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CR submanifolds in holomorphic statistical manifolds

(正則統計多様体のＣＲ部分多様体論)

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ABSTRACT

CR submanifolds in Kähler geometry are introduced by A. Bejancu in 1978 as a generalization of both complex submanifolds and totally real submanifolds. Works of K. Yano and M. Kon ([30]) and A. Bejancu (1986) gather the significant results on this topic. M. Djoric and M. Okumura have been intensively investigated CR submanifolds in the complex projective space, with focus on the case when CR submanifolds are of maximal CR dimension. They generalized many important results on real hypersufaces (Y. Tashiro and S. Tachibana in [24], R. Takagi in [26], among others), which are typical examples of CR submanifolds of maximal CR dimension. Soon after CR submanifolds in Kaehler manifolds are introduced, their research is extended to other ambient spaces. There is a large amount of literature when the ambient manifold is a nearly Kaehler or locally conformal Kaehler manifold. We initiate investigation of CR submanifolds in holomorphic statistical manifolds, which are new objects originating from information geometry. Statistical manifolds may be considered as manifolds consisting of certain probability density functions. They are geometrically formulated as Riemannian manifolds with a certain affine connection. Their complex version, i.e. holomorphic statistical manifolds are defined by T. Kurose in 2004.

After obtaining the fundamental equations for CR submanifolds in holomorphic statistical manifolds, we investigate CR submanifolds with umbilical shape operators, semi-parallel totally real submanifolds and real hypersurfaces. We show that the results obtained by O. Kassabov ([14]), M. Djoric and M. Okumura ([8]) and Y. Tashiro and S. Tachibana ([24]) hold in the theory of CR submanifolds in holomorphic statistical manifolds, by finding the corresponding conditions to the ones in the original theory. Our main results are the following.

When M is a CR submanifold of maximal CR dimension in a holomorphic statistical manifold, we naturally get a special normal vector field, which is called the distinguished normal vector field of M. Accordingly, in this setting we have two shape operators A, A*, which are the shape operator in the distinguished normal vector field direction with respect to the affine connection of the ambient space, and the one with respect to the dual connection. Let I denote the identity operator of the tangent space of M. We then obtain

Theorem. *Let M be an n-dimensional CR submanifold of maximal CR dimension in an $(n+p)$-dimensional holomorphic statistical manifold of constant holomorphic sectional curvature c, and let $p < n$. If the shape operators of the*

distinguished normal vector field of M are given as $A = \alpha I$, $A^* = \beta I$ for functions α and β, then $c = 0$.

Real hypersurfaces are typical examples of CR submanifolds of maximal CR dimension. When M is a real hypersurface, the distinguished normal vector field is a unit normal vector field to a hypersurface.

On the other hand, we obtain the following on semi-parallel totally real submanifolds of constant curvature. Let (∇, g) be the statistical structure on such a submanifold induced from the ambient space, and f the f-structure of the normal bundle induced from the complex structure of the ambient space. Let D and D^* be the normal connections with respect to the affine connection of the ambient space and the dual connection, respectively.

Theorem. *Let (M', ∇', g', J) be a holomorphic statistical manifold and M a totally real submanifold of M'. Suppose:*
(1) $D_X(fV) = fD^*_X V$ for any tangent vector field X and normal vector field V of M.
(2) (∇, g) is of constant curvature $c \neq 0$.
If M is semi-parallel for ∇', then M is totally geodesic for ∇'.

Finally, we construct holomorphic statistical structures on a domain of $\mathbb{R}^4$, dependent on eight functions, using a g-natural metric. Moreover, holomorphic statistical manifolds of constant holomorphic sectional curvature are also constructed.

1. INTRODUCTION

An n-dimensional submanifold M of a complex manifold $\overline{M}$ with complex structure J is said to be Cauchy-Riemannian, or CR, if on it there exists distributions $\mathfrak{D} := TM \cap JTM$ and $\mathfrak{D}^{\perp} := \{v \in T_xM |\ g(v,\ w) = 0,\ w \in \mathfrak{D}_x\}$ such that $J_x\mathfrak{D}_x^{\perp} \subset T_x^{\perp}M$ for any $x \in M$. The constant complex dimension of D_xM is called the CR dimension of M. A CR submanifold M is said to be of maximal CR dimension if $\dim H_xM = \frac{n-1}{2}$ ([8]), and totally real if $\dim H_xM = 0$. Typical examples of CR submanifolds of maximal CR dimension are real hypersurfaces. We now consider the case that $\overline{M}$ is a Kähler manifold, in particular, a complex space form. In 1963, Tashiro and Tachibana proved the nonexistence of umbilical hypersurfaces in a complex projective space $CP^{\overline{m}}$ and in a complex hyperbolic space $CH^{\overline{m}}$ ([24]). In $CP^{\overline{m}}$ it has been shown that for the shape operator A of a real hypersurface $\|\nabla A\| \geq 4(n-1)$, from which it follows the nonexistence of real hypersurfaces with parallel shape operator ([17]). The classification of homogeneous real hypersurfaces is obtained by Takagi in 1973 ([25]). There are five types of these hypersurfaces and they can have two, three or five principal curvatures ([26]). The structure vector U, defined by $U = -J\xi$, where ξ is the unit normal vector field to a hypersurface, is very important in extrinsic geometry of hypersurfaces. When U is an eigenvector of the shape operator A, it is called a Hopf hypersurface. In this case, Maeda proved that the eigenvalue of A corresponding to U is constant ([17]).
The results on real hypersurfaces are generalized to CR submanifolds of maximal CR dimension by Djorić and Okumura, among others. For example, in [8], it has been proven that if the shape operator with respect to the distinguished normal vector field of a CR submanifold of maximal CR dimension M has exactly two eigenvalues, then there exists a geodesic hypersphere S of CP^n such that M lies on S.

A submanifold is said to be semi-parallel, if

$$\begin{aligned}(R^{\widetilde{\nabla}^{\overline{g}}}(X,\ Y)h^{\overline{g}})(Z,\ W) &:= (\widetilde{\nabla}^{\overline{g}}_X(\widetilde{\nabla}^{\overline{g}}_Y h^{\overline{g}}))(Z,\ W) - (\widetilde{\nabla}^{\overline{g}}_Y(\widetilde{\nabla}^{\overline{g}}_X h^{\overline{g}}))(Z,\ W)\\ &\quad - (\widetilde{\nabla}^{\overline{g}}_{[X,\ Y]}h^{\overline{g}})(Z,\ W)\\ &= 0\end{aligned}$$

for tangent vector fields X, Y, Z, W of M. Here $\widetilde{\nabla}^{\overline{g}}$ denotes the connection of van der Waerden-Bortolotti. Semi-parallel hypersurfaces of a Euclidean space have been classified by Deprez ([10]). On the other hand, Kassabov investigated totally real semi-parallel submanifolds in Kählerian manifolds in 1986

([14]). Kassabov proved that these submanifolds of constant curvature are totally geodesic or flat.

We research totally real submanifolds, real hypersurfaces and CR submanifolds of maximal CR dimension in holomorphic statistical manifolds, which are new objects originating from information geometry. The field of information theory is based on the works of Amari ([3]), and it is used in fields such as statistical inference, neural networks and control systems. Statistical manifolds may be considered as manifolds consisting of certain probability density functions. These manifolds have applications in document classification, face recognition, image analysis, clustering and so on. Mathematically, a statistical manifold is defined as a triple $(\overline{M},\ \overline{\nabla},\ \overline{g})$, where $\overline{M}$ is a C^∞-manifold, $\overline{\nabla}$ is a connection of torsion free, and $\overline{g}$ is a Riemannian metric satisfying the Codazzi equation (see Definition 6). Furthermore, a quadruple $(\overline{M},\ \overline{\nabla},\ \overline{g},\ J)$ is called a holomorphic statistical manifold if the triple $(\overline{M},\ \overline{\nabla},\ \overline{g})$ is a statistical manifold and the Kähler form for $\overline{g}$ and J is $\overline{\nabla}$-parallel (see Definition 12). On a statistical manifold the dual connection $\overline{\nabla}^*$ of $\overline{\nabla}$ with respect to $\overline{g}$ is naturally obtained by $X\overline{g}(Y,Z) = \overline{g}(\overline{\nabla}_X Y, Z) + \overline{g}(Y, \overline{\nabla}^*_X Z)$. Accordingly, for submanifolds in statistical manifolds we have pairs of induced connections ∇, ∇^*, second fundamental forms h, h^*, shape operators A, A^*, and normal connections D, D^* satisfying equations analogous to the Gauss and the Weingarten ones for $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively, though the induced metric g is unique. We want to see whether the results in the classical submanifold theory, for example Kassabov's theorems and the results of Djorić and Okumura, mentioned above, hold for our setting or not. It appears that, if the ambient space is a holomorphic statistical manifold, the condition that the induced f-structure f is parallel in the normal bundle with respect to $\overline{\nabla}$ or $\overline{\nabla}^*$ could not give the expected results. Therefore, we needed to find the corresponding condition in our setting to the condition $D^{\overline{g}} f = 0$ in Kähler manifolds. Since in our setting we have dual connections $\overline{\nabla}$ and $\overline{\nabla}^*$, we give the definitions of semi-parallel submanifold with respect to $\overline{\nabla}$ ($\overline{\nabla}^*$) and of totally geodesic submanifold with respect to $\overline{\nabla}$ ($\overline{\nabla}^*$). In [18], the author proves that totally real semi-parallel submanifolds with respect to $\overline{\nabla}$ ($\overline{\nabla}^*$) are totally geodesic with respect to $\overline{\nabla}$ ($\overline{\nabla}^*$) or flat (Theorem 6). Furthermore, after obtaining the fundamental equations for CR submanifolds of maximal CR dimension in holomorphic statistical manifolds, we research umbilical shape operators A and A^* of the distinguished normal vector field. In this case we obtain that the ambient manifold is flat (Theorem 8).

The notion of a g natural metric is introduced in [15]. One of the examples of g-natural metrics is the Cheeger-Gromoll metric, proposed by Musso and Tricerri in [21]. Abassi and Sarih proved that every g-natural metric G on TM has the following properties: if $(TM,\ G)$ is flat, or locally symmetric, or of constant sectional curvature, or of constant scalar curvature, or an Einstein manifold, respectively, then $(M,\ g)$ possesses the same property, respectively ([1]). In [23], Oproiu considered Kähler structures defined on TM of a Rieman-

nian manifold $(M,\ g)$ of constant sectional curvature obtained by adapting an idea of Calabi ([5]), to define a hyperKähler structure on the cotangent bundle of a Kähler manifold having constant positive holomorphic sectional curvature. The found structures are defined by functions on TM, depending explicitly on the energy density (kinetic energy) on TM, only. We use Oproiu's Kählerian structures on TM to construct holomorphic statistical structures on $\mathbb{R}^4$, and show that these structures are dependent on eight functions. After that, by choosing eight functions properly, we construct examples of holomorphic statistical manifolds of constant holomorphic sectional curvature and of totally real submanifolds.

The thesis is organized as follows. In Chapter 2 we review the theory of Riemannian submanifolds, Kähler manifolds and CR submanifolds. Also, we give fundamental equations for statistical submanifolds ([27]). In Chapter 3 we exhibit the results on real hypersurfaces ([22]), CR submanifolds of maximal CR dimension ([8]) and on totally real submanifolds ([14]) in Kählerian manifolds. In Chapter 4 the definitions and the properties of holomorphic statistical manifolds are given ([12]). We then prove the original results on semi-parallel totally real statistical submanifolds ([18]), and on CR statistical submanifolds ([19]). A g-natural metric is used in Chapter 5 for the construction of four dimensional holomorphic statistical manifolds ([18]). In the end we give examples of totally real statistical submanifolds ([18]) and of holomorphic statistical manifolds of constant holomorphic sectional curvature ([19]).

2. PRELIMINARIES

2.1 Riemannian submanifolds

Let $\overline{M}$ be an $\overline{m}$ dimensional C^∞-manifold. We denote by $T_p\overline{M}$ the tangent space at a point $p \in \overline{M}$, and by $T\overline{M}$ the tangent bundle over $\overline{M}$. By $\Gamma(E)$ we denote the set of all C^∞ sections of a vector bundle $E \to \overline{M}$. For example, $\Gamma(T\overline{M}^{(p,\,q)})$ means the set of tensor fields of type $(p,\,q)$ on $\overline{M}$.

Definition 1. *(i) A connection or a covariant derivative on $\overline{M}$ is an operator $\overline{\nabla} : \Gamma(T\overline{M}) \times \Gamma(T\overline{M}) \to \Gamma(T\overline{M})$ that satisfies the following axioms:*
(1) $\overline{\nabla}_{X_1 + X_2}Y = \overline{\nabla}_{X_1}Y + \overline{\nabla}_{X_2}Y$.
(2) $\overline{\nabla}_X(Y_1 + Y_2) = \overline{\nabla}_X Y_1 + \overline{\nabla}_X Y_2$.
(3) $\overline{\nabla}_{fX}Y = f\overline{\nabla}_X Y$, where $f : \overline{M} \to \mathbb{R}$ is any smooth function.
(4) $\overline{\nabla}_X(fY) = (Xf)Y + f\overline{\nabla}_X Y$, where $f : \overline{M} \to \mathbb{R}$ is any smooth function.
(ii) A connection $\overline{\nabla}$ is said to be of torsion free if the torsion tensor, $T^{\overline{\nabla}}(X,\,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,\,Y]$ equals zero.

Definition 2. *For a connection $\overline{\nabla}$ the curvature tensor $\overline{R}$ is defined as*

$$\overline{R}(X,\,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,\,Y]}Z, \quad X,\,Y,\,Z \in \Gamma(T\overline{M}).$$

On a Riemannian manifold $(\overline{M},\,\overline{g})$ there exists the *Levi-Civita* connection, denoted by $\overline{\nabla}^{\overline{g}}$, which is the unique connection of torsion free compatible with the Riemannian metric $\overline{g}$, i.e. $\overline{\nabla}^{\overline{g}}_X\overline{g} = 0$ for all $X \in \Gamma(T\overline{M})$. The curvature tensor of the Levi-Civita connection $\overline{\nabla}^{\overline{g}}$ is called the *Riemannian curvature tensor*, and it is denoted by $R^{\overline{g}}$. We recall that the Riemannian curvature tensor has the following properties:

$$\overline{R}^{\overline{g}}(X,\,Y) = \overline{R}^{\overline{g}}(Y,\,X), \tag{2.1}$$

$$\overline{g}(\overline{R}^{\overline{g}}(X,\,Y)Z,\,U) = -\,\overline{g}(\overline{R}^{\overline{g}}(X,\,Y)U,\,Z), \tag{2.2}$$

$$\overline{g}(\overline{R}^{\overline{g}}(X,\,Y)Z,\,U) = \overline{g}(\overline{R}^{\overline{g}}(U,\,Z)Y,\,X), \tag{2.3}$$

$$\overline{R}^{\overline{g}}(X,\,Y)Z + \overline{R}^{\overline{g}}(Y,\,Z)X + \overline{R}^{\overline{g}}(Z,\,X)Y = 0. \tag{2.4}$$

$$(\overline{\nabla}^{\overline{g}}_X \overline{R}^{\overline{g}})(Y,\ Z)\ +\ (\overline{\nabla}^{\overline{g}}_Y \overline{R}^{\overline{g}})(Z,\ X)\ +\ (\overline{\nabla}^{\overline{g}}_Z \overline{R}^{\overline{g}})(X,\ Y)\ =\ 0. \tag{2.5}$$

A Riemannian manifold $(\overline{M},\ \overline{g})$ is said to be of *constant curvature* $c\ \in\ \mathbb{R}$ if

$$R^{\overline{g}}(X,\ Y)Z\ =\ c\{\overline{g}(Y,\ Z)X\ -\ \overline{g}(X,\ Z)Y\}. \tag{2.6}$$

Let M be an m-dimensional manifold and $\iota :\ M\ \to\ \overline{M}$ an immersion, i.e. $\iota_{*p}\ =\ d\iota_p :\ T_pM\ \to\ T_{\iota(p)}\overline{M}$ is injective for each point $p\ \in\ M$. On the submanifold M there is an induced metric g defined by $g\ :=\ \iota^*\overline{g}$, where $\iota^*\overline{g}$ denotes the pullback on M defined by $(\iota^*g)(v,\ w)\ =\ \overline{g}(d\iota(v),\ d\iota(w))$ for $u,\ v\ \in\ T_pM$.
We denote by ∇^g the Levi-Civita connection on M for the metric g. The connection ∇^g is the tangential component of the connection $\overline{\nabla}^{\overline{g}}$ when applied to vector fields in $\Gamma(TM)$, that is we have the following *Gauss equation*:

$$\overline{\nabla}^{\overline{g}}_X Y\ =\ \nabla^g_X Y\ +\ h^{\overline{g}}(X,\ Y), \tag{2.7}$$

for $X,\ Y\ \in\ \Gamma(TM)$. The normal part $h^{\overline{g}}(X,\ Y)$ to TM defines the tensor field h called the second fundamental form of M in $\overline{M}$. By $T^{\perp}_pM$ we denote the normal space of M at p, i.e. $T^{\perp}_pM\ =\ \{v\ \in\ T_p\overline{M}|\ \overline{g}(v,\ w)\ =\ 0,\ w\ \in\ T_pM\}$, and by $T^{\perp}M$ the normal bundle to M. For $V\ \in\ \Gamma(T^{\perp}M)$ and $X\ \in\ \Gamma(TM)$ we have the following Weingarten equation:

$$\overline{\nabla}^{\overline{g}}_X V\ =\ -A^{\overline{g}}_V X\ +\ D^{\overline{g}}_X V, \tag{2.8}$$

where the first term is tangential to M and defines the shape operator $A^{\overline{g}}$, which is related to the second fundamental form by

$$g(A^{\overline{g}}_V X,\ Y)\ =\ \overline{g}(h^{\overline{g}}(X,\ Y),\ V). \tag{2.9}$$

The *mean curvature* vector field is defined by

$$H\ :=\ \frac{1}{m} tr_g h^{\overline{g}}, \tag{2.10}$$

where tr_g denotes the trace with respect to g.
A submanifold M is called *minimal* if $H\ =\ 0$. The Gauss and Codazzi equations are given by:

$$\begin{aligned} g(R^g(X,\ Y)Z,\ W)\ &=\ \overline{g}(R^{\overline{g}}(X,\ Y)Z,\ W)\ +\ \overline{g}(h^{\overline{g}}(X,\ W),\ h^{\overline{g}}(Y,\ Z)) \\ &\quad -\ \overline{g}(h^{\overline{g}}(X,\ Z),\ h^{\overline{g}}(Y,\ W)), \end{aligned} \tag{2.11}$$

$$\{R^{\overline{g}}(X,\ Y)Z\}^{\perp}\ =\ (\widetilde{\nabla}^{\overline{g}}_X h^{\overline{g}})(Y,\ Z)\ -\ (\widetilde{\nabla}^{\overline{g}}_Y h^{\overline{g}})(X,\ Z), \tag{2.12}$$

respectively. Here $X,\ Y,\ Z,\ W\ \in\ \Gamma(TM)$, $\{R^{\overline{g}}(X,\ Y)Z\}^{\perp}$ denotes the normal part of $R^{\overline{g}}(X,\ Y)Z$ for M, and

$$(\widetilde{\nabla}^{\overline{g}}_X h^{\overline{g}})(Y,\ Z)\ :=\ D^{\overline{g}}(h^{\overline{g}}(Y,\ Z))\ -\ h^{\overline{g}}(\nabla^g_X Y,\ Z)\ -\ h^{\overline{g}}(Y,\ \nabla^g_X Z). \tag{2.13}$$

A Riemannian submanifold M is called *totally geodesic* in $\overline{M}$ if every geodesic of M is a geodesic of $\overline{M}$. Equivalently, M is a totally geodesic submanifold in $\overline{M}$ if $h^{\overline{g}}\ =\ 0$.

2.2 Kähler manifolds

The map $J \in \Gamma(T\overline{M}^{(1,\ 1)})$ is called an almost complex structure of $\overline{M}$ if $J^2 = -\ Id$. A differentiable manifold with an almost complex structure is called an almost complex manifold. Moreover, it is called a complex manifold if the almost complex structure is integrable. A complex manifold is even dimensional. If the almost structure is compatible with the Riemannian metric, i. e.

$$\overline{g}(X,\ Y) \ =\ \overline{g}(JX,\ JY), \tag{2.14}$$

for any $X,\ Y\ \in\ \Gamma(T\overline{M})$, $\overline{g}$ is called a *Hermitian metric* and the complex manifold $(\overline{M},\ J)$ with the Hermitian metric $\overline{g}$ is called a Hermitian manifold. By (2.14), J is skew-symmetric, that is

$$\overline{g}(JX,\ Y) \ =\ \overline{g}(J^2X,\ JY) \ =\ -\overline{g}(X,\ JY).$$

We recall that on any complex manifold there exists a Hermitian metric given by:

$$\overline{g}(X,\ Y) \ =\ \frac{1}{2}\{\overline{g}'(X,\ Y)\ +\ \overline{g}'(JX,\ JY)\},$$

where $\overline{g}'$ is the Riemannian metric on the manifold. The *Kähler form* $\overline{\omega}$ of a Hermitian manifold $(\overline{M},\ \overline{g},\ J)$ is defined by:

$$\overline{\omega}(X,\ Y) \ =\ \overline{g}(X,\ JY),$$

for $X,\ Y\ \in\ \Gamma(T\overline{M})$. If ω is closed, i. e. $d\omega\ =\ 0$, then $\overline{M}$ is called a *Kähler manifold* and the metric $\overline{g}$ is called a *Kähler metric*. A Hermitian manifold $\overline{M}$ is a Kähler manifold if and only if $\overline{\nabla}^{\overline{g}}_X J\ =\ 0$ for $X\ \in\ \Gamma(T\overline{M})$.

Definition 3. *(1) The holomorphic sectional curvature $H(X)$ of a Kähler manifold is defined by:*

$$H(X) \ =\ K_{X,\ JX} \ =\ \frac{\overline{g}(R^{\overline{g}}(X,\ JX)JX,\ X)}{\overline{g}(X,\ X)^2}.$$

(2) A Kähler manifold $\overline{M}$ is called a complex space form if it has constant holomorphic sectional curvature, namely, if $H(X)$ is a constant for all $X\ \in\ T_x\overline{M}$, $x\ \in\ \overline{M}$.

Theorem 1. *(cf. [8]) The Riemannian curvature tensor $R^{\overline{g}}$ of a complex space form is given by:*

$$\begin{aligned} R^{\overline{g}}(X,\ Y)Z \ =\ c\{&\overline{g}(Y,\ Z)X\ -\ \overline{g}(X,\ Z)Y\ +\ \overline{g}(JY,\ Z)JX \\ &-\ \overline{g}(JX,\ Z)JY\ -\ 2\overline{g}(JX,\ Y)JZ\}, \end{aligned} \tag{2.15}$$

where $c\ =\ \frac{H(X)}{4}$.

Proof. Suppose that $H(X) = 4c$. Then, by Definition 3, we have

$$\overline{g}(R^{\overline{g}}(X,\ JX)JX,\ X) = 4c\overline{g}(X,\ X)^2, \quad X \in \Gamma(T\overline{M}),$$

from which it follows

$$\sum_P \overline{g}(R^{\overline{g}}(X,\ JY)JZ,\ W) = 4c\sum_P \overline{g}(X,\ Y)\overline{g}(Z,\ W), \tag{2.16}$$

where $\sum_P$ denotes the sum of all permutations with respect to $X,\ Y,\ Z,\ W \in \Gamma(T\overline{M})$. Since $\overline{\nabla}^{\overline{g}}_X J = 0$, we have

$$R^{\overline{g}}(X,\ Y)JZ = JR^{\overline{g}}(X,\ Y)Z.$$

Hence,

$$\begin{aligned}\overline{g}(R^{\overline{g}}(X,\ Y)JZ,\ W) &= \overline{g}(JR^{\overline{g}}(X,\ Y)Z,\ W)\\ &= -\overline{g}(R^{\overline{g}}(X,\ Y)Z,\ JW) = \overline{g}(R^{\overline{g}}(X,\ Y)JW,\ Z).\end{aligned} \tag{2.17}$$

Now, relation (2.16) and repeated application of the property (2.3) and relation (2.17) imply

$$\begin{aligned}&\overline{g}(R^{\overline{g}}(X,\ JY)JW,\ Z) + \overline{g}(R^{\overline{g}}(X,\ JZ)JY,\ W) + \overline{g}(R^{\overline{g}}(X,\ JW)JY,\ Z)\\ &= 4c\{\overline{g}(X,\ Y)\overline{g}(Z,\ W) + \overline{g}(X,\ Z)\overline{g}(Y,\ W) + \overline{g}(X,\ W)\overline{g}(Y,\ Z)\}.\end{aligned}$$

Substituting JY and JW for Y and W in the above equation, respectively, we get

$$\begin{aligned}&\overline{g}(R^{\overline{g}}(X,\ Y)W,\ Z) - \overline{g}(R^{\overline{g}}(X,\ JZ)Y,\ JW) + \overline{g}(R^{\overline{g}}(X,\ W)Y,\ Z)\\ &= 4c\{\overline{g}(X,\ JY)\overline{g}(Z,\ JW) + \overline{g}(X,\ Z)\overline{g}(Y,\ W) + \overline{g}(X,\ JW)\overline{g}(JY,\ Z)\}.\end{aligned}$$

Taking the skew symmetric part of this equation with respect to Z and W, and using the Bianchi identity (2.4), we obtain

$$\begin{aligned}&2\overline{g}(R^{\overline{g}}(X,\ Y)W,\ Z) + \overline{g}(R^{\overline{g}}(X,\ JZ)JW,\ Y) - \overline{g}(R^{\overline{g}}(X,\ JW)JZ,\ Y)\\ &+ \overline{g}(R^{\overline{g}}(X,\ W)Y,\ Z) + \overline{g}(R^{\overline{g}}(X,\ Z)W,\ Y)\\ &= 4c\{-2\overline{g}(X,\ JY)\overline{g}(JZ,\ W) - \overline{g}(X,\ W)\overline{g}(Y,\ Z) + \overline{g}(X,\ Z)\overline{g}(Y,\ W)\\ &- \overline{g}(X,\ JZ)\overline{g}(JY,\ W) + \overline{g}(X,\ JW)\overline{g}(JY,\ Z)\}\end{aligned} \tag{2.18}$$

On the other hand, from (2.17) it follows

$$\begin{aligned}&\overline{g}(R^{\overline{g}}(X,\ JZ)JW,\ Y) - \overline{g}(R^{\overline{g}}(X,\ JW)JZ,\ Y)\\ &= \overline{g}(R^{\overline{g}}(X,\ JZ)JW,\ Y) + \overline{g}(R^{\overline{g}}(X,\ JW)Y,\ JZ)\\ &= -\overline{g}(R^{\overline{g}}(X,\ Y)JZ,\ JW) = -\overline{g}(JR^{\overline{g}}(X,\ Y)Z,\ JW)\\ &= -\overline{g}(R^{\overline{g}}(X,\ Y)Z,\ W).\end{aligned} \tag{2.19}$$

Using (2.19), relation (2.18) becomes

$$\begin{aligned}\overline{g}(R^{\overline{g}}(X,\ Y)Z,\ W)\ =\ c\{&\overline{g}(X,\ W)\overline{g}(Y,\ Z)\ +\ \overline{g}(X,\ Z)\overline{g}(Y,\ W)\\ &+\ \overline{g}(JY,\ Z)\overline{g}(JX,\ W)\ -\ \overline{g}(JX,\ Z)\overline{g}(JY,\ W)\\ &-\ 2\overline{g}(JX,\ Y)\overline{g}(JZ,\ W)\},\end{aligned}\tag{2.20}$$

that is, the Riemannian curvature tensor $R^{\overline{g}}$ of a complex space form is given by (2.15). □

Any complex space form of holomorphic sectional curvature c is locally isometric to one of the following spaces:
(1) Complex Euclidean space $\mathbb{C}$ ($c\ =\ 0$),
(2) Complex projective space CP^n ($c\ >\ 0$),
(3) Complex hyperbolic space CH^n ($c\ <\ 0$).
For more detailed reading we refer to [16].

2.3 CR submanifolds in Kähler manifolds

A submanifold M of a Kähler manifold $(\overline{M},\ \overline{g},\ J)$ is said to be *totally real* if $J(TM)$ is orthogonal to TM. A totally real submanifold M of $\overline{M}$ is called *Lagrangian* if $2\dim M\ =\dim\overline{M}$. There is considerable literature on the existence and properties of such submanifolds in Kählerian manifolds. Here we give some results obtained by Yano and Kon in 1983 (cf. [28]), and by Kassabov in 1986 (cf. [14]).
We decompose J into tangential and normal parts with respect to the submanifold M as follows:

$$JX\ =\ PX\ +\ FX,\quad X\ \in\ \Gamma(TM),\tag{2.21}$$

$$JV\ =\ tV\ +\ fV,\quad V\ \in\ \Gamma(T^{\perp}M).\tag{2.22}$$

Here, $P\ \in\ \Gamma(EndTM^{(1,\ 1)})$, $F\ \in\ \Gamma((TM)^*\ \otimes\ T^{\perp}M)$,
$t\ \in\ \Gamma((T^{\perp}M)^*\ \otimes\ TM)$, and $f\ \in\ \Gamma(EndT^{\perp}M)$.

Proposition 1. *Let M be a submanifold of a Kähler manifold $\overline{M}$, and P, F, t and f defined by (2.21) and (2.22). Then*

$$\overline{g}(FX,\ V)\ =\ -g(X,\ tV),\tag{2.23}$$

$$P^2\ =\ -Id_{TM}\ -\ tF,\quad FP\ +\ fF\ =\ 0,\tag{2.24}$$

$$Pt\ +\ tf\ =\ 0,\quad f^2\ =\ -Id_{TM}\ -\ Ft.\tag{2.25}$$

Proof. For $X\ \in\ \Gamma(TM)$ and $V\ \in\ \Gamma(T^{\perp}M)$ we have

$$\overline{g}(JX,\ V)\ =\ \overline{g}(PX\ +\ FX,\ V)\ =\ \overline{g}(FX,\ V).\tag{2.26}$$

On the other hand,

$$\overline{g}(JX,\ V)\ =\ -\overline{g}(X,\ JV)\ =\ -\overline{g}(X,\ tV\ +\ fV)\ =\ -g(X,\ tV). \tag{2.27}$$

From (2.26) and (2.27) we get (2.23). Now we calculate,

$$-X\ =\ J^2X\ =\ J(PX\ +\ FX)\ =\ P^2X\ +\ FPX\ +\ tFX\ +\ fFX. \tag{2.28}$$

Comparing the tangential and the normal parts in (2.28), we get (2.24). Similarly, using $-V\ =\ J^2V$, we get (2.25). □

Definition 4. *Let M be a submanifold of a Kähler manifold $\overline{M}$.*
(1) We define:

$$\begin{aligned}
\mathfrak{D}_x\ &:=\ T_xM\ \cap\ J_x(T_xM),\quad x\ \in\ M,\\
\mathfrak{D}_x^{\perp}\ &:=\ \{v\ \in\ T_xM|\ g(v,\ w)\ =\ 0,\ w\ \in\ \mathfrak{D}_x\},\\
N_x\ &:=\ \{\xi\ \in\ T_x^{\perp}M|\ \overline{g}(\xi,\ J_xw)\ =\ 0,\ w\ \in\ T_xM\}.
\end{aligned}$$

(2) M is called a CR submanifold of $\overline{M}$ if
(i) $\mathfrak{D}$ defines a C^∞ distribution on M,
(ii) $J_x\mathfrak{D}_x^{\perp}\ \subset\ T_x^{\perp}M,\quad x\ \in\ M$.

From Definition 4 it follows that the tangent space $T_x\overline{M}$ splits as:

$$T_x\overline{M}\ =\ T_xM\ \oplus\ T_x^{\perp}M\ =\ (\mathfrak{D}_x\ \oplus\ \mathfrak{D}_x^{\perp})\ \oplus\ (J\mathfrak{D}_x^{\perp}\ \oplus\ N_x).$$

Proposition 2. *Let M be a CR submanifold of a Kähler manifold $\overline{M}$. For $X\ \in\ \Gamma(TM)$ and $V\ \in\ \Gamma(T^{\perp}M)$ we have:*
(1) $X\ \in\ \Gamma(\mathfrak{D})$ if and only if $FX\ =\ 0$, $X\ \in\ \Gamma(\mathfrak{D}^{\perp})$ if and only if $PX\ =\ 0$,
(2) $V\ \in\ \Gamma(J\mathfrak{D}^{\perp})$ if and only if $fV\ =\ 0$, $V\ \in\ \Gamma(N)$ if and only if $tV\ =\ 0$,
(3) $FP\ =\ 0$, $fF\ =\ 0$, $tf\ =\ 0$, $Pt\ =\ 0$,
(4) $P^3\ +\ P\ =\ 0$, $f^3\ +\ f\ =\ 0$.

Proof. The assertions (1) and (2) follow immediately from Definition 4.
(3) We denote by l and $l^{\perp}$ the projection operators on $\mathfrak{D}$ and $\mathfrak{D}^{\perp}$, respectively. Then we have $l\ +\ l^{\perp}\ =\ 1$, $l^2\ =\ l$, $l^{\perp 2}\ =\ l^{\perp}$ and $ll^{\perp}\ =\ l^{\perp}l\ =\ 0$. From (2.21) we have $JlX\ =\ PlX\ +\ FlX$, from which it follows

$$l^{\perp}Pl\ =\ 0,\ Fl\ =\ 0, \tag{2.29}$$

since $\mathfrak{D}$ is J invariant. Similarly, we have $Jl^{\perp}X\ =\ Pl^{\perp}X\ +\ Fl^{\perp}X$, from which it follows $Pl^{\perp}\ =\ 0$, since $\mathfrak{D}^{\perp}$ is anti-invariant. From the last equation we have

$$Pl\ =\ P, \tag{2.30}$$

since $l^{\perp}\ =\ 1\ -\ l$. Now using (2.24), (2.29) and (2.30) we get that $FP\ =\ 0$, and consequently $fF\ =\ 0$. On the other hand, from the skew-symmetry of f and (2.23), we get $tf\ =\ 0$ and consequently $Pt\ =\ 0$, because of (2.25).

(4) Applying J to $JPX = P^2X$ we get $-PX = P^3X$, $X \in \Gamma(TM)$. Similarly, applying J to $JfV = f^2V$, we get $-fV = f^3V$, $V \in \Gamma(T^\perp M)$. □

Definition 5. *Let M be a CR submanifold of a Kähler manifold $\overline{M}$. M is a totally real submanifold of $\overline{M}$ if and only if $\dim \mathfrak{D}_x = 0$ for any $x \in M$. Equivalently, M is a totally real submanifold of $\overline{M}$ if and only if $P = 0$.*

We note that when M is a Lagrangian submanifold of $\overline{M}$ then $\dim N_x = 0$ for any $x \in M$, and $f = 0$.
The map f in the equation (2.22) is an endomorphism of the normal bundle $T^\perp M$ and it defines an f-structure in the normal bundle. In 1963 Yano defined an f-structure on a $\overline{m}$-dimensional manifold $\overline{M}$, as a tensor field of type $(1, 1)$ satisfying $f^3 + f = 0$. The rank r of f is constant and necessarily even. The tangent bundle $T\overline{M}$ splits into two complementary subbundles $\mathrm{Im}f$ and $\mathrm{Ker}f$, and the restriction of f to $\mathrm{Im}f$ determines a complex structure on such a subbundle. Almost complex and almost contact structures are special cases according as $r = \overline{m}$ and $r = \overline{m} - 1$ respectively. If $D^{\overline{g}}_X f = 0$ for $X \in \Gamma(TM)$ then the f-structure f in the normal bundle is called parallel, where $(D^{\overline{g}}_X f)V := D^{\overline{g}}_X(fV) - f(D^{\overline{g}}_X V)$ for $V \in \Gamma(T^\perp M)$.

Proposition 3. *Let M be a totally real submanifold of a Kähler manifold $\overline{M}$. If the f-structure f in the normal bundle is parallel then*

$$h^{\overline{g}}(X, Y) = JA^{\overline{g}}_{JX}Y = JA^{\overline{g}}_{JY}X, \tag{2.31}$$

$$D^{\overline{g}}_X JY = J\nabla^g_X Y, \tag{2.32}$$

$$A^{\overline{g}}_\xi = 0, \tag{2.33}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$.

Proof. Differentiating (2.22) we get

$$J\overline{\nabla}^{\overline{g}}_X V = \overline{\nabla}^{\overline{g}}_X tV + \overline{\nabla}^{\overline{g}}_X fV, \quad X \in \Gamma(TM),\ V \in \Gamma(T^\perp M).$$

Using the Gauss and Weingarten equations (2.7) and (2.8), from the last equation we get

$$-JA^{\overline{g}}_V X + tD^{\overline{g}}_X V + fD^{\overline{g}}_X V = \nabla^g_X tV + h^{\overline{g}}(X, tV) - A^{\overline{g}}_{fV}X + D^{\overline{g}}_X fV.$$

Equating the normal parts in the last equation we get

$$(D^{\overline{g}}_X f)V = -h^{\overline{g}}(X, tV) - JA^{\overline{g}}_V X. \tag{2.34}$$

Putting $V = \xi \in \Gamma(N)$ in (2.34) we get $A^{\overline{g}}_{\xi}X = 0$, and putting $V = JY$, $Y \in \Gamma(TM)$, in (2.34) we get $h^{\overline{g}}(X, Y) = JA^{\overline{g}}_{JX}Y = JA^{\overline{g}}_{JY}X$. On the other hand, for $X, Y \in \Gamma(TM)$ we have

$$\overline{\nabla}^{\overline{g}}_X JY = J\overline{\nabla}^{\overline{g}}_X Y = J(\nabla^g_X Y + h^{\overline{g}}(X, Y)),$$

from which it follows

$$-A^{\overline{g}}_{JY}X + D^{\overline{g}}_X JY = J\nabla^g_X Y - A^{\overline{g}}_{JY}X,$$

i. e.

$$D^{\overline{g}}_X JY = J\nabla^g_X Y.$$

□

Let $R^{\overline{g}\perp}$ be the curvature tensor of the normal connection $D^{\overline{g}}$:

$$R^{\overline{g}\perp}(X, Y)V := D^{\overline{g}}_X D^{\overline{g}}_Y V - D^{\overline{g}}_Y D^{\overline{g}}_X V - D^{\overline{g}}_{[X, Y]}V,$$

for $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proposition 4. *Under the assumptions of Proposition 3 we have*

$$R^{\overline{g}\perp}(X, Y)JZ = JR^g(X, Y)Z, \tag{2.35}$$

for $Z \in \Gamma(TM)$.

Proof. For $X, Y, Z \in \Gamma(TM)$ we have

$$\begin{aligned} R^{\overline{g}\perp}(X, Y)JZ &= D^{\overline{g}}_X D^{\overline{g}}_Y(JZ) - D^{\overline{g}}_Y D^{\overline{g}}_X(JZ) - D^{\overline{g}}_{[X, Y]}(JZ) \\ &= D^{\overline{g}}_X(J\nabla^g_Y Z) - D^{\overline{g}}_Y(J\nabla^g_X Z) - J\nabla^g_{[X, Y]}Z \\ &= J(\nabla^g_X\nabla^g_Y Z) - J(\nabla^g_Y\nabla^g_X Z) - J\nabla^g_{[X, Y]}Z \\ &= JR^g(X, Y)Z, \end{aligned}$$

where we used (2.32). □

A submanifold M is said to be *parallel* if $\widetilde{\nabla}^{\overline{g}}h^{\overline{g}} = 0$, where $\widetilde{\nabla}^{\overline{g}}$ is the Van der Waerden-Bortolotti connection defined by (2.13). In [4] the classification of parallel submanifolds in real space forms is obtained. *Semi-parallel* submanifolds are generalization of parallel submanifolds and they are defined as submanifolds satisfying

$$\begin{aligned} 0 = (R^{\widetilde{\nabla}^{\overline{g}}}(X, Y)h^{\overline{g}})(Z, W) &:= (\widetilde{\nabla}^{\overline{g}}_X(\widetilde{\nabla}^{\overline{g}}_Y h^{\overline{g}}))(Z, W) - (\widetilde{\nabla}^{\overline{g}}_Y(\widetilde{\nabla}^{\overline{g}}_X h^{\overline{g}}))(Z, W) \\ &\quad - (\widetilde{\nabla}^{\overline{g}}_{[X, Y]}h^{\overline{g}})(Z, W). \end{aligned}$$

Equivalently,

$$\begin{aligned} (R^{\widetilde{\nabla}^{\overline{g}}}(X, Y)h^{\overline{g}})(Z, W) &= R^{\overline{g}\perp}(X, Y)h^{\overline{g}}(Z, W) - h^{\overline{g}}(R^g(X, Y)Z, W) \\ &\quad - h^{\overline{g}}(Z, R^g(X, Y)W). \end{aligned}$$

Semi-parallel hypersurfaces of a Euclidean space have been classified by J. Deprez in [10]. On the other hand, in 1986, O. Kassabov investigated totally real semi-parallel submanifolds in Kählerian manifolds ([14]).

Proposition 5. *(cf. [14]) Let M be a totally real submanifold with parallel f-structure f in the normal bundle of a Kählerian manifold $\overline{M}$. Then M is semi-parallel if and only if*

$$R^g(X,\ Y)A_{JZ}W \ =\ A^{\overline{g}}_{JZ}R^g(X,\ Y)W \ +\ A^{\overline{g}}_{JW}R^g(X,\ Y)Z \tag{2.36}$$

for $X,\ Y,\ Z,\ W\ \in\ \Gamma(TM)$.

Proof. From the definition of semi-parallel submanifolds we have

$$R^{\overline{g}\perp}(X,\ Y)h^{\overline{g}}(Z,\ W)\ =\ h^{\overline{g}}(R^g(X,\ Y)Z,\ W)\ +\ h^{\overline{g}}(Z,\ R^g(X,\ Y)W),$$

from which it follows

$$JR^g(X,\ Y)A^{\overline{g}}_{JZ}W\ =\ JA^{\overline{g}}_{JW}R^g(X,\ Y)Z\ +\ JA^{\overline{g}}_{JZ}R^g(X,\ Y)W,$$

where we used (2.31) and (2.35). On the other hand, when (2.36) is satisfied, we get

$$R^g(X,\ Y)Jh^{\overline{g}}(Z,\ W)\ =\ Jh^{\overline{g}}(Z,\ R^g(X,\ Y)W)\ +\ Jh^{\overline{g}}(W,\ R^g(X,\ Y)Z),$$

because of (2.31). Now, because of Proposition 4, we have

$$JR^{\overline{g}\perp}(X,\ Y)h^{\overline{g}}(Z,\ W)\ =\ Jh^{\overline{g}}(Z,\ R^g(X,\ Y)W)\ +\ Jh^{\overline{g}}(W,\ R^g(X,\ Y)Z),$$

that is M is semi-parallel. □

A submanifold M is said to have *parallel mean curvature vector field* if $D^{\overline{g}}H\ =\ 0$ (for the definition of H see (2.10)). B. Y. Chen ([6]) and S. T. Yau ([29]) classified all the surfaces with parallel mean curvature vector in space forms. They proved that such surfaces are surfaces with constant mean curvature vector field of three dimensional umbilical hypersurfaces. A generalization of the submanifolds with parallel mean curvature vector field are submanifolds with *semi-parallel mean curvature vector field*, defined by $R^{\overline{g}\perp}(X,\ Y)H\ =\ 0$. Also, the class of submanifolds with semi-parallel mean curvature vector includes the semi-parallel submanifolds.

Proposition 6. *(cf. [14]) Let M be an n-dimensional $(n\ >\ 1)$ totally real submanifold of constant curvature c, with parallel f-structure f in the normal bundle of a Kählerian manifold $\overline{M}$. If the mean curvature vector field H of M is semi-parallel then M is minimal or flat.*

Proof. Because of (2.31) the mean curvature vector field $H\ \in\ \Gamma(JTM)$. Now, using (2.35) we obtain

$$0\ =\ R^{\overline{g}\perp}(X,\ Y)H\ =\ -JR^g(X,\ Y)JH,$$

i. e.

$$R^g(X,\ Y)JH\ =\ 0.$$

Since M is of constant curvature c, this implies that

$$c\{g(Y,\ JH)X\ -\ g(X,\ JH)Y\}\ =\ 0, \tag{2.37}$$

for $X,\ Y\ \in\ \Gamma(TM)$. Let $c\ \neq\ 0$. Putting $Y\ =\ JH$ in (2.37) and X orthogonal to Y we obtain $H\ =\ 0$. □

Theorem 2. *(cf. [14]) Let M be an n-dimensional $(n\ >\ 1)$ totally real semi-parallel submanifold of constant curvature c with parallel f-structure f in the normal bundle of a Kählerian manifold $\overline{M}$. Then M is flat, i. e. $c\ =\ 0$ or M is a totally geodesic submanifold of $\overline{M}$.*

Proof. Since M is of constant curvature c, (2.6) holds. Now, using Proposition 5 we have

$$\begin{aligned} c\{g(Y,\ A_{JZ}U)X\ -\ g(X,\ A_{JZ}U)Y\}\ =\ c\{&g(Y,\ Z)A_{JU}X\ -\ g(X,\ Z)A_{JU}Y\ +\\ &g(Y,\ U)A_{JZ}X\ -\ g(X,\ U)A_{JZ}Y\}, \end{aligned} \tag{2.38}$$

for $X,\ Y,\ Z,\ U\ \in\ \Gamma(TM)$. Let $c\ \neq\ 0$ and $\{E_1,\ldots,E_n\}$ be an orthonormal basis of TM. We put $X\ =\ U\ =\ E_i$ in (2.38) and we add for $i\ =\ 1,\ldots,n$, using (2.31) and Proposition 6. We get

$$(n\ +\ 1)cA_{JZ}Y\ =\ 0.$$

Hence, the assertion follows because of (2.33). □

2.4 Statistical submanifolds

Let $\overline{M}$ be a C^∞ manifold of dimension $\overline{m}\ \geq\ 2$, $\overline{\nabla}$ an affine connection on $\overline{M}$, and $\overline{g}$ a Riemannian metric on $\overline{M}$.

Definition 6. *1) $(\overline{M},\ \overline{\nabla},\ \overline{g})$ is called a statistical manifold if*
(i) $\overline{\nabla}$ is of torsion free and
(ii) $(\overline{\nabla}_X\overline{g})(Y,Z) = (\overline{\nabla}_Y\overline{g})(X,Z)$ for $X,\ Y,\ Z\in\Gamma(T\overline{M})$.
2) $\overline{\nabla}^*$ is called the dual connection of $\overline{\nabla}$ with respect to $\overline{g}$ if

$$X\overline{g}(Y,Z)\ =\ \overline{g}(\overline{\nabla}_XY,Z)\ +\ \overline{g}(Y,\overline{\nabla}^*_XZ),\quad X,\ Y,\ Z\ \in\ \Gamma(T\overline{M})$$

Proposition 7. *A triple $(\overline{M},\ \overline{\nabla},\ \overline{g})$ is a statistical manifold if and only if $(\overline{M},\ \overline{\nabla}^*,\ \overline{g})$ is a statistical manifold.*

Proof. Let $(\overline{M},\ \overline{\nabla},\ \overline{g})$ be a statistical manifold. For $X,\ Y,\ Z\ \in\ \Gamma(T\overline{M})$ we have

$$X\overline{g}(Y,\ Z)\ =\ \overline{g}(\overline{\nabla}_X Y,\ Z)\ +\ \overline{g}(Y,\ \overline{\nabla}^*_X Z),$$

that is

$$(\overline{\nabla}_X\overline{g})(Y,\ Z)\ +\ \overline{g}(\overline{\nabla}_X Y,\ Z)\ +\ \overline{g}(Y,\ \overline{\nabla}_X Z)\ =\ \overline{g}(\overline{\nabla}_X Y,\ Z)\ +\ \overline{g}(Y,\ \overline{\nabla}^*_X Z). \tag{2.39}$$

Interchanging X and Z in (2.39), we get

$$(\overline{\nabla}_Z\overline{g})(Y,\ Z)\ +\ \overline{g}(Y,\ \overline{\nabla}_Z X)\ =\ \overline{g}(Y,\ \overline{\nabla}^*_Z X). \tag{2.40}$$

Subtracting (2.39) and (2.40), we get $[X,\ Z]\ =\ \overline{\nabla}^*_X Z\ -\ \overline{\nabla}^*_Z X$, i.e. $\overline{\nabla}^*$ is of torsion free.
Next, we have

$$X\overline{g}(Y,\ Z)\ =\ (\overline{\nabla}^*_X\overline{g})(Y,\ Z)\ +\ \overline{g}(\overline{\nabla}^*_X Y,\ Z)\ +\ \overline{g}(Y,\ \overline{\nabla}^*_X Z),$$

i.e.

$$\overline{g}(\overline{\nabla}_X Y,\ Z)\ +\ \overline{g}(Y,\ \overline{\nabla}^*_X Z)\ =\ (\overline{\nabla}^*_X\overline{g})(Y,\ Z)\ +\ \overline{g}(\overline{\nabla}^*_X Y,\ Z)\ +\ \overline{g}(Y,\ \overline{\nabla}^*_X Z). \tag{2.41}$$

Interchanging X and Y in (2.41), we get

$$\overline{g}(\overline{\nabla}_Y X,\ Z)\ =\ (\overline{\nabla}^*_Y\overline{g})(X,\ Z)\ +\ \overline{g}(\overline{\nabla}^*_Y X,\ Z). \tag{2.42}$$

Subtracting (2.41) and (2.42), we have

$$\overline{g}([X,\ Y],\ Z)\ =\ (\overline{\nabla}^*_X\overline{g})(Y,\ Z)\ -\ (\overline{\nabla}^*_Y\overline{g})(X,\ Z)\ +\ \overline{g}([X,\ Y],\ Z),$$

i.e.

$$(\overline{\nabla}^*_X\overline{g})(Y,\ Z)\ =\ (\overline{\nabla}^*_Y\overline{g})(X,\ Z).$$

Hence, we have proved that $(\overline{M},\ \overline{\nabla}^*,\ \overline{g})$ is a statistical manifold. The opposite direction follows in the same fashion. □

Definition 7. *A statistical manifold $(\overline{M},\ \overline{\nabla},\ \overline{g})$ is said to be of constant curvature $c \in \mathbb{R}$ if*

$$R^{\overline{\nabla}}(X,Y)Z = c\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\},\quad X,\ Y,\ Z\ \in\ \Gamma(T\overline{M}),$$

where $R^{\overline{\nabla}}$ is the curvature tensor of $\overline{\nabla}$.

A statistical structure $(\overline{\nabla},\ \overline{g})$ of constant curvature 0 is called a Hessian structure.

Proposition 8. *A statistical manifold* $(\overline{M},\ \overline{\nabla},\ \overline{g})$ *is of constant curvature* c *if and only if* $(\overline{M},\ \overline{\nabla}^*,\ \overline{g})$ *is of constant curvature* c.

Proof. Let $(\overline{M},\ \overline{\nabla},\ \overline{g})$ be of constant curvature c. For $X,\ Y,\ Z,\ W\ \in\ \Gamma(T\overline{M})$, we have

$$\begin{aligned}
\overline{g}(R^{\overline{\nabla}^*}(X,\ Y)Z,\ W) &= \overline{g}(\overline{\nabla}^*_X\overline{\nabla}^*_Y Z,\ W)\ -\ \overline{g}(\overline{\nabla}^*_Y\overline{\nabla}^*_X Z,\ W)\ -\ \overline{g}(\overline{\nabla}^*_{[X,\ Y]}Z,\ W)\\
&= X\overline{g}(\overline{\nabla}^*_Y Z,\ W)\ -\ \overline{g}(\overline{\nabla}^*_Y Z,\ \overline{\nabla}_X W)\ -\ Y\overline{g}(\overline{\nabla}^*_X Z,\ W)\\
&+\ \overline{g}(\overline{\nabla}^*_X Z,\ \overline{\nabla}_Y W)\ -\ [X,\ Y]\overline{g}(Z,\ W)\ +\ \overline{g}(Z,\ \overline{\nabla}_{[X,\ Y]}W)\\
&= X\{Y\overline{g}(Z,\ W)\ -\ \overline{g}(Z,\ \overline{\nabla}_Y W)\}\ -\ Y\overline{g}(Z,\ \overline{\nabla}_X W)\\
&+\ \overline{g}(Z,\ \overline{\nabla}_Y\overline{\nabla}_X W)\ -\ Y\{X\overline{g}(Z,\ W)\ -\ \overline{g}(Z,\ \overline{\nabla}_X W)\}\\
&+\ X\overline{g}(Z,\ \overline{\nabla}_Y W)\ -\ \overline{g}(Z,\ \overline{\nabla}_X\overline{\nabla}_Y W)\ -\ [X,\ Y]\overline{g}(Z,\ W)\\
&+\ \overline{g}(Z,\ \overline{\nabla}_{[X,\ Y]}W)\ =\ -\ \overline{g}(Z,\ R^{\overline{\nabla}}(X,\ Y)W).
\end{aligned}$$

From this equation, we conclude that $(\overline{M},\ \overline{\nabla}^*,\ \overline{g})$ is of constant curvature c. □

For a statistical manifold $(\overline{M},\ \overline{\nabla},\ \overline{g})$, we set

$$\overline{R}^{(\overline{\nabla},\ \overline{g})}(X,\ Y,\ Z,\ W)\ =\ \overline{g}(\overline{R}^{\overline{\nabla}}(Z,\ W)Y,\ X),$$

and denote $\overline{R}^{(\overline{\nabla},\ \overline{g})}$ by $\overline{R}$, and $\overline{R}^{(\overline{\nabla}^*,\ \overline{g})}$ by $\overline{R}^*$ for short.

Lemma 1. *For a statistical manifold* $(\overline{M},\ \overline{\nabla},\ \overline{g})$, *the following hold for* $X,\ Y,\ Z,\ W\ \in\ \Gamma(T\overline{M})$:

$$\overline{R}(X,\ Y,\ W,\ Z)\ =\ -\ \overline{R}(X,\ Y,\ Z,\ W), \tag{2.43}$$

$$\overline{R}^*(X,\ Y,\ W,\ Z)\ =\ -\ \overline{R}^*(X,\ Y,\ Z,\ W), \tag{2.44}$$

$$\overline{R}(Y,\ X,\ W,\ Z)\ =\ -\ \overline{R}^*(X,\ Y,\ Z,\ W), \tag{2.45}$$

$$\overline{R}(X,\ Y,\ Z,\ W)\ +\ \overline{R}(X,\ Z,\ W,\ Y)\ +\ \overline{R}(X,\ W,\ Y,\ Z)\ =\ 0, \tag{2.46}$$

$$\overline{R}^*(X,\ Y,\ Z,\ W)\ +\ \overline{R}^*(X,\ Z,\ W,\ Y)\ +\ \overline{R}^*(X,\ W,\ Y,\ Z)\ =\ 0, \tag{2.47}$$

Proof. The formulas (2.43) and (2.44) follow directly from the definition of the curvature tensor field. By Definition 6 (2), we have $\overline{g}(\overline{R}^{\overline{\nabla}}(X,\ Y)Z,\ W)\ =\ -\ \overline{g}(Z,\ \overline{R}^{\overline{\nabla}^*}(X,\ Y)W)$, which means (2.45). The first Bianchi identity implies the formulas (2.46) and (2.47) because $\overline{\nabla}$ and $\overline{\nabla}^*$ are of torsion free. □

Definition 8. *For a statistical manifold* $(\overline{M},\ \overline{\nabla},\ \overline{g})$ *we define*

$$\overline{S}^{(\overline{\nabla},\ \overline{g})}(X,\ Y)Z\ =\ \frac{1}{2}\{\overline{R}^{\overline{\nabla}}(X,\ Y)Z\ +\ \overline{R}^{\overline{\nabla}^*}(X,\ Y)Z\},$$

and

$$\overline{S}(X,\ Y,\ Z,\ W)\ =\ \overline{g}(\overline{S}^{(\overline{\nabla},\ \overline{g})}(Z,\ W)Y,\ X),\quad X,\ Y,\ Z,\ W\ \in\ \Gamma(T\overline{M}).$$

Proposition 9. *Let* $(\overline{M},\ \overline{\nabla},\ \overline{g})$ *be a statistical manifold. The tensor field* $\overline{S}\ \in\ \Gamma(T\overline{M}^{(0,\ 4)})$ *in Definition 8 satisfies*

$$\overline{S}(X,\ Y,\ W,\ Z)\ =\ -\ \overline{S}(X,\ Y,\ Z,\ W), \tag{2.48}$$

$$\overline{S}(Y,\ X,\ W,\ Z)\ =\ -\ \overline{S}(X,\ Y,\ Z,\ W), \tag{2.49}$$

$$\overline{S}(X,\ Y,\ Z,\ W)\ +\ \overline{S}(X,\ Z,\ W,\ Y)\ +\ \overline{S}(X,\ W,\ Y,\ Z)\ =\ 0, \tag{2.50}$$

$$\overline{S}(Z,\ W,\ X,\ Y)\ =\ \overline{S}(X,\ Y,\ Z,\ W). \tag{2.51}$$

Proof. The formulas (2.48), (2.49) and (2.51) follow from Lemma 1. Denoting by $T(X,\ Y,\ Z,\ W)$ the left hand side of (2.50), we calculate from (2.48) and (2.49) that
$0\ =\ T(X,\ Y,\ Z,\ W) - T(Y,\ Z,\ W,\ X) - T(Z,\ W,\ X,\ Y) + T(W,\ X,\ Y,\ Z)\ =$
$\cdots =\ 2S(X,\ Y,\ Z, W)\ -\ 2S(Z,\ W,\ X,\ Y)$, which implies (2.51). □
Next, we give the definition of a statistical manifold of constant curvature, different then the definition of Kurose (see Definition 7).

Definition 9. *(cf. [13]) The sectional curvature of a statistical manifold* $(\overline{M},\ \overline{\nabla},\ \overline{g})$ *is constant* c *if and only if*

$$\overline{S}^{(\overline{\nabla},\ \overline{g})}(X,\ Y)Z\ =\ c\{\overline{g}(Y,\ Z)X\ -\ \overline{g}(X,\ Z)Y\}$$

for $X,\ Y,\ Z\ \in\ \Gamma(T\overline{M})$.

Definition 10. *Let* $(\overline{M},\ \overline{\nabla},\ \overline{g})$ *be a statistical manifold, and* M *a submanifold of* $\overline{M}$. *By* $T_x^{\perp}M$ *we denote the normal space of* M, *i. e.*
$T_x^{\perp}M\ :=\ \{v\ \in\ T_x\overline{M}\ |\ \overline{g}(v,\ w)\ =\ 0,\ w\ \in\ T_xM\ \}$, *and by* g *the induced metric on* M. *We define*

$$\begin{aligned}
&\nabla,\ \nabla^*\ :\ \Gamma(TM)\ \times\ \Gamma(TM)\ \to\ \Gamma(TM),\\
&h,\ h^*\ :\ \Gamma(TM)\ \times\ \Gamma(TM)\ \to\ \Gamma(T^{\perp}M),\\
&A,\ A^*\ :\ \Gamma(T^{\perp}M)\ \times\ \Gamma(TM)\ \to\ \Gamma(TM),
\end{aligned}$$

and

$$D,\ D^*\ :\ \Gamma(TM)\ \times\ \Gamma(T^{\perp}M)\ \to\ \Gamma(T^{\perp}M)$$

by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \overline{\nabla}_X V = -A_V X + D_X V,$$
$$\overline{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y), \ \overline{\nabla}^*_X V = -A^*_V X + D^*_X V,$$

for $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^\perp M)$.

Proposition 10. *Let* M, g, ∇, ∇^*, h, h^*, A, A^*, D *and* D^* *be as in Definition 10. Then:*
1) (M, ∇, g) *and* (M, ∇^*, g) *are statistical manifolds, and* ∇^* *is dual of* ∇ *with respect to* g.
2) D *and* D^* *are connections of* $T^\perp M \to M$. *Denoting by* $g^\perp$ *the metric of* $T^\perp M$ *induced from* $\overline{g}$, *we have*

$$Xg^\perp(U, V) = g^\perp(D_X U, V) + g^\perp(U, D^*_X V), \quad U, V \in \Gamma(T^\perp M),$$
$$X \in \Gamma(TM).$$

3) h, h^*, A *and* A^* *are tensor fields satisfying*

$$h(X, Y) = h(Y, X), \ h^*(X, Y) = h^*(Y, X),$$
$$g(A_V X, Y) = \overline{g}(h^*(X, Y), V), \ g(A^*_V X, Y) = \overline{g}(h(X, Y), V),$$
$$X, Y \in \Gamma(TM), V \in \Gamma(T^\perp M).$$

Proof. For $X, Y, Z \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, we have:
(1) Since $\overline{\nabla}$ is of torsion free, we get

$$\begin{aligned} 0 &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] \\ &= \nabla_X Y + h(X, Y) - \nabla_Y X - h(Y, X) - [X, Y]. \end{aligned} \tag{2.52}$$

Equating the tangent parts in (2.52), we conclude that

$$0 = \nabla_X Y - \nabla_Y X - [X, Y],$$

i. e. ∇ is of torsion free. Similarly, we get that ∇^* is of torsion free. Next, denoting by $\iota : M \to \overline{M}$ an immersion, we have

$$(\nabla_X g)(Y, Z) = (\overline{\nabla}_X \overline{g})(\iota_* Y, \iota_* Z),$$

from which it follows

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z).$$

Similarly,

$$(\nabla^*_X g)(Y, Z) = (\nabla^*_Y g)(X, Z).$$

∇ and ∇^* are dual connections with respect to g because

$$X\overline{g}(\iota_* Y, \iota_* Z) = \overline{g}(\overline{\nabla}_X \iota_* Y, \iota_* Z) + \overline{g}(\iota_* Y, \overline{\nabla}^*_X \iota_* Z), \tag{2.53}$$

from which it follows that

$$Xg(Y,\ Z)\ =\ g(\nabla_X Y,\ Z)\ +\ g(Y,\ \nabla^*_X Z), \tag{2.54}$$

by definition of the induced connection.
(2) D (D^*) is a connection. Next, we have

$$g^\perp(U,\ V)\ :=\ \overline{g}(U,\ V),\quad U,\ V\ \in\ \Gamma(T^\perp M).$$

The duality of D and D^* follows from

$$\begin{aligned} X\overline{g}(U,\ V)\ &=\ \overline{g}(\overline{\nabla}_X U,\ V)\ +\ \overline{g}(U,\ \overline{\nabla}^*_X V)\\ &=\ \overline{g}(D_X U,\ V)\ +\ \overline{g}(U,\ D^*_X V),\quad U,\ V\ \in\ \Gamma(T^\perp M),\ X\ \in\ \Gamma(TM). \end{aligned}$$

(3) Equating the normal parts in (2.52), we get

$$h(X,\ Y)\ =\ h(Y,\ X).$$

Similarly,

$$h^*(X,\ Y)\ =\ h^*(Y,\ X),\quad X,\ Y\ \in\ \Gamma(TM).$$

Now, for $X,\ Y\ \in\ \Gamma(TM)$ and $V\ \in\ \Gamma(T^\perp M)$, we have

$$X\overline{g}(Y,\ V)\ =\ 0,$$

from which it follows

$$\overline{g}(\overline{\nabla}_X Y,\ V)\ +\ \overline{g}(Y,\ \overline{\nabla}^*_X V)\ =\ 0.$$

Using the Gauss and Weingarten equations, from the last equation we get

$$\overline{g}(\nabla_X Y\ +\ h(X,\ Y),\ V)\ +\ \overline{g}(Y,\ -A^*_V X\ +\ D^*_X V)\ =\ 0,$$

that is

$$\overline{g}(h(X,\ Y),\ V)\ =\ g(Y,\ A^*_V X).$$

Similarly,

$$\overline{g}(h^*(X,\ Y),\ V)\ =\ g(Y,\ A_V X).\qquad \square$$

Proposition 11. *Let M be a submanifold of a statistical manifold $(\overline{M},\ \overline{\nabla},\ \overline{g})$. For simplicity, we denote by $\overline{R}\ :=\ R^{\overline{\nabla}}$, $\overline{R}^*\ :=\ R^{\overline{\nabla}^*}$, $R\ :=\ R^{\nabla}$, $R^*\ :=\ R^{\nabla^*}$, $R^\perp\ :=\ R^D$ and $R^{\perp *}\ :=\ R^{D^*}$ the curvature tensor fields of $\overline{\nabla}$, $\overline{\nabla}^*$, ∇, ∇^*, D and D^*, respectively. For $X,\ Y,\ Z\ \in\ \Gamma(TM)$, $V\ \in\ \Gamma(T^\perp M)$ we have*

$$\{\overline{R}(X,\ Y)Z\}^\top\ =\ R(X,\ Y)Z\ -\ A_{h(Y,\ Z)}X\ +\ A_{h(X,\ Z)}Y, \tag{2.55}$$

$$\{\overline{R}(X,\ Y)Z\}^\perp\ =\ (\widetilde{\nabla}_X h)(Y,\ Z)\ -\ (\widetilde{\nabla}_Y h)(X,\ Z), \tag{2.56}$$

$$\{\overline{R}(X,\ Y)V\}^\top\ =\ (\widetilde{\nabla}_Y A)_V X\ -\ (\widetilde{\nabla}_X A)_V Y, \tag{2.57}$$

$$\{\overline{R}(X,\ Y)V\}^\perp\ =\ R^\perp(X,Y)V\ +\ h(Y,\ A_V X)\ -\ h(X,\ A_V Y), \tag{2.58}$$

and their duals, i. e.,

$$\{\overline{R}^*(X,\ Y)Z\}^\top = R^*(X,\ Y)Z - A^*_{h^*(Y,\ Z)}X + A^*_{h^*(X,\ Z)}Y, \quad (2.55)^*$$
$$\{\overline{R}^*(X,\ Y)Z\}^\perp = (\widetilde{\nabla}^*_X h^*)(Y,\ Z) - (\widetilde{\nabla}^*_Y h^*)(X,\ Z), \quad (2.56)^*$$
$$\{\overline{R}^*(X,\ Y)V\}^\top = (\widetilde{\nabla}^*_Y A^*)_V X - (\widetilde{\nabla}^*_X A^*)_V Y, \quad (2.57)^*$$
$$\{\overline{R}^*(X,\ Y)V\}^\perp = R^{\perp *}(X,Y)V + h^*(Y,\ A^*_V X) - h^*(X,\ A^*_V Y), \quad (2.58)^*$$

where $\{\ \}^\top$ *(*$\{\ \}^\perp$*) denotes the tangential (normal) part of* $\{\ \}$ *for* M*, and*

$$(\widetilde{\nabla}_X h)(Y,\ Z) := D_X h(Y,\ Z) - h(\nabla_X Y,\ Z) - h(Y,\ \nabla_X Z),$$
$$(\widetilde{\nabla}_Y A)_V X := \nabla_Y (A_V X) - A_{D_Y V} X - A_V \nabla_Y X.$$

Proof. We prove only We prove only (2.55) and (2.56). Formulas (2.57) - (2.57)* are obtained in a similar way. For $X,\ Y \in \Gamma(TM)$ we have

$$\begin{aligned}
\overline{R}(X,\ Y)Z &= \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,\ Y]} Z \\
&= \overline{\nabla}_X (\nabla_Y Z + h(Y,\ Z)) - \overline{\nabla}_Y (\nabla_X Z + h(X,\ Z)) \\
&- (\nabla_{[X,\ Y]} Z + h([X,\ Y],\ Z)) = R(X,\ Y)Z + h(X,\ \nabla_Y Z) \\
&- h(Y,\ \nabla_X Z) - h([X,\ Y],\ Z) + \overline{\nabla}_X h(Y,\ Z) - \overline{\nabla}_Y h(X,\ Z) \\
&= R(X,\ Y)Z - A_{h(Y,\ Z)} X + A_{h(X,\ Z)} Y + h(X,\ \nabla_Y Z) \\
&- h(Y,\ \nabla_X Z) - h([X,\ Y],\ Z) + D_X h(Y,\ Z) - D_Y h(X,\ Z).
\end{aligned}$$

From this equation, (2.55) and (2.56) follow. □

Remark 1. *Equation* $(2.55)^*$ *is considered as the dual of* (2.55)*. In the similar way, if it is clear, we omit the dual equation and put * to the number of original equation.*

3. CR SUBMANIFOLDS OF MAXIMAL CR DIMENSION IN KÄHLER MANIFOLDS

3.1 Real hypersurfaces

Let $(\overline{M},\ \overline{g},\ J)$ be a complex space form of holomorphic sectional curvature $4c$ and of real dimension $2\overline{m}$. For a hypersurface M in $\overline{M}$, we take a unit normal vector field $\xi\ \in\ \Gamma(T^{\perp}M)$. The Gauss and Weingarten equations are given by:

$$\overline{\nabla}^{\overline{g}}_X Y\ =\ \nabla^{g}_X Y\ +\ g(A^{\overline{g}}X,\ Y)\xi,$$

and

$$\overline{\nabla}^{\overline{g}}_X \xi\ =\ -\ A^{\overline{g}}X,$$

where $X,\ Y\ \in\ \Gamma(TM)$. The structure vector field U is defined by:

$$U\ :=\ -\ J\xi. \tag{3.1}$$

It is obvious that $U\ \in\ \Gamma(TM)$, and $g(U,\ U)\ =\ 1$. The vector field JX, for $X\ \in\ \Gamma(TM)$, is decomposed into the tangent part and the normal part as follows:

$$JX\ =\ PX\ +\ u(X)\xi, \tag{3.2}$$

where P is an endomorphism on TM and u is one-form on M.
Using (2.14) and (3.2), we obtain

$$\begin{aligned} g(PX,\ Y)\ &=\ \overline{g}(JX,\ Y)\ =\ -\ \overline{g}(X,\ JY) \\ &=\ -\ \overline{g}(X,\ PY\ +\ u(Y)\xi)\ =\ -\ g(X,\ PY). \end{aligned} \tag{3.3}$$

On the other hand, since

$$-\ X\ =\ J^2X\ =\ J(PX\ +\ u(X)\xi)\ =\ P^2X\ +\ u(PX)\xi\ +\ u(X)J\xi,$$

it follows that

$$P^2X\ =\ -\ X\ +\ u(X)U, \tag{3.4}$$

and

$$u(PX)\ =\ 0. \tag{3.5}$$

In a similar way, from (2.14) it follows

$$\overline{g}(JX,\ JY)\ =\ \overline{g}(PX\ +\ u(X)U,\ PY\ +\ u(Y)U),$$

that is

$$g(PX,\ PY)\ =\ g(X,\ Y)\ -\ u(X)u(Y). \tag{3.6}$$

Putting $X\ =\ U$ in (3.2) gives

$$PU\ =\ 0. \tag{3.7}$$

A differentiable manifold M' is said to have an almost contact structure if it admits a vector field U, a one-form u and a $(1,\ 1)$-tensor field P satisfying (3.4), (3.5) and (3.7). The tensor field P is called the almost contact tensor field. Since $P^2\ =\ -\ Id$ on the space $U^{\perp}\ =\ \{X\ \in\ \Gamma(TM)|\ g(X,\ U)\ =\ 0\}$, it follows that rank$P\ =\ 2\overline{m}\ -\ 2$ and that ker$P\ =$ span$\{U\}$. Using (2.14) and (3.1), we get

$$\begin{aligned} g(X,\ U)\ &=\ -\ \overline{g}(X,\ J\xi)\ =\ \overline{g}(JX,\ \xi) \\ &=\ \overline{g}(PX,\ \xi)\ +\ u(X)\overline{g}(\xi,\ \xi)\ =\ u(X), \end{aligned}$$

i.e.

$$g(U,\ X)\ =\ u(X). \tag{3.8}$$

The Gauss and Codazzi equations are given by:

$$\begin{aligned} R^g(X,\ Y)Z\ &=\ g(A^{\overline{g}}Y,\ Z)A^{\overline{g}}X\ -\ g(A^{\overline{g}}X,\ Z)A^{\overline{g}}Y \\ &+\ c\{g(Y,\ Z)X\ -\ g(X,\ Z)Y\ +\ g(PY,\ Z)PX \\ &-\ g(PX,\ Z)PY\ +\ 2g(X,\ PY)PZ\}, \end{aligned} \tag{3.9}$$

and

$$(\nabla^g_X A^{\overline{g}})Y\ -\ (\nabla^g_Y A^{\overline{g}})X\ =\ c\{g(X,\ U)PY\ -\ g(Y,\ U)PX\ +\ 2g(X,\ PY)U\}. \tag{3.10}$$

A real hypersurface is said to be *totally umbilical* if the shape operator $A^{\overline{g}}$ satisfies $A^{\overline{g}}\ =\ \lambda Id$. In 1963, Tashiro and Tachibana proved the nonexistence of totally umbilical hypersurfaces. Their result is given in the next theorem.

Theorem 3. *(cf. [24]) Let M be a hypersurface in a complex space form $\overline{M}$ of holomorphic sectional curvature $4c\ \neq\ 0$. Then the shape operator $A^{\overline{g}}$ can not be parallel. Also, no identity of the form $A^{\overline{g}}\ =\ \lambda Id$ can hold, even with λ non constant. In particular, $\overline{M}$ has no umbilical hypersurfaces.*

Proof. Let $A^{\overline{g}}\ =\ \lambda Id$. Then, the Codazzi equation (3.10) becomes

$$(X\lambda)Y\ -\ (Y\lambda)X\ =\ c\{g(X,\ U)PY\ -\ g(Y,\ U)PX\ +\ 2g(X,\ PY)U\}.$$

Now, we put $Y = U$ in this equation. The resulting equation is

$$(X\lambda)U - (U\lambda)X = -cPX.$$

For the vector field $X \neq 0$ orthogonal to U, the set $\{X, PX, U\}$ is linearly independent, and so $c = 0$, which contradicts the hypothesis. Now, suppose that $\nabla^g A^{\overline{g}} = 0$. Take the vector field $X \neq 0$ orthogonal to U and $Y = U$ in the Codazzi equation, to get $-cPX = 0$, which is a contradiction. □

Now we give more known results on real hypersurfaces in complex space forms. Let π denote the canonical projection, $\overline{M}' \to \overline{M}$, where $(\overline{M}', \overline{g}')$ represents a $(2\overline{m}+1)$-sphere or an anti-de Sitter space, and $\overline{M}$ represents a complex projective space $P^{\overline{m}}(\mathbb{C})$ or a complex hyperbolic space $H^{\overline{m}}(\mathbb{C})$, respectively. For a hypersurface M in $\overline{M}$, $M' = \pi^{-1}M$ is an S^1-invariant hypersurface in $\overline{M}'$. For a vector fields X, X^L denotes the horizontal lift of X.

Lemma 2. *(cf. [22], Theorem 1.8, p. 241) Let M be a real hypersurface in a complex space form of holomorphic sectional curvature $4c \neq 0$. Then the shape operator A' of $M' = \pi^{-1}M$ satisfies:*

$$\pi_*((\nabla'_{X^L}A')Y^L) = (\nabla^g_X A^{\overline{g}})Y + c\{g(PX, Y)U + g(Y, U)PX\},$$

for $X, Y \in \Gamma(TM)$.

Let ξ' be a unit normal vector field for M'. The induced connection ∇' and the shape operator A' for M' satisfy

$$\overline{\nabla}^{\overline{g}}_X Y = \nabla'_X Y + g(A'X, Y)\xi',$$

$$\overline{\nabla}^{\overline{g}}_X \xi' = -A'X,$$

and the Codazzi equation:

$$(\nabla'_X A')Y = (\nabla'_Y A')X.$$

Proposition 12. *(cf. [22], Lemma 1. 10, p. 242) Let $A^{\overline{g}}$ be the shape operator of a hypersurface M in a complex space form $\overline{M}$ of holomorphic sectional curvature $4c$. Assume that there are constants α and β such that $A^{\overline{g}2} = \alpha A^{\overline{g}} + \beta Id$. If $\alpha^2 + 4\beta \neq 0$, then $\nabla^g A^{\overline{g}} = 0$.*

Proof. Differentiating the quadratic condition yields

$$(\nabla^g_X A^{\overline{g}})A^{\overline{g}} + A^{\overline{g}}(\nabla^g_X A^{\overline{g}}) = \alpha(\nabla^g_X A^{\overline{g}}), \tag{3.11}$$

from which it follows

$$A^{\overline{g}}(\nabla^g_X A^{\overline{g}})A^{\overline{g}} = (\alpha A^{\overline{g}} - (A^{\overline{g}})^2)\nabla^g_X A^{\overline{g}} = -\beta\nabla^g_X A^{\overline{g}}. \tag{3.12}$$

On the other hand, the equation (3.11) can be written as

$$g((\nabla^g_Z A^{\overline{g}})Y,\ A^{\overline{g}}X)\ +\ g((\nabla^g_Z A^{\overline{g}})X,\ A^{\overline{g}}Y)\ =\ \alpha g((\nabla^g_Z A^{\overline{g}})X,\ Y).$$

Applying the Codazzi equation and then replacing Z by $A^{\overline{g}}Z$ yields

$$g((\nabla^g_Y A^{\overline{g}})A^{\overline{g}}Z,\ A^{\overline{g}}X)\ +\ g((\nabla^g_X A^{\overline{g}})A^{\overline{g}}Z,\ A^{\overline{g}}Y)\ =\ \alpha g((\nabla^g_X A^{\overline{g}})A^{\overline{g}}Z,\ Y).$$

Using the Codazzi equation and (3.12), gives

$$-\ 2\beta g((\nabla^g_X A^{\overline{g}})Y,\ Z)\ =\alpha g(A^{\overline{g}}(\nabla^g_X A^{\overline{g}})Y,\ Z)\ =\ \alpha g((\nabla^g_X A^{\overline{g}})A^{\overline{g}}Z,\ Y).$$

Since the left side of this equation is symmetric in Y and Z, it follows

$$\alpha(\nabla^g_X A^{\overline{g}})A^{\overline{g}}\ =\ \alpha A^{\overline{g}}(\nabla^g_X A^{\overline{g}})\ =\ -\ 2\beta\nabla^g_X A^{\overline{g}}.$$

Because of (3.11), from the last equation it follows

$$(\alpha^2\ +\ 4\beta)\nabla^g_X A^{\overline{g}}\ =\ 0. \qquad \square$$

Using Theorem 2, the following result can be proven.

Proposition 13. *(cf. [22], Theorem 1.11, p. 243) Let M be a real hypersurface in a complex space form $\overline{M}$ of holomorphic sectional curvature $4c\ \neq\ 0$, and of real dimension $2\overline{m}$. Then the shape operator $A^{\overline{g}}$ satisfies $|\nabla^g A^{\overline{g}}|^2\ \geq\ 4c^2(\overline{m}\ -\ 1)$.*

Proof. Let

$$T(X,\ Y)\ =\ (\nabla^g_X A^{\overline{g}})Y\ +\ c\{g(PX,\ Y)U\ +\ g(Y,\ U)PX\}.$$

Then,

$$0\ \leq\ |T|^2\ =\ |\nabla^g A^{\overline{g}}|^2\ +\ 2ck_1\ +\ c^2k_2,$$

where k_1 is the sum of

$$g((\nabla^g_X A^{\overline{g}})Y,\ g(PX,\ Y)U\ +\ g(Y,\ U)PX)$$

as X and Y range over an orthonormal basis, while k_2 is the sum of

$$g(PX,\ Y)^2\ +\ g(Y,\ U)^2|PX|^2$$

over the same range of X and Y. After summing over Y and using the fact that $\nabla^g A^{\overline{g}}$ is symmetric, we get that k_1 is equal to the sum over X of

$$g(PX,\ (\nabla^g_X A^{\overline{g}})U)\ +\ g(U,\ (\nabla^g_X A^{\overline{g}})PX)\ =\ 2g(PX,\ (\nabla^g_X A^{\overline{g}})U).$$

Using the Codazzi equation, from the last equation it follows:

$$g(PX,\ (\nabla^g_X A^{\overline{g}})U)\ =\ g(PX,\ (\nabla^g_U A^{\overline{g}})X)\ -\ cg(PX,\ PX).$$

Because $\mathrm{trace}((\nabla^g_U A^{\overline{g}})P)\ =\ 0$ and $\mathrm{trace}P^2\ =\ -\ 2(n\ -\ 1)$, it follows that $k_1\ =\ -\ 4c(n\ -\ 1)$ and $k_2\ =\ 4(n\ -\ 1)$. Therefore,

$$|\nabla^g A^{\overline{g}}|^2\ -\ 8c^2(n\ -\ 1)\ +\ 4c^2(n\ -\ 1)\ \geq\ 0. \qquad \square$$

Corollary 1. *Let M be a real hypersurface in a complex space form $\overline{M}$ of holomorphic sectional curvature $4c$. If the shape operator $A^{\overline{g}}$ of M is parallel, i.e. $\nabla^g A^{\overline{g}} = 0$, then $c = 0$.*

Eigenvalues of the shape operator $A^{\overline{g}}$ are called *principal curvatures.*

Remark 2. *([7]) Let M be a connected real hypersurface in a complex projective space, with at most two distinct principal curvatures at each point. Then M is an open subset of geodesic hypersphere.*

3.2 CR submanifolds of maximal CR dimension

Definition 11. *An m-dimensional CR submanifold M of an $\overline{m}(= \frac{m+p}{2})$-dimensional Kähler manifold is said to be of maximal CR dimension if the real dimension of $JTM \cap TM$ is $m - 1$.*

It is easy to show that M is odd-dimensional and there exists a unit vector field $\xi \in \Gamma(T^\perp M)$ such that

$$JTM \subset TM \oplus span\{\xi\}. \tag{3.13}$$

Hence, for $X \in \Gamma(TM)$,

$$JX = PX + u(X)\xi, \tag{3.14}$$

where P is an endomorphism acting on TM and u is one-form on M, and the equation (3.3) holds.
Let $\eta \in \Gamma(T^\perp M)$ and η is orthogonal to ξ. Then,

$$\overline{g}(J\eta, X) = -\overline{g}(\eta, JX) = -\overline{g}(\eta, PX + u(X)\xi) = 0, \tag{3.15}$$

which shows that $J\eta \in \Gamma(T^\perp M)$. On the other hand,

$$\begin{aligned} 0 = \overline{g}(X, \eta) &= \overline{g}(JX, J\eta) \\ &= \overline{g}(PX, J\eta) + u(X)\overline{g}(\xi, J\eta) = u(X)\overline{g}(\xi, J\eta). \end{aligned}$$

Let $u_x(X) = 0$, for $X \in \Gamma(TM)$, $x \in M$. Then, using (3.14),

$$JX = PX, \quad X \in \Gamma(TM).$$

This means that TM is J-invariant and consequently M is even-dimensional, which is a contradiction. Therefore,

$$\overline{g}(\xi, J\eta) = 0. \tag{3.16}$$

i.e., the subbundle

$$T_1^\perp M = \{\eta \in \Gamma(T^\perp M) | \, \overline{g}(\eta, \xi) = 0\}$$

is J-invariant. That is, the following lemma holds.

Lemma 3. *Let M be a CR submanifold of maximal CR dimension in a Kähler manifold. A local orthonormal basis of $T^{\perp}M$ can be chosen in the following way:*

$$\xi,\ \xi_1,\ldots,\xi_q,\ \xi_{1^*},\ldots,\xi_{q^*},$$

where $\xi_{a^*} = J\xi_a$, $a = 1,\ldots,q$ and $q = \frac{p-1}{2}$.

Since $\overline{g}(J\xi,\ \eta) = -\overline{g}(\xi,\ J\eta) = 0$, it follows that $J\xi$ is a tangent vector field, which is denoted by:

$$J\xi = -U. \tag{3.17}$$

From (3.14), (3.17) it follows that

$$g(U,\ U) = 1, \tag{3.18}$$

and the equations (3.4), (3.5), (3.6), (3.7), (3.8) hold. Moreover,

$$D^{\overline{g}}_X\xi = \sum_{a=1}^{q}\{s^{\overline{g}}_a(X)\xi_a + s^{\overline{g}}_{a^*}(X)\xi_{a^*}\}. \tag{3.19}$$

The Weingarten equations for the basis in Lemma 3, are given by:

$$\begin{aligned}\overline{\nabla}^{\overline{g}}_X\xi &= -A^{\overline{g}}X + D^{\overline{g}}_X\xi\\ &= -A^{\overline{g}}X + \sum_{a=1}^{q}\{s^{\overline{g}}_a(X)\xi_a + s^{\overline{g}}_{a^*}(X)\xi_{a^*}\},\end{aligned} \tag{3.20}$$

$$\begin{aligned}\overline{\nabla}^{\overline{g}}_X\xi_a &= -A^{\overline{g}}_aX + D^{\overline{g}}_X\xi_a\\ &= -A^{\overline{g}}_aX - s^{\overline{g}}_a(X)\xi + \sum_{b=1}^{q}\{s^{\overline{g}}_{ab}(X)\xi_b + s^{\overline{g}}_{ab^*}(X)\xi_{b^*}\},\end{aligned} \tag{3.21}$$

$$\begin{aligned}\overline{\nabla}^{\overline{g}}_X\xi_{a^*} &= -A^{\overline{g}}_{a^*}X + D^{\overline{g}}_X\xi_{a^*}\\ &= -A^{\overline{g}}_{a^*}X - s^{\overline{g}}_{a^*}(X)\xi + \sum_{b=1}^{q}\{s^{\overline{g}}_{a^*b}(X)\xi_b + s^{\overline{g}}_{a^*b^*}(X)\xi_{b^*}\},\end{aligned} \tag{3.22}$$

Here, $A^{\overline{g}}$, $A^{\overline{g}}_a$, $A^{\overline{g}}_{a^*}$ are the shape operators for the normal vector fields ξ, ξ_a, ξ_{a^*}, respectively, and s's are the coefficients of the normal connection D. They are related to the second fundamental form by:

$$h^{\overline{g}}(X,\ Y) = g(A^{\overline{g}}X,\ Y) + \sum_{a=1}^{q}\{g(A^{\overline{g}}_aX,\ Y)\xi_a + g(A^{\overline{g}}_{a^*}X,\ Y)\xi_{a^*}\}. \tag{3.23}$$

When the ambient manifold $\overline{M}$ is a complex space form of holomorphic sectional curvature $4c$, then the Gauss equation and the Codazzi equation for the normal vector field ξ are:

$$\begin{aligned} R(X,\ Y)Z\ =\ & c\{g(Y,\ Z)X\ -\ g(X,\ Z)Y\ +\ g(PY,\ Z)PX\ -\ g(PX,\ Z)PY \\ & -\ 2g(PX,\ Y)PZ\}\ +\ g(A^{\overline{g}}Y,\ Z)A^{\overline{g}}X\ -\ g(A^{\overline{g}}X,\ Z)A^{\overline{g}}Y \\ & +\ \sum_{a=1}^{q}\{g(A_a^{\overline{g}}Y,\ Z)A_a^{\overline{g}}X\ -\ g(A_a^{\overline{g}}X,\ Z)A_a^{\overline{g}}Y\} \\ & +\ \sum_{a=1}^{q}\{g(A_{a^*}^{\overline{g}}Y,\ Z)A_{a^*}^{\overline{g}}X\ -\ g(A_{a^*}^{\overline{g}}X,\ Z)A_{a^*}^{\overline{g}}Y\}, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} (\nabla_X^g A^{\overline{g}})Y\ -\ (\nabla_Y^g A^{\overline{g}})X\ =\ & c\{u(X)PY\ -\ u(Y)PX\ -2g(PX,Y)U\} \\ & +\ \sum_{a\ =\ 1}^{q}\{s_a(X)A_a^{\overline{g}}Y\ -\ s_a(Y)A_a^{\overline{g}}X\} \\ & +\ \sum_{a\ =\ 1}^{q}\{s_{a^*}(X)A_{a^*}^{\overline{g}}Y\ -\ s_{a^*}(Y)A_{a^*}^{\overline{g}}X\}, \end{aligned} \tag{3.25}$$

respectively.
CR submanifolds of maximal CR dimension in a complex projective space are intensively studied by Djorić and Okumura. Here we give some of their results. Under the assumptions that M is an $n\ (\geq\ 3)$-dimensional submanifold of maximal CR dimension in a complex space form $\overline{M}$ with holomorphic sectional curvature $4c$, and that the distinguished normal vector field ξ is parallel with respect to the normal connection, the following assertions are obtained.

Theorem 4. *([8]) Let M be an $n\ (\geq\ 3)$-dimensional submanifold of maximal CR dimension in a complex space form $\overline{M}$ with holomorphic sectional curvature $4c$, and let the distinguished normal vector field ξ is parallel with respect to the normal connection. If the shape operator $A^{\overline{g}}$ for ξ has only one eigenvalue, then $\overline{M}$ is a complex Euclidian space.*

Proof. Because $D_X^{\overline{g}}\xi\ =\ 0$, the Codazzi equation becomes

$$(\nabla_X^g A^{\overline{g}})Y\ -\ (\nabla_Y^g A^{\overline{g}})X\ =\ c\{u(X)PY\ -\ u(Y)PX\ -2g(PX,Y)U\}. \tag{3.26}$$

According to the assumption, $A\ =\ 0$ or $AX\ =\ \alpha X$, for $X\ \in\ \Gamma(TM)$. In both cases the Codazzi equation (3.26) implies

$$(X\alpha)Y\ -\ (Y\alpha)X\ =\ c\{u(X)PY\ -\ u(Y)PX\ -\ 2g(PX,\ Y)U\},$$

for $X,\ Y\ \in\ \Gamma(TM)$. Putting $Y\ =\ U$, the last equation becomes

$$(U\alpha)X\ -\ (X\alpha)U\ =\ cPX.$$

Since $\dim M \geq 3$, there exist linearly independent vector fields U, X and PX. Therefore, $c = 0$. □

Remark 3. *([8]) Let M be an n (≥ 3)-dimensional submanifold of maximal CR dimension in a complex space form $\overline{M}$ with holomorphic sectional curvature $4c \neq 0$, and let the distinguished normal vector field ξ is parallel with respect to the normal connection. If the shape operator $A^{\overline{g}}$ for ξ has exactly two distinct eigenvalues, then it follows that U is an eigenvector of $A^{\overline{g}}$.*

Proposition 14. *([8]) Let M be an n (≥ 3)-dimensional submanifold of maximal CR dimension in a complex space form $\overline{M}$ with holomorphic sectional curvature $4c$, and let the distinguished normal vector field ξ is parallel with respect to the normal connection. If $A^{\overline{g}}P + PA^{\overline{g}} = 0$ holds at a point of the submanifold M, then c is non positive.*

Proof. From the assumption of the proposition and equation (**??**), it follows that U is an eigenvector of the shape operator $A^{\overline{g}}$, that is, $A^{\overline{g}}U = \alpha U$. Differentiating this equation and using the Codazzi equation (3.26), it follows

$$\begin{aligned} 2cg(PX, Y) + 2g(A^{\overline{g}}PA^{\overline{g}}X, Y) &= (X\alpha)u(Y) - (Y\alpha)u(X) \\ &\quad + \alpha((PA^{\overline{g}} + A^{\overline{g}}P)X, Y). \end{aligned} \tag{3.27}$$

Putting $Y = U$, equation (3.27) becomes

$$X\alpha = u(X)U\alpha, \tag{3.28}$$

since U is an eigenvector of the shape operator $A^{\overline{g}}$. Using (3.27) and (3.28), it follows

$$2cg(PX, Y) + 2g(A^{\overline{g}}PA^{\overline{g}}X, Y) = \alpha g((PA^{\overline{g}} + A^{\overline{g}}P)X, Y), \tag{3.29}$$

from which $cg(PY, X) = g(A^{\overline{g}}X, PA^{\overline{g}}Y)$. Putting $X = PY$ in the last equation, it follows

$$cg(PY, PY) = g(A^{\overline{g}}PY, PA^{\overline{g}}Y) = -g(PA^{\overline{g}}Y, PA^{\overline{g}}Y) \leq 0,$$

and therefore $c \leq 0$, since $\mathrm{rank}P = n - 1$. □

Remark 4. *(cf. [8]) Let M be an n (≥ 3)-dimensional submanifold of maximal CR dimension in a complex space form $\overline{M}$ with holomorphic sectional curvature $4c > 0$, and let the distinguished normal vector field ξ is parallel with respect to the normal connection. If the shape operator $A^{\overline{g}}$ has exactly two distinct eigenvalues, then they are constant.*

Remark 5. *(cf. [9]) Let M be an m-dimensional ($m > 2p - 1$, $p \geq 2$) CR submanifold of CR dimension $\frac{m-1}{2}$ of a complex projective space $P^{\frac{m+p}{2}}(\mathbb{C})$. If the shape operator $A^{\overline{g}}$ with respect to the distinguished normal vector field ξ has exactly two distinct eigenvalues, and if ξ is parallel with respect to the normal connection, then there exists a geodesic hypersphere S of $P^{\frac{m+p}{2}}(\mathbb{C})$ such that M lies on S.*

4. HOLOMORPHIC STATISTICAL MANIFOLDS

4.1 Definitions and properties

Definition 12. *Let* $(\overline{M},\ J,\ \overline{g}\ )$ *be a Kähler manifold and* $\overline{\nabla}$ *an affine connection of* $\overline{M}$. $(\overline{M},\ \overline{\nabla},\ \overline{g},\ J\ )$ *is called a holomorphic statistical manifold if*
1) $(\overline{M},\ \overline{\nabla},\ \overline{g}\ )$ *is a statistical manifold and*
2) $\overline{\omega}\ :=\ \overline{g}\ (\ *,\ J*\ )$ *is a* $\overline{\nabla}$ *- parallel* 2 *- form on* $\overline{M}$.

Lemma 4. *(cf. [12]) Let* $(\overline{M},\ \overline{g},\ J)$ *be a Kähler manifold. If we define a connection* $\overline{\nabla}$ *as* $\overline{\nabla} := \nabla^{\overline{g}} + K$, *where* K *is a* $(1,2)$*-tensor field satisfying the next three conditions:*

$$K(X,\ Y)\ =\ K(Y,\ X), \tag{4.1}$$
$$\overline{g}(K(X,\ Y),\ Z)\ =\ \overline{g}(Y,\ K(X,\ Z)), \tag{4.2}$$
$$K(X,\ JY)\ =\ -JK(X,\ Y), \tag{4.3}$$

for $X,\ Y,\ Z \in \Gamma(TM)$, *then* $(\overline{M},\ \overline{\nabla},\ \overline{g},\ J)$ *is a holomorphic statistical manifold.*

Proof. For $X,\ Y,\ Z\ \in\ \Gamma(T\overline{M})$, we have:
(i)

$$\begin{aligned}\overline{\nabla}_X Y\ -\ \overline{\nabla}_Y X\ &=\ \overline{\nabla}^{\overline{g}}_X Y\ +\ K(X,\ Y)\ -\ \overline{\nabla}^{\overline{g}}_Y X\ -\ K(Y,\ X)\\ &=\ \overline{\nabla}^{\overline{g}}_X Y\ -\ \overline{\nabla}^{\overline{g}}_Y X\ =\ [X,\ Y],\end{aligned}$$

because of (4.1) and because the Levi-Civita connection $\overline{\nabla}^{\overline{g}}_X Y$ is of torsion free.
(ii)

$$\begin{aligned}&(\overline{\nabla}_X\overline{g})(Y,\ Z)\ =\ X\overline{g}(Y,\ Z)\ -\ \overline{g}(\overline{\nabla}_X Y,\ Z)\ -\ \overline{g}(Y,\ \overline{\nabla}_X Z)\\ &=\ \overline{g}(\overline{\nabla}^{\overline{g}}_X Y,\ Z)\ +\ \overline{g}(Y,\ \overline{\nabla}^{\overline{g}}_X Z)\ -\ \overline{g}(\overline{\nabla}_X Y,\ Z)\ -\ \overline{g}(Y,\ \overline{\nabla}_X Z)\\ &=\ \overline{g}(\overline{\nabla}^{\overline{g}}_X Y,\ Z)\ +\ \overline{g}(Y,\ \overline{\nabla}^{\overline{g}}_X Z)\ -\ \overline{g}(\overline{\nabla}^{\overline{g}}_X Y\ +\ K(X,\ Y),\ Z)\ -\ \overline{g}(Y,\ \overline{\nabla}^{\overline{g}}_X Z\ +\ K(X,\ Z))\\ &=\ -\ \overline{g}(K(X,\ Y),\ Z)\ -\ \overline{g}(Y,\ K(X,\ Z))\\ &=\ -\ 2\overline{g}(K(X,\ Y),\ Z),\end{aligned} \tag{4.4}$$

because of (4.2).
Interchanging X and Y in (4.4), we get:

$$(\overline{\nabla}_Y\overline{g})(X,\ Z)\ =\ -\ 2\overline{g}(K(Y,\ X),\ Z). \tag{4.5}$$

Subtracting (4.4) and (4.5), we get:

$$(\overline{\nabla}_X\overline{g})(Y,\ Z)\ -\ (\overline{\nabla}_Y\overline{g})(X,\ Z)\ =\ -\ 2\overline{g}(K(X,\ Y),\ Z) \\ +\ 2\overline{g}(K(Y,\ X),\ Z)\ =\ 0, \qquad (4.6)$$

because of (4.2).
(iii)

$$\begin{aligned}
&(\overline{\nabla}_X\omega)(Y,\ Z)\ =\ X\omega(Y,\ Z)\ -\ \omega(\overline{\nabla}_XY,\ Z)\ -\ \omega(Y,\ \overline{\nabla}_XZ)\\
&=\ X\overline{g}(Y,\ JZ)\ -\ \overline{g}(\overline{\nabla}_XY,\ JZ)\ -\ \overline{g}(Y,\ J\overline{\nabla}_XZ)\\
&=\ \overline{g}(\overline{\nabla}^{\overline{g}}_XY,\ JZ)\ +\ \overline{g}(Y,\ \overline{\nabla}^{\overline{g}}_XJZ)\ -\ \overline{g}(\overline{\nabla}^{\overline{g}}_XY,\ JZ)\ -\ \overline{g}(K(X,\ Y),\ JZ)\ -\ \overline{g}(Y,\ J\overline{\nabla}^{\overline{g}}_XZ)\\
&\quad -\ \overline{g}(Y,\ JK(X,\ Z))\\
&=\ -\ \overline{g}(Y,\ J\overline{\nabla}^{\overline{g}}_XZ)\ -\ \overline{g}(K(X,\ Y),\ JZ)\ -\ \overline{g}(Y,\ J\overline{\nabla}^{\overline{g}}_XZ)\ -\ \overline{g}(Y,\ JK(X,\ Z))\\
&=\ -\ \overline{g}(K(X,\ Y),\ JZ)\ +\ \overline{g}(Y,\ K(X,\ JZ))\ -\ \overline{g}(K(X,\ Y),\ JZ)\ +\ \overline{g}(K(X,\ Y),\ JZ)\\
&=\ 0,
\end{aligned}$$

because of (4.2) and (4.3).
From (i), (ii) and (iii), we conclude that $(\overline{M}, \overline{\nabla}\ :=\ \overline{\nabla}^{\overline{g}}\ +\ K,\ \overline{g},\ J)$ is a holomorphic statistical manifold. □

Lemma 5. *Let $(\overline{M},\ \overline{\nabla},\ \overline{g},\ J)$ be a holomorphic statistical manifold. Then, $\overline{\nabla}_XJY\ =\ J\overline{\nabla}^*_XY$, for $X,\ Y\ \in\ \Gamma(T\overline{M})$, where $\overline{\nabla}^*$ is the dual connection of $\overline{\nabla}$ with respect to $\overline{g}$ (cf. Definition 6).*

Proof. For $X,\ Y,\ Z\ \in\ \Gamma(T\overline{M})$, we have

$$X\omega(Y,\ J^{-1}Z)\ =\ X\overline{g}(Y,\ Z),$$

from which it follows

$$\begin{aligned}
&(\overline{\nabla}_X\omega)(Y,\ J^{-\ 1}Z)\ +\ \omega(\overline{\nabla}_XY,\ J^{-\ 1}Z)\ +\ \omega(Y,\ \overline{\nabla}_X(J^{-\ 1}Z))\\
&=\ \overline{g}(\overline{\nabla}_XY,\ Z)\ +\ \overline{g}(Y,\ \overline{\nabla}^*_XZ),
\end{aligned}$$

that is

$$0\ =\ \overline{g}(\overline{\nabla}_XY,\ Z)\ +\ \overline{g}(Y,\ \overline{\nabla}^*_XZ)\ -\ \omega(\overline{\nabla}_XY,\ J^{-\ 1}Z)\ -\ \omega(Y,\ \overline{\nabla}_X(J^{-\ 1}Z)),$$

from which we conclude that

$$0\ =\ \overline{g}(Y,\ \overline{\nabla}^*_XZ)\ -\ \overline{g}(Y,\ J\overline{\nabla}_X(J^{-\ 1}Z)),$$

i.e.

$$\overline{\nabla}^*_XZ\ =\ -\ J\overline{\nabla}_X(JZ). \qquad \square$$

Example 1. *Let* $\mathbb{C}^2 = (\mathbb{R}^4, g_0, J_0)$ *be the complex Euclidean space, that is* $g_0 = \sum_{i=1}^4 dx^i \otimes dx^i$ *and* $J_0 \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^{i+2}}$, $i = 1,\ 2$, $J_0 \frac{\partial}{\partial x^i} = -\frac{\partial}{\partial x^{i-2}}$, $i = 3,\ 4$. *For functions* α_j *on* $\mathbb{R}^4$, $j = 1, .., \ 8$, *define a* $(1,2)$*-tensor field* $K = \sum_{i,j,l=1}^4 k_{ij}^l \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j$ *on* $\mathbb{C}^2$ *as follows:*

$$
\begin{aligned}
&k_{11}^1 = \alpha_1,\ k_{13}^3 = k_{31}^3 = k_{33}^1 = -\alpha_1,\ k_{11}^2 = k_{12}^1 = k_{21}^1 = \alpha_2,\\
&k_{13}^4 = k_{31}^4 = k_{14}^3 = k_{41}^3 = k_{23}^3 = k_{32}^3 = k_{34}^1 = k_{43}^1 = k_{33}^2 = -\alpha_2,\\
&k_{11}^3 = k_{13}^1 = k_{31}^1 = \alpha_3,\ k_{33}^3 = -\alpha_3,\\
&k_{11}^4 = k_{14}^1 = k_{41}^1 = k_{13}^2 = k_{31}^2 = k_{12}^3 = k_{21}^3 = k_{23}^1 = k_{32}^1 = \alpha_4,\\
&k_{33}^4 = k_{34}^3 = k_{43}^3 = -\alpha_4,\ k_{12}^2 = k_{21}^2 = k_{22}^1 = \alpha_5,\\
&k_{14}^4 = k_{41}^4 = k_{44}^1 = k_{23}^4 = k_{32}^4 = k_{34}^2 = k_{43}^2 = k_{24}^3 = k_{42}^3 = -\alpha_5,\\
&k_{22}^2 = \alpha_6,\ k_{24}^4 = k_{42}^4 = k_{44}^2 = -\alpha_6,\ k_{22}^4 = k_{24}^2 = k_{42}^2 = \alpha_7,\ k_{44}^4 = -\alpha_7,\\
&k_{12}^4 = k_{21}^4 = k_{14}^2 = k_{41}^2 = k_{24}^1 = k_{42}^1 = k_{23}^2 = k_{32}^2 = k_{22}^3 = \alpha_8,\\
&k_{34}^4 = k_{43}^4 = k_{44}^3 = -\alpha_8.
\end{aligned}
$$

Then K *satisfies the conditions in Lemma 4, and we get a holomorphic statistical manifold* $\overline{M} = (\mathbb{R}^4,\ \overline{\nabla} := \nabla^{g_0} + K,\ g_0,\ J_0)$. $\triangle$

Definition 13. *(cf. [13]) A holomorphic statistical manifold* $(\ \overline{M},\ \overline{\nabla},\ \overline{g},\ J\ )$ *is said to be of constant holomorphic sectional curvature* $c \in \mathbb{R}$ *if*

$$
\begin{aligned}
\overline{S}^{(\overline{\nabla},\ \overline{g})}(X,\ Y)Z \ =\ \frac{c}{4}\{&\overline{g}(Y,\ Z)X \ -\ \overline{g}(X,\ Z)Y\\
&+\ \overline{g}(JY,\ Z)JX \ -\ \overline{g}(JX,\ Z)JY\\
&+\ 2\overline{g}(X,\ JY)JZ\},
\end{aligned}
$$

for $X,\ Y,\ Z \in \Gamma(T\overline{M})$.

Example 2. *(cf. [13]) For* $c \in \mathbb{R}$, *let* O *be an interval in* $\{t > 0 |\ 1 - 2ct^3 > 0\}$, *and set a domain* $\Omega = O \times \mathbb{R}$ *in the* $(u^1,\ u^2)$*-plane* $\mathbb{R}^2$. J_0 *denotes the standard complex structure on* Ω, *determined by* $J_0 \frac{\partial}{\partial u^1} = \frac{\partial}{\partial u^2}$. *Define a Riemannian metric* g *and an affine connection* $\widetilde{\nabla}$ *on* Ω *by*

$$
\begin{aligned}
g \ &=\ u^1\{(du^1)^2 \ +\ (du^2)^2\},\\
\widetilde{\nabla}_{\frac{\partial}{\partial u^1}} \frac{\partial}{\partial u^1} \ &=\ -\ \frac{1}{2}\phi(u^1)^{-1}\frac{\partial}{\partial u^1},\\
\widetilde{\nabla}_{\frac{\partial}{\partial u^1}} \frac{\partial}{\partial u^2} \ &=\ \widetilde{\nabla}_{\frac{\partial}{\partial u^2}} \frac{\partial}{\partial u^1} \ =\ (u^1)^{-1}(1 \ +\ \frac{1}{2}\phi(u^1))\frac{\partial}{\partial u^2},\\
\widetilde{\nabla}_{\frac{\partial}{\partial u^2}} \frac{\partial}{\partial u^2} \ &=\ \frac{1}{2}\phi(u^1)^{-1}\frac{\partial}{\partial u^2},
\end{aligned}
$$

where $\phi(t) \ =\ -\ 1 \ \pm\ \sqrt{1 \ -\ 2ct^3}$. *Then* $(\Omega,\ \widetilde{\nabla},\ g,\ J)$ *is a holomorphic statistical manifold of constant holomorphic sectional curvature* c.

Lemma 6. *Let* $(\overline{M},\ \overline{\nabla},\ \overline{g},\ J)$ *be a holomorphic statistical manifold and* M *a submanifold of* $\overline{M}$*. Let* $\nabla,\ \nabla^*,\ h,\ h^*,\ A,\ A^*,\ D$ *and* D^* *be as in Definition 10, and* $P,\ F,\ t$ *and* f *as in (2.21) and (2.22). For* $X,\ Y\ \in\ \Gamma(TM)$*,* $V\ \in\ \Gamma(T^{\perp}M)$ *we have*

$$A_{FY}X\ +\ th^*(X,\ Y)\ =\ \nabla_X(PY)\ -\ P\nabla^*_XY, \tag{4.7}$$

$$-\ h(X,\ PY)\ +\ fh^*(X,\ Y)\ =\ D_X(FY)\ -\ F\nabla^*_XY, \tag{4.8}$$

$$A_{fV}X\ -\ PA^*_VX\ =\ \nabla_X(tV)\ -\ tD^*_XV, \tag{4.9}$$

$$-\ FA^*_VX\ -\ h(X,\ tV)\ =\ D_X(fV)\ -\ fD^*_XV, \tag{4.10}$$

and their duals (4.7)* - (4.10)* *hold.*

Proof. For $X,\ Y\ \in\ \Gamma(TM)$ we have

$$\begin{aligned}\overline{\nabla}_X(JY)\ &=\ \overline{\nabla}_X(PY)\ +\ \overline{\nabla}_X(FY)\\ &=\ \nabla_X(PY)\ -\ A_{FY}X\ +\ h(X,\ PY)\ +\ D_X(FY)\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}J\overline{\nabla}^*_XY\ &=\ J(\nabla^*_XY\ +\ h^*(X,\ Y))\\ &=\ P\nabla^*_XY\ +\ th^*(X,\ Y)\ +\ F\nabla^*_XY\ +\ fh^*(X,\ Y).\end{aligned} \tag{4.12}$$

Subtracting (4.11) and (4.12) and using Lemma 5, we get (4.7) and (4.8). In a similar way we get the other formulas. □

Definition 14. *Let* $(\overline{M},\ \overline{\nabla},\ \overline{g},\ J)$ *be a holomorphic statistical manifold and let* M *be its submanifold.* M *is said to be a totally real submanifold of* $\overline{M}$ *if* $JTM\ \subset\ T^{\perp}M$*.*

Proposition 15. *Let* $(\overline{M},\ \overline{\nabla},\ \overline{g},\ J)$ *be a holomorphic statistical manifold.*
(1) Let M *be a totally real submanifold of* $\overline{M}$*. Then*

$$A_{JY}X\ +\ th^*(X,\ Y)\ =\ 0, \tag{4.13}$$

$$fh^*(X,\ Y)\ =\ D_X(JY)\ -\ J\nabla^*_XY, \tag{4.14}$$

$$A_{fV}X\ =\ \nabla_X(tV)\ -\ tD^*_XV, \tag{4.15}$$

$$-\ JA^*_VX\ -\ h(X,\ tV)\ =\ D_X(fV)\ -\ fD^*_XV, \tag{4.16}$$

and their duals hold.
(2) Let M *be a totally real submanifold of* $\overline{M}$ *with* $D_X(fV)\ =\ fD^*_XV$*. Then*

$$A^*_{\xi}\ =\ 0,\ \xi\ \in\ \Gamma(N), \tag{4.17}$$

$$h(X,\ Y)\ =\ JA^*_{JY}X, \tag{4.18}$$

$$D_X(JY)\ =\ J\nabla^*_XY, \tag{4.19}$$

and their duals hold, where N *is given in Definition 4.*

Proof. Formulas (4.13) - (4.16) follow directly from (4.7) - (4.10), since $P = 0$. From (4.16) it follows that

$$-JA^*_V X = h(X, tV). \tag{4.20}$$

If $V = \xi \in \Gamma(N)$, then from (4.20) we conclude that $A^*_\xi X = 0$, because of Proposition 2. If $V = JY \in \Gamma(J\mathcal{D}^\perp)$ then from (4.20) it follows $-JA^*_{JY}X = h(X, tJY) = h(X, tFY) = -h(X, Y)$, because of (2.24). Hence, we prove (4.17) and (4.18). To prove (4.19) we use (4.17)* and Proposition 10 (3), from which it follows that $h^*(X, Y) \in \Gamma(J\mathcal{D}^\perp)$, i.e. $fh^*(X, Y) = 0$ (cf. Proposition 2 (2)). Now, from (4.14) the equation (4.19) follows. □

Remark 6. *Let M be a totally real submanifold of $\overline{M}$. $D_X(fV) = fD^*_X V$ holds if dim $\overline{M} = 2$ dim M, because $f = 0$.*

Proposition 16. *Let $(\overline{M}, \overline{\nabla}, \overline{g}, J)$ be a holomorphic statistical manifold. We define $I(X, Y) := (\overline{\nabla}_X J)Y$, $I^*(X, Y) := (\overline{\nabla}^*_X J)Y$ for $X, Y \in \Gamma(T\overline{M})$. Then $I(X, Y) = I(Y, X)$, $I(X, Y) = -I^*(X, Y)$ and $I(X, JY) = -JI(X, Y)$.*

Proof. For $X, Y \in \Gamma(T\overline{M})$ we have

$$I(X, Y) = (\overline{\nabla}_X J)Y = \overline{\nabla}_X(JY) - J\overline{\nabla}_X Y = J\overline{\nabla}^*_X Y - J\overline{\nabla}_X Y. \tag{4.21}$$

Interchanging X and Y in (4.21), we get

$$I(Y, X) = J\overline{\nabla}^*_Y X - J\overline{\nabla}_Y X. \tag{4.22}$$

Subtracting (4.21) and (4.22), we get $I(X, Y) - I(Y, X) = 0$, since $\overline{\nabla}^*$ and $\overline{\nabla}$ are of torsion free. Next, we calculate

$$\begin{aligned} I(X, Y) &= (\overline{\nabla}_X J)Y = \overline{\nabla}_X(JY) - J\overline{\nabla}_X Y = J\overline{\nabla}^*_X Y - \overline{\nabla}^*_X(JY) \\ &= -(\overline{\nabla}^*_X J)Y = -I^*(X, Y), \end{aligned}$$

and

$$\begin{aligned} I(X, JY) &= (\overline{\nabla}_X J)JY = \overline{\nabla}_X(JJY) - J\overline{\nabla}_X(JY) \\ &= -J(\overline{\nabla}_X J)Y = -JI(X, Y). \qquad \square \end{aligned}$$

Theorem 5. *Let $(\overline{M}, \overline{\nabla}, \overline{g}, J)$ be a holomorphic statistical manifold and M a totally real submanifold of $\overline{M}$ with $dim\overline{M} = 2dimM$. Then $I(X, Y) \in \Gamma(T^\perp M)$ for $X, Y \in \Gamma(TM)$ if and only if $h = h^*$, i.e. if and only if $A = A^*$.*

Proof. For $X, Y \in \Gamma(TM)$, we calculate

$$\begin{aligned} (\overline{\nabla}_X J)Y &= \nabla_X(PY) - P\nabla_X Y - A_{FY}X - th(X, Y) \\ &\quad + D_X(FY) - F\nabla_X Y + h(X, PY) - fh(X, Y). \end{aligned} \tag{4.23}$$

Comparing the tangential parts in (4.23), we have

$$A_{FY}X \ + \ th(X,\ Y) \ = \ \nabla_X(PY) \ - \ P\nabla_X Y. \tag{4.24}$$

Since $P \ = \ 0$, from (4.24), it follows

$$A_{JY}X \ = \ - \ th(X,\ Y). \tag{4.25}$$

From (4.25) and (4.13), we get

$$th(X,\ Y) \ = \ th^*(X,\ Y),$$

i.e.

$$Jh(X,\ Y) \ = \ Jh^*(X,\ Y),$$

since $f \ = \ 0$. $\square$

Example 3. *Let $z = (z_1,\ z_2)$, with $z_j = x_j \ + \ \sqrt{-1}y_j$, $j = 1,\ 2$, denote the complex coordinates on $\mathbb{C}^2$. An immersion $\iota : \mathbb{R}^2 \to \mathbb{C}^2$ given by $\iota : (x_1,\ x_2) \mapsto (x_1,\ x_2)$ is totally real. Suppose $\alpha_3 = \alpha_4 = \alpha_7 = \alpha_8 = 0$ for $\overline{M} = (\mathbb{R}^4,\ \overline{\nabla} \ := \ \nabla^{g_0} + K,\ g_0,\ J_0)$ in Example 1. Then $I(X,\ Y)$ is orthogonal to $\mathbb{R}^2$ for $X,\ Y \ \in \ \Gamma(T\mathbb{R}^2)$. We can easily check this by calculating $\overline{g}(K(J.,\ .),\ .) \ = \ 0$ on the basis vectors $\frac{\partial}{\partial x_i}$, $i \ = \ 1,\ 2$, where $\overline{g}$ denotes the Euclidean metric.* $\triangle$

4.2 Semiparallel statistical submanifolds of constant curvature

Definition 15. *Let M be a submanifold of a holomorphic statistical manifold $\overline{M}$.*
(1) M is said to be totally geodesic for the connection $\overline{\nabla}$ if $h \ = \ 0$.
(2) M is said to be totally geodesic for the connection $\overline{\nabla}^*$ if $h^* \ = \ 0$.

Definition 16. *Let M be a submanifold of a holomorphic statistical manifold $\overline{M}$.*
(1) M is said to have parallel second fundamental form h with respect to the connection $\overline{\nabla}$ if $\widetilde{\nabla}h \ = \ 0$. Here,

$$(\widetilde{\nabla}_X h)(Y,\ Z) \ := \ D_X h(Y,\ Z) \ - \ h(\nabla_X Y,\ Z) \ - \ h(Y,\ \nabla_X Z).$$

(2) M is called a semi-parallel submanifold for the connection $\overline{\nabla}$ if $R^{\widetilde{\nabla}}(X,\ Y)h \ = \ 0$, where $(R^{\widetilde{\nabla}}(X,\ Y)h)(Z,\ W) \ :=$ $(\widetilde{\nabla}_X(\widetilde{\nabla}_Y h))(Z,\ W) \ - \ (\widetilde{\nabla}_Y(\widetilde{\nabla}_X h))(Z,\ W) \ - \ (\widetilde{\nabla}_{[X,\ Y]}h)(Z,\ W)$.

Proposition 17. *Let $(\overline{M},\ \overline{\nabla},\ \overline{g},\ J)$ be a holomorphic statistical manifold and M a totally real submanifold of $\overline{M}$. Suppose:*
1) $D_X(fV) \ = \ fD^*_X V$ for $X \ \in \ \Gamma(TM)$ and $V \ \in \ \Gamma(T^\perp M)$,
2) $(\nabla,\ g)$ is of constant curvature $c \ \neq \ 0$, in a sense of Definition 7.

Then, for $Z \in \Gamma(TM)$, *from* $R^{\perp}(X, Y)JZ = 0$ *for* $X, Y \in \Gamma(TM)$, *it follows* $Z = 0$.
In particular, if $R^{\perp}(X, Y)H = 0$ *for* $X, Y \in \Gamma(TM)$, *then* $H = 0$. *Here* H *denotes the mean curvature vector field of* M *with respect to* $\overline{\nabla}$, *that is* $H = \frac{1}{m} tr_g h$.

Proof. For $X, Y, Z \in \Gamma(TM)$ we have

$$\begin{aligned} 0 &= R^{\perp}(X, Y)JZ = D_X D_Y(JZ) - D_Y D_X(JZ) - D_{[X, Y]}(JZ) \\ &= D_X J\nabla^*_Y Z - D_Y J\nabla^*_X Z - J\nabla^*_{[X, Y]}Z \\ &= J\nabla^*_X\nabla^*_Y Z - J\nabla^*_Y\nabla^*_X Z - J\nabla^*_{[X, Y]}Z \\ &= JR^*(X, Y)Z \\ &= cg(Y, Z)JX - cg(X, Z)JY. \end{aligned} \tag{4.26}$$

Putting $Y = Z$ and X orthogonal to Y in the last equation, we obtain $Z = 0$. Next, using (4.18), we get $H = J(\frac{1}{m}\sum_{i=1}^{m} A^*_{Je_i} e_i)$, where $\{e_i\}_{i=1}^m$ is an orthonormal basis with respect to g. Therefore we conclude that from $R^{\perp}(X, Y)H = 0$ it follows $H = 0$. □
The dual assertion of Proposition 17 holds, which means that $R^{\perp *}(X, Y)JZ = 0$ implies $Z = 0$ under the same assumption.

Theorem 6. *Let* $(\overline{M}, \overline{\nabla}, \overline{g}, J)$ *be a holomorphic statistical manifold and* M *a totally real submanifold of* $\overline{M}$. *Suppose:*
1) $D_X(fV) = fD^*_X V$,
2) (∇, g) *is of constant curvature* $c \neq 0$, *in a sense of Definition 7.*
If M *is semi-parallel for* $\overline{\nabla}$, *then* M *is totally geodesic for* $\overline{\nabla}$.

Proof. For $X, Y \in \Gamma(TM)$ we have

$$\begin{aligned} mR^{\perp}(X, Y)H &= \sum_{i=1}^{m} R^{\perp}(X, Y)h(e_i, e_i) \\ &= (\overline{R}(X, Y)h)(e_i, e_i) + h(R(X, Y)e_i, e_i) + h(e_i, R(X, Y)e_i) \\ &= 2\sum_{i=1}^{m} h(R(X, Y)e_i, e_i) \\ &= 2c\sum_{i=1}^{m} h(g(Y, e_i)X - g(X, e_i)Y, e_i) \\ &= 2c\sum_{i=1}^{m} \{g(Y, e_i)h(X, e_i) - g(X, e_i)h(Y, e_i)\} \\ &= 2c\{h(X, Y) - h(Y, X)\} = 0, \end{aligned}$$

i.e.

$$R^{\perp}(X, Y)H = 0. \tag{4.27}$$

On the other hand, by (4.18) and (4.26),

$$\begin{aligned}
0 &= R^{\widetilde{\nabla}}(X,\ Y)h \\
&= R^{\perp}(X,\ Y)h(Z,\ W) \ -\ h(R(X,\ Y)Z,\ W) \ -\ h(Z,\ R(X,\ Y)W) \\
&= R^{\perp}(X,\ Y)JA^*_{JZ}W \ -\ JA^*_{JW}R(X,\ Y)Z \ -\ JA^*_{JZ}R(X,\ Y)W \\
&= cJ\{g(Y,\ A^*_{JZ}W)X \ -\ g(X,\ A^*_{JZ}W)Y \ -\ g(Y,\ Z)A^*_{JW}X \\
&\quad +\ g(X,\ Z)A^*_{JW}Y \ -\ g(Y,\ W)A^*_{JZ}X \ +\ g(X,\ W)A^*_{JZ}Y\}. \qquad (4.28)
\end{aligned}$$

Now we will put $X = W = e_i$ in (4.28) and add for $i = 1, \ldots, m$, we get:

$$\begin{aligned}
0 &= \sum_{i=1}^{m}\{g(A^*_{JZ}e_i,\ Y)e_i \ -\ g(A^*_{JZ}e_i,\ e_i)Y \ -\ g(Y,\ Z)A^*_{Je_i}e_i \\
&\quad +\ g(e_i,\ Z)A^*_{Je_i}Y \ -\ g(Y,\ e_i)A^*_{JZ}e_i \ +\ g(e_i,\ e_i)A^*_{JZ}Y\} \\
&= \sum_{i=1}^{m}\{g(A^*_{JZ}Y,\ e_i)e_i \ -\ g(A^*_{Je_i}e_i,\ Z)Y \ -\ g(Y,\ Z)A^*_{Je_i}e_i \\
&\quad +\ A^*_{g(e_i,\ Z)Je_i}Y \ -\ A^*_{JZ}g(Y,\ e_i)e_i \ +\ g(e_i,\ e_i)A^*_{JZ}Y\} \\
&= A^*_{JZ}Y \ -\ g(-mJH,\ Z)Y \ +\ g(Y,\ Z)mJH \ +\ A^*_{JZ}Y \ -\ A^*_{JZ}Y \\
&\quad +\ mA^*_{JZ}Y \\
&= mg(JH,\ Z)Y \ +\ mg(Y,\ Z)JH \ +\ (m\ +\ 1)A^*_{JZ}Y. \qquad (4.29)
\end{aligned}$$

From (4.27) and Proposition 17 it follows $H = 0$, i.e. $A^*_{JZ}Y = 0$, because of (4.29). □

Corollary 2. *Let $(\overline{M},\ \overline{\nabla},\ \overline{g},\ J)$ be a holomorphic statistical manifold and M a Lagrangian submanifold of $\overline{M}$. If $(\nabla,\ g)$ is of constant curvature $c \neq 0$ and M is semi-parallel for $\overline{\nabla}$, then M is totally geodesic with respect to $\overline{\nabla}$.*

4.3 CR statistical submanifolds

Let $(\overline{M},\ \overline{\nabla},\ \overline{g},\ J)$ be a $2m$-dimensional holomorphic statistical manifold and M a $(2m\ -\ 1)$-dimensional submanifold of $\overline{M}$, i.e. a hypersurface of $\overline{M}$. Let ξ be a unit normal vector field of M. The Gauss and Weingarten equations are given by:

$$\begin{aligned}
&\overline{\nabla}_X Y = \nabla_X Y + h(X,\ Y), \quad \overline{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,\ Y), \\
&\overline{\nabla}_X \xi = -AX + s(X)\xi, \quad \overline{\nabla}^*_X \xi = -A^*X + s^*(X)\xi,
\end{aligned}$$

respectively. Here,

$$g(A^*X,\ Y) = \overline{g}(h(X,\ Y),\ \xi), \quad g(AX,\ Y) = \overline{g}(h^*(X,\ Y),\ \xi).$$

Since

$$\begin{aligned}
0 &= X\overline{g}(\xi,\ \xi) = \overline{g}(\overline{\nabla}_X\xi,\ \xi) + \overline{g}(\xi,\ \overline{\nabla}^*_X\xi) \\
&= \overline{g}(-AX + s(X)\xi,\ \xi) + \overline{g}(\xi,\ -A^*X + s^*(X)\xi) \\
&= s(X) + s^*(X),
\end{aligned}$$

we conclude that

$$s(X) \;=\; -\,s^*(X).$$

The structure vector U is defined by

$$U \;:=\; -\,J\xi \;\in\; \Gamma(TM).$$

For $X \in \Gamma(TM)$, JX decomposes to the tangent and the normal part as:

$$JX \;=\; PX \;+\; g(X,\,U)\xi. \tag{4.30}$$

The Gauss and Codazzi equations are derived in the following way.

$$\begin{aligned}
\overline{R}(X,\,Y)Z \;&=\; \overline{\nabla}_X\overline{\nabla}_Y Z \;-\; \overline{\nabla}_Y\overline{\nabla}_X Z \;-\; \overline{\nabla}_{[X,\,Y]}Z \\
&=\; \overline{\nabla}_X(\nabla_Y Z \;+\; h(Y,\,Z)) \;-\; \overline{\nabla}_Y(\nabla_X Z \;+\; h(X,\,Z)) \;-\; \nabla_{[X,\,Y]}Z \\
&\;-\; h([X,\,Y],\,Z) \\
&=\; \overline{\nabla}_X(\nabla_Y Z \;+\; g(A^*Y,\,Z)\xi) \\
&\;-\; \overline{\nabla}_Y(\nabla_X Z \;+\; g(A^*X,\,Z)\xi) \;-\; \nabla_{[X,\,Y]}Z \;-\; h([X,\,Y],\,Z) \\
&=\; \nabla_X\nabla_Y Z \;+\; h(X,\,\nabla_Y Z) \;+\; \overline{\nabla}_X(g(A^*Y,\,Z)\xi) \;-\; \nabla_Y\nabla_X Z \\
&\;-\; h(Y,\,\nabla_X Z) \;-\; \overline{\nabla}_Y(g(A^*X,\,Z)\xi) \;-\; \nabla_{[X,\,Y]}Z \;-\; g(A^*[X,\,Y],\,Z)\xi \\
&=\; \nabla_X\nabla_Y Z \;+\; g(A^*X,\,\nabla_Y Z)\xi \;+\; (\nabla_X g)(A^*Y,\,Z) \;+\; g((\nabla^*_X A^*)Y,\,Z)\xi \\
&\;+\; g(A^*\nabla^*_X Y,\,Z)\xi \;+\; g(A^*Y,\,\nabla_X Z)\xi \;+\; g(A^*Y,\,Z)\overline{\nabla}_X\xi \;-\; \nabla_Y\nabla_X Z \\
&\;-\; g(A^*Y,\,\nabla_X Z)\xi \;-\; g((\nabla^*_Y A^*)X,\,Z)\xi \\
&\;-\; g(A^*\nabla^*_Y X,\,Z)\xi \;-\; g(A^*X,\,\nabla_Y Z)\xi \;-\; g(A^*X,\,Z)\overline{\nabla}_Y\xi \;-\; \nabla_{[X,\,Y]}Z \\
&\;-\; g(A^*[X,\,Y],\,Z)\xi \\
&=\; \nabla_X\nabla_Y Z \;+\; g(A^*X,\,\nabla_Y Z)\xi \;+\; g((\nabla^*_X A^*)Y,\,Z)\xi \\
&\;+\; g(A^*\nabla^*_X Y,\,Z)\xi \;+\; g(A^*Y,\,\nabla_X Z)\xi \;-\; g(A^*Y,\,Z)AX \;+\; g(A^*Y,\,Z)s(X)\xi \\
&\;-\; \nabla_Y\nabla_X Z \;-\; g(A^*Y,\,\nabla_X Z)\xi \;-\; g((\nabla^*_Y A^*)X,\,Z)\xi \;-\; g(A^*\nabla^*_Y X,\,Z)\xi \\
&\;-\; g(A^*X,\,\nabla_Y Z)\xi \;+\; g(A^*X,\,Z)AY \;-\; g(A^*X,\,Z)s(Y)\xi \;-\; \nabla_{[X,\,Y]}Z \\
&\;-\; g(A^*[X,\,Y],\,Z)\xi \\
&=\; R(X,\,Y)Z \;-\; \{g(A^*Y,\,Z)AX \;-\; g(A^*X,\,Z)AY\} \\
&\;+\; g((\nabla^*_X A^*)Y,\,Z)\xi \;-\; g((\nabla^*_Y A^*)X,\,Z)\xi \;+\; g(A^*Y,\,Z)s(X)\xi \;-\; g(A^*X,\,Z)s(Y)\xi.
\end{aligned}$$

The dual equation of the last equation is:

$$\begin{aligned}
\overline{R}^*(X,\,Y)Z \;=\; & R^*(X,\,Y)Z \;-\; \{g(AY,\,Z)A^*X \;-\; g(AX,\,Z)A^*Y\} \;+\; g((\nabla_X A)Y,\,Z)\xi \\
& -\; g((\nabla_Y A)X,\,Z)\xi \;+\; g(AY,\,Z)s^*(X)\xi \;-\; g(AX,\,Z)s^*(Y)\xi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
S^{(\overline{\nabla},\,\overline{g})}(X,\,Y)Z \;=\; & S^{(\nabla,\,g)}(X,\,Y)Z \;+\; \frac{1}{2}\{g(A^*X,\,Z)AY \;-\; g(A^*Y,\,Z)AX \\
& +\; g(AX,\,Z)A^*Y \;-\; g(AY,\,Z)A^*X \;+\; g((\nabla^*_X A^* \;+\; \nabla_X A)Y,\,Z)\xi \\
& -\; g((\nabla^*_Y A^* \;+\; \nabla_Y A)X,\,Z)\xi \;+\; g(A^*Y,\,Z)s(X)\xi \;-\; g(A^*X,\,Z)s(Y)\xi \\
& +\; g(AY,\,Z)s^*(X)\xi \;-\; g(AX,\,Z)s^*(Y)\xi\}.
\end{aligned}$$

From the last equation we get the Gauss equation:

$$\begin{aligned}\overline{g}(S^{(\overline{\nabla},\,\overline{g})}(X,\,Y)Z,\,W) &= g(S^{(\nabla,\,g)}(X,\,Y)Z,\,W) + \frac{1}{2}\{g(A^*X,\,Z)g(AY,\,W)\\ &\quad - g(A^*Y,\,Z)g(AX,\,W) + g(AX,\,Z)g(A^*Y,\,W)\\ &\quad - g(AY,\,Z)g(A^*X,\,W)\},\end{aligned} \tag{4.31}$$

and the Codazzi equation:

$$\begin{aligned}\overline{g}(S^{(\overline{\nabla},\,\overline{g})}(X,\,Y)Z,\,\xi) &= \frac{1}{2}\{g((\nabla^*_XA^* + \nabla_XA)Y,\,Z) - g((\nabla^*_YA^* + \nabla_YA)X,\,Z)\\ &\quad + g((A^* - A)Y,\,Z)s(X) - g((A^* - A)X,\,Z)s(Y)\},\end{aligned} \tag{4.32}$$

where $X,\,Y,\,Z,\,W \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$.
When $\overline{M}$ is of constant holomorphic sectional curvature c, the Gauss and the Codazzi equations are:

$$\begin{aligned}S^{(\nabla,\,g)}(X,\,Y)Z &= \frac{1}{2}\{g(A^*Y,\,Z)AX - g(A^*X,\,Z)AY + g(AY,\,Z)A^*X\\ &\quad - g(AX,\,Z)A^*Y\} + \frac{c}{4}\{g(Y,\,Z)X - g(X,\,Z)Y\\ &\quad + g(PY,\,Z)PX - g(PX,\,Z)PY + 2g(X,\,PY)PZ\},\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}&\frac{c}{2}\{g(X,\,U)PY - g(Y,\,U)PX + 2g(X,\,PY)U\}\\ &= (\nabla^*_XA^* + \nabla_XA)Y - (\nabla^*_YA^* + \nabla_YA)X\\ &\quad + s(X)(A^* - A)Y - s(Y)(A^* - A)X,\end{aligned} \tag{4.34}$$

respectively.

Proposition 18. *Let M be a real hypersurface of a holomorphic statistical manifold $\overline{M}$. For $X,\,Y \in \Gamma(TM)$, the following relations hold:*
(1) $\nabla_XU = PA^*X + s^*(X)U = PA^*X - s(X)U$,
(2) $(\nabla_XP)Y = u(Y)AX - g(AX,\,Y)U - P\nabla_XY + P\nabla^*_XY$.

Proof. (1)

$$\begin{aligned}\nabla_XU &= \overline{\nabla}_XU - h(X,\,U)\\ &= -\overline{\nabla}_X(J\xi) - h(X,\,U)\\ &= -J\overline{\nabla}^*_X\xi - h(X,\,U)\\ &= J(A^*X - s^*(X)\xi) - h(X,\,U)\\ &= PA^*X + g(A^*X,\,U)\xi + s^*(X)U - h(X,\,U)\\ &= PA^*X + s^*(X)U.\end{aligned}$$

(2)

$$\begin{aligned}
0 &= \overline{\nabla}_X JY - J\overline{\nabla}^*_X Y = \overline{\nabla}_X(PY + u(Y)\xi) - J(\nabla^*_X Y + g(AX, Y)\xi) \\
&= \nabla_X(PY) + g(A^*X, PY)\xi + Xu(Y)\xi + u(Y)(-AX + s(X)\xi) \\
&\quad - P\nabla^*_X Y - u(\nabla^*_X Y)\xi + g(AX, Y)U \\
&= (\nabla_X P)Y + P\nabla_X Y + g(A^*X, PY)\xi \\
&\quad + g(\nabla_X U, Y)\xi + g(U, \nabla^*_X Y)\xi - u(Y)AX \\
&\quad + g(U, Y)s(X)\xi - P\nabla^*_X Y - g(\nabla^*_X Y, U)\xi + g(AX, Y)U \\
&= (\nabla_X P)Y + P\nabla_X Y - u(Y)AX + g(AX, Y)U - P\nabla^*_X Y. \qquad \square
\end{aligned}$$

Theorem 7. *Let M be a real hypersurface of a holomorphic statistical manifold $\overline{M}$ of constant holomorphic sectional curvature c. If for the shape operators A, A^* of M, and functions α and β, $AX = \alpha X$ and $A^*X = \beta X$, then $c = 0$.*

Proof. Differentiating $AX = \alpha X$, we get

$$(\nabla_Y A)X + A\nabla_Y X = (Y\alpha)X + \alpha\nabla^*_Y X,$$

i.e.

$$(\nabla_Y A)X = (Y\alpha)X. \tag{4.35}$$

Similarly, we get that

$$(\nabla^*_Y A^*)X = (Y\beta)X. \tag{4.36}$$

From the Codazzi equation (4.34), (4.35) and (4.36), we have

$$\begin{aligned}
&X(\alpha + \beta)Y - Y(\alpha + \beta)X + (\beta - \alpha)(s(X)Y - s(Y)X) \\
&= \frac{c}{2}\{g(X, U)PY - g(Y, U)PX + 2g(X, PY)U\}.
\end{aligned} \tag{4.37}$$

Now, we put $Y = U$ in (4.37). We get

$$X(\alpha + \beta)U - U(\alpha + \beta)X + (\beta - \alpha)(s(X)U - s(U)X) = -\frac{c}{2}PX.$$

From this equation we get

$$\begin{aligned}
&X(\alpha + \beta)g(U, PX) - U(\alpha + \beta)g(X, PX) \\
&\quad + (\beta - \alpha)(s(X)g(U, PX) - s(U)g(X, PX)) = -\frac{c}{2}g(PX, PX),
\end{aligned}$$

that is, $c = 0$. $\square$

Proposition 19. *Let M be a real hypersurface of a holomorphic statistical manifold $\overline{M}$ of constant holomorphic sectional curvature c. If for the shape operators A, A^* of M, and functions α and β, $AX = \alpha X$ and $A^*X = \beta X$, then $\alpha + \beta = const.$ if and only if ξ is parallel with respect to the normal connection D (D^*).*

Proof. From Theorem 7, using the Codazzi equation (4.34), we get

$$X(\alpha + \beta)U - U(\alpha + \beta)X + (\beta - \alpha)(s(X)U - s(U)X) = 0. \quad (4.38)$$

Next, we multiply the last equation by PY. The result is

$$U(\alpha + \beta)PX + (\beta - \alpha)PX = 0.$$

From the last equation it follows

$$U(\alpha + \beta) = (\alpha - \beta)s(U).$$

Now, from the last equation and (4.38), we get

$$X(\alpha + \beta) = (\alpha - \beta)s(X). \qquad \square$$

Let M be an n-dimensional CR submanifold of maximal CR dimension in a holomorphic statistical manifold $\overline{M}$ of dimension $n + p$. Using a basis $\xi,\ \xi_1, \ldots, \xi_q,\ \xi_{1^*}, \ldots, \xi_{q^*},\ q = \frac{p-1}{2}$, as in Lemma 3, its shape operators $A := A_\xi,\ A_1 := A_{\xi_1} \ldots, A_q := A_{\xi_q},\ A_{1^*} := A_{\xi_{1^*}}, \ldots, A_{q^*} := A_{\xi_{q^*}}$ for $\overline{\nabla}$ and its shape operators $A^* := A^*_\xi,\ A^*_1 := A^*_{\xi_1} \ldots, A^*_q := A^*_{\xi_q},\ A^*_{1^*} := A^*_{\xi_{1^*}}, \ldots, A^*_{q^*} := A^*_{\xi_{q^*}}$ for $\overline{\nabla}^*$, we can write $D_X\xi$ and $D^*_X\xi$ as:

$$D_X\xi = s(X)\xi + \sum_{a=1}^{q} \{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*}\}, \quad (4.39)$$

and

$$D^*_X\xi = s^*(X)\xi + \sum_{a=1}^{q} \{s^*_a(X)\xi_a + s^*_{a^*}(X)\xi_{a^*}\}. \quad (4.39)^*$$

Then, the Weingarten equations are:

$$\overline{\nabla}_X\xi = -AX + D_X\xi = -AX + s(X)\xi + \sum_{a=1}^{q} \{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*}\}, \quad (4.40)$$

$$\overline{\nabla}^*_X\xi = -A^*X + D^*_X\xi = -A^*X + s^*(X)\xi + \sum_{a=1}^{q} \{s^*_a(X)\xi_a + s^*_{a^*}(X)\xi_{a^*}\}, \quad (4.40)^*$$

$$\overline{\nabla}_X\xi_a = -A_aX + D_X\xi_a = -A_aX + s_{a0}(X)\xi + \sum_{b=1}^{q} \{s_{ab}(X)\xi_b + s_{ab^*}(X)\xi_{b^*}\}, \quad (4.41)$$

$$\overline{\nabla}^*_X\xi_a \;=\; -\,A^*_aX \;+\; D^*_X\xi_a \;=\; -\,A^*_aX \;+\; s^*_{a0}(X)\xi \;+\; \sum_{b\,=\,1}^{q}\{s^*_{ab}(X)\xi_b \;+\; s^*_{ab^*}(X)\xi_{b^*}\}, \tag{4.41)*}$$

$$\overline{\nabla}_X\xi_{a^*} \;=\; -\,A_{a^*}X \;+\; D_X\xi_{a^*} \;=\; -\,A_{a^*}X \;+\; s_{a^*0}(X)\xi \;+\; \sum_{b\,=\,1}^{q}\{s_{a^*b}(X)\xi_b \;+\; s_{a^*b^*}(X)\xi_{b^*}\}, \tag{4.42}$$

$$\overline{\nabla}^*_X\xi_{a^*} \;=\; -\,A^*_{a^*}X \;+\; D^*_X\xi_{a^*} \;=\; -\,A^*_{a^*}X \;+\; s^*_{a^*0}(X)\xi \;+\; \sum_{b\,=\,1}^{q}\{s^*_{a^*b}(X)\xi_b \;+\; s^*_{a^*b^*}(X)\xi_{b^*}\}, \tag{4.42)*}$$

Using (4.40), (4.40)*, (4.41), (4.41)*, (4.42) and (4.42)*, we obtain

$$\begin{aligned} 0 \;&=\; X\overline{g}(\xi_a,\;\xi_b) \;=\; \overline{g}(\overline{\nabla}_X\xi_a,\;\xi_b) \;+\; \overline{g}(\xi_a,\;\overline{\nabla}^*_X\xi_b) \\ &=\; \overline{g}(\sum_{c\,=\,1}^{p} s_{ac}(X)\xi_c,\;\xi_b) \;+\; \overline{g}(\xi_a,\;\sum_{c\,=\,1}^{p} s^*_{bc}(X)\xi_c) \\ &=\; s_{ab}(X) \;+\; s^*_{ba}(X), \end{aligned}$$

that is

$$s_{ab} \;=\; -\,s^*_{ba}. \tag{4.43}$$

Differentiating the equation $\xi_{a^*} \;=\; J\xi_a$, and using (4.40), (4.41), (4.42) and their dual equations, we get

$$A_{a^*}X \;=\; PA^*_aX \;-\; s_{a0}(X)U, \quad A^*_{a^*}X \;=\; PA_aX \;-\; s^*_{a0}(X)U, \tag{4.44}$$

$$A_aX \;=\; -\,PA^*_{a^*}X \;+\; s_{a^*0}(X)U, \quad A^*_aX \;=\; -PA_{a^*}X \;+\; s^*_{a^*0}(X)U, \tag{4.45}$$

$$s^*_{a^*0}(X) \;=\; u(A^*_aX) \;=\; g(A^*_aX,\;U), \quad s_{a^*0}(X) \;=\; u(A_aX) \;=\; g(A_aX,\;U), \tag{4.46}$$

$$s^*_{a0}(X) \;=\; -\,u(A^*_{a^*}X) \;=\; -\,g(A^*_{a^*}X,\;U), \quad s_{a0}(X) \;=\; -\,u(A_{a^*}X) \;=\; -\,g(A_{a^*}X,\;U), \tag{4.47}$$

$$s_{a^*b} \;=\; -\,s^*_{ab^*}, \quad s_{a^*b^*} \;=\; s^*_{ab}, \quad s^*_{a^*b} \;=\; -\,s_{ab^*}, \quad s^*_{a^*b^*} \;=\; s_{ab}. \tag{4.48}$$

Using the equations (4.44) and (4.45), we get

$$trace\ A_{a^*} = -s_a(U), \quad trace\ A_a = s_{a^*}(U), \quad trace\ A^*_{a^*} = -s^*_a(U),$$
$$trace\ A^*_a = s^*_{a^*}(U). \tag{4.49}$$

Furthermore, from the equations (4.44)-(4.48) it follows:

$$\begin{aligned} g((A_aP + PA_a)X, Y) &= g(PX, A_aY) - g(A_aX, PY) \\ &= g(P^2X, A^*_{a^*}Y) - g(A^*_{a^*}X, P^2Y) \\ &= g(-X + g(X, U)U, A^*_{a^*}Y) - g(A^*_{a^*}X, -Y + g(Y, U)U) \\ &= -s^*_{a0}(Y)g(X, U) + s^*_{a0}(X)g(Y, U). \end{aligned} \tag{4.50}$$

In a similar way, we get the following equation.

$$g((A_{a^*}P + PA_{a^*})X, Y) = s^*_{a^*0}(X)g(Y, U) - s^*_{a^*0}(Y)g(X, U). \tag{4.51}$$

Now, suppose $\overline{M}$ is of constant holomorphic curvature c. Then, the Gauss equations and the Codazzi equation for $\xi \in \Gamma(T^\perp M)$ are:

$$\begin{aligned} S^{(\nabla, g)} &= \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(PY, Z)PX - g(PX, Z)PY \\ &\quad - 2g(PX, Y)PZ\} + \frac{1}{2}\{g(AY, Z)A^*X - g(AX, Z)A^*Y + g(A^*Y, Z)AX \\ &\quad - g(A^*X, Z)AY\} + \frac{1}{2}\sum_{b=1}^{q}\{g(A_bY, Z)A^*_bX - g(A_bX, Z)A^*_bY \\ &\quad + g(A^*_bY, Z)A_bX - g(A^*_bX, Z)A_bY\} + \frac{1}{2}\sum_{b=1}^{q}\{g(A_{b^*}Y, Z)A^*_{b^*}X \\ &\quad - g(A_{b^*}X, Z)A^*_{b^*}Y + g(A^*_{b^*}Y, Z)A_{b^*}X - g(A^*_{b^*}X, Z)A_{b^*}Y\}, \end{aligned} \tag{4.52}$$

and

$$\begin{aligned} &(\nabla^*_XA^* + \nabla_XA)Y - (\nabla^*_YA^*X + \nabla_YA)X = \frac{c}{2}\{u(X)PY - u(Y)PX \\ &- 2g(PX, Y)U\} + s(X)(A - A^*)Y - s(Y)(A - A^*)X \\ &+ \sum_{b=1}^{q}\{s_{b0}(X)(A_b - A^*_b)Y - s_{b0}(Y)(A_b - A^*_b)X\}, \end{aligned} \tag{4.53}$$

respectively.

Theorem 8. *Let M be an n-dimensional CR submanifold of maximal CR dimension in an $(n + p)$-dimensional holomorphic statistical manifold $\overline{M}$ of constant holomorphic sectional curvature c and let $p < n$. If for the shape operators A, A^* of the distinguished normal vector field ξ, and functions α and β, $AX = \alpha X$ and $A^*X = \beta X$, then $c = 0$.*

Proof. Differentiating $AX = \alpha X$, $A^*X = \beta X$, and using the Codazzi equation (4.53), we get

$$X(\alpha + \beta)Y - Y(\alpha + \beta)X = \frac{c}{2}\{u(X)PY - u(Y)PX - 2g(PX, Y)U\}$$
$$+ s(X)(\alpha - \beta)Y - s(Y)(\alpha - \beta)X + \sum_{b=1}^{q}\{s^*_{b0}(X)A^*_bY - s^*_{b0}(Y)A^*_bX\}$$
$$+ \sum_{b=1}^{q}\{s_{b0}(X)A_bY - s_{b0}(Y)A_bX\}. \tag{4.54}$$

If we multiply (4.54) by U, we get

$$X(\alpha + \beta)g(Y, U) - Y(\alpha + \beta)g(X, U) = -cg(PX, Y)$$
$$+ s(X)(\alpha - \beta)u(Y) - s(Y)(\alpha - \beta)u(X)$$
$$+ 2\sum_{b=1}^{q}\{s_{b0}(X)s_{b^*0}(Y) - s_{b0}(Y)s_{b^*0}(X)\}. \tag{4.55}$$

If we put $Y = U$ in (4.55), we get

$$X(\alpha + \beta) - U(\alpha + \beta)u(X) = (\alpha - \beta)(s(X) - s(U)u(X))$$
$$+ 2\sum_{b=1}^{q}\{s_{b0}(X)s_{b^*0}(U) - s_{b0}(U)s_{b^*0}(X)\}. \tag{4.56}$$

Next, we put $Y = U$ in (4.54), we obtain

$$X(\alpha + \beta)U - U(\alpha + \beta)X = -\frac{c}{2}PX + s(X)(\alpha - \beta)U - s(U)(\alpha - \beta)X$$
$$+ \sum_{b=1}^{q}\{s_{b0}(X)(A_b - A^*_b)U - s_{b0}(U)(A_b - A^*_b)X\}. \tag{4.57}$$

From (4.56) and (4.57), we have

$$U(\alpha + \beta)u(X)U - (\alpha - \beta)s(U)u(X)U + 2\sum_{b=1}^{q}\{s_{b0}(X)s_{b^*0}(U)U - s_{b0}(U)s_{b^*0}(X)U\}$$
$$= U(\alpha + \beta)X - \frac{c}{2}PX - s(U)(\alpha - \beta)X$$
$$+ \sum_{b=1}^{q}\{s_{b0}(X)(A_b - A^*_b)U - s_{b0}(U)(A_b - A^*_b)X\}. \tag{4.58}$$

Now, we apply P to (4.58), we get

$$- U(\alpha + \beta)PX + \frac{c}{2}P^2X + s(U)(\alpha - \beta)PX$$
$$= \sum_{b=1}^{q}\{- s_{b0}(X)P^2A^*_{b^*}U + s_{b0}(X)P^2A_{b^*}U + s_{b0}(U)P^2A^*_{b^*}X - s_{b0}(U)P^2A_{b^*}X\}, \tag{4.59}$$

where we used (4.44) and (4.45). From (4.59) and (**??**), we get

$$\begin{aligned} &- U(\alpha + \beta)PX + \frac{c}{2}P^2X + s(U)(\alpha - \beta)PX \\ &= \sum_{b=1}^{q} \{s_{b0}(X)A^*_{b^*}U - s_{b0}(X)A_{b^*}U - s_{b0}(U)A^*_{b^*}X + s_{b0}(U)A_{b^*}X\}. \end{aligned} \tag{4.60}$$

When we multiply (4.60) by Y and interchange X and Y in the obtained equation, and then subtract the obtained two equations, we get

$$U(\alpha + \beta) = s(U)(\alpha - \beta). \tag{4.61}$$

From (4.61) and (4.56), we obtain

$$X(\alpha + \beta) = (\alpha - \beta)s(X) + 2\sum_{b=1}^{q} \{s_{b0}(X)s_{b^*0}(U) - s_{b0}(U)s_{b^*0}(X)\}. \tag{4.62}$$

From (4.55) and (4.62), we conclude that

$$\begin{aligned} &2\sum_{b=1}^{q} \{s_{b0}(X)s_{b^*0}(U)U - s_{b0}(U)s_{b^*0}(X)U\} - 2\sum_{b=1}^{q} \{-s_{b^*0}(U)u(X)A_{b^*}U - s_{b0}(U)u(X)A_bU\} \\ &= -cPX + 2\sum_{b=1}^{q} \{s_{b0}(X)A_bU + s_{b^*0}(X)A_{b^*}U\}. \end{aligned} \tag{4.63}$$

We put $X = PX$ in (4.63) and apply P to the obtained equation. The resulting equation is

$$cP^3X = 2\sum_{b=1}^{q} \{s_{b0}(PX)PA_bU + s_{b^*0}(PX)PA_{b^*}U\}. \tag{4.64}$$

Since $p < n$, there exists a vector field Y orthogonal to U and $\mathrm{span}\{A_bU, A_{b^*}U\}$, $b = 1, \ldots, q$. If we take the inner product of (4.64) by Y, we get that $c = 0$. □

5. FOUR DIMENSIONAL HOLOMORPHIC STATISTICAL MANIFOLDS

5.1 g-natural metrics

Here, we give Oproiu's construction of Kähler structure on the tangent bundle $T\overline{M}$ of a manifold $(\overline{M},\ \overline{g})$. We recall that $T\overline{M}$ has a structure of $2n$-dimensional smooth manifold induced from the smooth manifold structure of $\overline{M}$. A local chart $(U,\ \phi)\ =\ (U,\ x^1,\ldots,x^n)$ on M induces a local chart $(\tau^{-1}(U),\ \Phi)\ =\ (\tau^{-1}(U),\ x^1,\ldots,x^n,\ y^1,\ldots,y^n)$ on $T\overline{M}$, where $\tau : T\overline{M}\ \rightarrow\ \overline{M}$ is the projection map, and the local coordinates $x^i,\ y^i,\ i\ =\ 1,\ldots,n$ are defined as follows. The first n local coordinates $x^i\ =\ x^i\ \circ\ \tau,\ i\ =\ 1,\ldots,n$, on $T\overline{M}$ are the local coordinates in the local chart $(U,\ \phi)$ of the base point of a tangent vector from $\tau^{-1}U$. The last n local coordinates $y^i,\ i\ =\ 1,\ldots,n$ are the vector space coordinates of the tangent vector, with respect to the natural local basis in the corresponding tangent space, defined by the local chart $(U,\ \phi)$. We recall that the Levi-Civita connection g defines a direct sum decomposition

$$T T\overline{M}\ =\ VT\overline{M}\ \oplus\ HT\overline{M}, \tag{5.1}$$

of the tangent bundle to $TT\overline{M}$ into the vertical distribution $VT\overline{M}\ =\ Ker\tau_*$ and the horizontal distribution $HT\overline{M}$ defined by the Levi-Civita connection $\overline{\nabla}^{\overline{g}}$ of $\overline{g}$. The vector fields $(\frac{\partial}{\partial y^1},\ldots,\frac{\partial}{\partial y^n})$ define local frame fields for $VT\overline{M}$, and for $HT\overline{M}$ we have the local frame fields $(\frac{\delta}{\delta x^1},\ldots,\frac{\delta}{\delta x^n})$, defined as

$$\frac{\delta}{\delta x^i}\ =\ \frac{\partial}{\partial x^i}\ -\ \Gamma^h_{i0}\frac{\partial}{\partial y^h},$$

where $\Gamma^h_{i0}\ =\ \Gamma^h_{ik}y^k$ and $\Gamma^h_{ik}(x)$ are the Christoffel symbols defined by the Riemannian metric g. The distributions $VT\overline{M}$ and $HT\overline{M}$ are isomorphic to each other and it is possible to derive an almost complex structure on $T\overline{M}$ which together with the Sasaki metric determine a structure of almost Kählerian manifold on $T\overline{M}$ (cf. [11]). Now, we consider the energy density

$$t\ =\ \frac{1}{2}\overline{g}_{ik}(x)y^iy^k\ =\ \frac{1}{2}\|y\|^2\ =\ \frac{1}{2}\overline{g}_{\tau(y)}(y,\ y), \tag{5.2}$$

defined on $T\overline{M}$ by the Riemannian metric $\overline{g}$ of $\overline{M}$, where $\overline{g}_{ik}$ are the components of $\overline{g}$ in the local chart $(U,\ \phi)$. Let $u,\ v:\ [0,\ \infty)\ \rightarrow\ \mathbb{R}$ be two real

smooth functions. We assume that u and $u + 2tv$ have positive values. Then we may consider the following:

$$G_{ij} = u(t)\overline{g}_{ij} + v(t)\overline{g}_{0i}\overline{g}_{0j}, \tag{5.3}$$

where $\overline{g}_{0i} = \overline{g}_{hi}y^h$. The matrix (G_{ij}) is symmetric. If $u > 0$ and $u + 2tv > 0$, then the quadratic form $G_{ij}z^iz^j$ is positive for all nonzero vectors $(z^1, \ldots, z^n)$. The inverse of the matrix (G_{ij}) has the entries

$$H^{kl} = \frac{1}{u(t)}\overline{g}^{kl} + w(t)y^ky^l, \tag{5.4}$$

where $\overline{g}^{kl}$ are the components of the inverse of the matrix $(\overline{g}_{ij})$ and

$$w(t) = -\frac{v(t)}{u(t)(u(t) + 2tv(t))}. \tag{5.5}$$

The components $H^{kl}(x, y)$ are well defined on $T\overline{M}$. With $H_{ij}(x, y)$ a symmetric M-tensor field of type $(0, 2)$ obtained from the components H^{kl}, is denoted:

$$H_{ij} = \overline{g}_{ik}H^{kl}\overline{g}_{lj} = \frac{1}{u(t)}\overline{g}_{ij} + w(t)\overline{g}_{0i}\overline{g}_{0j}. \tag{5.6}$$

Also,

$$\begin{aligned} G^{kl} &= \overline{g}^{ki}G_{ij}\overline{g}^{jl} = u(t)\overline{g}^{kl} + v(t)y^ky^l, \\ G^i_k &= G^{ih}\overline{g}_{hk} = G_{kh}\overline{g}^{hi} = u(t)\delta^i_k + v(t)y^i\overline{g}_{0k}, \\ H^i_k &= H^{ih}\overline{g}_{hk} = H_{kh}\overline{g}^{hi} = \frac{1}{u(t)}\delta^i_k + w(t)y^i\overline{g}_{0k}. \end{aligned} \tag{5.7}$$

Here, (H^i_k) is the inverse of the matrix (G^i_k).
On $T\overline{M}$, the following Riemannian metric is considered:

$$G = G_{ij}dx^idx^j + H_{ij}\delta y^i\delta y^j, \tag{5.8}$$

where $\delta y^i = dy^i + \Gamma^i_{j0}dx^j$ is the absolute differential of y^i with respect to the Levi-Civita connection $\overline{\nabla}^{\overline{g}}$ of $\overline{g}$. Equivalently,

$$\begin{aligned} G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) &= G_{ij}, \\ G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) &= H_{ij}, \\ G(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) &= G(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}) = 0. \end{aligned}$$

The vertical bundle $VT\overline{M}$ and the horizontal bundle $HT\overline{M}$, in (5.1), are orthogonal to each other with respect to G, but the Riemannian metrics

induced from G on $T\overline{M}$ is not of Sasaki type. Also, the system of 1-forms $(dx^1,\dots,dx^n,\ \delta y^1,\dots,\ \delta y^n)$, defines a local frame of $T^*T\overline{M}$, dual to the local frame $(\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^n},\ \frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^n})$ adapted to the direct sum decomposition in (5.1). An almost complex structure, J, on $T\overline{M}$ is defined by:

$$J\frac{\delta}{\delta x^i} \ = \ G^i_k\frac{\partial}{\partial y^k}, \quad J\frac{\partial}{\partial y^i} \ = \ -\ H^k_i\frac{\delta}{\delta x^k}. \tag{5.9}$$

Theorem 9. *(cf. [23]) $(T\overline{M},\ G,\ J)$ is an almost Kähler manifold.*

The associated 2-form Ω is given by

$$\Omega \ = \ \overline{g}_{ij}\delta y^i \ \wedge \ dx^j. \tag{5.10}$$

Ω is closed since it coincides with the 2-form associated to the Sasaki metric on $T\overline{M}$ (cf. [11]).

Theorem 10. *(cf. [23]) The almost complex structure J on $T\overline{M}$ is integrable if $(\overline{M},\ \overline{g})$ has constant sectional curvature c, and the function v is given by*

$$v \ = \ \frac{c\ -\ uu'}{2tu'\ -\ u}, \tag{5.11}$$

and so

$$w \ = \ w(t) \ = \ \frac{uu'\ -\ c}{u(2tc\ -\ u^2)}. \tag{5.12}$$

5.2 Construction theorems

We use Oproiu's theorems (Theorem 9 and Theorem 10) to construct holomorphic statistical structures on a $\mathbb{R}^2\ \times\ \Omega$ on $\mathbb{R}^4$. Let (x_1,x_2,y_1,y_2) be a standard coordinate system on $\mathbb{R}^4$. Take a function $a:[0,\infty)\to(0,\infty)$. Set $b(t):=-a(t)a'(t)(2ta'(t)-a(t))^{-1}$ and assume $a(t)+2tb(t)>0$ for $t\geq 0$. Define $t:=(y_1^2+y_2^2)/2$ and functions u,v on $\mathbb{R}^4$ by $u(x_1,x_2,y_1,y_2):=a(t),\ v(x_1,x_2,y_1,y_2):=b(t)$. Oproiu defined a g-natural metric G on $\mathbb{R}^2\ \times\ \Omega$, and a complex structure J by:

$$\begin{aligned} G \ = \ & (u+vy_1^2)dx_1dx_1 \ + \ 2vy_1y_2dx_1dx_2 \ + \ (u\ +\ vy_2^2)dx_2dx_2 \\ & + \ \frac{u+vy_2^2}{u(u+2tv)}dy_1dy_1 - 2\frac{vy_1y_2}{u(u+2tv)}dy_1dy_2 \ + \ \frac{u+vy_1^2}{u(u+2tv)}dy_2dy_2, \end{aligned} \tag{5.13}$$

$$\begin{aligned} &J\frac{\partial}{\partial x_1} = (u+vy_1^2)\frac{\partial}{\partial y_1} + vy_1y_2\frac{\partial}{\partial y_2},\ J\frac{\partial}{\partial x_2} = vy_1y_2\frac{\partial}{\partial y_1} + (u+vy_2^2)\frac{\partial}{\partial y_2}, \\ &J\frac{\partial}{\partial y_1} = -\frac{u+vy_2^2}{u(u+2tv)}\frac{\partial}{\partial x_1} + \frac{vy_1y_2}{u(u+2tv)}\frac{\partial}{\partial x_2}, \\ &J\frac{\partial}{\partial y_2} = \frac{vy_1y_2}{u(u+2tv)}\frac{\partial}{\partial x_1} - \frac{u+vy_1^2}{u(u+2tv)}\frac{\partial}{\partial x_2}. \end{aligned} \tag{5.14}$$

The metric G and the complex structure J are constructed so that $(\mathbb{R}^2 \times \Omega,\ G,\ J)$ is Kählerian. For these metric G and complex structure J we construct a tensor field K that satisfies the conditions of Lemma 4. That is, the following theorem holds.

Theorem 11. *Let G be a g-natural metric defined by (5.13) and J a complex structure defined by (5.14). Then there exists a tensor field K defined by eight functions on $\mathbb{R}^2 \times \Omega$ such that $\overline{M} = (\mathbb{R}^2 \times \Omega,\ \overline{\nabla} := \nabla^G + K,\ G,\ J)$ is a holomorphic statistical manifold.*

Proof. We calculate a $(1,\ 2)$-tensor field $K = \sum_{i,j,l=1}^{4} k_{ij}^{l} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j$ on $\mathbb{R}^2 \times \Omega$, in a way that it satisfies the conditions in Lemma 4. That is, K satisfies (4.1), (4.2) and (4.3). For the convenience we denote by $p := u + vy_1^2$, $q := u + vy_2^2$, $r := u + 2tv$, $s := vy_1y_2$. We get the

following.

$$
\begin{aligned}
&k^1_{14} = k^1_{41} = k^2_{13} = k^2_{31} = -k^3_{34} = -k^3_{43} = \alpha_1, \quad k^4_{11} = k^3_{12} = k^3_{21} = \alpha_2,\\
&k^4_{12} = k^4_{21} = k^3_{22} = \alpha_3, \quad k^1_{24} = k^1_{42} = k^2_{23} = k^2_{32} = -k^4_{34} = -k^4_{43} = \alpha_4,\\
&k^2_{22} = -k^4_{24} = -k^4_{42} = \alpha_5, \quad k^1_{11} = k^1_{12} = k^1_{21} = -k^3_{23} = -k^3_{32} = \alpha_6,\\
&k^2_{12} = k^2_{21} = -k^3_{24} = -k^3_{42} = \alpha_6\frac{s-p}{s} + \alpha_{10}\frac{q}{s}, \quad k^1_{33} = \alpha_7,\\
&k^2_{11} = \frac{-p^2 + p(u+r)}{s^2+ur}\alpha_6,\\
&k^3_{14} = k^3_{41} = \alpha_8, \quad k^2_{14} = k^2_{41} = -\alpha_2\frac{s}{urq} + \alpha_3\frac{p}{urq} - \alpha_1\frac{s}{q},\\
&k^1_{23} = k^1_{32} = \alpha_2\frac{q}{urp} - \alpha_4\frac{s}{p} - \alpha_3\frac{s}{urp},\\
&k^3_{11} = \alpha_1\frac{2s^4 - u^2r^2}{sq} + \alpha_2\frac{ur+2s^2}{sq} - \alpha_3\frac{p}{q},\\
&k^4_{22} = -\alpha_2\frac{q}{p} - \alpha_4\frac{u^2r^2}{sp} + \alpha_3\frac{ur+2s^2}{sp},\\
&k^1_{13} = k^1_{31} = \alpha_2\frac{q}{sur} - \alpha_3\frac{1}{ur} - \alpha_1\frac{q}{s},\\
&k^4_{33} = -\alpha_2\frac{q}{urp} + \alpha_3\frac{s}{urp} + \alpha_4\frac{s}{p}, \quad k^4_{44} = -k^2_{24} = \alpha_2\frac{1}{ur} - \alpha_3\frac{p}{sur} + \alpha_4\frac{p}{s},\\
&k^3_{44} = \alpha_2\frac{s}{qur} - \alpha_3\frac{p}{qur} + \alpha_1\frac{s}{q}, \quad k^3_{33} = -\alpha_2\frac{q}{sur} + \alpha_3\frac{1}{ur} + \alpha_1\frac{q}{s},\\
&k^1_{22} = -k^4_{23} = -\alpha_5\frac{s}{p} + \alpha_6\frac{sq - s^2 + uq + qr}{sp},\\
&k^1_{34} = k^1_{43} = -\alpha_7\frac{s}{q} + \alpha_6\Big(\frac{sq - 2ur - s^2}{urqp} + \frac{-p^2 + p(u+r)}{s^2+ur}\frac{q}{urp}\Big) - \alpha_5\frac{s^2}{urpq},\\
&k^1_{44} = \alpha_7\frac{s^2}{q^2} + \alpha_6\Big(\frac{s}{q^2p} - \frac{(s-p)(2s^2+ur)}{surpq} - \frac{-p^2 + p(u+r)}{s^2+ur}\frac{2s^2+ur}{surp}\Big) + \alpha_5\frac{s(2s^2+ur)}{urpq^2},\\
&k^3_{13} = k^3_{31} = \frac{p^2s - ps(u+r) - s^2p - urp}{p(s^2+ur)}\alpha_6 - \alpha_8\frac{s}{p}, \quad k^4_{14} = k^4_{41} = \alpha_6\frac{p-s}{s} + \alpha_8\frac{q}{s},\\
&k^4_{13} = k^4_{31} = \Big(\frac{-p^2 + p(u+r)}{s^2+ur}\frac{ur}{p^2} - \frac{u+r}{p}\Big)\alpha_6\frac{u+r}{p} - \alpha_8\frac{s^2}{p^2},\\
&k^2_{33} = -\alpha_8\frac{q}{urp} + \alpha_6\Big(\frac{p^2 - p(u+r)}{s^2+ur}\frac{q}{pur} + \frac{s-q}{ur}\Big) - \alpha_7\frac{p}{s},\\
&k^2_{34} = k^2_{43} = -\alpha_8\frac{s}{urp} + \alpha_7\frac{p}{q} + \alpha_5\frac{s}{urq} + \alpha_6\Big(\frac{p^2 - p(u+r)}{s^2+ur}\frac{1}{ps} + \frac{1}{qs}\Big),\\
&k^2_{44} = -\alpha_5\frac{pq+s^2}{urq^2} - \alpha_8\frac{s^2}{urpq} - \alpha_7\frac{sp}{q^2} + \alpha_6\Big(\frac{-p^2 + p(u+r)}{s^2+ur}\frac{2ur+s^2}{urpq} - \frac{1}{q^2} + \frac{s-p}{urq}\Big),
\end{aligned}
$$

where $\alpha_1, \ldots, \alpha_8$ are functions. Then a manifold
$\overline{M} = (\mathbb{R}^2 \times \Omega, \overline{\nabla} := \nabla^G + K, G, J)$ is holomorphic statistical. To show

this, we calculate the next, for $\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial y_1}\ \in\ \Gamma(T\overline{M})$.
(1)

$$\overline{\nabla}_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial y_1} = \nabla^G_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial y_1} + K(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial y_1}) = \nabla^G_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial y_1} + k^1_{13}\frac{\partial}{\partial x_1} + k^2_{13}\frac{\partial}{\partial x_2} + k^3_{13}\frac{\partial}{\partial y_1} + k^4_{13}\frac{\partial}{\partial y_2},$$

and

$$\begin{aligned}\overline{\nabla}_{\frac{\partial}{\partial y_1}}\frac{\partial}{\partial x_1} &= \nabla^G_{\frac{\partial}{\partial y_1}}\frac{\partial}{\partial x_1} + k^1_{31}\frac{\partial}{\partial x_1} + k^2_{31}\frac{\partial}{\partial x_2} + k^3_{31}\frac{\partial}{\partial y_1} + k^4_{31}\frac{\partial}{\partial y_2}\\ &= \nabla^G_{\frac{\partial}{\partial y_1}}\frac{\partial}{\partial x_1} + k^1_{13}\frac{\partial}{\partial x_1} + k^2_{13}\frac{\partial}{\partial x_2} + k^3_{13}\frac{\partial}{\partial y_1} + k^4_{13}\frac{\partial}{\partial y_2}.\end{aligned}$$

Subtracting the last two equations, we get

$$\overline{\nabla}_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial y_1} - \overline{\nabla}_{\frac{\partial}{\partial y_1}}\frac{\partial}{\partial x_1} = \nabla^G_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial y_1} - \nabla^G_{\frac{\partial}{\partial y_1}}\frac{\partial}{\partial x_1} = [\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial y_1}],$$

since ∇^G is of torsion free, as being a Levi-Civita connection.
(2)

$$\begin{aligned}(\overline{\nabla}_{\frac{\partial}{\partial x_1}}G)(\frac{\partial}{\partial x_2},\ \frac{\partial}{\partial y_1}) &= G(\nabla^G_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2},\ \frac{\partial}{\partial y_1}) + G(\frac{\partial}{\partial x_2},\ \nabla^G_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial y_1})\\ &\quad - G(\overline{\nabla}_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2},\ \frac{\partial}{\partial y_1}) - G(\frac{\partial}{\partial x_2},\ \overline{\nabla}_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial y_1})\\ &= -2G(k^1_{12}\frac{\partial}{\partial x_1} + k^2_{12}\frac{\partial}{\partial x_2} + k^3_{12}\frac{\partial}{\partial y_1} + k^4_{12}\frac{\partial}{\partial y_2},\ \frac{\partial}{\partial y_1})\\ &= k^3_{12}\frac{u + vy_2^2}{u(u + 2tv)} - k^4_{12}\frac{vy_1y_2}{u(u + 2tv)},\end{aligned}$$

and

$$\begin{aligned}(\overline{\nabla}_{\frac{\partial}{\partial x_2}}G)(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial y_1}) &= k^3_{21}\frac{u + vy_2^2}{u(u + 2tv)} - k^4_{21}\frac{vy_1y_2}{u(u + 2tv)}\\ &= k^3_{12}\frac{u + vy_2^2}{u(u + 2tv)} - k^4_{12}\frac{vy_1y_2}{u(u + 2tv)}.\end{aligned}$$

Therefore, $(\overline{\nabla}_{\frac{\partial}{\partial x_1}}G)(\frac{\partial}{\partial x_2},\ \frac{\partial}{\partial y_1}) = (\overline{\nabla}_{\frac{\partial}{\partial x_2}}G)(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial y_1})$.

(3)

$$\begin{aligned}(\overline{\nabla}_{\frac{\partial}{\partial x_1}}\omega)(\frac{\partial}{\partial x_2},\ \frac{\partial}{\partial y_1}) &= -G(K(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2}),\ J\frac{\partial}{\partial y_1}) + G(\frac{\partial}{\partial x_2},\ K(\frac{\partial}{\partial x_1},\ J\frac{\partial}{\partial y_1}))\\ &- G(K(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2}),\ J\frac{\partial}{\partial y_1}) + G(K(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2}),\ J\frac{\partial}{\partial y_1})\\ &= -\frac{q}{ur}s(-\alpha_8\frac{q}{s} + \alpha_6\frac{p}{s} + \alpha_6\frac{s-p}{s} + \alpha_8\frac{q}{s}) - \alpha_8\frac{q^2}{ur}\\ &+ \alpha_6\frac{s^2}{ur} + (\alpha_6\frac{s-p}{s} + \alpha_8\frac{q}{s})\frac{s}{ur}q + \frac{q}{ur}pk^1_{11} - \frac{s^2}{ur}k^1_{11}\\ &= -\alpha_6 + k^1_{11} = 0.\end{aligned}$$

Therefore, the conditions in Definition 12 are satisfied for the chosen basis elements of $\Gamma(T\overline{M})$ $\square$

Let the function u in the proof of Theorem 11 be defined as $u := \frac{1+\sqrt{1+4t}}{2}$ and p, q, r, s be defined in the same way. And let

$$\alpha_6 = \alpha_8 = 0,\ \alpha_2 = \frac{1}{2}s(u_{y_1}+2y_1) + \frac{1}{2}qu_{y_2}, \tag{5.15}$$

and assume that α_1 and α_3 satisfy the following equation:

$$(\alpha_2\frac{q}{sur} - \alpha_3\frac{1}{ur} - \alpha_1\frac{q}{s})\frac{pur}{q} + \alpha_1\frac{sur}{q} + \alpha_2\frac{s}{q} = \frac{1}{2}p(u_{y_1}+2y_1) + \frac{1}{2}su_{y_2}, \tag{5.16}$$

where $u_{y_1} := \frac{\partial u}{\partial y_1}$ and $u_{y_2} := \frac{\partial u}{\partial y_2}$.

Corollary 3. *Let K_0 be a tensor field constructed as in the proof of Theorem 11. If K_0 also satisfies the conditions (5.15) and (5.16) then $\overline{M} = (\mathbb{R}^2 \times \Omega,\ \overline{\nabla} := \nabla^G + K_0,\ G,\ J)$ is a special Kähler manifold.*

Sketch of proof. For $\frac{\partial}{\partial_{x_1}},\ \frac{\partial}{\partial_{x_2}},\ \frac{\partial}{\partial_{y_1}} \in \Gamma(T\overline{M})$, we calculate that $\overline{R}(\frac{\partial}{\partial_{x_1}},\ \frac{\partial}{\partial_{x_2}})\frac{\partial}{\partial_{y_1}} = 0$. On all the others combinations of the basis elements we get that the curvature tensor $\overline{R}$ vanishes. $\square$

5.3 Examples

Motivated by Theorem 6 and Theorem 5, here we give examples of two dimensional submanifolds M of holomorphic statistical manifolds defined in Theorem 11 which satisfy the condition

$$I(X,\ Y) \in \Gamma(T^\perp M),\ X,\ Y \in \Gamma(TM). \tag{5.17}$$

Example 4. *Let $\overline{M}$ be the holomorphic statistical manifold in Theorem 11 with functions $\alpha_1, \ldots, \alpha_9$ satisfying $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ and α_6 is a function of variables y_1 and y_2. Let M be a submanifold of $\overline{M}$ such that*

$A_1 = \frac{\partial}{\partial x_1}$ is a unit vector field tangent to M. In this case $A_2 = \alpha_6 \frac{s}{p} \frac{\partial}{\partial x_1} - \alpha_6 \frac{\partial}{\partial x_2}$ is a unit vector field orthogonal to A_1 and tangent to M. Calculating the Lie bracket of A_1 and A_2 we get $[A_1, A_2] = 0$, therefore a submanifold M tangent to A_1 and A_2 exists.
Consider now that M is given by

$$f_1(x_1, x_2, y_1, y_2) = 0,$$
$$f_2(x_1, x_2, y_1, y_2) = 0,$$

where f_1 and f_2 are smooth functions on $\mathbb{R}^4$ satisfying rank $\frac{D(f_1,f_2)}{D(x_1,x_2,y_1,y_2)} = 2$. Since A_1 and A_2 are tangent to M we have

$$\frac{\partial f_i}{\partial x_1} = 0,$$
$$\alpha_6 \frac{s}{p} \frac{\partial f_i}{\partial x_1} - \alpha_6 \frac{\partial f_i}{\partial x_2} = 0, \ i = 1, 2.$$

Then it follows $\frac{\partial f_i}{\partial x_2} = 0$, hence $f_i = f_i(y_1, y_2)$, $i = 1, 2$. Consequently, M is given by $y_1 = constant$ and $y_2 = constant$, that is, M is a portion of a 2-plane. We remark here that in this case the condition $I(X, Y) \in T^\perp M$ is satisfied for $X, Y \in \Gamma(TM)$. Also, vectors

$$B_1 = \alpha_6 \frac{ur}{p} \frac{\partial}{\partial y_2},$$
$$B_2 = p \frac{\partial}{\partial y_1} + s \frac{\partial}{\partial y_2},$$

are unit, orthogonal and generate the normal bundle of M. We can calculate the mean curvature vector field with respect to $\overline{\nabla}$ as

$$\begin{aligned} H \ = \ & \{-\frac{1}{2\alpha_6} u_{y_2} \ + \ \alpha_6 (-\frac{1}{2} \frac{s^2}{p^2} u_{y_2} + \frac{y_1 s}{p} - \ \frac{1}{2} u_{y_2} - y_2)\} B_1 \\ & + \ \{\frac{1}{2}(-u_{y_1} - 2y_1) - \frac{1}{2} u_{y_2} \frac{s}{p} \ + \ \frac{\alpha_6^2}{2} \frac{s^2}{p^2} (-u_{y_1} - 2y_1) \ - \frac{\alpha_6^2}{2} \frac{s^3}{p^3} u_{y_2} \\ & + \ \frac{\alpha_6^2 s}{p^2} (p y_2 + y_1 s) \ - \ \frac{\alpha_6^2}{2p} (p u_{y_1} \ + \ s(u_{y_2} + 2y_2))\} B_2. \quad \triangle \end{aligned}$$

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