha-(t-r) & \geq t-r+\beta \geq t-r-k \text{ and } t-r-\beta \geq t-r \end{aligned}$$

hold in the domain of the integral above. By making use of (2.8.4), we have

$$\begin{aligned} I_2(r, t) & \geq \frac{CC_g^p(t-r)^{(n-3)/2}(t-r-k)^{(n-3)/2}}{2^{n-1-(n-1)p/2}r^{(3n-7)/2}} \varepsilon^p \int_{-k_1}^{-k_0} d\beta \times \\ & \times \int_{2(t-r)+\beta}^{t+r} (\alpha-\beta)^{1-(n-1)p/2} (t+r-\alpha)^{(n-3)/2} d\alpha \\ & \geq \frac{CC_g^p(t-r)^{(n-2)-(n-1)p/2}}{2^{(n-1)p/2-(3n-9)/2}r^{(3n-7)/2}} \varepsilon^p \int_{-k_1}^{-k_0} d\beta \times \\ & \times \int_{2(t-r)+\beta}^{3(t-r)} \{3(t-r)-\alpha\}^{(n-3)/2} d\alpha \end{aligned}$$

in  $\Sigma_0$ . The second term of the right-hand side of (2.8.5) follows from this inequality. Therefore, the proof of Lemma 2.8.3 is ended.  $\square$

**[The 2nd Step] Comparison argument.**

Let us consider a solution  $w$  of

$$\begin{aligned} w(t-r) = & \frac{C2^{(n-5)/2}(t-r)^{(n-1)/2}}{r^{(3n-7)/2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-3)/2} d\beta \\ & \times \int_{2(t-r)+\beta}^{t+r} (t+r-\alpha)^{(n-3)/2} |w(\beta)|^p d\alpha \\ & + \frac{E_1(t-r)^{(3n-5)/2-(n-1)p/2}}{2r^{(3n-7)/2}} \varepsilon^p. \end{aligned} \quad (2.8.6)$$

Then we have the following comparison lemma.

**Lemma 2.8.4** *Let  $u$  be a solution of (2.8.2) and  $w$  be a solution of (2.8.6). Then,  $u$  and  $w$  satisfy*

$$u > w \quad \text{in } \Sigma_0.$$

**Proof.** Fix a point  $(r_0, t_0) \in \Sigma_0$ . Define

$$\Lambda(r, t) = \{(\lambda, \tau) \in D(r, t) : 2k \leq \tau - \lambda \leq \lambda\},$$

where we set

$$D(r, t) = \{(\lambda, \tau) : t-r \leq \tau + \lambda \leq t+r, -k \leq \tau - \lambda \leq t-r\}$$

which is the domain of the integral in (2.5.17). Let us consider  $u$  and  $v$  in  $\Lambda(r_0, t_0)$ . Note that  $u > w$  on  $\tau - \lambda = 2k$  and at  $(2k, 4k)$  which is an edge of  $\Sigma_0$ . By compactness of the closure of  $\Lambda(r_0, t_0)$ , we have  $u > w$  in a neighborhood of  $\tau - \lambda = 2k$  and  $\lambda \geq 2k$ .

Assume that there exist a point  $(r_1, t_1)$  with  $u(r_1, t_1) = w(t_1 - r_1)$ , which is nearest to  $(2k, 4k)$  in such a neighborhood. Since  $u > w$  in  $R(r_1, t_1)$ , we have

$$\begin{aligned} & \frac{C2^{(n-3)/2}(t_1 - r_1)^{(n-1)/2}}{r_1^{(3n-7)/2}} \iint_{R(r_1, t_1)} (t_1 - r_1 - \tau + \lambda)^{(n-3)/2} \times \\ & \times (t_1 + r_1 - \tau - \lambda)^{(n-3)/2} |u(\lambda, \tau)|^p d\lambda d\tau + \frac{E_1(t_1 - r_1)^{(3n-5)/2-(n-1)p/2}}{r_1^{(3n-7)/2}} \varepsilon^p \\ & > \frac{C2^{(n-3)/2}(t_1 - r_1)^{(n-1)/2}}{r_1^{(3n-7)/2}} \iint_{R(r_1, t_1)} (t_1 - r_1 - \tau + \lambda)^{(n-3)/2} \\ & \times (t_1 + r_1 - \tau - \lambda)^{(n-3)/2} |w(\tau - \lambda)|^p d\lambda d\tau + \frac{E_1(t_1 - r_1)^{(3n-5)/2-(n-1)p/2}}{2r_1^{(3n-7)/2}} \varepsilon^p, \end{aligned}$$

In view of (2.8.5) and (2.8.6), this inequality implies that  $u > w$  at  $(r_1, t_1)$ , which is a contradiction to the definition of  $(r_1, t_1)$ . Therefore, we have  $u > w$  in  $\Lambda(r_0, t_0)$ .  $(r_0, t_0)$  stands for any point in  $\Sigma_0$ , so that  $\Lambda(r_0, t_0)$  covers all of  $\Sigma_0$ . The proof is completed.  $\square$

We note that Lemma 2.8.4 implies that the lifespan of  $w$  is greater than the one of  $u$ , so that it is sufficient to look for the lifespan of  $w$  in  $\Sigma_0$ . By definition of  $w$  in (2.8.6), we have

$$w(\xi) \geq \frac{C\xi^{3-n}}{2^{n-1}} \int_{2k}^{\xi} (\xi - \beta)^{(n-3)/2} |w(\beta)|^p d\beta \\ \times \int_{2\xi+\beta}^{3\xi} (3\xi - \alpha)^{(n-3)/2} d\alpha + \frac{E_1}{2^{(3n-5)/2}} \xi^{-q-(n-1)/2} \varepsilon^p$$

in  $\Gamma_0$ , where we set

$$\xi = \frac{r}{2}, \quad \Gamma_0 = \{t - r = \xi, r \geq 4k\}.$$

Hence we obtain that

$$w(\xi) \geq \frac{C\xi^{3-n}}{2^{n-2}(n-1)} \int_{2k}^{\xi} (\xi - \beta)^{n-2} |w(\beta)|^p d\beta + \frac{E_1 \xi^{-q-(n-1)/2}}{2^{(3n-5)/2}} \varepsilon^p$$

for  $\xi \geq 2k$ . Then, it follows from the setting

$$W(\xi) = \xi^{q+(n-1)/2} w(\xi)$$

that

$$W(\xi) \geq D_n \xi^{q-(n-5)/2} \int_{2k}^{\xi} \frac{(\xi - \beta)^{n-2} |W(\beta)|^p d\beta}{\beta^{(n-1)p/2+pq}} + E_2 \varepsilon^p \quad \text{for } \xi \geq 2k, \quad (2.8.7)$$

where we set

$$D_n = \frac{C}{2^{n-3}(n-1)}, \quad E_2 = \frac{E_1}{2^{(3n-5)/2}}.$$

Therefore we obtain the iteration frame in this section,

$$W(\xi) \geq D_n \int_{2k}^{\xi} \left( \frac{\xi - \beta}{\xi} \right)^{n-2} \frac{|W(\beta)|^p}{\beta^{pq}} d\beta + E_2 \varepsilon^p \quad \text{for } \xi \geq 2k. \quad (2.8.8)$$

### [The 3rd step] Slicing method in the iteration.

Let us define a blow-up domain as follows. Let us set

$$\Gamma_j = \{\xi \geq l_j k\}, \quad l_j = 2 + \frac{1}{2} + \cdots + \frac{1}{2^j} \quad (j \in \mathbf{N}).$$

We shall use the fact that a sequence  $\{l_j\}$  is monotonously increasing and bounded,  $2 < l_j < 3$ , so that  $\Gamma_{j+1} \subset \Gamma_j$ . Assume an estimate of the form

$$W(\xi) \geq C_j \left( \log \frac{\xi}{l_j k} \right)^{a_j} \quad \text{in } \Gamma_j \quad (2.8.9)$$

where  $a_j \geq 0$  and  $C_j > 0$ . Putting (2.8.9) into (2.8.8) and recalling that  $pq = 1$ , we get an estimate in  $\Gamma_{j+1}$  such as

$$W(\xi) \geq D_n C_j^p \int_{l_j k}^{\xi} \left( \frac{\xi - \beta}{\xi} \right)^{n-2} \left( \log \frac{\beta}{l_j k} \right)^{pa_j} \frac{d\beta}{\beta}.$$

Noting that  $\frac{l_j}{l_{j+1}}\xi \geq l_j k$  in  $\Gamma_{j+1}$ , we have

$$\begin{aligned} W(\xi) &\geq D_n C_j^p \int_{l_j k}^{l_j \xi / l_{j+1}} \left( \frac{\xi - \beta}{\xi} \right)^{n-2} \left( \log \frac{\beta}{l_j k} \right)^{pa_j} \frac{d\beta}{\beta} \\ &\geq D_n C_j^p \left( 1 - \frac{l_j}{l_{j+1}} \right)^{n-2} \int_{l_j k}^{l_j \xi / l_{j+1}} \left( \log \frac{\beta}{l_j k} \right)^{pa_j} \frac{d\beta}{\beta} \\ &= \frac{D_n C_j^p}{pa_j + 1} \left( 1 - \frac{l_j}{l_{j+1}} \right)^{n-2} \left( \log \frac{\xi}{l_{j+1} k} \right)^{pa_j + 1}. \end{aligned}$$

By monotonicity of  $\{l_j\}$  and

$$1 - \frac{l_j}{l_{j+1}} = \frac{l_{j+1} - l_j}{l_{j+1}} = \frac{1}{2^{j+1} l_{j+1}} \geq \frac{1}{3 \cdot 2^{j+1}},$$

we finally obtain

$$W(\xi) \geq C_{j+1} \left( \log \frac{\xi}{l_{j+1} k} \right)^{pa_j + 1} \quad \text{in } \Gamma_{j+1}, \quad (2.8.10)$$

where we set

$$C_{j+1} = \frac{D_n C_j^p}{3^{n-2} \cdot 2^{(j+1)(n-2)} (pa_j + 1)}.$$

Now, we are in a position to define sequences in the iteration. In view of (2.8.8), the first estimate is  $W(\xi) \geq E_2 \varepsilon^p$ , so that, with the help of (2.8.9) and (2.8.10), a sequence  $\{a_j\}$  should be defined by

$$a_1 = 0, \quad a_{j+1} = pa_j + 1 \quad (j \in \mathbf{N}).$$

Also a sequence  $\{C_j\}$  should be defined by

$$C_1 = E_2 \varepsilon^p, \quad C_{j+1} = \frac{D_n C_j^p}{3^{n-2} \cdot 2^{(j+1)(n-2)} (pa_j + 1)} \quad (j \in \mathbf{N}).$$

One can easily check that

$$a_j = \frac{p^{j-1} - 1}{p - 1} \quad (j \in \mathbf{N}),$$

which gives us

$$\frac{1}{pa_j + 1} \geq \frac{p - 1}{p^j}.$$

Thus one can find that

$$C_{j+1} \geq E \frac{C_j^p}{(2^{n-2}p)^j} \quad (j \in \mathbf{N}),$$

where  $E$  is a positive constant defined by

$$E = \frac{D_n(p - 1)}{6^{n-2}}.$$

Hence we inductively obtain, for  $j \geq 2$ , that

$$\log C_j \geq p^{j-1} \left\{ \log C_1 + \sum_{k=0}^{j-2} \frac{p^k \log E - (j - 1 - k)p^k \log(2^{n-2}p)}{p^{j-1}} \right\}.$$

The sum part of above inequality converges as  $j \rightarrow \infty$  by d'Alembert's criterion. It follows from this fact that there exist a constant  $S$  independent of  $j$  such that

$$C_j \geq \exp\{p^{j-1}(\log C_1 + S)\} \quad \text{for } j \geq 2.$$

Combining all the estimates above and making use of the monotonicity of  $\Gamma_j$ , we have the final inequality

$$\begin{aligned} W(\xi) &\geq \exp\{p^{j-1}(\log C_1 + S)\} \left( \log \frac{\xi}{3k} \right)^{(p^{j-1}-1)/(p-1)} \\ &= \exp\{p^{j-1}I(\xi)\} \left( \log \frac{\xi}{3k} \right)^{-1/(p-1)} \end{aligned}$$

in  $\Gamma_\infty = \{\xi \geq 3k\}$ , where we set

$$I(\xi) = \log \left( e^S E_2 \varepsilon^p \left( \log \frac{\xi}{3k} \right)^{1/(p-1)} \right).$$

If there exist a point  $\xi_0 \in \Gamma_\infty \subset \Gamma_j$  ( $j \geq 1$ ) such that  $I(\xi_0) > 0$ , we get  $W(\xi_0) \rightarrow \infty$  as  $j \rightarrow \infty$ . Note that  $I(\xi_0) > 0$  is equivalent to

$$\xi_0 > 3k \exp\{(e^S E_2)^{-(p-1)} \varepsilon^{-p(p-1)}\}.$$

It is trivial that there exists a positive constant  $\varepsilon_0 = \varepsilon_0(g, n, p, k)$  such that

$$\exp\{(e^S E_2)^{-(p-1)} \varepsilon^{-p(p-1)}\} \geq 1 \quad \text{for } 0 < \varepsilon \leq \varepsilon_0.$$

Since there exists  $(r_0, t_0) \in \Sigma_0$  such that  $t_0 - r_0 = \xi_0 > 3k$ , we obtain the desired conclusion;

$$T > 3k \exp\{(e^S E_2)^{-(p-1)} \varepsilon^{-p(p-1)}\}.$$

Therefore the proof of the critical case is now completed with a minor modification on  $\varepsilon_0$ .  $\square$

## 2.9 Upper bound of the lifespan for the subcritical case in odd dimensions

By a similar argument to the previous section, we prove the blow-up result for (2.3.9) in the subcritical case in odd space dimensions. Note that we do not have to make use of the slicing method.

**Proposition 2.9.1** *Suppose that the assumptions of Theorem 2.3.2 are fulfilled. Let  $u$  be a  $C^0$ -solution of (2.3.9) in  $\mathbf{R}^n \times [0, T]$ . Then, there exists a positive constant  $\varepsilon_0 = \varepsilon_0(g, n, p, k)$  such that  $T$  cannot be taken as*

$$T > c\varepsilon^{-2p(p-1)/\gamma(p,n)} \quad \text{if } 1 < p < p_0(n) \quad (2.9.1)$$

for  $0 < \varepsilon \leq \varepsilon_0$ , where  $c$  is a positive constant independent of  $\varepsilon$ .

**Proof.** Because of the fact that  $-(n-1)p/2 - pq < 0$  for  $n \geq 5$ , (2.8.7) yields

$$W(\xi) \geq D_n \xi^{-(n-2)-pq} \int_{2k}^{\xi} (\xi - \beta)^{n-2} |W(\beta)|^p d\beta + E_2 \varepsilon^p \quad \text{for } \xi \geq 2k. \quad (2.9.2)$$

This is our iteration frame in this case.

Assume an estimate of the form

$$W(\xi) \geq C_j \frac{(\xi - 2k)^{a_j}}{\xi^{b_j}} \quad \text{in } \Gamma_0, \quad (2.9.3)$$

where  $a_j, b_j \geq 0$  and  $C_j > 0$ . Then, putting (2.9.3) into (2.9.2), we get

$$W(\xi) \geq \frac{D_n C_j^p}{\xi^{(n-2)+pq+pb_j}} \int_{2k}^{\xi} (\xi - \beta)^{n-2} (\beta - 2k)^{pa_j} d\beta \quad \text{in } \Gamma_0.$$



Applying the integration by parts to  $\beta$ -integral  $(n-2)$  times, we obtain that

$$\begin{aligned}
& \frac{n-2}{pa_j+1} \int_{2k}^{\xi} (\xi-\beta)^{n-3} (\beta-2k)^{pa_j+1} d\beta \\
& \geq \frac{(n-2)(n-3)}{(pa_j+2)^2} \int_{2k}^{\xi} (\xi-\beta)^{n-4} (\beta-2k)^{pa_j+2} d\beta \\
& \dots \\
& \geq \frac{(n-2)!}{(pa_j+n-1)^{n-1}} (\xi-2k)^{pa_j+n-1}.
\end{aligned}$$

Therefore we finally get

$$W(\xi) \geq C_{j+1} \frac{(\xi-2k)^{pa_j+n-1}}{\xi^{pb_j+n-2+pq}} \text{ in } \Gamma_0, \quad (2.9.4)$$

where we set

$$C_{j+1} = \frac{D_n C_j^p (n-2)!}{(pa_j+n-1)^{n-1}}.$$

Now, we are in a position to define sequences in the iteration. In view of (2.9.2), the first estimate is  $W(\xi) \geq E_2 \varepsilon^p$ , so that, with the help of (2.9.3) and (2.9.4), sequences  $\{a_j\}$  and  $\{b_j\}$  should be defined by

$$a_1 = 0, \quad a_{j+1} = pa_j + n - 1 \quad (j \in \mathbf{N})$$

and

$$b_1 = 0, \quad b_{j+1} = pb_j + n - 2 + pq \quad (j \in \mathbf{N}).$$

Also a sequence  $\{C_j\}$  should be defined by

$$C_1 = E_2 \varepsilon^p, \quad C_{j+1} = \frac{D_n C_j^p (n-2)!}{(pa_j+n-1)^{n-1}} \quad (j \in \mathbf{N}).$$

One can readily check that

$$a_j = \frac{n-1}{p-1} (p^{j-1} - 1), \quad b_j = \frac{pq+n-2}{p-1} (p^{j-1} - 1) \quad (j \geq \mathbf{N}),$$

which gives us

$$\frac{1}{pa_j+n-1} \geq \frac{p-1}{p^j(n-1)}.$$

Hence one can find that

$$C_{j+1} \geq F \frac{C_j^p}{p^{(n-1)j}} \quad (j \in \mathbf{N}),$$

where  $F$  is a positive constant defined by

$$F = \frac{D_n(n-2)!(p-1)^{n-1}}{(n-1)^{n-1}}.$$

Due to the induction argument again, we obtain, for  $j \geq 2$ , that

$$\log C_j \geq p^{j-1} \left\{ \log C_1 + \sum_{k=0}^{j-2} \frac{p^k \log F - (j-1-k)p^k \log(p^{n-1})}{p^{j-1}} \right\}.$$

As before, this inequality yields that there exist a constant  $S$  independent of  $j$  such that

$$C_j \geq \exp\{p^{j-1}(\log C_1 + S)\} \quad \text{for } j \geq 2.$$

Combining all the estimates above, we reach the the final inequality

$$W(\xi) \geq \exp\{p^{j-1}I(\xi)\} \frac{\xi^{(pq+n-2)/(p-1)}}{(\xi-2k)^{(n-1)/(p-1)}} \quad \text{for } \xi \geq 2k,$$

where we set

$$I(\xi) = \log \left( E_2 e^S \varepsilon^p (\xi - 2k)^{(n-1)/(p-1)} \xi^{-(pq+n-2)/(p-1)} \right).$$

Note that

$$\frac{n-1}{p-1} - \frac{pq+n-2}{p-1} = \frac{1-pq}{p-1}.$$

If there exist a point  $\xi_0 \in \{\xi \geq 4k\} \subset \{\xi \geq 2k\}$  such that  $I(\xi_0) > 0$ , the desired conclusion can be established by the same argument as in the previous section.  $I(\xi_0) > 0$  is equivalent to

$$\xi_0 > 2^{(n-1)/(1-pq)} (e^S E_2)^{-(p-1)/(1-pq)} \varepsilon^{-2p(p-1)/\gamma(p,n)}$$

in this case, so that the proof of the subcritical case is now completed.  $\square$

## 2.10 Upper bound of the lifespan for the critical case in even dimensions

In this section, we prove the blow-up theorem in the critical case in even dimensions. The proof is based on the one in odd dimensional case. However, Huygens' principle for  $u^0$  is no longer available. Therefore the blow-up domain to ensure the positivity of the linear part is modified.

**Proposition 2.10.1** *Suppose that of the assumption of Theorem 2.3.2 are fulfilled. Let  $u$  be a  $C^0$ -solution of (2.3.9) if  $n > 4$  and  $p = p_0(n)$ , or  $u$  be a classical solution of (2.3.10) if  $n = 4$  and  $p = p_0(4)$  in  $\mathbf{R}^n \times [0, T]$ . Then, there exists a positive constant  $\varepsilon_0 = \varepsilon_0(g, n, p, k)$  such that  $T$  cannot be taken as*

$$T > \exp(c\varepsilon^{-p(p-1)}) \quad \text{if } p = p_0(n) \quad (2.10.1)$$

for  $0 < \varepsilon \leq \varepsilon_0$ , where  $c$  is a positive constant independent of  $\varepsilon$ .

**Proof.** Similarly to the odd dimensional case, we may assume that the solution of (2.3.9) is radially symmetric without loss of the generality. Let  $u = u(r, t)$  be a  $C^0$ -solution of

$$u \geq \varepsilon u^0 + L_{\text{even},1}(|u|^p) \quad \text{in } (0, \infty) \times [0, T], \quad (2.10.2)$$

where  $L_{\text{even},1}$  is defined by (2.5.18) and  $u^0 = u^0(r, t)$  is a solution of (2.8.3).

**[The 1st step] Estimate of  $u^0$ .**

We shall employ the following representation of  $u^0$ .

**Lemma 2.10.1 (Rammaha [26])** *Let  $n = 4, 6, 8, \dots$  and  $u^0$  be a solution of (2.8.3). Then,  $u^0$  is represented by*

$$u^0(r, t) = \frac{2}{\pi r^{(n-2)/2}} \int_0^t \frac{\rho d\rho}{\sqrt{t^2 - \rho^2}} \times \\ \times \int_{|r-\rho|}^{r+\rho} \frac{\lambda^{(n-2)/2} g(\lambda) T_{(n-4)/2}((\lambda^2 + r^2 - \rho^2)/(2r\lambda)) d\lambda}{\sqrt{\lambda^2 - (r - \rho)^2} \sqrt{(r + \rho)^2 - \lambda^2}},$$

where  $T_k$  is Tschebyscheff polynomials of degree  $k$  defined by

$$T_k(z) = \frac{(-1)^k}{(2k-1)!!} (1-z^2)^{1/2} \frac{d^k}{dz^k} (1-z^2)^{k-(1/2)}.$$

See (6b) on 681p. in [26] for the proof. This lemma implies the following estimate.

**Lemma 2.10.2 (Rammaha [26])** *Let  $n = 4, 6, 8, \dots$ . Assume (3.1.3). Then there exists a positive constant  $C_g$  such that, for  $t + k_0 < r < t + k_1$  and  $t \geq k_2$ ,*

$$u^0(r, t) \geq \frac{C_g}{r^{(n-1)/2}}, \quad (2.10.3)$$

where  $k_2 = k - k_0$ .

See Lemma 2 on 682p. in [26] for the proof.

Our frame in the iteration argument is obtained by the following lemma.

**Lemma 2.10.3** *Let  $u$  be a  $C^0$ -solution of (2.10.2). Assume (3.1.3). Then  $u$  in  $\Sigma_0 = \{(r, t) : 2k \leq t - r \leq r\}$  satisfies*

$$\begin{aligned} u(r, t) &\geq \frac{C2^{(n-3)/2}(t-r)^{(n-1)/2}}{(n-1)r^{(3n-5)/2}} \times \\ &\times \iint_{R(r,t)} \{(t-r-\tau+\lambda)(t+r-\tau-\lambda)\}^{(n-2)/2} |u(\lambda, \tau)|^p d\lambda d\tau + \quad (2.10.4) \\ &+ \frac{F_1(t-r)^{(3n-3)/2-(n-1)p/2}}{r^{(3n-5)/2}} \varepsilon^p + \varepsilon u^0(r, t), \end{aligned}$$

where  $C$  is the one in (2.5.18) and

$$F_1 = \frac{CC_g^p(k_1 - k_0)2^{(9-3n)/2-(n-1)p/2}}{n(n-1)}.$$

**Proof.** In view of (2.5.18), we have that

$$\begin{aligned} L_{even,1}(|u|^p)(r, t) &\geq \frac{C}{r^{n-2}} \int_0^t (t-\tau)^{2-n} d\tau \int_{|t-r-\tau|}^{t+r-\tau} \lambda |u(\lambda, \tau)|^p d\lambda \times \\ &\times \int_{|\lambda-r|}^{t-\tau} \frac{\rho h(\lambda, \rho, r)}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho \end{aligned}$$

in  $\Sigma_0$ . Noticing that  $(\lambda+r)^2 - \rho^2 \geq (\lambda+r)^2 - (t-\tau)^2$  for  $\rho \leq t-\tau$ , we get

$$\begin{aligned} L_{even,1}(|u|^p)(r, t) &\geq \frac{C}{r^{n-2}} \int_0^t (t-\tau)^{2-n} d\tau \times \\ &\times \int_{|t-r-\tau|}^{t+r-\tau} \frac{\lambda |u(\lambda, \tau)|^p \{(\lambda+r)^2 - (t-\tau)^2\}^{(n-3)/2} d\lambda}{\{(t-\tau)^2 - (\lambda-r)^2\}^{1/2}} \times \\ &\times \int_{|\lambda-r|}^{t-\tau} \rho \{\rho^2 - (\lambda-r)^2\}^{(n-3)/2} d\rho \end{aligned}$$

in  $\Sigma_0$ . Since the  $\rho$ -integral above is

$$\frac{1}{n-1} \{(t-\tau)^2 - (\lambda-r)^2\}^{(n-1)/2},$$

we obtain that

$$\begin{aligned} L_{even,1}(|u|^p)(r, t) &\geq \frac{C}{r^{n-2}(n-1)} \int_0^t (t-\tau)^{2-n} d\tau \times \\ &\times \int_{|t-r-\tau|}^{t+r-\tau} \{(\lambda+r)^2 - (t-\tau)^2\}^{(n-3)/2} \times \\ &\times \{(t-\tau)^2 - (\lambda-r)^2\}^{(n-2)/2} \lambda |u(\lambda, \tau)|^p d\lambda \\ &\geq J_1 + J_2 \end{aligned}$$

in  $\Sigma_0$ , where we set

$$J_1(r, t) = \frac{C}{(n-1)r^{n-2}} \int_{R(r,t)} (t-\tau)^{2-n} \{(\lambda+r)^2 - (t-\tau)^2\}^{(n-3)/2} \times \\ \times \{(t-\tau)^2 - (\lambda-r)^2\}^{(n-2)/2} \lambda |u(\lambda, \tau)|^p d\lambda d\tau, \\ J_2(r, t) = \frac{C}{(n-1)r^{n-2}} \int_{S(r,t)} (t-\tau)^{2-n} \{(\lambda+r)^2 - (t-\tau)^2\}^{(n-3)/2} \times \\ \times \{(t-\tau)^2 - (\lambda-r)^2\}^{(n-2)/2} \lambda |u(\lambda, \tau)|^p d\lambda d\tau.$$

Changing variables by (2.5.28) in  $J_1$ , we have that

$$J_1(r, t) \geq \frac{C}{2(n-1)r^{n-2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-2)/2} (t+r-\beta)^{(n-3)/2} d\beta \times \\ \times \int_{2(t-r)+\beta}^{t+r} \{\alpha - (t-r)\}^{(n-3)/2} (t+r-\alpha)^{(n-2)/2} \times \\ \times \{t - (\alpha + \beta)/2\}^{2-n} (\alpha - \beta) |u(\lambda, \tau)|^p d\alpha$$

in  $\Sigma_0$ . Note that

$$t+r-\beta \geq 2r, \quad t - \frac{\alpha + \beta}{2} \leq r, \\ \alpha - \beta \geq 2(t-r), \quad \alpha - (t-r) \geq t-r+\beta \geq t-r$$

hold in the domain of the integral above. Hence we get

$$J_1(r, t) \geq \frac{C 2^{(n-5)/2} (t-r)^{(n-1)/2}}{(n-1)r^{(3n-5)/2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-2)/2} d\beta \times \\ \times \int_{2(t-r)+\beta}^{t+r} (t+r-\alpha)^{(n-2)/2} |u(\lambda, \tau)|^p d\alpha$$

in  $\Sigma_0$ . Therefore, we obtain the first term of the right-hand side in (2.10.4).

Similarly to the above,  $J_2(r, t)$  is bounded from below by

$$\frac{C}{4(n-1)r^{n-2}} \int_{-k_1}^{-k_0} (t-r-\beta)^{(n-2)/2} (t+r-\beta)^{(n-3)/2} d\beta \\ \times \int_{2(t-r)+\beta}^{t+r} \{\alpha - (t-r)\}^{(n-3)/2} (t+r-\alpha)^{(n-2)/2} \\ \times \{t - (\alpha + \beta)/2\}^{2-n} (\alpha - \beta) |u(\lambda, \tau)|^p d\alpha$$

in  $\Sigma_0$ . Note that

$$t+r-\beta \geq r, \quad t - \frac{\alpha + \beta}{2} \leq 2r, \quad \alpha - (t-r) \geq t-r-k \\ \text{and } t-r-\beta \geq t-r$$

hold in the domain of the integral above. Hence (2.10.3) yields that  $J_2(r, t)$  in  $\Sigma_0$  is estimated from below by

$$\begin{aligned}
&\geq \frac{\varepsilon^p C C_g^p 2^{(n-1)p/2-n} (t-r)^{(n-2)/2} (t-r-k)^{(n-3)/2}}{(n-1)r^{(3n-5)/2}} \int_{-k_1}^{-k_0} d\beta \\
&\quad \times \int_{2(t-r)+\beta}^{t+r} (\alpha-\beta)^{1-(n-1)p/2} (t+r-\alpha)^{(n-2)/2} d\alpha \\
&\geq \frac{\varepsilon^p C C_g^p 2^{2-n-(n-1)p} (t-r)^{n/2-(n-1)p/2} (t-r-k)^{(n-3)/2}}{(n-1)r^{(3n-5)/2}} \int_{-k_1}^{-k_0} d\beta \times \\
&\quad \times \int_{2(t-r)+\beta}^{3(t-r)} \{3(t-r)-\alpha\}^{(n-2)/2} d\alpha.
\end{aligned}$$

The second term of the right-hand side of (2.10.4) follows from this inequality. Therefore, the proof of Lemma 2.10.3 is ended.  $\square$

Next, we shall show the positivity of the right-hand side of (2.10.4). Under the condition (3.1.3), (2.4.6) yields that

$$\begin{aligned}
\varepsilon u^0(r, t) &\geq \frac{-C_{n,k,0,g}\varepsilon}{(t+r+2k)^{(n-1)/2}(t-r+2k)^{(n-1)/2}} \\
&\geq \frac{-C_{n,k,0,g}\varepsilon}{r^{(n-1)/2}(t-r)^{(n-1)/2}}
\end{aligned}$$

for  $t-r > 0$ . Let us define a domain

$$\Sigma_1 = \left\{ (r, t) \in (0, \infty)^2 : r \geq t-r \geq \frac{r}{2}, r \geq K\varepsilon^{-L} \right\},$$

where we set

$$\begin{aligned}
K &= \left( 2^{2n-(n-1)p/2-1} F_1^{-1} C_{n,k,0,g} \right)^{1/(n-(n-1)p/2)}, \\
L &= \frac{p-1}{n-(n-1)p/2} > 0.
\end{aligned}$$

Taking  $\varepsilon$  to satisfy

$$K\varepsilon^{-L} \geq 4k$$

and setting

$$A(r, t) = \frac{F_1(t-r)^{(3n-3)/2-(n-1)p/2}}{r^{(3n-5)/2}} > 0,$$

we obtain that, in  $\Sigma_1$ ,

$$\begin{aligned}
&\frac{A(r, t)}{2} \varepsilon^p + \varepsilon u^0(r, t) \\
&\geq \frac{F_1 \varepsilon^p (t-r)^{2n-2-(n-1)p/2} r^{(n-1)/2} - 2C_{n,k,0,g} \varepsilon r^{(3n-5)/2}}{2r^{2n-3} (t-r)^{(n-1)/2}} \geq 0.
\end{aligned}$$

Making use of this inequality, we obtain that

$$u(r, t) \geq \frac{C2^{(n-5)/2}(t-r)^{(n-1)/2}}{(n-1)r^{(3n-5)/2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-2)/2} d\beta \times \\ \times \int_{2(t-r)+\beta}^{t+r} (t+r-\alpha)^{(n-2)/2} |u(\lambda, \tau)|^p d\alpha + \frac{A(r, t)}{2} \varepsilon^p$$

in  $\Sigma_1$ . Cutting the domain of the integral, we get

$$u(r, t) > \frac{C2^{(n-5)/2}(t-r)^{(n-1)/2}}{(n-1)r^{(3n-5)/2}} \int_{K\varepsilon^{-L}/2}^{t-r} (t-r-\beta)^{(n-2)/2} d\beta \times \\ \times \int_{3(t-r)}^{t+r} (t+r-\alpha)^{(n-2)/2} |u(\lambda, \tau)|^p d\alpha + \frac{A(r, t)}{4} \varepsilon^p$$

in  $\Sigma_1$ . Here we introduce a change of variables  $(\alpha, \beta)$  to  $(\xi, \eta)$  by

$$\xi = \alpha, \quad \eta = \frac{\alpha + \beta}{2} - \frac{3}{2} \cdot \frac{\alpha - \beta}{2} = \frac{5\beta - \alpha}{4}.$$

Then, cutting the domain of the integral again, we get

$$u(r, t) > \frac{C2^{(n-1)/2}(t-r)^{(n-1)/2}}{5(n-1)r^{(3n-5)/2}} \int_{K\varepsilon^{-L}/2}^{t-3r/2} \left\{ t-r - \left( \frac{4\eta + \xi}{5} \right) \right\}^{(n-2)/2} d\eta \times \\ \times \int_{3(t-r)}^{t+r} (t+r-\xi)^{(n-2)/2} |u(\lambda, \tau)|^p d\xi + B(r, t) \varepsilon^p$$

in  $\Sigma_2$ , where we set

$$\Sigma_2 = \left\{ (r, t) \in (0, \infty)^2 : \frac{r}{2} \geq t - \frac{3}{2}r \geq \frac{K\varepsilon^{-L}}{2} \right\}$$

and

$$B(r, t) = \frac{A(r, t)}{4}.$$

Therefore we obtain that, in  $\Sigma_2$ ,

$$u(r, t) > \frac{C2^{(3n-5)/2}(t-r)^{(n-1)/2}}{5^{n/2}(n-1)r^{(3n-5)/2}} \int_{K\varepsilon^{-L}/2}^{t-3r/2} \left( t - \frac{3}{2}r - \eta \right)^{(n-2)/2} d\eta \times \\ \times \int_{3(t-r)}^{t+r} (t+r-\alpha)^{(n-2)/2} |u(\lambda, \tau)|^p d\alpha + B(r, t) \varepsilon^p. \quad (2.10.5)$$

**[The 2nd Step] Comparison argument.**

Let us consider a solution  $y$  of

$$\begin{aligned} y\left(t - \frac{3}{2}r\right) &= \frac{C2^{(3n-3)/2}(t-r)^{(n-1)/2}}{5^{n/2}(n-1)r^{(3n-5)/2}} \int_{K\varepsilon^{-L}/2}^{t-3r/2} \left(t - \frac{3}{2}r - \eta\right)^{(n-2)/2} d\eta \times \\ &\quad \times \int_{3(t-r)}^{t+r} (t+r-\alpha)^{(n-2)/2} |y(\eta)|^p d\alpha + B(r, t)\varepsilon^p. \end{aligned} \quad (2.10.6)$$

Then we have the following comparison lemma.

**Lemma 2.10.4** *Let  $u$  be a solution of (2.10.2) and  $y$  be a solution of (2.10.6). Then,  $u$  and  $y$  satisfy*

$$u > y \quad \text{in } \Sigma_2.$$

**Proof.** Fix a point for any  $(r_0, t_0) \in \Sigma_2$ . Define

$$\Lambda(r, t) = \left\{ (\lambda, \tau) \in D(r, t) : \frac{K\varepsilon^{-L}}{2} \leq \tau - \frac{3}{2}\lambda \leq \frac{\lambda}{2} \right\},$$

where

$$D(r, t) = \{(\lambda, \tau) : t - r \leq \tau + \lambda \leq t + r, -k \leq \tau - \lambda \leq t - r\}.$$

Let us consider  $u$  and  $y$  in  $\Lambda(r_0, t_0)$ . Note that  $u > y$  on  $\tau - \frac{3}{2}\lambda = \frac{K\varepsilon^{-L}}{2}$  and at  $(K\varepsilon^{-L}, 2K\varepsilon^{-L})$  which is an edge point of  $\Sigma_2$ . By compactness of the closure of  $\Lambda(r_0, t_0)$ , we have  $u > y$  in a neighborhood of  $\tau - \frac{3}{2}\lambda = \frac{K\varepsilon^{-L}}{2}$  and  $\lambda \geq K\varepsilon^{-L}$ .

Assume that there exist a point  $(r_1, t_1)$  with  $u(r_1, t_1) = y(t_1 - 3r_1/2)$  which is nearest to  $(K\varepsilon^{-L}, 2K\varepsilon^{-L})$  in such a neighborhood. Since  $u > y$  in  $R'(r_1, t_1)$ , we have

$$\begin{aligned} &\frac{C2^{(3n-7)/2}(t_1 - r_1)^{(n-1)/2}}{5^{(n-2)/2}(n-1)r_1^{(3n-5)/2}} \iint_{R'(r_1, t_1)} \left(t_1 - \frac{3}{2}r_1 - \tau + \frac{3}{2}\lambda\right)^{(n-2)/2} \times \\ &\quad \times (t_1 + r_1 - \tau - \lambda)^{(n-2)/2} |u(\lambda, \tau)|^p d\lambda d\tau + B(r_1, t_1)\varepsilon^p \\ &> \frac{C2^{(3n-7)/2}(t_1 - r_1)^{(n-1)/2}}{5^{(n-2)/2}(n-1)r_1^{(3n-5)/2}} \iint_{R'(r_1, t_1)} \left(t_1 - \frac{3}{2}r_1 - \tau + \frac{3}{2}\lambda\right)^{(n-2)/2} \times \\ &\quad \times \left(t_1 + r_1 - \tau - \lambda\right)^{(n-2)/2} \left|y\left(\tau - \frac{3}{2}\lambda\right)\right|^p d\lambda d\tau + B(r_1, t_1)\varepsilon^p, \end{aligned}$$

where we set

$$R'(r, t) = \left\{ (\lambda, \tau) : 3(t - r) \leq \tau + \lambda \leq t + r, \frac{K\varepsilon^{-L}}{2} \leq \tau - \frac{3}{2}\lambda \leq t - \frac{3}{2}r \right\}.$$



In view of (2.10.5) and (2.10.6), this inequality yield that  $u > y$  at  $(r_1, t_1)$ , which is a contradiction to the definition of  $(r_1, t_1)$ . Therefore, the proof of Lemma 2.10.4 is now established by the same argument as the one for Lemma 2.8.4.  $\square$

We note that Lemma 2.10.4 implies that the lifespan of  $y$  is greater than the lifespan of  $u$ , so that it is sufficient to look for the lifespan of  $y$  in  $\Sigma_2$ . By definition  $y$  in (2.10.6), we have

$$y(\xi) = \frac{C3^{(n-1)/2}\xi^{2-n}}{2^{(3n-5)/2}5^{n/2}(n-1)} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi - \eta)^{(n-2)/2} |y(\eta)|^p d\eta \\ \times \int_{9\xi}^{11\xi} (11\xi - \alpha)^{(n-2)/2} d\alpha + \frac{\varepsilon^p F_1 \xi^{-q-(n-1)/2}}{2^{(3n-3)}3^{(n-1)p/2-(3n-3)/2}}$$

in  $\Gamma_1$ , where we set

$$\xi = \frac{r}{4}, \quad \Gamma_1 = \left\{ t - \frac{3}{2}r = \xi, r \geq 2K\varepsilon^{-L} \right\} \subset \Sigma_2.$$

Hence, we obtain that

$$y(\xi) \geq \frac{C3^{(n-1)/2}\xi^{(4-n)/2}}{2^{(2n-7)/2}5^{n/2}n(n-1)} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi - \eta)^{(n-2)/2} |y(\eta)|^p d\eta \\ + \frac{\varepsilon^p F_1 \xi^{-q-(n-1)/2}}{2^{(3n-3)}3^{(n-1)p/2-(3n-3)/2}}$$

for  $\xi \geq K\varepsilon^{-L}/2$ . Then it follows from the setting

$$Y(\xi) = \xi^{q+(n-1)/2} y(\xi)$$

that

$$Y(\xi) \geq E_n \xi^{q+3/2} \int_{K\varepsilon^{-L}/2}^{\xi} \frac{(\xi - \eta)^{(n-2)/2} |Y(\eta)|^p d\eta}{\eta^{(n-1)p/2+pq}} + F_2 \varepsilon^p, \quad (2.10.7)$$

where we set

$$E_n = \frac{C3^{(n-1)/2}}{2^{(2n-7)/2}5^{n/2}n(n-1)}, \quad F_2 = \frac{F_1}{2^{(3n-3)}3^{(n-1)p/2-(3n-3)/2}}.$$

Therefore we obtain the iteration frame in this section,

$$Y(\xi) \geq E_n \int_{K\varepsilon^{-L}/2}^{\xi} \left( \frac{\xi - \eta}{\xi} \right)^{(n-2)/2} \frac{|Y(\eta)|^p}{\eta^{pq}} d\eta + F_2 \varepsilon^p \quad \text{for } \xi \geq \frac{K\varepsilon^{-L}}{2}. \quad (2.10.8)$$

**[The 3rd step] Slicing method with the iteration.**

Let us define a blow-up domain as follows. Let us set

$$\Gamma_j = \{\xi \geq l_j K \varepsilon^{-L}\}, \quad l_j = \frac{1}{2} + \cdots + \frac{1}{2^j} \quad (j \in \mathbf{N}).$$

We shall use the fact that a sequence  $\{l_j\}$  is monotonously increasing and bounded as  $\frac{1}{2} < l_j < 1$ , so that  $\Gamma_{j+1} \subset \Gamma_j$ . Assume an estimate of the form

$$Y(\xi) \geq C_j \left( \log \frac{\xi}{l_j K \varepsilon^{-L}} \right)^{a_j} \quad \text{in } \Gamma_j, \quad (2.10.9)$$

where  $a_j \geq 0$  and  $C_j > 0$ . Putting (2.10.9) into (2.10.8) and recalling  $pq = 1$ , we get an estimate in  $\Gamma_{j+1}$  such as

$$Y(\xi) \geq E_n C_j^p \int_{l_j K \varepsilon^{-L}}^{\xi} \left( \frac{\xi - \eta}{\xi} \right)^{(n-2)/2} \left( \log \frac{\eta}{l_j K \varepsilon^{-L}} \right)^{pa_j} \frac{d\eta}{\eta}.$$

Noting that  $\frac{l_j}{l_{j+1}} \xi \geq l_j K \varepsilon^{-L}$  in  $\Gamma_{j+1}$ , we have

$$\begin{aligned} Y(\xi) &\geq E_n C_j^p \left( 1 - \frac{l_j}{l_{j+1}} \right)^{(n-2)/2} \int_{l_j K \varepsilon^{-L}}^{l_j \xi / l_{j+1}} \left( \log \frac{\eta}{l_j K \varepsilon^{-L}} \right)^{pa_j} \frac{d\eta}{\eta} \\ &= \frac{E_n C_j^p}{pa_j + 1} \left( 1 - \frac{l_j}{l_{j+1}} \right)^{(n-2)/2} \left( \log \frac{\xi}{l_{j+1} K \varepsilon^{-L}} \right)^{pa_j + 1}. \end{aligned}$$

By monotonicity of  $\{l_j\}$  and

$$1 - \frac{l_j}{l_{j+1}} = \frac{l_{j+1} - l_j}{l_{j+1}} = \frac{1}{2^{j+1} l_{j+1}} \geq \frac{1}{2^{j+1}},$$

we finally obtain

$$Y(\xi) \geq C_{j+1} \left( \log \frac{\xi}{l_{j+1} K \varepsilon^{-L}} \right)^{pa_j + 1} \quad \text{in } \Gamma_{j+1}, \quad (2.10.10)$$

where we set

$$C_{j+1} = \frac{E_n C_j^p}{2^{(n-2)(j+1)/2} (pa_j + 1)}.$$

Now, we are in a position to define sequences in the iteration. In view of (2.10.8), the first estimate is  $Y(\xi) \geq F_2 \varepsilon^p$ , so that, with the help of (2.10.9) and (2.10.10), a sequence  $\{a_j\}$  should be defined by

$$a_1 = 0, \quad a_{j+1} = pa_j + 1 \quad (j \in \mathbf{N}).$$

Also a sequence  $\{C_j\}$  should be defined by

$$C_1 = F_2 \varepsilon^p, \quad C_{j+1} = \frac{E_n C_j^p}{2^{(n-2)(j+1)/2} (pa_j + 1)} \quad (j \in \mathbf{N}).$$

One can easily check that

$$a_j = \frac{p^{j-1} - 1}{p - 1} \quad (j \in \mathbf{N})$$

which gives us

$$\frac{1}{pa_j + 1} \geq \frac{p - 1}{p^j}.$$

Thus one can find that

$$C_{j+1} \geq E \frac{C_j^p}{(2^{(n-2)/2} p)^j} \quad (j \in \mathbf{N}),$$

where  $E$  is a positive constant defined by

$$E = \frac{E_n(p - 1)}{2^{(n-2)/2}}.$$

Hence, we inductively obtain, for  $j \geq 2$ , that

$$\log C_j \geq p^{j-1} \left\{ \log C_1 + \sum_{k=0}^{j-2} \frac{p^k \log E - (j - 1 - k) p^k \log(2^{(n-2)/2} p)}{p^{j-1}} \right\}.$$

This inequality yields that there exist a constant  $S$  independent of  $j$  such that

$$C_j \geq \exp\{p^{j-1}(\log C_1 + S)\} \quad \text{for } j \geq 2.$$

Combining all the estimates above and making use of the monotonicity of  $\Gamma_j$ , we obtain the final inequality,

$$\begin{aligned} Y(\xi) &\geq \exp\{p^{j-1}(\log C_1 + S)\} \left( \log \frac{\xi}{K \varepsilon^{-L}} \right)^{(p^{j-1}-1)/(p-1)} \\ &= \exp\{p^{j-1} I(\xi)\} \left( \log \frac{\xi}{K \varepsilon^{-L}} \right)^{-1/(p-1)}. \end{aligned}$$

in  $\Gamma_\infty = \{\xi \geq K \varepsilon^{-L}\}$ , where we set

$$I(\xi) = \log \left( e^S F_2 \varepsilon^p \left( \log \frac{\xi}{K \varepsilon^{-L}} \right)^{1/(p-1)} \right).$$

If there exist a point  $\xi_0 \in \Gamma_\infty \subset \Gamma_j$  ( $j \geq 1$ ) such that  $I(\xi_0) > 0$ , we get the desired conclusion by the same argument in the end of section 7. In this case,  $I(\xi_0) > 0$  is equivalent to

$$\xi_0 > \exp\{(e^S F_2)^{-(p-1)} \varepsilon^{-p(p-1)}\} K \varepsilon^{-L}.$$

Therefore the proof of the critical case is now completed.  $\square$

## 2.11 Upper bound of the lifespan for the subcritical case in even dimensions

Similarly to the previous section, we prove the blow-up result of solution for (2.3.9) in the subcritical case in even space dimensions. Note that we do not have to make use of the slicing method.

**Proposition 2.11.1** *Suppose that the same assumption of Theorem 2.3.2 are fulfilled. Let  $u$  be a  $C^0$ -solution of (2.3.9)  $\mathbf{R}^n \times [0, T]$ . Then there exists a positive constant  $\varepsilon_0 = \varepsilon_0(g, n, p, k)$  such that  $T$  cannot be taken as*

$$T > c \varepsilon^{-2p(p-1)/\gamma(p,n)} \text{ if } 1 < p < p_0(n) \quad (2.11.1)$$

for  $0 < \varepsilon \leq \varepsilon_0$ , where  $c$  is a positive constant independent of  $\varepsilon$ .

**Proof.** Because of the fact that  $-(n-1)p/2 - pq < 0$  for  $n \geq 4$ , (2.10.7) yields

$$Y(\xi) \geq E_n \xi^{-(n-2)/2-pq} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi - \eta)^{(n-2)/2} |Y(\eta)|^p d\eta + F_2 \varepsilon^p \quad (2.11.2)$$

for  $\xi \geq K\varepsilon^{-L}/2$ . This is our iteration frame in this case.

Assume an estimate of the form

$$Y(\xi) \geq C_j \frac{\left(\xi - \frac{K\varepsilon^{-L}}{2}\right)^{a_j}}{\xi^{b_j}} \text{ in } \Gamma_1, \quad (2.11.3)$$

where  $a_j, b_j \geq 0$  and  $C_j > 0$ . Then, putting (2.11.3) into (2.11.2), we get the following estimate in  $\Gamma_1$  of the form

$$Y(\xi) \geq E_n C_j^p \xi^{-(n-2)/2-pq-pb_j} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi - \eta)^{(n-2)/2} \left(\eta - \frac{K\varepsilon^{-L}}{2}\right)^{pa_j} d\eta.$$

Applying the integration by parts to  $\beta$ -integral  $(n-2)/2$  times, we obtain that

$$\begin{aligned}
& \frac{(n-2)}{2(pa_j+1)} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi-\eta)^{(n-4)/2} \left(\eta - \frac{K\varepsilon^{-L}}{2}\right)^{pa_j+1} d\eta \\
& \geq \frac{(n-2)(n-4)}{(pa_j+2)^2 \cdot 2} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi-\eta)^{(n-6)/2} \left(\eta - \frac{K\varepsilon^{-L}}{2}\right)^{pa_j+2} d\eta \\
& \dots \\
& \geq \frac{((n-2)/2)!}{\left(pa_j + \frac{n}{2}\right)^{n/2}} \left(\xi - \frac{K\varepsilon^{-L}}{2}\right)^{pa_j+n/2}.
\end{aligned}$$

Therefore we finally get

$$Y(\xi) \geq \frac{C_{j+1} \left(\xi - \frac{K\varepsilon^{-L}}{2}\right)^{pa_j+n/2}}{\xi^{pb_j+(n-2)/2+pq}} \text{ in } \Gamma_1, \quad (2.11.4)$$

where

$$C_{j+1} = \frac{((n-2)/2)! E_n C_j^p}{\left(pa_j + \frac{n}{2}\right)^{n/2}}.$$

Now, we are in a position to define sequences in the iteration. In view of (2.11.2), the first estimate is  $Y(\xi) \geq F_2 \varepsilon^p$ , so that, with the help of (2.11.3) and (2.11.4), sequences  $\{a_j\}$  and  $\{b_j\}$  should be defined by

$$a_1 = 0, \quad a_{j+1} = pa_j + \frac{n}{2} \quad (j \in \mathbf{N})$$

and

$$b_1 = 0, \quad b_{j+1} = pb_j + pq + \frac{n-2}{2} \quad (j \in \mathbf{N}).$$

Also a sequence  $\{C_j\}$  should be defined by

$$C_1 = F_2 \varepsilon^p, \quad C_{j+1} = \frac{((n-2)/2)! E_n C_j^p}{\left(pa_j + \frac{n}{2}\right)^{n/2}} \quad (j \in \mathbf{N}).$$

One can readily check that

$$a_j = \frac{n}{2(p-1)}(p^{j-1} - 1), \quad b_j = \frac{pq + (n-2)/2}{p-1}(p^{j-1} - 1) \quad (j \in \mathbf{N}),$$

which gives us

$$\frac{1}{pa_j + n/2} \geq \frac{2(p-1)}{p^j n}.$$

Hence one can find that

$$C_{j+1} \geq F \frac{C_j^p}{p^{(n/2)j}} \quad (j \in \mathbf{N}),$$

where  $F$  is a positive constant defined by

$$F = \frac{E_n((n-2)/2)! \{2(p-1)\}^{n/2}}{n^{n/2}}.$$

Due to the induction argument again, we obtain, for  $j \geq 2$ , that

$$\log C_j \geq p^{j-1} \left\{ \log C_1 + \sum_{k=0}^{j-2} \frac{p^k \log F - (j-1-k)p^k \log(p^{n/2})}{p^{j-1}} \right\}.$$

This inequality yields that there exist a constant  $S$  independent of  $j$  such that

$$C_j \geq \exp\{p^{j-1}(\log C_1 + S)\} \quad \text{for } j \geq 2.$$

Combining all the estimates, we reach the final inequality

$$Y(\xi) \geq \exp\{p^{j-1}I(\xi)\} \frac{\xi^{(pq+(n-2)/2)/(p-1)}}{\left(\xi - \frac{K\varepsilon^{-L}}{2}\right)^{n/2(p-1)}} \quad \text{for } \xi \geq \frac{K\varepsilon^{-L}}{2},$$

where we set

$$I(\xi) = \log \left( F_2 e^S \varepsilon^p \left( \xi - \frac{K\varepsilon^{-L}}{2} \right)^{n/2(p-1)} \xi^{-(pq+(n-2)/2)/(p-1)} \right).$$

Note that

$$\frac{n}{2(p-1)} - \frac{pq + (n-2)/2}{p-1} = \frac{1-pq}{p-1}.$$

If there exist a point  $\xi_0 \in \{\xi \geq K\varepsilon^{-L}\} \subset \{\xi \geq (K\varepsilon^{-L})/2\}$  such that  $I(\xi_0) > 0$ , we get the desired conclusion as before.  $I(\xi_0) > 0$  is equivalent to

$$\xi_0 > 2^{n/2(1-pq)} (e^S F_2)^{-(1-p)/(1-pq)} \varepsilon^{-2p(p-1)/\gamma(p,n)}.$$

Therefore the proof of the subcritical case is completed.  $\square$

# Chapter 3

## The lifespan of classical solutions to one dimensional wave equations with weighted nonlinear terms

In this chapter, we introduce the result of Wakasa [34].

### 3.1 Introduction

In this chapter, we are concerned with the estimates of the lifespan of classical solutions to the following Cauchy problem:

$$\begin{cases} u_{tt} - u_{xx} = H(x, u(x, t)), & (x, t) \in \mathbf{R} \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbf{R}, \end{cases} \quad (3.1.1)$$

where  $u = u(x, t)$  is a scalar unknown function of space-time variables,  $(f, g) \in C^2(\mathbf{R}) \times C^1(\mathbf{R})$  and  $\varepsilon > 0$  is a “small” parameter. The nonlinear term,  $H$  is given by

$$H(x, u) = \frac{F(u(x, t))}{(1 + x^2)^{(1+a)/2}}, \quad (3.1.2)$$

where  $a \geq -1$ , and  $F(u) = |u|^p$  or  $|u|^{p-1}u$  with  $p > 1$ . Let us define the lifespan  $T_\varepsilon$  of  $C^2$ -solution (classical solution) of (3.1.1) by

$$T_\varepsilon \equiv T_\varepsilon(f, g) := \sup\{T \in [0, \infty) : \text{There exists a unique solution } u \in C^2(\mathbf{R} \times [0, T]) \text{ of (3.1.1)}\}$$

with arbitrarily fixed  $(f, g)$ .

Let us recall known results for the case  $a = -1$  in general spatial dimensions from the section 2.2 in Chapter 2:

$$\begin{cases} u_{tt} - \Delta u = |u|^p & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & \text{for } x \in \mathbf{R}^n, \end{cases}$$

where  $n \geq 1$ . When  $n \geq 2$ , let  $p_0(n)$  is a positive root of (2.2.2). Then,  $T_\varepsilon = \infty$  holds for “small”  $\varepsilon$  with compact support if  $p > p_0(n)$ , and  $T_\varepsilon < \infty$  holds for “positive”  $(f, g)$  if  $1 < p \leq p_0(n)$ .

On the other hand, when  $n = 1$ , and  $(f, g)$  has a compact support and satisfies some positivity assumption, Kato [17] showed that  $T_\varepsilon < \infty$  for any  $p > 1$ . The difference between the cases  $n \geq 2$  and  $n = 1$  comes from the fact that the solutions to the homogeneous wave equations has a decay estimate,  $|u(x, t)| \leq (t + 1)^{-(n-1)/2}$ . Especially, the solution does not have decay property when  $n = 1$ .

The result due to [17] motivates one to introduce a weight function  $(1 + x^2)^{-(1+a)/2}$  in the nonlinearity for getting a global solution. Actually, Suzuki [31] showed that  $T_\varepsilon = \infty$  with  $F(u) = |u|^{p-1}u$  for  $p > (1 + \sqrt{5})/2$  and  $pa > 1$  if  $f$  and  $g$  are odd functions and  $\varepsilon$  is small enough, and Kubo & Osaka & Yazici [20] have obtained the same conclusion for any  $p > 1$  satisfying  $pa > 1$ . On the other hand, they showed that  $T_\varepsilon < \infty$  for  $F(u) = |u|^p$  with  $p > 1$  and  $a \geq -1$  if  $(f, g)$  satisfies  $f \equiv 0$ ,  $g(x) \geq 0$  for  $x \in \mathbf{R}$ , and  $\int_{\delta/2}^\delta g(y)dy > 0$  with some  $0 < \delta < 1$ . Also, they obtained an upper bound of the lifespan,  $T_\varepsilon \leq C\varepsilon^{-p^2}$ , where  $C$  is a positive constant independent of  $\varepsilon$ . However, this estimate is not sharp at least in the case of  $a = -1$  by Zhou’s result (2.2.4).

Our purpose is to extend Zhou’s result to the case where  $a > -1$ . To obtain a blow-up result, we require the following assumptions on the data:

$$\begin{aligned} &\text{Let } f \equiv 0 \text{ and } g \in C^1(\mathbf{R}) \text{ does not vanish identically.} \\ &\text{Assume } g(x) \geq 0 \text{ for all } x \in \mathbf{R} \text{ and } \int_{-1}^1 g(y)dy > 0. \end{aligned} \tag{3.1.3}$$

Then, we have the following blow-up theorem.

**Theorem 3.1.1** *Let  $a \geq -1$  and  $F(u) = |u|^{p-1}u$  or  $|u|^p$  with  $p > 1$ . Assume (3.1.3). Then, there exist positive constants  $\varepsilon_0 = \varepsilon_0(g, a, p)$  and  $C = C(g, a, p)$  such that*

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-(p-1)/(1-a)} & \text{if } -1 \leq a < 0, \\ \phi^{-1}(C\varepsilon^{-(p-1)}) & \text{if } a = 0, \\ C\varepsilon^{-(p-1)} & \text{if } a > 0, \end{cases} \tag{3.1.4}$$



holds for any  $\varepsilon$  with  $0 < \varepsilon \leq \varepsilon_0$ , where  $\phi = \phi(s)$  is a function defined by  $\phi(s) = s \log(2 + s)$  for  $s \geq 0$ .

The proof of this theorem done by an iteration argument concerning point-wise estimates. Such kind of framework was introduced by John [13] in three space dimensions. The first step of the iteration argument comes from the linear estimate of the solution to the homogeneous wave equation from below. Kubo & Osaka & Yazici [20] obtained such an estimate only in a strip domain,  $\{0 \leq x - t \leq \delta/2\}$ , where  $0 < \delta < 1$  is a constant. On the other hand, we are able to show a similar estimate in unbounded domain,  $\{t - x \geq 1\}$ . This improvement enable us to establish sharp upper bound of  $T_\varepsilon$ . See Lemma 3.3.2 and Remark 3.3.1 for details.

To show the optimality of the upper bounds in Theorem 3.1.1, we require the following assumptions on  $(f, g)$

$$\begin{aligned} f \in C^2(\mathbf{R}) \text{ and } g \in C^1(\mathbf{R}) \text{ satisfy } \|f\|_{L^\infty(\mathbf{R})} < \infty \\ \text{and } \|g\|_{L^1(\mathbf{R})} < \infty. \end{aligned} \quad (3.1.5)$$

Then, we have the following theorem.

**Theorem 3.1.2** *Let  $a \geq -1$  and  $F(u) = |u|^{p-1}u$  or  $|u|^p$  with  $p > 1$ . Assume (3.1.5). Then, there exists a positive constant  $c = c(f, g, a, p)$  such that*

$$T_\varepsilon \geq \begin{cases} c\varepsilon^{-(p-1)/(1-a)} & \text{if } -1 \leq a < 0, \\ \phi^{-1}(c\varepsilon^{-(p-1)}) & \text{if } a = 0, \\ c\varepsilon^{-(p-1)} & \text{if } a > 0, \end{cases} \quad (3.1.6)$$

holds for  $\varepsilon > 0$ , where  $\phi$  is the function in Theorem 3.1.1.

**Remark 3.1.1** *One can easily generalize the assumption on  $F$  in Theorem 3.1.2 as follows:*

$$\begin{aligned} F \in C^1(\mathbf{R}) \text{ satisfies } F(0) = F'(0) = 0 \text{ and} \\ |F'(s)| \leq pA|s|^{p-1} \text{ for } s \in \mathbf{R}, \text{ where } p > 1 \text{ and } A > 0. \end{aligned} \quad (3.1.7)$$

This chapter is organized as follows. In the next section, we prepare some notations. The upper bounds of the lifespan and lower bounds of the lifespan are obtained in Section 3.2 and Section 3.3, respectively.

## 3.2 Notations

In this section, we give some notations and definitions.

We define

$$u^0(x, t) = \frac{1}{2}\{f(x+t) + f(x-t)\} + \frac{1}{2} \int_{x-t}^{x+t} g(y) dy \quad (3.2.1)$$

and

$$L(V)(x, t) = \frac{1}{2} \iint_{D(x, t)} V(y, s) dy ds \quad (3.2.2)$$

for  $V \in C(\mathbf{R} \times [0, \infty))$ , where

$$D(x, t) = \{(y, s) \in \mathbf{R} \times [0, \infty) : 0 \leq s \leq t, x-t+s \leq y \leq x+t-s\}.$$

For  $(f, g) \in C^2(\mathbf{R}) \times C^1(\mathbf{R})$ , if  $u \in C(\mathbf{R} \times [0, \infty))$  is a solution of

$$u(x, t) = \varepsilon u^0(x, t) + L(H(\cdot, u))(x, t), \quad (x, t) \in \mathbf{R} \times [0, \infty), \quad (3.2.3)$$

then  $u \in C^2(\mathbf{R} \times [0, \infty))$  is the solution to the initial value problem (3.1.1).

For  $T > 0$ , we define the following domains:

$$\begin{aligned} \Gamma_1 &= \{(x, t) \in [0, \infty) \times [0, T] : t - x \geq 1\}, \\ \Gamma_2 &= \{(x, t) \in [0, \infty) \times [0, T] : x \geq t - x \geq 1\}, \\ \Sigma_j &= \{(x, t) \in [0, \infty) \times [0, T] : t - x \geq l_j\}, \end{aligned} \quad (3.2.4)$$

where

$$\begin{cases} l_1 = 3 \\ l_j = l_1 + \sum_{k=1}^{j-1} 2^{-(k-1)} = l_1 + 2 \left(1 - \frac{1}{2^{j-1}}\right) \quad \text{for } j \geq 2. \end{cases} \quad (3.2.5)$$

### 3.3 Upper bound of the lifespan

In this section, we prove Theorem 3.1.1. It is sufficient to show that the solution to the integral equation,

$$u(x, t) = \varepsilon u^0(x, t) + \frac{1}{2} \iint_{D(x, t)} \frac{|u(y, s)|^p dy ds}{(1 + y^2)^{(1+a)/2}}, \quad (x, t) \in \mathbf{R} \times [0, \infty), \quad (3.3.1)$$

blows up in finite time. Because, if  $u \in C(\mathbf{R} \times [0, \infty))$  is a solution of (3.3.1), then  $u$  satisfies  $u(x, t) \geq 0$  for  $(x, t) \in \mathbf{R} \times [0, \infty)$  by the assumptions in (3.1.3). Therefore, this  $u$  must solve the equation (3.2.3) with  $F(u) = |u|^{p-1}u$  by the uniqueness of solutions to (3.1.1).

Before proving Theorem 3.1.1, we prepare the following lemmas:

**Lemma 3.3.1** *Let  $p > 1$ ,  $a \geq -1$  and let us define a sequence*

$$\begin{cases} C_{a,j} = \exp\{p^{j-1}(\log(C_{a,1}F_{p,a}^{-S_j}E_{p,a}^{1/(p-1)})) - \log E_{p,a}^{1/(p-1)}\} \quad (j \geq 2), \\ C_{a,1} = c_0^p k_a \varepsilon^p, \end{cases} \quad (3.3.2)$$

where

$$E_{p,a} = \begin{cases} (p-1)^2/(2^{a+5}p^2), & \text{if } -1 \leq a < 0, \\ (p-1)^2/(2p^2), & \text{if } a = 0, \\ (p-1)/(2^{a+2}p), & \text{if } a > 0, \end{cases} \quad (3.3.3)$$

$$F_{p,a} = \begin{cases} p^2, & \text{if } -1 \leq a \leq 0, \\ 2p, & \text{if } a > 0, \end{cases} \quad (3.3.4)$$

$$k_a = \begin{cases} 2^{-(a+4)}, & \text{if } -1 \leq a < 0, \\ 2^{-1}, & \text{if } a = 0, \\ 2^{-(a+2)}, & \text{if } a > 0, \end{cases} \quad (3.3.5)$$

and

$$S_j = \sum_{i=1}^{j-1} \frac{i}{p^i}. \quad (3.3.6)$$

Then, we have the following relation:

$$C_{a,j+1} = \frac{C_{a,j}^p E_{p,a}}{F_{p,a}^j} \quad (j \in \mathbf{N}). \quad (3.3.7)$$

**Proof.** First, we shall show (3.3.7) for  $j = 1$ . One can easily get

$$\begin{aligned} \log \left( \frac{C_{a,1}^p E_{p,a}}{F_{p,a}} \right) &= p \log(C_{a,1} F_{p,a}^{-1/p}) + \log E_{p,a} \\ &= p \log(C_{a,1} F_{p,a}^{-1/p} E_{p,a}^{1/(p-1)}) - \log E_{p,a}^{1/(p-1)} = \log C_{a,2}. \end{aligned}$$

Hence (3.3.7) holds for  $j = 1$ . Next, we shall show (3.3.7) for  $j \geq 2$ . Note that (3.3.7) is equivalent to

$$\log C_{a,j+1} = p \log C_{a,j} - j \log F_{p,a} + \log E_{p,a}.$$

By (3.3.2) and the expression of  $S_j$  in (3.3.6), the right-hand side of this identity is equal to

$$\begin{aligned} & p^j \{\log(C_{a,1} F_{p,a}^{-S_j} E_{p,a}^{1/(p-1)})\} - p \log E_{p,a}^{1/(p-1)} - j \log F_{p,a} + \log E_{p,a} \\ &= p^j \{\log(C_{a,1} F_{p,a}^{-S_{j+1}} E_{p,a}^{1/(p-1)})\} + p^j \log F_{p,a}^{j/p^j} - j \log F_{p,a} - \log E_{p,a}^{1/(p-1)} \\ &= p^j \{\log(C_{a,1} F_{p,a}^{-S_{j+1}} E_{p,a}^{1/(p-1)})\} - \log E_{p,a}^{1/(p-1)}. \end{aligned}$$

Hence, we obtain (3.3.7) by (3.3.2) with  $j$  replaced by  $j + 1$ . This completes the proof.  $\square$

Next, we derive a lower bound of the solution to (3.3.1) which is a starting point of our iteration argument.

**Lemma 3.3.2** *Suppose that the assumptions in Theorem 3.1.1 are fulfilled. Let  $u \in C(\mathbf{R} \times [0, T])$  be the solution of (3.3.1). Then,  $u$  satisfies*

$$u(x, t) \geq \varepsilon c_0 \quad \text{for } (x, t) \in \Gamma_1, \quad (3.3.8)$$

where  $c_0 = \frac{1}{2} \int_{-1}^1 g(y) dy > 0$  and  $\Gamma_1 = \{(x, t) \in [0, \infty) \times [0, T] : t - x \geq 1\}$  is the one in (3.2.4).

**Proof.** By (3.1.3) and (3.2.1), we get

$$\varepsilon u^0(x, t) = \frac{\varepsilon}{2} \int_{x-t}^{x+t} g(y) dy \geq \varepsilon c_0 \quad \text{for } (x, t) \in \Gamma_1.$$

Making use of the positivity of the second term of right-hand side in (3.3.1), we have (3.3.8). This completes the proof.  $\square$

**Remark 3.3.1** *In three space dimensions, the following estimate which is necessary to get the first step of the iteration argument was obtained by John [13] in a strip domain: For  $(x, t) \in S$ , we have*

$$u^0(x, t) \geq Cr^{-1},$$

where  $r = |x|$ ,  $C$  is a positive constant and  $S = \{(r, t) \in (0, \infty) \times [0, \infty) : \delta \leq t - r \leq \delta'\}$ , with some  $\delta', \delta$  ( $\delta' > \delta > 0$ ).

On the contrary, our estimate holds in some domain without any restriction of upper bound for  $t - x$ . This is the key point to obtain sharp upper bound of  $T_\varepsilon$ .

Our iteration argument will be done by using the following estimates.

**Proposition 3.3.1** *Suppose that the assumptions in Theorem 3.1.1 are fulfilled. Let  $j \in \mathbf{N}$  and let  $u \in C(\mathbf{R} \times [0, T])$  be the solution of (3.3.1). Then,  $u$  satisfies*

$$u(x, t) \geq C_{a,j} \{(t - x)^{-(a+1)} (t - x - 1)^2\}^{a_j} \quad \text{if } -1 \leq a < 0, \quad (3.3.9)$$

for  $(x, t) \in \Gamma_2$ ,

$$u(x, t) \geq C_{0,j} \{(t - x - 1) \log(1 + x)\}^{a_j} \quad \text{if } a = 0, \quad (3.3.10)$$

for  $(x, t) \in \Gamma_1$ , and

$$u(x, t) \geq C_{a,j} (t - x - l_j)^{a_j} \quad \text{if } a > 0, \quad (3.3.11)$$

for  $(x, t) \in \Sigma_j$ , where  $\Gamma_1$ ,  $\Gamma_2$  and  $\Sigma_j$  are defined in (3.2.4). Here  $C_{a,j}$  is the one in (3.3.2) with  $c_0 = \frac{1}{2} \int_{-1}^1 g(y) dy > 0$  and  $a_j$  is defined by

$$a_j = \frac{p^j - 1}{p - 1} \quad (j \in \mathbf{N}). \quad (3.3.12)$$

**Proof.** We shall show (3.3.9), (3.3.10) and (3.3.11) by induction. Noticing that  $u^0(x, t) \geq 0$  for  $(x, t) \in \mathbf{R} \times [0, \infty)$  and  $(1 + y^2)^{1/2} \leq 1 + |y|$ , we get

$$u(x, t) \geq \frac{1}{2} \iint_{D(x, t)} \frac{|u(y, s)|^p}{(1 + |y|)^{1+a}} dy ds \quad \text{in } \mathbf{R} \times [0, \infty). \quad (3.3.13)$$

(i) **Estimate in the case of  $-1 \leq a < 0$ .**

Let  $(x, t) \in \Gamma_2$ . Define

$$T_1(x, t) := \{(y, s) \in D(x, t) : 1 \leq s - y \leq t - x \leq y, s + y \leq t + x\}.$$

Changing the variables in the integral of (3.3.13) by

$$\alpha = s + y, \quad \beta = s - y, \quad (3.3.14)$$

(Jacobian,  $\partial(s, y)/\partial(\alpha, \beta)$  is  $1/2$ .) and replacing the domain of integration by  $T_1(x, t)$ , we get

$$u(x, t) \geq \frac{1}{4} \int_1^{t-x} d\beta \int_{2(t-x)+\beta}^{t+x} \frac{|u(y, s)|^p}{\{1 + (\alpha - \beta)/2\}^{1+a}} d\alpha \quad \text{in } \Gamma_2. \quad (3.3.15)$$

Making use of (3.3.8) and  $T_1(x, t) \subset \Gamma_1$  for  $(x, t) \in \Gamma_2$ , we have

$$u(x, t) \geq \frac{c_0^p \varepsilon^p}{4} \int_1^{t-x} d\beta \int_{2(t-x)+\beta}^{t+x} \frac{d\alpha}{\{1 + (\alpha - \beta)/2\}^{1+a}} \quad \text{in } \Gamma_2.$$

Note that  $x \geq t - x$  is equivalent to  $t + x \geq 3(t - x)$ , we get

$$u(x, t) \geq \frac{c_0^p \varepsilon^p}{4} \int_1^{t-x} d\beta \int_{2(t-x)+\beta}^{3(t-x)} \frac{d\alpha}{\{1 + (\alpha - \beta)/2\}^{1+a}} \quad \text{in } \Gamma_2.$$

It follows from

$$1 + \frac{\alpha - \beta}{2} \leq 1 + \frac{3(t - x) - 1}{2} \leq 2(t - x) \quad (3.3.16)$$

for  $\alpha \leq 3(t-x)$ ,  $\beta \geq 1$  and  $t-x \geq 1$  that

$$u(x, t) \geq \frac{c_0^p \varepsilon^p}{2^{a+3}(t-x)^{1+a}} \int_1^{t-x} (t-x-\beta) d\beta = C_{a,1} \frac{(t-x-1)^2}{(t-x)^{1+a}} \quad \text{in } \Gamma_2.$$

Therefore, (3.3.9) holds for  $j = 1$ .

Assume that (3.3.9) holds. Noticing that  $T_1(x, t) \subset \Gamma_2$  for  $(x, t) \in \Gamma_2$  and putting (3.3.9) into (3.3.15), we have

$$u(x, t) \geq \frac{C_{a,j}^p}{4} \int_1^{t-x} \frac{(\beta-1)^{2pa_j}}{\beta^{p(a+1)a_j}} d\beta \int_{2(t-x)+\beta}^{t+x} \frac{d\alpha}{\{1 + (\alpha - \beta)/2\}^{1+a}} \quad \text{in } \Gamma_2.$$

It follows from (3.3.16) that

$$u(x, t) \geq \frac{C_{a,j}^p}{2^{a+3}(t-x)^{a+1}} \int_1^{t-x} \frac{(\beta-1)^{2pa_j}}{\beta^{p(a+1)a_j}} d\beta \int_{2(t-x)+\beta}^{3(t-x)} d\alpha.$$

Noticing that  $\beta \leq t-x$ , we have

$$\begin{aligned} u(x, t) &\geq \frac{C_{a,j}^p}{2^{a+3}(t-x)^{(a+1)(pa_j+1)}} \int_1^{t-x} (\beta-1)^{2pa_j} d\beta \int_{2(t-x)+\beta}^{3(t-x)} d\alpha \\ &= \frac{C_{a,j}^p}{2^{a+3}(t-x)^{(a+1)(pa_j+1)}} \int_1^{t-x} (\beta-1)^{2pa_j} (t-x-\beta) d\beta \end{aligned}$$

in  $\Gamma_2$ . Making use of integration by parts to the integral above, we have

$$\begin{aligned} \int_1^{t-x} (\beta-1)^{2pa_j} (t-x-\beta) d\beta &= \frac{1}{2pa_j+1} \int_1^{t-x} (\beta-1)^{2pa_j+1} d\beta \\ &= \frac{(t-x-1)^{2(pa_j+1)}}{2(2pa_j+1)(pa_j+1)} \geq \frac{(t-x-1)^{2(pa_j+1)}}{2^2(pa_j+1)^2}. \end{aligned}$$

Hence we get

$$u(x, t) \geq \frac{C_{a,j}^p (t-x-1)^{2(pa_j+1)}}{2^{a+5}(pa_j+1)^2 (t-x)^{(a+1)(pa_j+1)}} \quad \text{in } \Gamma_2.$$

Recalling the definition of  $a_j$ , we have

$$a_{j+1} = pa_j + 1 \leq \frac{p^{j+1}}{p-1}. \quad (3.3.17)$$

Making use of (3.3.7), we get

$$u(x, t) \geq \frac{C_{a,j}^p (p-1)^2}{2^{a+5} p^{2(j+1)}} \cdot \frac{(t-x-1)^{2a_{j+1}}}{(t-x)^{(a+1)a_{j+1}}} = C_{a,j+1} \frac{(t-x-1)^{2a_{j+1}}}{(t-x)^{(a+1)a_{j+1}}}$$

in  $\Gamma_2$ . Therefore, (3.3.9) holds for all  $j \in \mathbf{N}$ .

**(ii) Estimate in the case of  $a = 0$ .**

Let  $(x, t) \in \Gamma_1$ . Define

$$T_2(x, t) := \{(y, s) \in D(x, t) : 1 \leq s - y \leq t - x, s + y \leq t + x, y \geq 0\}.$$

Changing the variables by (3.3.14) in the integral of (3.3.13) and replacing the domain of integration by  $T_2(x, t)$ , we get

$$u(x, t) \geq \frac{1}{4} \int_1^{t-x} d\beta \int_\beta^{t+x} \frac{|u(y, s)|^p}{1 + (\alpha - \beta)/2} d\alpha \quad \text{in } \Gamma_1. \quad (3.3.18)$$

By making use of (3.3.8) and  $T_2(x, t) \subset \Gamma_1$  for  $(x, t) \in \Gamma_1$ , we get

$$u(x, t) \geq \frac{c_0^p \varepsilon^p}{4} \int_1^{t-x} d\beta \int_\beta^{t+x} \frac{d\alpha}{1 + (\alpha - \beta)/2} \quad \text{in } \Gamma_1.$$

Noticing that

$$\int_\beta^{t+x} \frac{d\alpha}{1 + (\alpha - \beta)/2} = 2 \log \left( 1 + \frac{t + x - \beta}{2} \right) \geq 2 \log(1 + x),$$

for  $\beta \leq t - x$ , we obtain

$$u(x, t) \geq \frac{c_0^p \varepsilon^p}{2} \log(1 + x) \int_1^{t-x} d\beta = C_{0,1}(t - x - 1) \log(1 + x) \quad \text{in } \Gamma_1.$$

Therefore, (3.3.10) holds for  $j = 1$ .

Assume that (3.3.10) holds. Noticing that  $T_2(x, t) \subset \Gamma_1$  for  $(x, t) \in \Gamma_1$  and putting (3.3.10) into (3.3.18), we have

$$u(x, t) \geq \frac{C_{0,j}^p}{4} \int_1^{t-x} (\beta - 1)^{pa_j} d\beta \int_\beta^{t+x} \frac{\{\log(1 + (\alpha - \beta)/2)\}^{pa_j} d\alpha}{1 + (\alpha - \beta)/2} \quad \text{in } \Gamma_1.$$

The  $\alpha$ -integral yields

$$\int_\beta^{t+x} \frac{\{\log(1 + (\alpha - \beta)/2)\}^{pa_j} d\alpha}{1 + (\alpha - \beta)/2} = \frac{2}{pa_j + 1} \left\{ \log \left( 1 + \frac{t + x - \beta}{2} \right) \right\}^{pa_j + 1}.$$

Then, we get

$$\begin{aligned} u(x, t) &\geq \frac{C_{0,j}^p}{2(pa_j + 1)} \int_1^{t-x} (\beta - 1)^{pa_j} \left\{ \log \left( 1 + \frac{t + x - \beta}{2} \right) \right\}^{pa_j + 1} d\beta \\ &\geq \frac{C_{0,j}^p \{\log(1 + x)\}^{pa_j + 1}}{2(pa_j + 1)} \int_1^{t-x} (\beta - 1)^{pa_j} d\beta \end{aligned}$$

in  $\Gamma_1$ . It follows from (3.3.17) and (3.3.7) that

$$\begin{aligned} u(x, t) &\geq \frac{C_{0,j}^p (p-1)^2}{2p^{2(j+1)}} \cdot \{(t-x-1) \log(1+x)\}^{a_{j+1}} \\ &= C_{0,j+1} \{(t-x-1) \log(1+x)\}^{a_{j+1}} \end{aligned}$$

in  $\Gamma_1$ . Therefore, (3.3.10) holds for all  $j \in \mathbb{N}$ .

**(iii) Estimate in the case of  $a > 0$ .**

Let  $(x, t) \in \Sigma_1$ . Define

$$L_1(x, t) := \{(y, s) \in D(x, t) : 1 \leq s - y \leq t - x - 2, 0 \leq y \leq 1\}.$$

Changing the variables by (3.3.14) in the integral of (3.3.13) and replacing the domain of integration by  $L_1(x, t)$ , we get

$$u(x, t) \geq \frac{1}{4} \int_1^{t-x-2} d\beta \int_\beta^{2+\beta} \frac{|u(y, s)|^p d\alpha}{\{1 + (\alpha - \beta)/2\}^{1+a}} \quad \text{in } \Sigma_1.$$

By making use of (3.3.8) and  $L_1(x, t) \subset \Gamma_1$  for  $(x, t) \in \Sigma_1$ , we have

$$u(x, t) \geq \frac{c_0^p \varepsilon^p}{4} \int_1^{t-x-2} d\beta \int_\beta^{2+\beta} \frac{d\alpha}{\{1 + (\alpha - \beta)/2\}^{1+a}} \quad \text{in } \Sigma_1.$$

It follows from  $1 + (\alpha - \beta)/2 \leq 2$  for  $\alpha \leq 2 + \beta$  that

$$u(x, t) \geq \frac{c_0^p \varepsilon^p}{2^{a+2}} \int_1^{t-x-2} d\beta = C_{a,1} (t - x - 3) \quad \text{in } \Sigma_1.$$

Therefore, (3.3.11) holds for  $j = 1$ .

Assume that (3.3.11) holds. Let  $(x, t) \in \Sigma_{j+1}$ . Define

$$L_j(x, t) := \{(y, s) \in D(x, t) : l_j \leq s - y \leq t - x - 2^{-(j-1)}, 0 \leq y \leq 2^{-j}\}$$

for  $j \geq 1$ , where  $l_j$  is defined in (3.2.5). Making use of (3.3.14) and replacing the domain of integration in (3.3.13) by  $L_j(x, t)$ , we have

$$u(x, t) \geq \frac{1}{4} \int_{l_j}^{t-x-2^{-(j-1)}} d\beta \int_\beta^{2^{-(j-1)}+\beta} \frac{|u(y, s)|^p d\alpha}{\{1 + (\alpha - \beta)/2\}^{1+a}} \quad \text{in } \Sigma_{j+1}.$$

Noticing that  $L_j(x, t) \subset \Sigma_j$  for  $(x, t) \in \Sigma_{j+1}$  and putting (3.3.11) into the integral above, we have

$$u(x, t) \geq \frac{C_{a,j}^p}{4} \int_{l_j}^{t-x-2^{-(j-1)}} (\beta - l_j)^{pa_j} d\beta \int_\beta^{2^{-(j-1)}+\beta} \frac{d\alpha}{\{1 + (\alpha - \beta)/2\}^{1+a}}$$



in  $\Sigma_{j+1}$ . Note that

$$1 + \frac{\alpha - \beta}{2} \leq 1 + \frac{1}{2^j} \leq 2$$

for  $\alpha \leq 2^{-(j-1)} + \beta$ , we get

$$u(x, t) \geq \frac{C_{a,j}^p}{2^{a+2+j}} \int_{l_j}^{t-x-2^{-(j-1)}} (\beta - l_j)^{pa_j} d\beta \quad \text{in } \Sigma_{j+1}.$$

It follows from  $l_j + 2^{-(j-1)} = l_{j+1}$ , (3.3.17) and (3.3.7) that

$$u(x, t) \geq \frac{(p-1)C_{a,j}^p}{2^{a+2+j}p^{j+1}} \cdot (t-x-l_{j+1})^{a_{j+1}} = C_{a,j+1}(t-x-l_{j+1})^{a_{j+1}}$$

in  $\Sigma_{j+1}$ . Therefore, (3.3.11) holds for all  $j \in \mathbf{N}$ . The proof of Proposition 3.3.1 is now completed.  $\square$

**End of the proof of Theorem 3.1.1.** Let  $u \in C(\mathbf{R} \times [0, T])$  be the solution of the integral equation, (3.3.1). Setting  $S = \lim_{j \rightarrow \infty} S_j$ , we see from (3.3.6) that  $S_j \leq S$  for all  $j \in \mathbf{N}$ . Therefore, (3.3.2) yields

$$\begin{aligned} C_{a,j} &\geq \exp\{p^{j-1}\{\log(C_{a,1}F_{p,a}^{-S}E_{p,a}^{1/(p-1)})\} - \log E_{p,a}^{1/(p-1)}\} \\ &= E_{p,a}^{-1/(p-1)} \exp\{p^{j-1}\{\log(C_{a,1}F_{p,a}^{-S}E_{p,a}^{1/(p-1)})\}\}. \end{aligned} \quad (3.3.19)$$

**(i) The lifespan in the case of  $-1 \leq a < 0$ .**

We take  $\varepsilon_0 = \varepsilon_0(g, a, p) > 0$  so small that

$$B_1 \varepsilon_0^{-(p-1)/(1-a)} \geq 4,$$

where we set

$$B_1 = (c_0^p 2^{-(a+4)+p(a-3)/(p-1)} p^{-2S} E_{p,a}^{1/(p-1)})^{-(p-1)/p(1-a)} > 0.$$

Next, for a fixed  $\varepsilon \in (0, \varepsilon_0]$ , we suppose that  $T$  satisfies

$$T > B_1 \varepsilon^{-(p-1)/(1-a)} (\geq 4). \quad (3.3.20)$$

Combining (3.3.19) with (3.3.9), we have

$$\begin{aligned} u(x, t) &\geq E_{p,a}^{-1/(p-1)} \exp\{p^{j-1}\{\log(C_{a,1}F_{p,a}^{-S}E_{p,a}^{1/(p-1)})\}\} \\ &\quad \times \left\{ \frac{(t-x-1)^2}{(t-x)^{(1+a)}} \right\}^{(p^j-1)/(p-1)} \end{aligned}$$

in  $\Gamma_2$ . Note that  $t - x - 1 \geq (t - x)/2$  is equivalent to  $t - x \geq 2$ . Furthermore, we have  $(t/2, t) \in \Gamma_2$  for  $t \in [4, T]$ . Hence we get

$$\begin{aligned} u(t/2, t) &\geq (2^{a-3}E_{p,a})^{-1/(p-1)} \exp\{p^{j-1}\{\log(2^{p(a-3)/(p-1)}C_{a,1}F_{p,a}^{-S}E_{p,a}^{1/(p-1)})\}\} \\ &\quad \times t^{(1-a)(p^j-1)/(p-1)} \\ &= (2^{a-3}E_{p,a})^{-1/(p-1)} \exp\{p^{j-1}K_1(t)\}t^{-(1-a)/(p-1)} \end{aligned}$$

for  $t \in [4, T]$ , where we set

$$K_1(t) = \log(\varepsilon^p c_0^p 2^{-(a+4)+p(a-3)/(p-1)} p^{-2S} E_{p,a}^{1/(p-1)} t^{p(1-a)/(p-1)})$$

(recall (3.3.4) and (3.3.5)).

By (3.3.20) and the definition of  $B_1$ , we have  $K_1(T) > 0$ . Therefore we get  $u(T/2, T) \rightarrow \infty$  as  $j \rightarrow \infty$ . Hence, (3.3.20) implies that  $T_\varepsilon \leq B_1 \varepsilon^{-(p-1)/(1-a)}$  for  $0 < \varepsilon \leq \varepsilon_0$ .

**(ii) The lifespan in the case of  $a = 0$ .**

We take  $\varepsilon_1 = \varepsilon_1(g, p) > 0$  so small that

$$\phi^{-1}(B_2 \varepsilon_1^{-(p-1)}) \geq 4,$$

where  $\phi$  is the one in Theorem 3.1.1 and

$$B_2 = (c_0^p 2^{-1-3p/(p-1)} p^{-2S} E_{p,0}^{1/(p-1)})^{-(p-1)/p} > 0.$$

Next, for a fixed  $\varepsilon \in (0, \varepsilon_1]$ , we suppose that  $T$  satisfies

$$T > \phi^{-1}(B_2 \varepsilon^{-(p-1)}) (\geq 4). \quad (3.3.21)$$

Combining the estimates (3.3.19) and (3.3.10), we have

$$\begin{aligned} u(t/2, t) &\geq (2^{-2}E_{p,0})^{-1/(p-1)} \exp\{p^{j-1}\{\log(\varepsilon^p c_0^p 2^{-1-2p/(p-1)} p^{-2S} E_{p,0}^{1/(p-1)})\}\} \\ &\quad \times \{t \log(1 + t/2)\}^{(p^j-1)/(p-1)} \end{aligned}$$

for  $4 \leq t \leq T$ . Noticing that

$$\log\left(1 + \frac{t}{2}\right) = \log(2 + t) - \log 2 \geq \frac{\log(2 + t)}{2} \quad \text{for } t \geq 2,$$

we get

$$u(t/2, t) \geq (2^{-3}E_{p,0})^{-1/(p-1)} \exp\{p^{j-1}K_2(t)\}\phi(t)^{-1/(p-1)}$$

for  $4 \leq t \leq T$ , where we set

$$K_2(t) = \log\left(\varepsilon^p c_0^p 2^{-1-3p/(p-1)} p^{-2S} E_{p,0}^{1/(p-1)} \{\phi(t)\}^{p/(p-1)}\right).$$

Analogously to the case of  $-1 \leq a < 0$ , we have  $K_2(T) > 0$  by (3.3.21) and the definition of  $B_2$ . Therefore we get  $u(T/2, T) \rightarrow \infty$  as  $j \rightarrow \infty$ . Hence, (3.3.21) implies that  $T_\varepsilon \leq \phi^{-1}(B_2 \varepsilon^{-(p-1)})$  for  $0 < \varepsilon \leq \varepsilon_1$ .

**(iii) The lifespan in the case of  $a > 0$ .**

We take  $\varepsilon_2 = \varepsilon_2(g, a, p) > 0$  so small that

$$B_3 \varepsilon_2^{-(p-1)} \geq 20,$$

where we set

$$B_3 = (c_0^p 2^{-(a+2)-2p/(p-1)} (2p)^{-S} E_{p,a}^{1/(p-1)})^{-(p-1)/p} > 0.$$

Next, for a fixed  $\varepsilon \in (0, \varepsilon_2]$ , we suppose that  $T$  satisfies

$$T > B_3 \varepsilon^{-(p-1)} (\geq 20). \quad (3.3.22)$$

Combining the estimates (3.3.19) with (3.3.11), we have

$$u(t/2, t) \geq (2^{-2} E_{p,a})^{-1/(p-1)} \exp\{p^{j-1} K_3(t)\} t^{-1/(p-1)}$$

for  $20 \leq t \leq T$ , where we set

$$K_3(t) = \log \left( \varepsilon^p c_0^p 2^{-(a+2)-2p/(p-1)} (2p)^{-S} E_{p,a}^{1/(p-1)} t^{p/(p-1)} \right).$$

Since  $K_3(T) > 0$ , by (3.3.22) and the definition of  $B_3$ , we get  $u(T/2, T) \rightarrow \infty$  as  $j \rightarrow \infty$ . Hence, (3.3.22) implies that  $T_\varepsilon \leq B_3 \varepsilon^{-(p-1)}$  for  $0 < \varepsilon \leq \varepsilon_2$ . Therefore, the proof of Theorem 3.1.1 is now completed.  $\square$

## 3.4 Lower bound of the lifespan

In this section, we prove Theorem 3.1.2. First of all, we introduce a Banach space

$$X = \{u \in C(\mathbf{R} \times [0, T]) : \|u\|_{L^\infty(\mathbf{R} \times [0, T])} < \infty\}, \quad (3.4.1)$$

which is equipped with a norm

$$\|u\|_{L^\infty(\mathbf{R} \times [0, T])} = \sup_{(x, t) \in \mathbf{R} \times [0, T]} |u(x, t)|. \quad (3.4.2)$$

We shall construct a solution of the integral equation (3.2.3) in  $X$  under suitable assumption on  $T$  such as (3.4.7) below. Define a sequence of functions  $\{u_n\}_{n \in \mathbf{N}}$  by

$$u_n = u_0 + L(H(\cdot, u_{n-1})), \quad u_0 = \varepsilon u^0, \quad (3.4.3)$$

where  $L$ ,  $H$  and  $u^0$  are given by (3.2.2), (3.1.2) and (3.2.1), respectively. Since  $\|u_0\|_{L^\infty(\mathbf{R} \times [0, T])} \leq M\varepsilon$ , where  $M = \|f\|_{L^\infty(\mathbf{R})} + \|g\|_{L^1(\mathbf{R})}$  by (3.2.1), we have  $u_0 \in X$ .

The following *a priori* estimate plays a key role in the proof of Theorem 3.1.2.

**Lemma 3.4.1** *Let  $V \in X$ ,  $a \geq -1$ , and let  $D = D(\tau)$  is a function defined by*

$$D(\tau) = \begin{cases} (1 + \tau)^{1-a} & \text{if } -1 \leq a < 0, \\ \phi(\tau) & \text{if } a = 0, \\ 1 + \tau & \text{if } a > 0, \end{cases} \quad (3.4.4)$$

for  $\tau \geq 0$ , where  $\phi$  is the one in Theorem 3.1.1. Then, there exists a positive constant  $C_a$  such that

$$\left\| L \left( \frac{V}{(1 + |\cdot|^2)^{(1+a)/2}} \right) \right\|_{L^\infty(\mathbf{R} \times [0, T])} \leq C_a D(T) \|V\|_{L^\infty(\mathbf{R} \times [0, T])}. \quad (3.4.5)$$

**Proof.** Noticing that  $(1 + y^2) \geq (1 + |y|)^2/2$ , the left-hand side in (3.4.5) is dominated by

$$C_a \|V\|_{L^\infty(\mathbf{R} \times [0, T])} \iint_{D(x, t)} \frac{dy ds}{\langle y \rangle^{1+a}},$$

where we set  $\langle y \rangle = 1 + |y|$ . Thus, it is enough to show the inequality,

$$I(x, t) \leq C_a D(T) \quad \text{for } (x, t) \in \mathbf{R} \times [0, T], \quad (3.4.6)$$

where we set

$$I(x, t) = \iint_{D(x, t)} \frac{dy ds}{\langle y \rangle^{1+a}}.$$

We may assume  $x \geq 0$ . Because  $I(x, t)$  is an even function with respect to  $x$ . When  $t \geq x \geq 0$ , we divide the integral domain  $D(x, t)$  into two parts  $D_j(x, t)$  ( $j = 1, 2$ ), where

$$\begin{aligned} D_1(x, t) &= \{(y, s) \in \mathbf{R} \times [0, \infty) : 0 \leq s \leq t - x, x - t + s \leq y \leq t - x - s\}, \\ D_2(x, t) &= \{(y, s) \in [0, \infty)^2 : 0 \leq s \leq t, |x - t + s| \leq y \leq x + t - s\}. \end{aligned}$$

Namely, we set

$$I_j(x, t) = \iint_{D_j(x, t)} \frac{1}{\langle y \rangle^{1+a}} dy ds \quad (j = 1, 2),$$

so that  $I(x, t) = I_1(x, t) + I_2(x, t)$ . We shall estimate  $I_1$ . Since  $\langle y \rangle$  is an even function, we obtain

$$I_1(x, t) = 2 \int_0^{t-x} ds \int_0^{t-x-s} \frac{dy}{(1 + y)^{1+a}} \quad \text{for } t \geq x \geq 0.$$

Then, the  $y$ -integral is dominated by

$$\begin{cases} -a^{-1}(1+t-x)^{-a} & \text{if } a < 0, \\ \log(1+t-x) & \text{if } a = 0, \\ a^{-1} & \text{if } a > 0. \end{cases}$$

Hence, we get

$$I_1(x, t) \leq C_a D(t-x) \leq C_a D(T) \quad \text{for } 0 \leq x \leq t \leq T.$$

Next, we shall estimate  $I_2$ . It follows that

$$I_2(x, t) = \int_0^t ds \int_{|x-t+s|}^{t-x-s} \frac{dy}{(1+y)^{1+a}} \leq \int_0^t ds \int_0^{t+x-s} \frac{dy}{(1+y)^{1+a}}$$

for  $t \geq x \geq 0$ , and that the  $y$ -integral is dominated by

$$\begin{cases} -a^{-1}(1+t+x)^{-a} & \text{if } a < 0, \\ \log(1+t+x) & \text{if } a = 0, \\ a^{-1} & \text{if } a > 0. \end{cases}$$

Noticing that

$$\log(1+2t) \leq \log 2 + \log(2+t) \leq 2 \log(2+t) \quad \text{for } t \geq 0,$$

we get

$$I_2(x, t) \leq C_a D(t+x) \leq C_a D(T) \quad \text{for } 0 \leq x \leq t \leq T.$$

When  $x \geq t$ , we have

$$I(x, t) \leq \int_0^t \frac{ds}{(1+s)^{1+a}} \int_{x-t+s}^{x+t-s} dy \leq 2t \int_0^t \frac{ds}{(1+s)^{1+a}} \leq C_a D(T).$$

Therefore, the proof of Lemma 3.4.1 is ended.  $\square$

Now, we move on to the proof of Theorem 3.1.2. First of all, we take  $T > 0$  such that

$$2^{p+1} p C_a D(T) M^{p-1} \varepsilon^{p-1} \leq 1, \quad (3.4.7)$$

where  $C_a$  is the one in Lemma 3.4.1. We shall show

$$\|u_n\|_{L^\infty(\mathbf{R} \times [0, T])} \leq 2M\varepsilon \quad (n \in \mathbf{N}), \quad (3.4.8)$$

by induction. Assume that  $\|u_{n-1}\|_{L^\infty(\mathbf{R} \times [0, T])} \leq 2M\varepsilon$  ( $n \geq 2$ ). It follows from (3.4.3) and Lemma 3.4.1 that

$$\begin{aligned} \|u_n\|_{L^\infty(\mathbf{R} \times [0, T])} &\leq \|u_0\|_{L^\infty(\mathbf{R} \times [0, T])} + \|L(H(\cdot, u_{n-1}))\|_{L^\infty(\mathbf{R} \times [0, T])} \\ &\leq M\varepsilon + C_a D(T) \|u_{n-1}\|_{L^\infty(\mathbf{R} \times [0, T])}^p. \end{aligned}$$

The assumption of the induction yields that

$$\|u_n\|_{L^\infty(\mathbf{R} \times [0, T])} \leq M\varepsilon + C_a(2M\varepsilon)^p D(T).$$

This inequality shows (3.4.8), provided (3.4.7) holds.

Next we shall estimate the differences of  $\{u_n\}_{n \in \mathbf{N}}$ . Since

$$\begin{aligned} |H(y, u_n) - H(y, u_{n-1})| &\leq \left| \frac{p}{(1+y^2)^{(1+a)/2}} \int_{u_{n-1}(y,s)}^{u_n(y,s)} |\eta|^{p-1} d\eta \right| \\ &\leq \frac{p}{(1+y^2)^{(1+a)/2}} (|u_{n-1}(y,s)|^{p-1} + |u_n(y,s)|^{p-1}) \\ &\quad \times |u_n(y,s) - u_{n-1}(y,s)| \end{aligned}$$

for  $(y, s) \in \mathbf{R} \times [0, \infty)$ , we see from Lemma 3.4.1 that

$$\begin{aligned} &\|u_{n+1} - u_n\|_{L^\infty(\mathbf{R} \times [0, T])} \\ &\leq pC_a D(T) (\|u_n\|_{L^\infty(\mathbf{R} \times [0, T])}^{p-1} + \|u_{n-1}\|_{L^\infty(\mathbf{R} \times [0, T])}^{p-1}) \|u_n - u_{n-1}\|_{L^\infty(\mathbf{R} \times [0, T])}. \end{aligned}$$

Making use of (3.4.8), we have

$$\|u_{n+1} - u_n\|_{L^\infty(\mathbf{R} \times [0, T])} \leq \frac{1}{2} \|u_n - u_{n-1}\|_{L^\infty(\mathbf{R} \times [0, T])} \quad \text{for } n \in \mathbf{N}$$

provided (3.4.7) holds. Hence, we obtain

$$\|u_{n+1} - u_n\|_{L^\infty(\mathbf{R} \times [0, T])} \leq \frac{1}{2^n} \|u_1 - u_0\|_{L^\infty(\mathbf{R} \times [0, T])} \quad \text{for } n \in \mathbf{N}.$$

Therefore,  $\{u_n\}_{n \in \mathbf{N}}$  is a Cauchy sequence in  $X$  provided (3.4.7) holds. Since  $X$  is complete, there exists  $u \in X$  such that  $u_n$  converges uniformly to  $u$  in  $X$ . Therefore, by taking limits under the integral sign,  $u$  satisfies the integral equation (3.2.3), so that  $u$  is the  $C^2$ -solution of (3.1.1). Hence, the proof of Theorem 3.1.2 is completed.  $\square$

# Appendix

The aim of this appendix is to derive a representation formula for the solution to the homogeneous wave equation. The proof refers to Courant and Hilbert [4] and Yajima [35].

From now on, we assume that  $n \geq 2$  and write  $n = 2m$ ,  $2m + 1$  ( $m = 1, 2, 3, \dots$ ). Because, the representation formula in one space dimension is written in (3.2.1). We consider the following initial value problem for the wave equation:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = 0, \quad u_t(x, 0) = g(x), & x \in \mathbf{R}^n. \end{cases} \quad (\text{A.1})$$

Then, we have the following representation formula for the solutions to (A.1).

**Theorem A.1** *Let  $g \in C_0^\infty(\mathbf{R}^n)$ . Then, the solution of (A.1) is represented as*

$$u(x, t) = \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t (t^2 - r^2)^{(n-3)/2} r Q_n(x, r) dr, \quad (\text{A.2})$$

where

$$Q_n(x, r) = \frac{1}{\omega_n} \int_{|\omega|=1} g(x + r\omega) dS_\omega, \quad (\text{A.3})$$

and  $\omega_n$  is the area of the unit sphere in  $\mathbf{R}^n$ , i.e.

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \begin{cases} \frac{2(2\pi)^m}{(2m-1)!!} & \text{for } n = 2m + 1, \\ \frac{2\pi^m}{(m-1)!} & \text{for } n = 2m. \end{cases}$$

Moreover, if  $n = 2m + 1$  we have

$$u(x, t) = \frac{1}{(n-2)!!} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-3)/2} \{ t^{n-2} Q_n(x, t) \}, \quad (\text{A.4})$$

and

$$u(x, t) = \frac{1}{(n-2)!!} \sum_{k=0}^{(n-3)/2} \alpha_{(n-3)/2, k} t^{k+1} \left( \frac{\partial}{\partial t} \right)^k Q_n(x, t), \quad (\text{A.5})$$

where  $\alpha_{(n-3)/2,k}$  is a suitable constant satisfying  $\alpha_{(n-3)/2,0} = (n-2)!!$ .

If  $n = 2m$ , we have

$$u(x, t) = \frac{1}{(n-1)!!} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-2)/2} \{t^{n-1} H_n(x, t)\}, \quad (\text{A.6})$$

where

$$H_n(x, t) = \frac{2}{\omega_{n+1}} \int_{|\xi| \leq 1} \frac{g(x + t\xi) d\xi}{\sqrt{1 - |\xi|^2}}, \quad (\text{A.7})$$

and

$$u(x, t) = \frac{1}{(n-1)!!} \sum_{k=0}^{(n-2)/2} \alpha_{(n-2)/2,k} t^{k+1} \left( \frac{\partial}{\partial t} \right)^k H_n(x, t), \quad (\text{A.8})$$

where  $\alpha_{(n-2)/2,k}$  is a suitable constant satisfying  $\alpha_{(n-2)/2,0} = (n-1)!!$ .

**Remark A.1** It is sufficient to consider the case of  $u(x, 0) \equiv 0$  by the following observation. Let  $u$  be the solution of (A.1). Set  $v(x, t) = \partial_t u(x, t)$ . Noticing that  $u_{tt}(x, 0) = \Delta u(x, 0) \equiv 0$ ,  $v$  satisfies

$$\begin{cases} v_{tt} - \Delta v = 0, & \text{in } \mathbf{R}^n \times [0, \infty), \\ v(x, 0) = g(x), \quad v_t(x, 0) = 0, & x \in \mathbf{R}^n. \end{cases}$$

Therefore, the solution of

$$\begin{cases} w_{tt} - \Delta w = 0, & \text{in } \mathbf{R}^n \times [0, \infty), \\ w(x, 0) = f(x), \quad w_t(x, 0) = g(x), & x \in \mathbf{R}^n \end{cases}$$

is represented by  $w(x, t) = \tilde{v}(x, t) + u(x, t)$ , where  $u$  is the solution of (A.1) and  $\tilde{v}(x, t)$  is a solution of

$$\begin{cases} \tilde{v}_{tt} - \Delta \tilde{v} = 0, & \text{in } \mathbf{R}^n \times [0, \infty), \\ \tilde{v}(x, 0) = f(x), \quad \tilde{v}_t(x, 0) = 0, & x \in \mathbf{R}^n. \end{cases}$$

Before proving the theorem, we state the following lemma:

**Lemma A.1 (F. John [12])** Let  $\phi \in C(\mathbf{R})$ . Then we have

$$\int_{|\omega|=1} \phi(y \cdot \omega) dS_\omega = \omega_{n-1} \int_{-1}^1 (1 - p^2)^{(n-3)/2} \phi(p|y|) dp \quad (\text{A.9})$$

for  $y \in \mathbf{R}^n$ .



See [12] for the proof of this lemma.

**Proof of (A.2).** We introduce the Fourier transform for  $\psi \in \mathcal{S}(\mathbf{R}^n)$  by

$$\hat{\psi}(\xi) = \int_{\mathbf{R}^n} \psi(x) e^{-ix \cdot \xi} dx. \quad (\text{A.10})$$

Applying the Fourier transform to (A.1) with respect to the variable  $x$ , we have

$$\begin{cases} \partial_t^2 \hat{u} + |\xi|^2 \hat{u} = 0, \\ \hat{u}(\xi, 0) = 0, \partial_t \hat{u}(\xi, 0) = \hat{g}(\xi). \end{cases} \quad (\text{A.11})$$

This is the initial value problem for an ordinary differential equation with parameter  $\xi$ . Thus, we get

$$\hat{u}(\xi, t) = C_1(\xi) \cos |\xi|t + C_2(\xi) \sin |\xi|t,$$

where  $C_1(\xi)$  and  $C_2(\xi)$  are constants independent of  $t$ . By the initial condition in (A.11), we see that  $C_1(\xi) = \hat{u}(\xi, 0) = 0$  and  $C_2(\xi) = \partial_t \hat{u}(\xi, 0)/|\xi| = \hat{g}(\xi)/|\xi|$ . Hence we get

$$\hat{u}(\xi, t) = \frac{\sin |\xi|t}{|\xi|} \hat{g}(\xi).$$

By using the inverse Fourier transform, we have

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix \cdot \xi} \frac{\sin |\xi|t}{|\xi|} \hat{g}(\xi) d\xi. \quad (\text{A.12})$$

Since  $g \in \mathcal{S}(\mathbf{R}^n)$ , we have

$$u(x, t) = \lim_{\varepsilon \rightarrow 0} \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix \cdot \xi - \varepsilon |\xi|} \frac{\sin |\xi|t}{|\xi|} \hat{g}(\xi) d\xi.$$

First we consider the case of  $n = 2m + 1$ . For  $\varepsilon > 0$ , we define

$$E_\varepsilon(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix \cdot \xi - \varepsilon |\xi|} \frac{\sin |\xi|t}{|\xi|} d\xi. \quad (\text{A.13})$$

Note that the integrand is an integrable function. Then, we get

$$u(x, t) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n} E_\varepsilon(x - y, t) g(y) dy.$$

Hence the proof of (A.2) reduce to the following proposition:

**Proposition A.1** *Let  $n = 2m + 1$  ( $m = 1, 2, 3, \dots$ ), and let  $E_\varepsilon$  is the function defined in (A.13). Then we have*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n} E_\varepsilon(x - y, t) g(y) dy \\ = \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t (t^2 - r^2)^{(n-3)/2} r Q_n(x, r) dr, \end{aligned} \quad (\text{A.14})$$

where  $Q_n$  is defined in (A.3).

**Proof.** Introducing the polar coordinates  $\xi = \rho\omega$  ( $|\omega| = 1$ ) in the integral of (A.13), we get

$$E_\varepsilon(x, t) = \frac{1}{(2\pi)^n} \int_{|\omega|=1} \left( \int_0^\infty e^{ix \cdot \rho\omega - \varepsilon\rho} \sin(t\rho) \rho^{n-2} d\rho \right) dS_\omega. \quad (\text{A.15})$$

It follows from

$$\sin(t\rho) \rho^{n-2} = (-1)^{(n-1)/2} \left( \frac{\partial}{\partial t} \right)^{n-2} \cos(t\rho)$$

that

$$E_\varepsilon(x, t) = \frac{(-1)^{(n-1)/2}}{(2\pi)^n} \left( \frac{\partial}{\partial t} \right)^{n-2} \left\{ \int_{|\omega|=1} \left( \int_0^\infty e^{ix \cdot \rho\omega - \varepsilon\rho} \cos(t\rho) d\rho \right) dS_\omega \right\}.$$

Noticing that

$$e^{ix \cdot \rho\omega - \varepsilon\rho} \cos(t\rho) = \frac{e^{i(x \cdot \omega + t + i\varepsilon)\rho} + e^{i(x \cdot \omega - t + i\varepsilon)\rho}}{2},$$

we get

$$\begin{aligned} E_\varepsilon(x, t) \\ = \frac{(-1)^{(n-1)/2}}{2(2\pi)^n} \left( \frac{\partial}{\partial t} \right)^{n-2} \left\{ \int_{|\omega|=1} \left( \int_0^\infty (e^{i(x \cdot \omega + t + i\varepsilon)\rho} + e^{i(x \cdot \omega - t + i\varepsilon)\rho}) d\rho \right) dS_\omega \right\}. \end{aligned}$$

Here the  $\rho$ -integral is equals to

$$i \left( \frac{1}{x \cdot \omega + t + i\varepsilon} + \frac{1}{x \cdot \omega - t + i\varepsilon} \right).$$

Making use of the rotational invariance ( $\omega \rightarrow -\omega$ ) of the surface integral, we have

$$\begin{aligned} E_\varepsilon(x, t) \\ = \frac{(-1)^{(n-1)/2}}{2(2\pi)^n} \left( \frac{\partial}{\partial t} \right)^{n-2} \int_{|\omega|=1} i \left( \frac{1}{x \cdot (-\omega) + t + i\varepsilon} + \frac{1}{x \cdot \omega - t + i\varepsilon} \right) dS_\omega \\ = \frac{(-1)^{(n-1)/2}}{(2\pi)^n} \left( \frac{\partial}{\partial t} \right)^{n-2} \int_{|\omega|=1} \frac{\varepsilon}{(x \cdot \omega - t)^2 + \varepsilon^2} dS_\omega. \end{aligned} \quad (\text{A.16})$$

Here we apply Lemma A.1 as  $\phi(s) = \varepsilon/\{(s-t)^2 + \varepsilon^2\}$  for  $s \in \mathbf{R}$  and  $t \geq 0$ . Then we get

$$\int_{|\omega|=1} \phi(x \cdot \omega) dS_\omega = \omega_{n-1} \int_{-1}^1 (1-p^2)^{(n-3)/2} \frac{\varepsilon}{(|x|p-t)^2 + \varepsilon^2} dp.$$

Changing the variables by  $s = |x|p$ , we have

$$\begin{aligned} \int_{|\omega|=1} \phi(x \cdot \omega) dS_\omega &= \omega_{n-1} \int_{-|x|}^{|x|} \left(1 - \frac{s^2}{|x|^2}\right)^{(n-3)/2} P_\varepsilon(s-t) |x|^{-1} ds \\ &= \frac{\omega_{n-1}}{|x|^{n-2}} \int_{-|x|}^{|x|} (|x|^2 - s^2)^{(n-3)/2} P_\varepsilon(s-t) ds, \end{aligned}$$

where we set

$$P_\varepsilon(s) = \frac{\varepsilon}{s^2 + \varepsilon^2} \quad \text{for } s \in \mathbf{R}.$$

Then (A.16) yields

$$E_\varepsilon(x, t) = \frac{(-1)^{(n-1)/2} \omega_{n-1}}{(2\pi)^n |x|^{n-2}} \int_{-|x|}^{|x|} (|x|^2 - s^2)^{(n-3)/2} P_\varepsilon(s-t) ds,$$

and that

$$\begin{aligned} \int_{\mathbf{R}^n} E_\varepsilon(x-y, t) g(y) dy &= \int_{\mathbf{R}^n} E_\varepsilon(y, t) g(x-y) dy \\ &= \frac{(-1)^{(n-1)/2} \omega_{n-1}}{(2\pi)^n} \int_{\mathbf{R}^n} \left(\frac{\partial}{\partial t}\right)^{n-2} \left( |y|^{2-n} \int_{-|y|}^{|y|} (|y|^2 - s^2)^{(n-3)/2} P_\varepsilon(s-t) ds \right) \\ &\quad \times g(x-y) dy. \end{aligned}$$

Noticing that

$$\left| \left(\frac{\partial}{\partial t}\right)^{n-2} P_\varepsilon(s-t) \right| \leq C_{n,\varepsilon} \quad \text{for } s \in \mathbf{R}, \quad n \geq 3,$$

where  $C_{n,\varepsilon}$  is a positive constant depending on  $n$  and  $\varepsilon$ , we have

$$\left| \left(\frac{\partial}{\partial t}\right)^{n-2} \left( |y|^{2-n} \int_{-|y|}^{|y|} (|y|^2 - s^2)^{(n-3)/2} P_\varepsilon(s-t) ds \right) g(x-y) \right| \leq C_{n,\varepsilon} |g(x-y)|$$

for  $y \in \mathbf{R}^n$ . We note that the right hand side of this inequality is an integrable function on  $\mathbf{R}^n$ , because of  $g \in C_0^\infty(\mathbf{R}^n)$ . Hence, making use of differentiation under the integral sign, we get

$$\int_{\mathbf{R}^n} E_\varepsilon(x-y, t) g(y) dy = \frac{(-1)^{(n-1)/2}}{(2\pi)^n} \left(\frac{\partial}{\partial t}\right)^{n-2} K_\varepsilon(x, t), \quad (\text{A.17})$$

where we set

$$K_\varepsilon(x, t) = \omega_{n-1} \int_{\mathbf{R}^n} |y|^{2-n} \int_{-|y|}^{|y|} (|y|^2 - s^2)^{(n-3)/2} P_\varepsilon(s - t) g(x - y) ds dy.$$

Introducing the polar coordinate  $y = r\omega$ , with  $r = |y|$ ,  $|\omega| = 1$ , and using the rotational invariance of the surface integral, we have

$$\begin{aligned} K_\varepsilon(x, t) &= \omega_{n-1} \int_0^\infty r^{2-n} \left( \int_{-r}^r (r^2 - s^2)^{(n-3)/2} P_\varepsilon(s - t) \int_{|\omega|=1} g(x + r\omega) dS_\omega ds \right) r^{n-1} dr. \end{aligned}$$

Making use of the definition of  $Q_n$  in (A.3), we get

$$K_\varepsilon(x, t) = \omega_{n-1} \omega_n \int_0^\infty r \left( \int_{-r}^r (r^2 - s^2)^{(n-3)/2} P_\varepsilon(s - t) Q_n(x, r) ds \right) dr.$$

Noticing that

$$\begin{aligned} &\omega_n \int_0^\infty r \left( \int_{-r}^r (r^2 - s^2)^{(n-3)/2} P_\varepsilon(s - t) |Q_n(x, r)| ds \right) dr \\ &\leq 2\varepsilon^{-1} \int_0^\infty r^{n-1} \left( \int_{|\omega|=1} |g(x + r\omega)| dS_\omega \right) dr = 2\varepsilon^{-1} \|g\|_{L^1(\mathbf{R}^n)} < \infty, \end{aligned}$$

we can invert the order of  $(s, r)$ -integral, and hence

$$K_\varepsilon(x, t) = \omega_{n-1} \omega_n \int_{-\infty}^\infty P_\varepsilon(s - t) \left( \int_{|s| \leq r} (r^2 - s^2)^{(n-3)/2} r Q_n(x, r) dr \right) ds.$$

It follows from

$$\Gamma\left(\frac{n-1}{2}\right) = \frac{(n-3)!!}{2^{(n-3)/2}} \quad \text{and} \quad \Gamma\left(\frac{n}{2}\right) = \frac{(n-2)!!}{2^{(n-1)/2}} \sqrt{\pi},$$

that

$$\begin{aligned} \omega_n \omega_{n-1} &= \frac{2\pi^{(n-1)/2} 2\pi^{n/2}}{\Gamma((n-1)/2) \Gamma(n/2)} \\ &= \frac{\pi^{n-1/2} 2^n}{(n-3)!!(n-2)!!\pi^{1/2}} = \frac{(2\pi)^n}{\pi(n-2)!}. \end{aligned}$$

Hence (A.17) yields

$$\begin{aligned} &\int_{\mathbf{R}^n} E_\varepsilon(x - y, t) g(y) dy \\ &= \frac{(-1)^{(n-1)/2}}{\pi(n-2)!} \left( \frac{\partial}{\partial t} \right)^{n-2} \int_{-\infty}^\infty P_\varepsilon(s - t) \left( \int_{|s| \leq r} (r^2 - s^2)^{(n-3)/2} r Q_n(x, r) dr \right) ds. \end{aligned}$$

Since  $(-1)^{(n-1)/2+(n-3)/2} = -1$ , we have

$$\begin{aligned} & \int_{\mathbf{R}^n} E_\varepsilon(x-y, t) g(y) dy \\ &= \frac{-1}{\pi(n-2)!} \left( \frac{\partial}{\partial t} \right)^{n-2} \int_{-\infty}^{\infty} P_\varepsilon(s-t) \left( \int_{|s| \leq r} (s^2 - r^2)^{(n-3)/2} r Q_n(x, r) dr \right) ds. \end{aligned}$$

We set

$$\Psi(s) = \frac{-1}{(n-2)!} \int_{|s| \leq r} (s^2 - r^2)^{(n-3)/2} r Q_n(x, r) dr$$

for  $s \in \mathbf{R}$ . We note that  $\Psi \in C_0^\infty(\mathbf{R})$  because  $g \in C_0^\infty(\mathbf{R}^n)$ . Then we get

$$\begin{aligned} \int_{\mathbf{R}^n} E_\varepsilon(x-y, t) g(y) dy &= \frac{1}{\pi} \left( \frac{\partial}{\partial t} \right)^{n-2} \int_{-\infty}^{\infty} P_\varepsilon(s-t) \Psi(s) ds \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} P_\varepsilon(s-t) \frac{d^{n-2}}{ds^{n-2}} \Psi(s) ds. \end{aligned} \quad (\text{A.18})$$

Noticing that  $P_\varepsilon$  is the Poisson kernel, we get

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} P_\varepsilon(s-t) \frac{d^{n-2}}{ds^{n-2}} \Psi(s) ds = \frac{d^{n-2}}{dt^{n-2}} \Psi(t). \quad (\text{A.19})$$

From the definition, we have

$$\begin{aligned} \frac{d^{n-2}}{dt^{n-2}} \Psi(t) &= \frac{-1}{(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_{|t|}^{\infty} (t^2 - r^2)^{(n-3)/2} r Q_n(x, r) dr \\ &= \frac{-1}{(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \left( \int_0^\infty - \int_0^{|t|} \right) (t^2 - r^2)^{(n-3)/2} r Q_n(x, r) dr. \end{aligned}$$

It follows from that  $(t^2 - r^2)^{(n-3)/2}$  is a polynomial of degree  $n-3$ , we have

$$\frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^\infty (t^2 - r^2)^{(n-3)/2} r Q_n(x, r) dr = 0. \quad (\text{A.20})$$

In fact, we obtain

$$\frac{\partial^{n-3}}{\partial t^{n-3}} (t^2 - r^2)^{(n-3)/2} r Q_n(x, r) = C_n r Q_n(x, r)$$

for  $r \geq 0$ , with a suitable constant  $C_n$ . It follows that

$$\begin{aligned} \omega_n \left| \int_0^\infty r Q_n(x, r) dr \right| &\leq \omega_n \int_0^1 r |Q_n(x, r)| dr + \omega_n \int_1^\infty r |Q_n(x, r)| dr \\ &\leq \int_0^1 \int_{|\omega|=1} |g(x + r\omega)| dS_\omega dr + \int_1^\infty r^{n-1} \int_{|\omega|=1} |g(x + r\omega)| dS_\omega dr \\ &\leq \|g\|_{L^\infty(\mathbf{R}^n)} + \|g\|_{L^1(\mathbf{R}^n)} < \infty, \end{aligned}$$

because  $g \in C_0^\infty(\mathbf{R}^n)$ . Thus we get (A.20).

Hence for  $t \geq 0$ , we obtain

$$\frac{d^{n-2}}{dt^{n-2}}\Psi(t) = \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t (t^2 - r^2)^{(n-3)/2} r Q_n(x, r) dr.$$

By (A.18) and (A.19), we finally get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n} E_\varepsilon(x - y, t) g(y) dy \\ = \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t (t^2 - r^2)^{(n-3)/2} r Q_n(x, r) dr. \end{aligned}$$

This completes the proof of Proposition A.1, and we get (A.2) in the case of  $n = 2m + 1$ .  $\square$

Next we consider the case of  $n = 2m$ . We use Hadamard's method of descent to get (A.2) in the case  $n = 2m$  from the case  $n = 2m + 1$ .

Define  $\tilde{x} = (x, x_{n+1}) \in \mathbf{R}^{n+1}$ , where  $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ . Suppose that  $u$  is a solution of (A.1). Define  $\tilde{u}(\tilde{x}, t) = u(x, t)$  and  $\tilde{g}(\tilde{x}) = g(x)$ . Then  $\tilde{u}$  is a solution of (A.1) in  $\mathbf{R}^{n+1}$ , where  $n + 1$  is odd. By (A.2), we have

$$\tilde{u}(\tilde{x}, t) = \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial t^{n-1}} \int_0^t (t^2 - r^2)^{(n-2)/2} r Q_{n+1}(\tilde{x}, r) dr. \quad (\text{A.21})$$

Noticing that

$$\begin{aligned} & \int_{|\tilde{\omega}|=1, \omega_{n+1}>0} \tilde{g}(\tilde{x} + r\tilde{\omega}) dS_{\tilde{\omega}} \\ &= \int_{|\omega|<1} g(x + r\omega) \sqrt{1 + \sum_{j=1}^n \left( \frac{d}{d\omega_j} \omega_{n+1} \right)^2} d\omega \\ &= \int_{|\omega|<1} \frac{g(x + r\omega)}{\sqrt{1 - |\omega|^2}} d\omega, \end{aligned}$$

we get

$$\begin{aligned} Q_{n+1}(\tilde{x}, r) &= \frac{1}{\omega_{n+1}} \int_{|\tilde{\omega}|=1} \tilde{g}(\tilde{x} + r\tilde{\omega}) dS_{\tilde{\omega}} \\ &= \frac{2}{\omega_{n+1}} \int_{|\xi|<1} \frac{g(x + r\xi)}{\sqrt{1 - |\xi|^2}} d\xi. \end{aligned} \quad (\text{A.22})$$

Here we introduce the polar coordinate  $\xi = \rho\omega$  ( $|\omega| = 1$ ) in the above integral. Then we get

$$Q_{n+1}(\tilde{x}, r) = \frac{2}{\omega_{n+1}} \int_0^1 \frac{\rho^{n-1}}{\sqrt{1 - \rho^2}} d\rho \int_{|\omega|=1} g(x + r\rho\omega) dS_\omega.$$

Changing the variables by  $r\rho = s$ , we have

$$\begin{aligned} Q_{n+1}(\tilde{x}, r) &= \frac{2}{\omega_{n+1}r^{n-1}} \int_0^r \frac{s^{n-1}}{\sqrt{r^2 - s^2}} ds \int_{|\omega|=1} g(x + s\omega) dS_\omega \\ &= \frac{2\omega_n}{\omega_{n+1}r^{n-1}} \int_0^r \frac{s^{n-1}}{\sqrt{r^2 - s^2}} Q_n(x, s) ds. \end{aligned}$$

Substituting this into (A.21), we get

$$\begin{aligned} u(x, t) &= \frac{1}{(n-1)! \omega_{n+1}} \frac{\partial^{n-1}}{\partial t^{n-1}} \int_0^t \frac{(t^2 - r^2)^{(n-2)/2}}{r^{n-1}} r dr \\ &\quad \times \int_0^r \frac{s^{n-1}}{\sqrt{r^2 - s^2}} Q_n(x, s) ds. \end{aligned} \tag{A.23}$$

It follows that

$$\begin{aligned} &\frac{\partial}{\partial t} \int_0^t \frac{(t^2 - r^2)^{(n-2)/2}}{r^{n-1}} r dr \int_0^r \frac{s^{n-1}}{\sqrt{r^2 - s^2}} Q_n(x, s) ds \\ &= \begin{cases} \int_0^t \frac{s}{\sqrt{t^2 - s^2}} Q_2(x, s) ds & \text{if } n = 2, \\ (n-2)t \int_0^t \frac{(t^2 - r^2)^{(n-4)/2}}{r^{n-2}} dr \int_0^r \frac{s^{n-1}}{\sqrt{r^2 - s^2}} Q_n(x, s) ds & \text{if } n \geq 4. \end{cases} \end{aligned}$$

Since  $2\omega_2/\omega_3 = 1$ , we get (A.2) in the case of  $n = 2$  from (A.23). From now on, we assume  $n \geq 4$ . Then (A.23) yields

$$\begin{aligned} u(x, t) &= \frac{n-2}{(n-1)! \omega_{n+1}} \frac{\partial^{n-2}}{\partial t^{n-2}} t \int_0^t \frac{(t^2 - r^2)^{(n-4)/2}}{r^{n-2}} dr \\ &\quad \times \int_0^r \frac{s^{n-1}}{\sqrt{r^2 - s^2}} Q_n(x, s) ds. \end{aligned}$$

Inverting the order of  $(s, r)$ -integral, we get

$$\begin{aligned} u(x, t) &= \frac{n-2}{(n-1)! \omega_{n+1}} \frac{\partial^{n-2}}{\partial t^{n-2}} t \int_0^t s^{n-1} Q_n(x, s) ds \\ &\quad \times \int_s^t \frac{(t^2 - r^2)^{(n-4)/2}}{r^{n-2} \sqrt{r^2 - s^2}} dr. \end{aligned}$$

Changing the variable by

$$1 - \frac{s^2}{r^2} = \left(1 - \frac{s^2}{t^2}\right) \sigma$$

in the above  $r$ -integral and noting that

$$\sigma = \frac{(r^2 - s^2)t^2}{(t^2 - s^2)r^2}, \quad 1 - \sigma = \frac{(t^2 - r^2)s^2}{(t^2 - s^2)r^2} \quad \text{and} \quad 2\frac{s^2}{r^3}dr = \left(1 - \frac{s^2}{t^2}\right) d\sigma,$$

we see that the  $r$ -integral equals to

$$\begin{aligned} & \int_0^1 \frac{(t^2 - s^2)^{(n-4)/2} (1 - \sigma)^{(n-4)/2} \left(\frac{r}{s}\right)^{n-4}}{r^{n-2} \frac{r}{t} \sqrt{t^2 - s^2} \sqrt{\sigma}} \cdot \frac{r^3 (t^2 - s^2)}{2s^2 t^2} d\sigma \\ &= \frac{(t^2 - s^2)^{(n-3)/2}}{2ts^{n-2}} B\left(\frac{1}{2}, \frac{n-2}{2}\right), \end{aligned}$$

where we set

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \quad (\alpha, \beta > 0).$$

Noticing that

$$\begin{aligned} & \frac{n-2}{(n-1)!} \frac{2\omega_n}{\omega_{n+1}} B\left(\frac{1}{2}, \frac{n-2}{2}\right) \\ &= \frac{n-2}{(n-1)!} \cdot \frac{2\pi^{n/2}/\Gamma(n/2)}{2\pi^{(n+1)/2}/\Gamma((n+1)/2)} \cdot \frac{\Gamma(1/2)\Gamma((n-2)/2)}{\Gamma((n-1)/2)} \\ &= \frac{n-2}{(n-1)!} \cdot \frac{2\pi^{n/2}\Gamma((n-1)/2)(n-1)/2}{2\pi^{(n+1)/2}\Gamma((n-2)/2)(n-2)/2} \cdot \frac{\pi^{1/2}\Gamma((n-2)/2)}{\Gamma((n-1)/2)} = \frac{1}{(n-2)!}, \end{aligned}$$

we get

$$u(x, t) = \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t (t^2 - s^2)^{(n-3)/2} s Q_n(x, s) ds.$$

Therefore we obtain (A.2) in the case of  $n = 2m$ .  $\square$

**Proof of (A.4).** Let  $\lambda$  be a fixed non-negative integer, and let  $\phi \in C^\lambda(\mathbf{R})$ . Define

$$U_\lambda(t) = \frac{1}{(2\lambda+1)!} \left(\frac{d}{dt}\right)^{2\lambda+1} \int_0^t (t^2 - r^2)^\lambda \phi(r) dr, \quad t \geq 0. \quad (\text{A.24})$$

Then, we have the following lemma:

**Lemma A.2** *Let  $U_\lambda$  be the function in (A.24). Then we have*

$$U_\lambda(t) = \frac{1}{(2\lambda+1)!!} \left(\frac{1}{t} \frac{d}{dt}\right)^\lambda (t^{2\lambda} \phi(t)) \quad (\text{A.25})$$

for  $\lambda = 0, 1, 2, \dots$ .



**Proof.** Set

$$V_\lambda(t) = \frac{1}{(2\lambda+1)!!} \left( \frac{1}{t} \frac{d}{dt} \right)^\lambda (t^{2\lambda} \phi(t)), \quad (\text{A.26})$$

which is the right hand side of (A.25). When  $\lambda = 0$ , we have

$$U_0(t) = \frac{d}{dt} \int_0^t \phi(r) dr = \phi(t) = V_0(t).$$

Hence (A.25) holds for  $\lambda = 0$ .

For  $\lambda \geq 1$ , (A.24) yields

$$\begin{aligned} U_\lambda(t) &= \frac{1}{(2\lambda+1)!} \left( \frac{d}{dt} \right)^{2\lambda} \left( 2\lambda t \int_0^t (t^2 - r^2)^{\lambda-1} \phi(r) dr \right) \\ &= \frac{1}{(2\lambda+1)!} \left\{ 2\lambda t \left( \frac{d}{dt} \right)^{2\lambda} \int_0^t (t^2 - r^2)^{\lambda-1} \phi(r) dr \right. \\ &\quad \left. + (2\lambda)^2 \left( \frac{\partial}{\partial t} \right)^{2\lambda-1} \int_0^t (t^2 - r^2)^{\lambda-1} \phi(r) dr \right\}. \end{aligned}$$

By (A.24), we have

$$U_\lambda(t) = \frac{1}{(2\lambda+1)!} \left\{ 2\lambda t (2\lambda-1)! U'_{\lambda-1}(t) + (2\lambda)^2 (2\lambda-1)! U_{\lambda-1}(t) \right\}.$$

Hence we get

$$U_\lambda(t) = \frac{1}{2\lambda+1} \left\{ t U'_{\lambda-1}(t) + 2\lambda U_{\lambda-1}(t) \right\}. \quad (\text{A.27})$$

On the other hand, (A.26) yields

$$\begin{aligned} V_\lambda(t) &= \frac{1}{(2\lambda+1)!!} \left( \frac{1}{t} \frac{d}{dt} \right)^{\lambda-1} (2\lambda t^{2\lambda-2} \phi(t) + t^{2\lambda-1} \phi'(t)) \\ &= \frac{1}{2\lambda+1} \left\{ \frac{2\lambda}{(2\lambda-1)!!} \left( \frac{1}{t} \frac{d}{dt} \right)^{\lambda-1} (t^{2\lambda-2} \phi(t)) \right. \\ &\quad \left. + \frac{1}{(2\lambda-1)!!} \left( \frac{1}{t} \frac{d}{dt} \right)^{\lambda-1} (t^{2\lambda-1} \phi'(t)) \right\}. \end{aligned}$$

By (A.26), we get

$$V_\lambda(t) = \frac{1}{2\lambda+1} \left\{ 2\lambda V_{\lambda-1}(t) + \frac{1}{(2\lambda-1)!!} \left( \frac{1}{t} \frac{d}{dt} \right)^{\lambda-1} (t^{2\lambda-1} \phi'(t)) \right\}.$$

Define  $\mathcal{D} = \frac{1}{t} \frac{d}{dt}$ . If one can show

$$\mathcal{D}^{\lambda-1} t^{2\lambda} \mathcal{D} w(t) = t^2 \mathcal{D}^\lambda t^{2(\lambda-1)} w(t) \quad (\text{A.28})$$

for  $w \in C^\lambda(\mathbf{R})$ , then

$$V_\lambda(t) = \frac{1}{2\lambda+1} \{2\lambda V_{\lambda-1}(t) + t V'_{\lambda-1}(t)\}$$

holds, and we get  $U_\lambda(t) = V_\lambda(t)$  by (A.27). In what follows we show (A.28) by induction.

Obviously (A.28) holds for  $\lambda = 1$ . Assume that (A.28) holds for  $\lambda \geq 1$ . Noticing that

$$\mathcal{D} t^2 w(t) = 2w(t) + t^2 \mathcal{D} w(t), \quad (\text{A.29})$$

we get

$$\begin{aligned} \mathcal{D}^\lambda t^{2\lambda+2} \mathcal{D} w(t) &= \mathcal{D}^\lambda t^{2\lambda} (\mathcal{D} t^2 - 2) w(t) \\ &= \{\mathcal{D}(\mathcal{D}^{\lambda-1} t^{2\lambda} \mathcal{D}) t^2 - 2 \mathcal{D}^\lambda t^{2\lambda}\} w(t). \end{aligned}$$

Making use of the assumption of induction, we get

$$\begin{aligned} \mathcal{D}^\lambda t^{2\lambda+2} \mathcal{D} w(t) &= \{\mathcal{D}(t^2 \mathcal{D}^\lambda t^{2(\lambda-1)}) t^2 - 2 \mathcal{D}^\lambda t^{2\lambda}\} w(t) \\ &= \{(\mathcal{D} t^2 - 2) \mathcal{D}^\lambda t^{2\lambda}\} w(t). \end{aligned}$$

By using (A.29), we get

$$\mathcal{D}^\lambda t^{2\lambda+2} \mathcal{D} w(t) = (t^2 \mathcal{D}) \mathcal{D}^\lambda t^{2\lambda} w(t) = t^2 \mathcal{D}^{\lambda+1} t^{2\lambda} w(t).$$

Hence (A.28) holds for all  $\lambda \geq 1$ . Therefore, the proof of Lemma A.2 is completed.  $\square$

We turn back to the proof of (A.4). Set  $\lambda = (n-3)/2$  and  $\phi(r) = r Q_n(x, r)$  in (A.24). Then by (A.2), we have  $u(x, t) = U_\lambda(t)$ . Making use of Lemma A.2, we get

$$u(x, t) = \frac{1}{(n-2)!!} \left( \frac{1}{t} \frac{d}{dt} \right)^{(n-3)/2} (t^{n-2} Q_n(x, t)).$$

Therefore, we get (A.4).

**Proof of (A.5).** We first prepare the following proposition.

**Proposition A.2** *Let  $l$  be a fixed non-negative integer, and let  $\phi \in C^l(\mathbf{R})$ . Then there exist constants  $\alpha_{l,k} > 0$  such that*

$$\left( \frac{1}{t} \frac{d}{dt} \right)^l (t^{2l+1} \phi(t)) = \sum_{k=0}^l \alpha_{l,k} t^{k+1} \frac{d^k}{dt^k} \phi(t). \quad (\text{A.30})$$

Moreover,  $\alpha_{l,0} = (2l+1)!!$  holds.

**Proof.** Let  $l = 0$ . Then (A.30) holds as  $\alpha_{0,0} = 1$ .

Suppose that

$$\left(\frac{1}{t} \frac{d}{dt}\right)^{l-1} (t^{2l-1} \phi(t)) = \sum_{k=0}^{l-1} \alpha_{l-1,k} t^{k+1} \frac{d^k}{dt^k} \phi(t).$$

holds for  $l \geq 1$ . Replacing  $\phi(t)$  by  $t^2 \phi(t)$  in the above, we get

$$(\Phi(t) :=) \left(\frac{1}{t} \frac{d}{dt}\right)^l (t^{2l+1} \phi(t)) = \frac{1}{t} \frac{d}{dt} \sum_{k=0}^{l-1} \alpha_{l-1,k} t^{k+1} \frac{d^k}{dt^k} t^2 \phi(t).$$

It follows that

$$\begin{aligned} \Phi(t) &= \frac{1}{t} \frac{d}{dt} \sum_{k=0}^{l-1} \alpha_{l-1,k} t^{k+1} \left\{ t^2 \frac{d^k}{dt^k} \phi(t) + 2kt \frac{d^{k-1}}{dt^{k-1}} \phi(t) + k(k-1) \phi(t) \right\} \\ &= \sum_{k=0}^{l-1} \alpha_{l-1,k} \frac{1}{t} \frac{d}{dt} \left\{ t^{k+3} \frac{d^k}{dt^k} \phi(t) \right\} + \sum_{k=1}^{l-1} 2k \alpha_{l-1,k} \frac{1}{t} \frac{d}{dt} \left\{ t^{k+2} \frac{d^{k-1}}{dt^{k-1}} \phi(t) \right\} \\ &\quad + \sum_{k=2}^{l-1} k(k-1) \alpha_{l-1,k} \frac{1}{t} \frac{d}{dt} \left\{ t^{k+1} \frac{d^{k-1}}{dt^{k-1}} \phi(t) \right\} \\ &= \sum_{k=0}^{l-1} (k+3) \alpha_{l-1,k} (k+3) t^{k+1} \frac{d^k}{dt^k} \phi(t) + \sum_{k=1}^l \alpha_{l-1,k-1} t^{k+1} \frac{d^k}{dt^k} \phi(t) \\ &\quad + \sum_{k=0}^{l-2} 2k(k+1)(k+3) \alpha_{l-1,k+1} t^{k+1} \frac{d^k}{dt^k} \phi(t) \\ &\quad + \sum_{k=1}^{l-1} 2k \alpha_{l-1,k} t^{k+1} t^{k+1} \frac{d^k}{dt^k} \phi(t) \\ &\quad + \sum_{k=0}^{l-3} (k+3)(k+2)(k+1) \alpha_{l-1,k+2} t^{k+1} \frac{d^k}{dt^k} \phi(t) \\ &\quad + \sum_{k=1}^{l-2} (k+1)k \alpha_{l-1,k+1} t^{k+1} \frac{d^k}{dt^k} \phi(t) \\ &= \{3\alpha_{l-1,0} + 6\alpha_{l-1,1} + 6\alpha_{l-1,2}\} t \phi(t) \\ &\quad + \sum_{k=0}^{l-3} \{3(k+1)\alpha_{l-1,k}(k+3) + \alpha_{l-1,k-1} + 3(k+2)(k+1)\alpha_{l-1,k+1} \\ &\quad + (k+3)(k+2)(k+1)\alpha_{l-1,k+2}\} t^{k+1} \frac{d^k}{dt^k} \phi(t) \\ &\quad + \{(3l-1)\alpha_{l-1,l-2} + \alpha_{l-1,l-3} + 3l(l-1)\alpha_{l-1,l-1}\} t^{l-1} \frac{d^{l-2}}{dt^{l-2}} \phi(t) \\ &\quad + \{3l\alpha_{l-1,l-1} + \alpha_{l-1,l-2}\} t^l \frac{d^{l-1}}{dt^{l-1}} \phi(t) + \alpha_{l-1,l-1} t^{l+1} \frac{d^l}{dt^l} \phi(t). \end{aligned}$$

Therefore, (A.30) holds for all  $l \geq 1$ . Moreover, if we set  $\phi(t) = 1$  in (A.30), then we get

$$\left(\frac{1}{t} \frac{d}{dt}\right)^l t^{2l+1} = \alpha_{l,0} t.$$

Hence we have  $\alpha_{l,0} = (2l+1)!!$ . This completes the proof.  $\square$

We turn back to the proof of (A.5). Setting  $l = (n-3)/2$  and  $\phi(t) = Q_n(x, t)$  in (A.30), we get (A.5) by (A.4). This completes the proof.  $\square$

**Proof of (A.6) and (A.8).** Again, we shall apply Hadamard's method of descent to get (A.6) and (A.8) from (A.4) and (A.5). In view of the proof of (A.2) in the case of  $n = 2m$ , we get

$$u(x, t) = \frac{1}{(n-1)!!} \left(\frac{1}{t} \frac{d}{dt}\right)^{(n-2)/2} (t^{n-1} Q_{n+1}(x, t)),$$

where

$$Q_{n+1}(x, r) = \frac{2}{\omega_{n+1}} \int_{|\xi| < 1} \frac{g(x + r\xi)}{\sqrt{1 - |\xi|^2}} d\xi.$$

Also, we get

$$u(x, t) = \frac{1}{(n-1)!!} \sum_{k=0}^{(n-2)/2} \alpha_{(n-2)/2, k} t^{k+1} \left(\frac{\partial}{\partial t}\right)^k Q_{n+1}(x, t).$$

Since  $Q_{n+1}(x, t) = H_n(x, t)$ , we get (A.6) and (A.8). Therefore, the proof of Theorem A.1 is completed.  $\square$

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