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Anisotropic convexified Gauss curvature flow of bounded open sets:  
stochastic approximation, weak solution and viscosity solution

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## 1 Introduction

Gauss curvature flow is known as a mathematical model of the wearing process of a convex stone rolling on a beach (see [2]).

In [3] we proposed and studied a two dimensional random crystalline algorithm for the curvature flow of smooth simple closed convex curves.

In [4] we studied a convexified Gauss curvature flow of compact sets by the level set approach in the theory of viscosity solutions.

In this talk we discuss a random crystalline algorithm of and PDE on an anisotropic convexified Gauss curvature flow of bounded open sets in  $\mathbf{R}^N$  for **any**  $N \geq 2$  (see [5]).

We introduce an assumption and a notation before we describe the PDE under consideration.

$$(A.1). \quad R \in L^1(\mathbf{S}^{N-1} : [0, \infty), d\mathcal{H}^{N-1}), \text{ and } \|R\|_{L^1(\mathbf{S}^{N-1})} = 1.$$

For  $p \in \mathbf{R}^N$  and a  $N \times N$ -symmetric real matrix  $X$ , put  $G(o, X) := 0$  and

$$G(p, X) := |p| \det_+ \left( -\left(I - \frac{p}{|p|} \otimes \frac{p}{|p|}\right) \frac{X}{|p|} \left(I - \frac{p}{|p|} \otimes \frac{p}{|p|}\right) + \frac{p}{|p|} \otimes \frac{p}{|p|} \right)$$

if  $p \neq o$ .

We study a weak solution and a viscosity solution of the following PDE in this talk:

$$\partial_t u(t, x) = \sigma^+(u, Du(t, x), t, x) R \left( \frac{Du(t, x)}{|Du(t, x)|} \right) G(Du(t, x), D^2u(t, x)) \quad (1.1)$$

$((t, x) \in (0, \infty) \times \mathbf{R}^N)$ . Here

$$\sigma^+(u, p, t, x) := \begin{cases} 1 & \text{if } u(t, \cdot) \leq u(t, x) \text{ on } H(p, x) \text{ and } p \in \mathbf{R}^N \setminus \{o\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$H(p, x) := \{y \in \mathbf{R}^N \setminus \{x\} \mid \langle y - x, p \rangle \leq 0\}.$$

To introduce the notion of a weak solution to (1.1), we give several notations.

Let  $F$  be a closed convex subset of  $\mathbf{R}^N$ . For  $x \in \partial F$ , put

$$N_F(x) := \{p \in \mathbf{S}^{N-1} \mid F \subset \{y \mid \langle y - x, p \rangle \leq 0\}\}.$$

**Definition 1** *Suppose that (A.1) holds. Let  $u : \mathcal{D}(u) (\subset \mathbf{R}^N) \mapsto \mathbf{R}$  be bounded and  $r \in \mathbf{R}$ . For any  $B \in B(\mathbf{R}^N)$ , put*

$$\omega_r(R, u, B) := \int_{N_{(co\ u^{-1}([r, \infty))} - (B \cap \partial(co\ u^{-1}([r, \infty))))} R(p) d\mathcal{H}^{N-1}(p),$$

$$\mathbf{w}(R, u, B) := \int_{\mathbf{R}} dr \omega_r(R, u, B),$$

provided the right hand side is well defined.

**Definition 2 (Weak Solutions)** *Suppose that (A.1) holds.*

(i) *A family of bounded open sets  $\{D(t)\}_{t \geq 0}$  in  $\mathbf{R}^N$  is called an anisotropic convexified Gauss curvature flow if*

$$D(t) = \begin{cases} (co\ D(t)) \cap D(0) & \text{for } t \in [0, Vol(D)), \\ \emptyset & \text{for } t \geq Vol(D) \end{cases} \quad (1.2)$$

; and for any  $\varphi \in C_o(\mathbf{R}^N)$  and any  $t \geq 0$ ,

$$\int_{\mathbf{R}^N} \varphi(x) (I_{D(0)}(x) - I_{D(t)}(x)) dx = \int_0^t ds \int_{\mathbf{R}^N} \varphi(x) \omega_1(I_{D(s)}(\cdot), dx). \quad (1.3)$$

(ii)  *$u \in C_b([0, \infty) \times \mathbf{R}^N)$  is called the weak solution to (1.1) if the following holds: for any  $\varphi \in C_o(\mathbf{R}^N)$  and any  $t \geq 0$ ,*

$$\int_{\mathbf{R}^N} \varphi(x) (u(0, x) - u(t, x)) dx = \int_0^t ds \int_{\mathbf{R}^N} \varphi(x) \mathbf{w}(u(s, \cdot), dx). \quad (1.4)$$

Before we introduce the notion of a viscosity solution to (1.1), we introduce notations

$f \in \mathcal{F}$  iff  $f \in C^2([0, \infty))$ ,  $f''(r) > 0$  on  $(0, \infty)$ , and  $f(r)/r^N \rightarrow 0$  as  $r \rightarrow 0$ .

Let  $\Omega$  be an open subset of  $(0, \infty) \times \mathbf{R}^N$ .  $f \in \mathcal{A}(\Omega)$  iff  $\varphi \in C^2(\Omega)$ , and for any  $(\hat{t}, \hat{x}) \in \Omega$  for which  $D\varphi$  vanishes, there exists  $f \in \mathcal{F}$  such that

$$|\varphi(t, x) - \varphi(\hat{t}, \hat{x}) - \partial_t \varphi(\hat{t}, \hat{x})(t - \hat{t})| \leq f(|x - \hat{x}|) + o(|t - \hat{t}|) \quad \text{as } (t, x) \rightarrow (\hat{t}, \hat{x}).$$

**Definition 3 (Viscosity solution)** (see [6]).

Let  $0 < T \leq \infty$  and set  $\Omega := (0, T) \times \mathbf{R}^N$ .

(i). A function  $u \in USC(\Omega)$  is called a viscosity subsolution of (1.1) in  $\Omega$  if whenever  $\varphi \in \mathcal{A}(\Omega)$ ,  $(s, y) \in \Omega$ , and  $u - \varphi$  attains a local maximum at  $(s, y)$ , then

$$\partial_t \varphi(s, y) + \sigma^-(u, D\varphi(s, y), s, y) R\left(\frac{D\varphi(s, y)}{|D\varphi(s, y)|}\right) G(D\varphi(s, y), D^2\varphi(s, y)) \leq 0, \quad (1.5)$$

where

$$\sigma^-(u, p, s, y) := \begin{cases} 1 & \text{if } u(s, \cdot) < u(s, y) \text{ on } H(p, y) \text{ and } p \in \mathbf{R}^N \setminus \{o\}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii). A function  $u \in LSC(\Omega)$  is called a viscosity supersolution of (1.1) in  $\Omega$  if whenever  $\varphi \in \mathcal{A}(\Omega)$ ,  $(s, y) \in \Omega$ , and  $u - \varphi$  attains a local minimum at  $(s, y)$ , then

$$\partial_t \varphi(s, y) + \sigma^+(u, D\varphi(s, y), s, y) R\left(\frac{D\varphi(s, y)}{|D\varphi(s, y)|}\right) G(D\varphi(s, y), D^2\varphi(s, y)) \geq 0. \quad (1.6)$$

(iii). A function  $u \in C(\Omega)$  is called a viscosity solution of (1.1) in  $\Omega$  if it is both a viscosity subsolution and a supersolution of (1.1) in  $\Omega$ .

Next we introduce a class of stochastic processes of which continuum limit becomes an anisotropic convexified Gauss curvature flow.

The following is an assumption on the initial set.

(A.2).  $D$  is a bounded open set in  $\mathbf{R}^N$  such that  $\text{Vol}(\partial D) = 0$ .

Take  $K > 0$  so that  $\text{co } D \subset [-K + 1, K - 1]^N$ . Put

$$\mathcal{S}_n := \{I_A : [-K, K]^N \cap (\mathbf{Z}^N/n) \mapsto \{0, 1\} \mid A \subset \mathbf{Z}^N/n\}.$$

For  $x, z \in \mathbf{Z}^N/n$  and  $v \in \mathcal{S}_n$ , put

$$v_{n,z}(x) := \begin{cases} v(x) & \text{if } x \neq z, \\ 0 & \text{if } x = z \end{cases}$$

; and for a bounded  $f : \mathcal{S}_n \mapsto \mathbf{R}$ , put

$$A_n f(v) := n^N \sum_{z \in [-K, K]^N \cap (\mathbf{Z}^N/n)} \omega_1(R, v, \{z\}) \{f(v_{n,z}) - f(v)\}.$$

Let  $\{Y_n(t, \cdot)\}_{t \geq 0}$  be a Markov process on  $\mathcal{S}_n$  ( $n \geq 1$ ), with the generator  $A_n$ , such that  $Y_n(0, z) = I_{D^c \cap (\mathbf{Z}^N/n)}(z)$ .

For  $(t, x) \in [0, \infty) \times [-K, K]^N$ , put also

$$D_n(t) := (\text{co } Y_n(t, \cdot)^{-1}(1))^o \cap D. \tag{1.7}$$

$$X_n(t, x) := I_{D_n(t)}(x). \tag{1.8}$$

Then  $\{X_n(t, \cdot)\}_{t \geq 0}$  is a stochastic process on

$$\mathcal{S} := \{f \in L^2([-K, K]^N) : \|f\|_{L^2([-K, K]^N)} \leq (2K)^N\}$$

which is a complete separable metric space by the metric

$$d(f, g) := \sum_{k=1}^{\infty} \frac{\max(|\langle f - g, e_k \rangle_{L^2([-K, K]^N)}|, 1)}{2^k}.$$

Here  $\{e_k\}_{k \geq 1}$  denotes a complete orthonormal basis of  $L^2([-K, K]^N)$ .

By definition, the following holds.

- (1)  $D_n(0) \rightarrow D$  in Hausdorff metric as  $n \rightarrow \infty$ .
- (2)  $\sum_{z \in (\mathbf{Z}^N/n) \cap [-K, K]^N} |I_{D_n(t)}(z) - I_{D_n(t-)}(z)| = 0$  or  $1$  for all  $t \geq 0$ .
- (3) If  $|I_{D_n(t)}(z) - I_{D_n(t-)}(z)| = 1$ , then  $z \in \partial(\text{co } D_n(t-))$ .
- (4)  $\sum_{z \in (\mathbf{Z}^N/n) \cap [-K, K]^N} |I_{D_n(t)}(z) - I_{D_n(t-)}(z)| = 1$  if and only if  $t = \sigma_{n,i}$  for some  $i$ , where  $0 < \sigma_{n,1} < \sigma_{n,1} < \dots$  are random variables such that  $\{\sigma_{n,i+1} - \sigma_{n,i}\}_{i > 0}$  are independent and that

$$P(\sigma_{n,i+1} - \sigma_{n,i} \in dt) = n^N \exp(-n^N t) dt.$$

- (5)  $P(I_{D_n(\sigma_{n,i})}(z) - I_{D_n(\sigma_{n,i}-)}(z) = 1) = E[\omega_1(R, I_{D_n(\sigma_{n,i}-)}, \{z\})]$ .

## 2 Main result

In this section we give our main result from [5].

The following theorem implies that  $D_n$  is a random crystalline approximation of an anisotropic convexified Gauss curvature flow.

**Theorem 1** *Suppose that (A.1)-(A.2) hold. Then there exists a unique anisotropic convexified Gauss curvature flow  $\{D(t)\}_{t \geq 0}$  with  $D(0) = D$ , and for any  $\gamma > 0$ ,*

$$\lim_{n \rightarrow \infty} P(\sup_{0 \leq t} \|X_n(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \geq \gamma) = 0. \quad (2.1)$$

Suppose in addition that  $D$  is convex. Then for any  $T \in [0, \text{Vol}(D))$  and  $\gamma > 0$ ,

$$\lim_{n \rightarrow \infty} P(\sup_{0 \leq t \leq T} d_H(D_n(t), D(t)) \geq \gamma) = 0, \quad (2.2)$$

where  $d_H$  denotes Hausdorff metric.

We introduce an additional assumption.

(A.3).  $h \in C_b(\mathbf{R}^N)$  and for any  $r \in \mathbf{R}$ , the set  $h^{-1}((r, \infty))$  is bounded or  $\mathbf{R}^N$ .

The following corollary implies that a level set of a continuous weak solution to (1.1) is determined by that at  $t = 0$ .

**Corollary 1** *Suppose that (A.1) and (A.3) hold. Then there exists a unique bounded continuous weak solution  $\{u(t, \cdot)\}_{t \geq 0}$  to (1.1) and for any  $r \in \mathbf{R}$ ,  $\{u(t, \cdot)^{-1}((r, \infty))\}_{t \geq 0}$  is a unique anisotropic convexified Gauss curvature flow with initial data  $u(0, \cdot)^{-1}((r, \infty))$ .*

We state properties of anisotropic convexified Gauss curvature flows.

**Theorem 2** *Suppose that (A.1)-(A.2) hold. Let  $\{D(t)\}_{t \geq 0}$  be a unique anisotropic convexified Gauss curvature flow  $\{D(t)\}_{t \geq 0}$  with  $D(0) = D$ . Then*

- (a)  $t \mapsto D(t)$  is nonincreasing on  $[0, \infty)$ .
- (b) For any  $t \leq T^* := \text{Vol}(D(0))$ ,

$$\text{Vol}(D(0) \setminus D(t)) = t. \quad (2.3)$$

(c) Let  $\{D_1(t)\}_{t \geq 0}$  be an anisotropic convexified Gauss curvature flow such that  $D_1(0)$  is a bounded, convex, open set which contains  $D$ . Then

$$D(t) \subset D_1(t) \quad \text{for all } t \geq 0, \quad (2.4)$$

where the equality holds if and only if  $D(0) = D_1(0)$ .

We give an additional assumption and state the result on viscosity solutions to (1.1).

(A.4).  $R \in C(S^{N-1} : [0, \infty))$ .

**Theorem 3** *Suppose that (A.2) and (A.4) hold. Let  $\{D(t)\}_{t \geq 0}$  be a unique anisotropic convexified Gauss curvature flow  $\{D(t)\}_{t \geq 0}$  with  $D(0) = D$ . Then  $I_{D(t)}(x)$  and  $I_{D(t)^-}(x)$  are a viscosity supersolution and a viscosity subsolution to (1.1), respectively.*

The following results imply that  $u \in C_b([0, \infty) \times \mathbf{R}^N)$  is a weak solution to (1.1) if and only if it is a viscosity solution to (1.1), in case  $R$  is constant.

**Corollary 2** *Suppose that (A.3)-(A.4) hold. Then a unique weak solution  $u \in C_b([0, \infty) \times \mathbf{R}^N)$  to (1.1) is a viscosity solution to it.*

**Corollary 3** *(see [4]) Suppose that (A.3) holds and that  $R$  is constant. Then a continuous viscosity solution to (1.1) is unique and is a weak solution to it.*

### 3 Sketch of Proof

(Idea of Proof of Theorem 1). We first show that  $\{X_n(t, \cdot)\}_{t \geq 0}$  is tight in  $D([0, \infty) : \mathcal{S})$ . By the weak convergence result on  $\omega_1$  by Bakelman [1], we show that any weak limit point of  $\{X_n(t, \cdot)\}_{t \geq 0}$  is a weak solution to (1.1).

The following lemma implies the uniqueness, and hence completes the proof.

**Lemma 1** *Suppose that (A.1) hold. If  $\{I_{D_i(t)}\}_{t \geq 0}$  ( $i = 1, 2$ ) are weak solutions to (1.1) for which  $D_1(0) \subset D_2(0)$ , then  $D_1(t) \subset D_2(t)$  for all  $t \geq 0$ . In particular,*

$$d(D_1(t), D_2(t)^c) \geq d(D_1(0), D_2(0)^c), \quad (3.1)$$

for  $t \leq \text{Vol}(D_1(0))$ .

(Sketch of Proof of Corollary 1). For  $r \in \mathbf{R}$ , let  $\{I_{D_r(t)}\}_{t \geq 0}$  denote a unique weak solution of (1.1) with  $D_r(0) = h^{-1}((r, \infty))$ .

Put

$$u(t, x) := \sup\{r \in \mathbf{R} \mid x \in D_r(t)\}.$$

Then  $u$  is continuous. In particular, for all  $t \geq 0$  and  $r \in \mathbf{R}$ ,

$$u(t, \cdot)^{-1}((r, \infty)) = D_r(t).$$

For  $n \geq 1$ , put  $k_{n,1} := [n \sup\{h(y) \mid y \in \mathbf{R}^N\}]$  and  $k_{n,0} := [n \inf\{h(y) \mid y \in \mathbf{R}^N\}]$ . Then for any  $\varphi \in C_o(\mathbf{R}^N)$  and any  $t \geq 0$ ,

$$\begin{aligned}
& \int_{\mathbf{R}^N} \varphi(x) \left[ \sum_{k_{n,0} \leq k \leq k_{n,1}} \frac{k}{n} (I_{D_{\frac{k}{n}}(t)^c}(x) - I_{D_{\frac{k+1}{n}}(t)^c}(x)) \right. \\
& \quad \left. - \sum_{k_{n,0} \leq k \leq k_{n,1}} \frac{k}{n} (I_{D_{\frac{k}{n}}(0)^c}(x) - I_{D_{\frac{k+1}{n}}(0)^c}(x)) \right] dx \\
&= \int_0^t ds \left[ \sum_{k_{n,0} < k \leq k_{n,1}} \frac{1}{n} \int_{\mathbf{R}^N} \varphi(x) \omega_0(R, I_{D_{\frac{k}{n}}(s)^c}(\cdot), dx) \right].
\end{aligned}$$

Letting  $n \rightarrow \infty$  in (3.4),  $u$  is shown to be a weak solution to (1.1).

The uniqueness of  $u$  follows from that of  $D_r(\cdot)$  for all  $r$ . In fact we can show that for a continuous weak solution  $v$  to (1.1),  $\{v(t, \cdot)^{-1}((r, \infty))\}_{t \geq 0}$  is an anisotropic convexified Gauss curvature flow.

Q. E. D.

We omit the proof of Theorems 2 and 3.

(Sketch of Proof of Corollary 2)

Let  $u$  be a weak solution to (1.1).

We first show that  $u$  is a viscosity supersolution to (1.1). Suppose that  $u$  is smooth in  $\Omega$  and that  $\varphi \in \mathcal{A}(\Omega)$ ,  $(s, y) \in \Omega$ , and  $u - \varphi$  attains a local maximum at  $(s, y)$ . Then

$$\partial_s(u - \varphi)(s, y) \geq 0.$$

$$\partial_s(u - \varphi^\varepsilon)(s, y) \geq 0$$

( $\varepsilon > 0$ ), where  $\varphi^\varepsilon := \varphi - \varepsilon$ . Hence formally, we have, in some neighborhood of  $(s, y)$ ,

$$\begin{aligned}
& \partial_s \varphi^\varepsilon(t, x) \\
& \leq \partial_s u(t, x) = -\mathbf{w}(u(t, \cdot), dx)/dx \\
& \leq -\mathbf{w}(\varphi^\varepsilon(t, \cdot), dx)/dx = -R\left(\frac{D\varphi(t, x)}{|D\varphi(t, x)|}\right)G(D\varphi(t, x), D^2\varphi(t, x)).
\end{aligned}$$

Then following lemma completes the proof.

Take  $\varphi \in C^2(\mathbf{R}^N : \mathbf{R})$  for which  $D\varphi(x_o) \neq 0$  for some  $x_o \in \mathbf{R}^N$ . For  $i = 1, \dots, N$ , put

$$y_i(x) := \left( -(1 - \delta_{ij}) \frac{\partial_j \varphi(x)}{|D\varphi(x)|} + \delta_{ij} \varphi(x) \right)_{j=1}^N.$$

Then

**Lemma 2** *Suppose that all eigenvalues of  $-D(D\varphi(x_o)/|D\varphi(x_o)|)$  are non-negative. Then, for  $i = 1, \dots, N$ ,*

$$\frac{\partial_i \varphi(x_o)}{|D\varphi(x_o)|} G(D\varphi(x_o), D^2\varphi(x_o)) = \det(Dy_i(x_o)). \quad (3.2)$$

Similarly one can show that  $u$  is a viscosity subsolution to (1.1).

Q. E. D.

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