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Title	Finite-element solution of anisotropic waveguides with arbitrary tensor permittivity
Author(s)	Koshiha, M.; 小柴, 正則; Hayata, K. et al.
Citation	Journal of Lightwave Technology, 4(2), 121-126
Issue Date	1986-02
Doc URL	https://hdl.handle.net/2115/6085
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Type	journal article
File Information	JLT4_2.pdf



Finite-Element Solution of Anisotropic Waveguides with Arbitrary Tensor Permittivity

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Abstract—An improved vector finite-element method has been used for the solution of general anisotropic waveguide problems. This method is formulated in terms of all three components of the magnetic field and is valid for arbitrary tensor permittivity. In the improved finite-element analysis, the spurious nonphysical solutions do not appear when the effective refractive index is larger than 1. Therefore, this method is very useful for the analysis of the surface-wave modes of optical waveguides. To show the validity and usefulness of the improved finite-element method, computed results are illustrated for anisotropic rectangular waveguides with optic axis in any orientation and gyrotropic rectangular waveguides.

I. INTRODUCTION

A VECTOR finite-element method in an axial-component (E_z - H_z) formulation [1]–[7] or in a three-component (the magnetic field H) formulation [8], [9] is widely used for the analysis of optical waveguides [2]–[4], [6], [7], [9]. The most serious difficulty in using the vector finite-element analysis is the appearance of the spurious nonphysical solutions [1], [5]–[9]. Recently, an improved finite-element method has been formulated in terms of all three components of H [10]–[15]. In this improved H -field formulation, the spurious solutions do not appear when the effective refractive index β/k_0 is larger than 1 [14], [15], where k_0 is the wavenumber of free space and β is the phase constant in the direction (z axis) of propagation. Therefore, this method is very useful for the analysis of the surface-wave modes of optical waveguides which correspond to the solutions in the region $\beta/k_0 \geq 1$. In addition, the H -field formulation is valid for general anisotropic problems. However, in [10]–[15], an isotropic dielectric waveguide is mainly studied and the application of the H -field formulation is restricted to an anisotropic dielectric waveguide composed of a uniaxial material whose optic axis lies in the plane (xy -plane) perpendicular to the z axis [10], [13], [15].

In this paper, to show the validity and usefulness of the improved H -field formulation for a variety of anisotropic waveguides, computed results are illustrated for anisotropic rectangular waveguides with optic axis in any orientation and gyrotropic rectangular waveguides.

II. FINITE-ELEMENT METHOD

A variational expression with the divergence-free constraint ($\nabla \cdot H = 0$) which is imposed in a least-square sense is known to be [10]–[16]

$$\begin{aligned} \tilde{F} = & \iint_{\Omega} (\nabla \times H)^* \cdot ([K]^{-1} \nabla \times H) d\Omega \\ & - k_0^2 \iint_{\Omega} H^* \cdot H d\Omega \\ & + \iint_{\Omega} (\nabla \cdot H)^* (\nabla \cdot H) d\Omega \end{aligned} \quad (1)$$

where Ω represents the cross section of a waveguide, the asterisk denotes a complex conjugate, and the relative permittivity tensor $[K]$ is given by

$$[K] = \begin{bmatrix} K_{xx} & K_{xy} & K_{zx}^* \\ K_{xy}^* & K_{yy} & K_{yz} \\ K_{zx} & K_{yz}^* & K_{zz} \end{bmatrix} \quad (2)$$

Here $[\cdot]$ denotes a matrix. As we consider loss-free materials, $[K]$ is Hermitian. In anisotropic dielectric cases, all elements of $[K]$ are real so that $[K]$ is a real symmetric matrix [17]. In gyrotropic cases, diagonal elements of $[K]$ are real, whereas off-diagonal ones in general are complex [17]. The former corresponds to materials with crystalline anisotropy or electrooptic cases, while the latter corresponds to the magneto-optic case [17].

Dividing the cross section Ω of the waveguide into a number of second-order triangular elements in Fig. 1, the magnetic fields H_x , H_y , and H_z within each element are defined in terms of the magnetic fields at the corner and midside nodal points:

$$H_x = \{N\}^T \{H_x\}_e \exp(-j\beta z) \quad (3)$$

$$H_y = \{N\}^T \{H_y\}_e \exp(-j\beta z) \quad (4)$$

$$H_z = j \{N\}^T \{H_z\}_e \exp(-j\beta z) \quad (5)$$

where $\{H_x\}_e$, $\{H_y\}_e$, and $\{H_z\}_e$ are magnetic field vectors corresponding to the nodal points within each element, $\{N\}$ is the shape function vector [10], [15], and T , $\{\cdot\}$, and $\{\cdot\}^T$ denote a transpose, a column vector, and a row

Manuscript received March 18, 1985; revised June 1, 1985.
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IEEE Log Number 8406199.

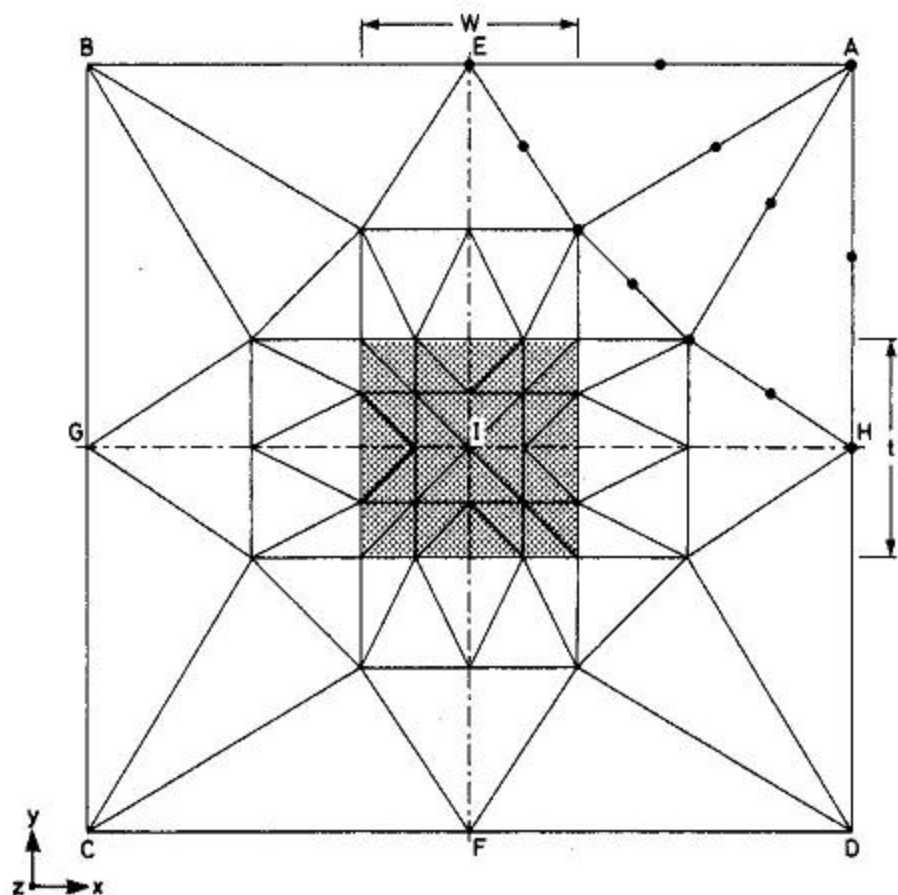


Fig. 1. Finite-element division of an anisotropic rectangular waveguide or a gyrotropic rectangular waveguide.

vector, respectively. Variation of (1) with respect to the nodal variables leads to the following eigenvalue problem:

$$([S] + [U]) \{H\} - k_0^2 [T] \{H\} = \{0\} \quad (6)$$

where $\{H\}$ is the nodal magnetic field vector, $\{0\}$ is a null vector, $[S]$, $[T]$, and $[U]$ are the matrices related to the first, second, and third terms on the right-hand side of (1), respectively, and $[T]$ is a real symmetric matrix. For loss-free materials, $[U]$ is a real symmetric matrix and $[S]$ is a complex Hermitian matrix. When $\text{Im}(K_{xy}) = \text{Re}(K_{yz}) = \text{Re}(K_{zx}) = 0$, $[S]$ becomes a real symmetric matrix [8], [13].

If the divergence-free constraint is neglected, the following eigenvalue equation is derived [8], [9], [18]:

$$[S] \{H\} - k_0^2 [T] \{H\} = \{0\}. \quad (7)$$

In the earlier finite-element analysis using (7), the spurious solutions appear and these solutions are scattered all over the propagation diagram [8]-[16], [18].

In the improved finite-element analysis using (6), the appearance of the spurious solutions is limited to the region $\beta/k_0 < 1$ [14], [15]. They do not appear when the effective refractive index β/k_0 is larger than 1. Therefore, (6) is very useful for the analysis of the surface-wave modes of optical waveguides which correspond to the solutions in the region $\beta/k_0 \geq 1$.

III. COMPUTED RESULTS

In this section, we present the computed results for anisotropic rectangular waveguides and gyrotropic rectangular waveguides. The guided modes are designated as E_{pq}^x or E_{pq}^y [19]. The main field components of the E_{pq}^x modes are E_x and H_y , while those of the E_{pq}^y modes are E_y and H_x [19]. A typical division of these waveguides into second-order triangular elements is shown in Fig. 1,

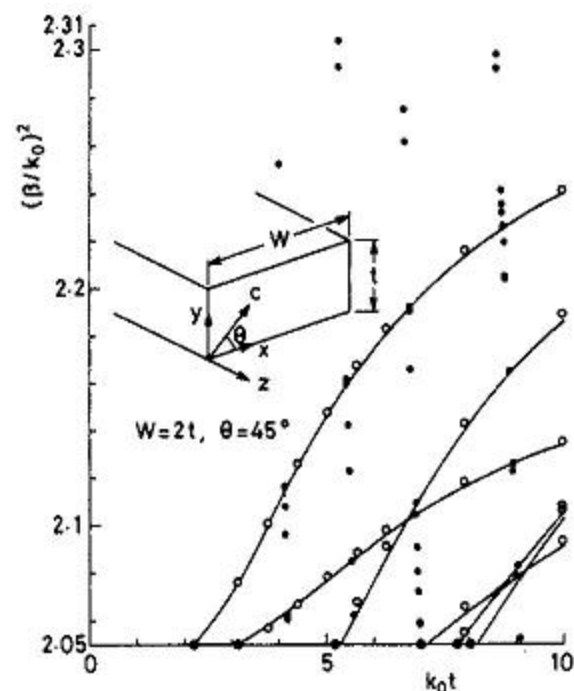


Fig. 2. Dispersion characteristics of an anisotropic rectangular waveguide. Solid lines, circles, and dots represent the solutions of (6), the results of the variational method [20], and the solutions of (7), respectively.

where the boundaries AB , BC , CD , and DA are assumed to be perfect electric conductors [15]. The convergence of the solution is checked by moving these conductors gradually away from a core region.

A. Anisotropic Rectangular Waveguides

We consider an anisotropic rectangular waveguide composed of a uniaxial material surrounded by an isotropic material of refractive index $\sqrt{2.05}$, where the ordinary and extraordinary refractive indices of a rectangular core are $\sqrt{2.31}$ and $\sqrt{2.19}$, respectively.

Fig. 2 shows the dispersion characteristics of the anisotropic rectangular waveguide with $W = 2t$ whose optic axis c lies in the xy -plane at an angle $\theta = 45^\circ$ from the x axis, where the whole region $ABCD$ in Fig. 1 should be divided into elements because of the lack of symmetry of the field. The solid lines in Fig. 2 represent the solutions of the improved finite-element program in (6). Comparison of our results with the results of the variational method [20] indicated by circles shows good agreement. This fact demonstrates the reliability of the present method. The dots in Fig. 2 represent the solutions of the earlier finite-element program in (7). It is found that when (7) is used numerous spurious solutions appear. The guided modes of the waveguide in Fig. 2 can not be designated as E_{pq}^x or E_{pq}^y because the main magnetic field component of these guided modes is not H_x or H_y [20].

Figs. 3 and 4 show the dispersion characteristics of the anisotropic square waveguides ($W = t$) whose optic axis lies in the xy -plane at an angle θ from the x axis. The waveguide with $\theta = 0^\circ$ in Fig. 3 has two planes of symmetry. Therefore, for the waveguide in Fig. 3, the region $AEIH$ in Fig. 1 is divided into elements. The E_{11}^y and the E_{11}^x modes in Fig. 3 are the fundamental and the first higher-order modes, respectively. In an isotropic square waveguide, the E_{11}^y and the E_{11}^x modes are degenerate [19]. The propagation diagram of the waveguide with $\theta = 45^\circ$ in Fig. 4 is very similar to that in Fig. 3. However, the eigenvectors for the guided modes of the waveguide

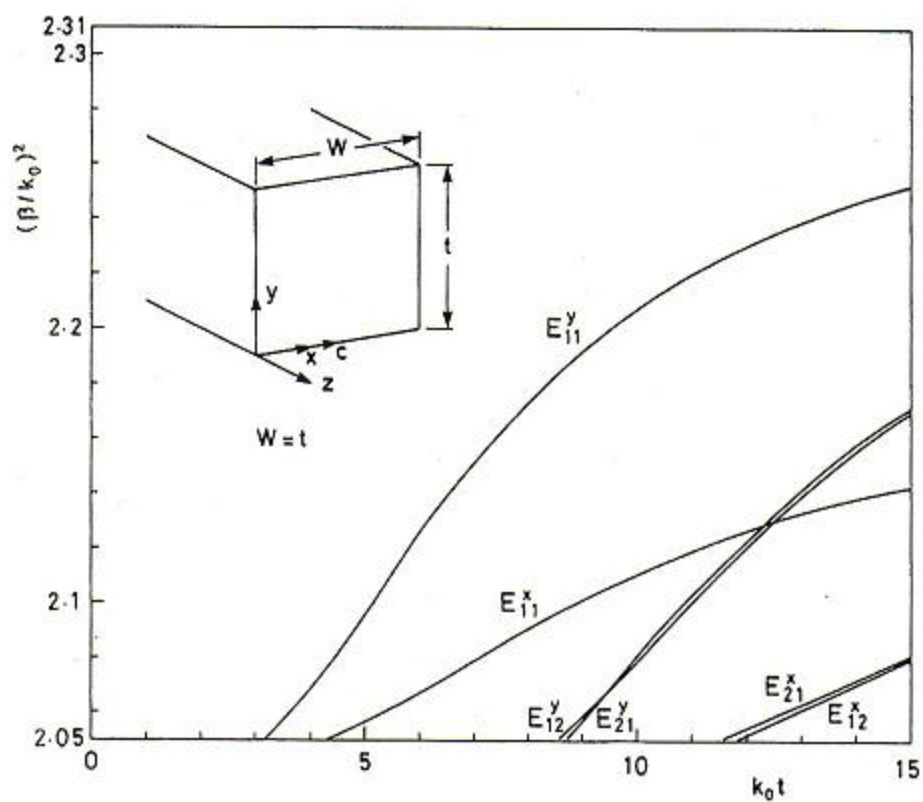


Fig. 3. Dispersion characteristics of an anisotropic square waveguide with the optic axis parallel to the x axis.

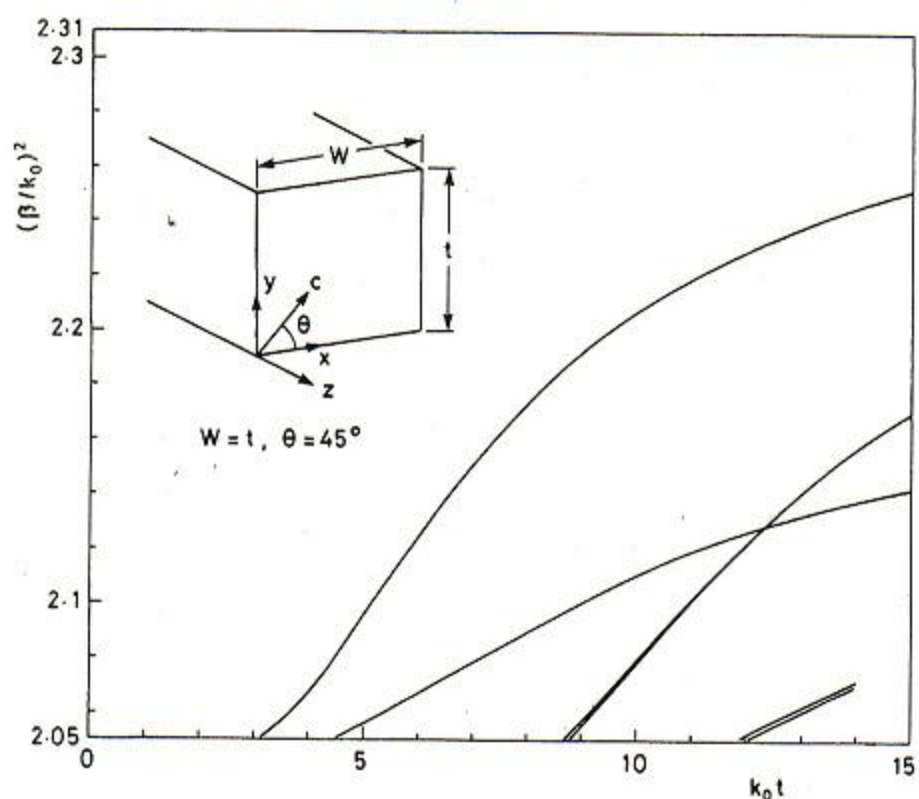


Fig. 4. Dispersion characteristics of an anisotropic square waveguide whose optic axis lies in the xy -plane at an angle $\theta = 45^\circ$ from the x axis.

in Fig. 4 are different from those in Fig. 3. For the fundamental and the first higher-order modes, the variations of the transverse components of the magnetic field vector in a core-region with θ are shown in Figs. 5 and 6, respectively, where $\beta t = 14$. The directions of the magnetic field for the fundamental and the first higher-order modes are almost parallel and almost normal to the optic axis, respectively. Therefore, the guided modes of the waveguide in Fig. 4 can not be designated as E_{pq}^x or E_{pq}^y .

Figs. 7 and 8 show the dispersion characteristics of the anisotropic square waveguides whose optic axis does not lie in the xy -plane. The optic axis lies in the yz -plane at an angle θ from the y axis. The waveguide with $\theta = 90^\circ$ in Fig. 7 has two planes of symmetry, and therefore the region $AEIH$ in Fig. 1 is divided into elements. On the other hand, the waveguide with $\theta = 45^\circ$ in Fig. 8 has one plane of symmetry, and therefore the region $AEFD$ in Fig. 1 is divided into elements. In the waveguide in Fig. 7, the E_{11}^x and the E_{11}^y modes are degenerate. When $\theta \neq 90^\circ$, these modes are no longer degenerate and the E_{11}^x

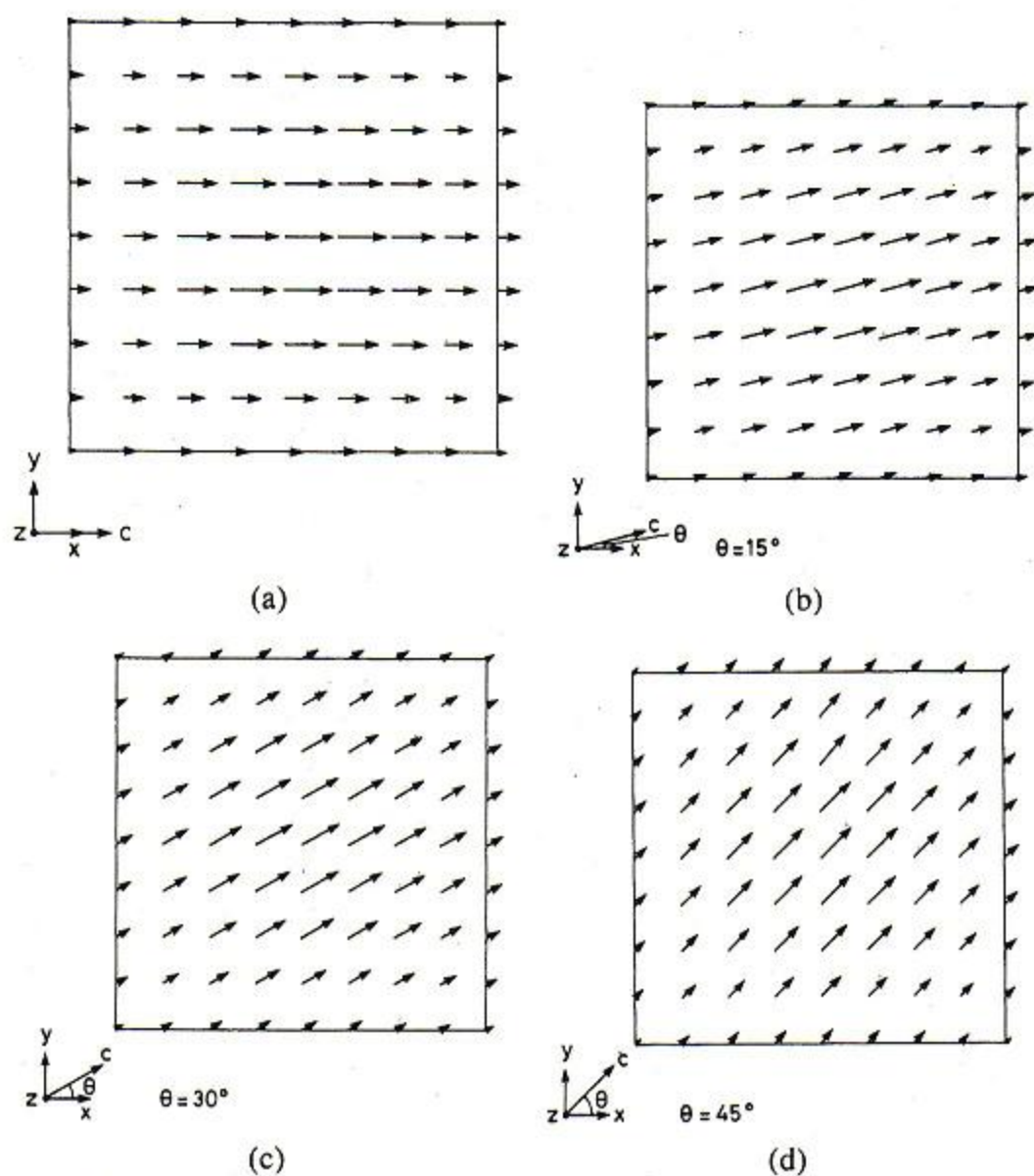


Fig. 5. Magnetic field vector for the fundamental mode of anisotropic square waveguides whose optic axis lies in the xy -plane at an angle θ from the x axis. (a) $\theta = 0^\circ$. (b) $\theta = 15^\circ$. (c) $\theta = 30^\circ$. (d) $\theta = 45^\circ$.

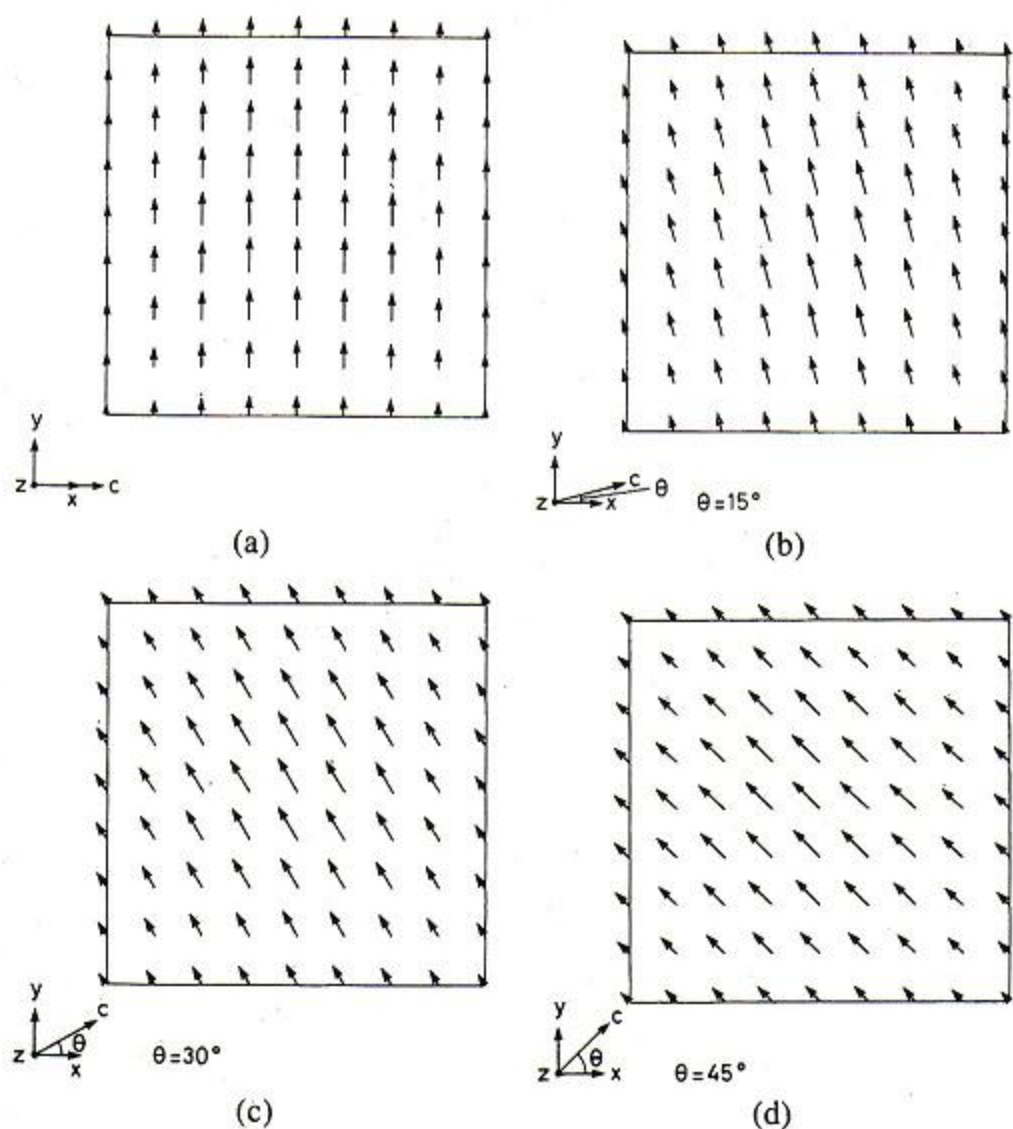


Fig. 6. Magnetic field vector for the first higher-order mode of anisotropic square waveguides whose optic axis lies in the xy -plane at an angle θ from the x axis. (a) $\theta = 0^\circ$. (b) $\theta = 15^\circ$. (c) $\theta = 30^\circ$. (d) $\theta = 45^\circ$.

mode is the fundamental mode. The effect of the orientation of the optic axis is much stronger on the E_{11}^y mode than the E_{11}^x mode.

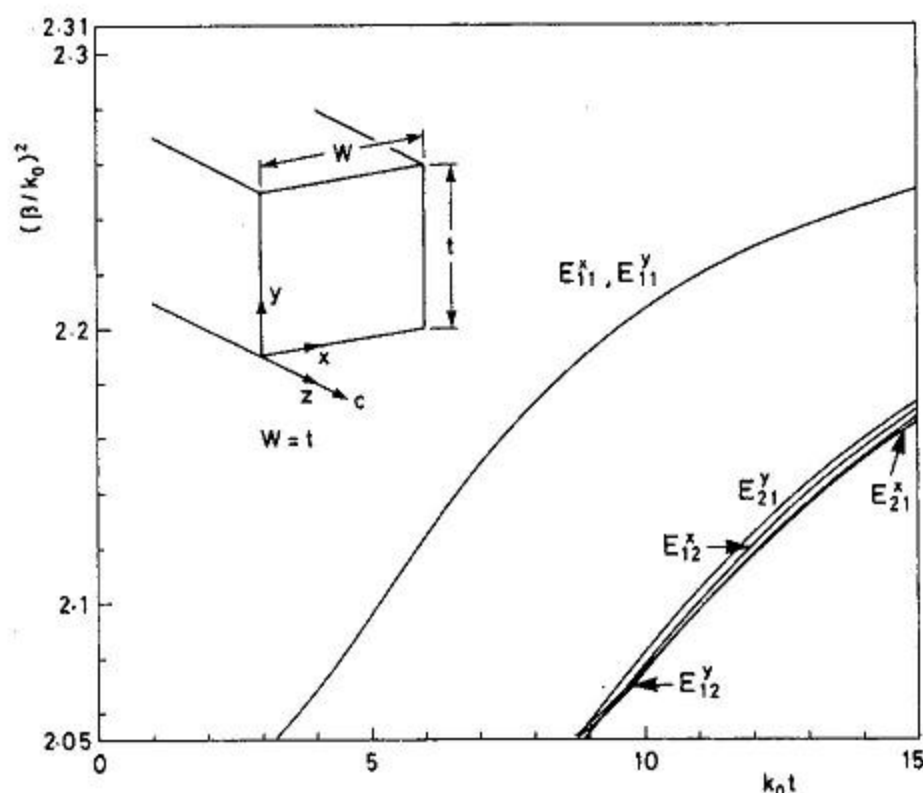


Fig. 7. Dispersion characteristics of an anisotropic square waveguide with the optic axis parallel to the z axis.

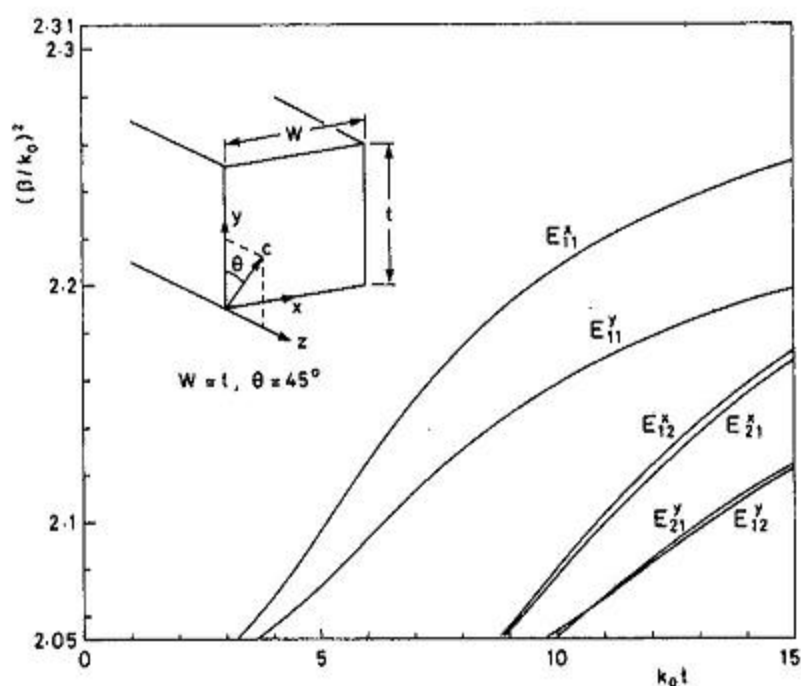


Fig. 8. Dispersion characteristics of an anisotropic square waveguide whose optic axis lies in the yz -plane at an angle $\theta = 45^\circ$ from the y axis.

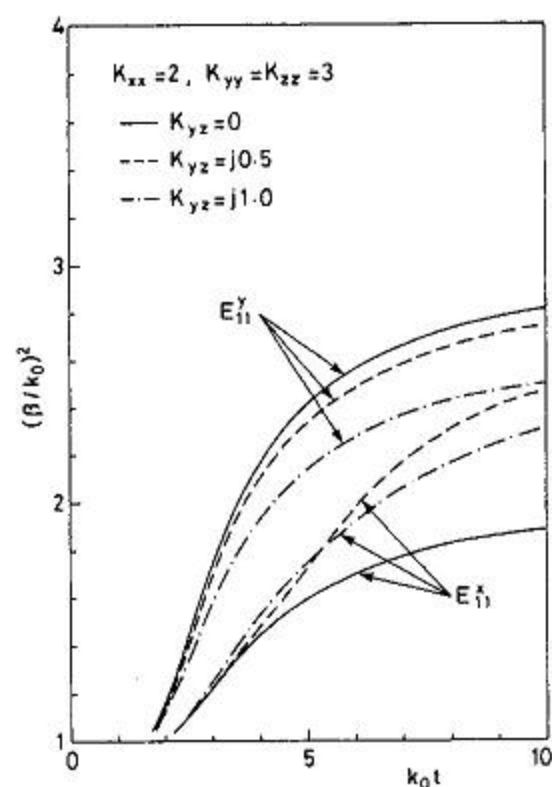
B. Gyrotropic Rectangular Waveguides

We consider a gyrotropic rectangular waveguide surrounded by air, where the relative permittivity tensor of the waveguide is given by

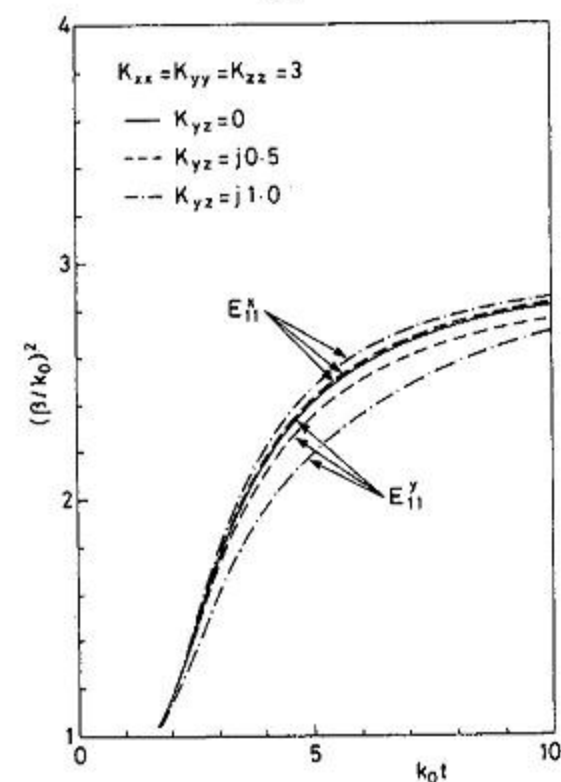
$$[K] = \begin{bmatrix} K_{xx} & 0 & 0 \\ 0 & K_{yy} & K_{yz} \\ 0 & K_{yz}^* & K_{zz} \end{bmatrix}. \quad (8)$$

Here $K_{yy} = K_{zz} = 3$ and K_{yz} is pure imaginary. This waveguide has one plane of symmetry, and therefore the region $AEFD$ in Fig. 1 is divided into elements.

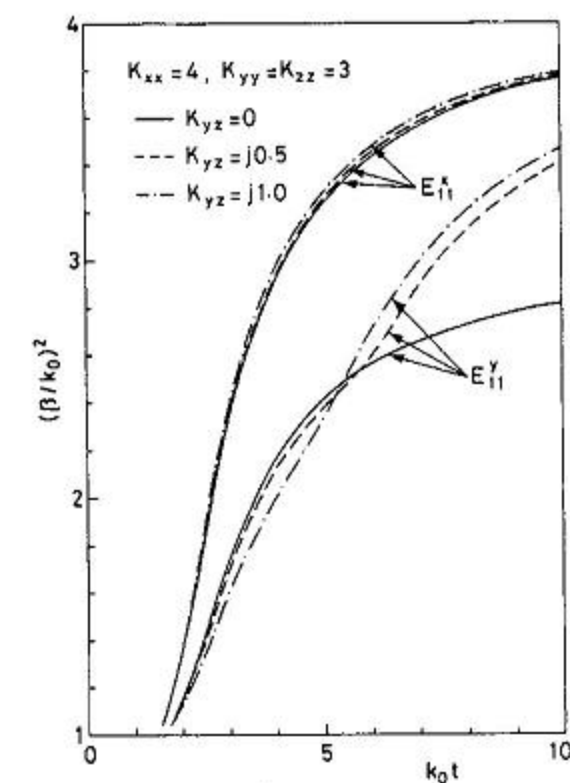
Fig. 9 shows the dispersion characteristics of the gyrotropic square waveguides ($W = t$). When $K_{xx} = 3$ and $K_{yz} = 0$, the E_{11}^x and the E_{11}^y modes are degenerate. In other cases, these modes are no longer degenerate. For $K_{xx} = 2$ ($K_{xx} < K_{yy} = K_{zz}$) or $K_{xx} = 4$ ($K_{xx} > K_{yy} = K_{zz}$), the fundamental mode is the E_{11}^y or the E_{11}^x mode, respectively. The effect of K_{xx} is much stronger on the E_{11}^x mode than the E_{11}^y mode. In the E_{11}^x mode, the effective refractive indices for $K_{yz} \neq 0$ is larger than those for $K_{yz} = 0$.



(a)



(b)



(c)

Fig. 9. Dispersion characteristics of gyrotropic square waveguides. (a) $K_{xx} = 2$. (b) $K_{xx} = 3$. (c) $K_{xx} = 4$.

In the E_{11}^y mode, on the other hand, the effective refractive indices for $K_{yz} \neq 0$ is smaller than those for $K_{yz} = 0$, except the higher frequency range for $K_{xx} = 4$.

For the gyrotropic square waveguide with the off-di-

agonal element K_{zx} ($K_{xy} = K_{yz} = 0$), the region $ABGH$ in Fig. 1 is divided into elements. The dispersion curves for the E_{11}^x or the E_{11}^y mode of this waveguide are the same as those for the E_{11}^y or the E_{11}^x mode of the waveguide in Fig. 9, respectively, where K_{xx} , K_{yy} , and K_{yz} in Fig. 9 should be replaced by K_{yy} , K_{xx} , and K_{zx} , respectively.

For the gyrotropic square waveguide with the off-diagonal element K_{xy} ($K_{yz} = K_{zx} = 0$), the region $ABCD$ in Fig. 1 is divided into elements. The guided modes of this waveguide may not be designated as E_{pq}^x or E_{pq}^y in the same manner as the cases of Figs. 2 and 4.

In Figs. 3, 4, and 7-9, the spurious solutions do not appear because $\beta/k_0 \geq 1$.

IV. CONCLUSIONS

Here we have shown the capability of the improved finite-element method in the three-component magnetic field formulation for solving general anisotropic waveguide problems. In this approach, the spurious solutions do not appear when the effective refractive index is larger than 1. Therefore, this method is very useful for the analysis of the surface-wave modes of optical waveguides. The validity of the method was confirmed by comparing numerical results for the anisotropic rectangular waveguide with the results of the variational method. The numerical results for the anisotropic square waveguides and the gyrotropic square waveguides were presented and the propagation characteristics of each waveguide was discussed.

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