Duality of singular paths for $(2, 3, 5)$-distributions

G. Ishikawa[*], Y. Kitagawa and W. Yukuno

Abstract

We show a duality which arises from distributions of Cartan type, having growth (2, 3, 5), from the view point of geometric control theory. In fact we consider the space of singular (or abnormal) paths on a given five dimensional space endowed with a Cartan distribution, which form another five dimensional space with a cone structure. We regard the cone structure as a control system and show that the space of singular paths of the cone structure is naturally identified with the original space. Moreover we observe an asymmetry on this duality in terms of singular paths.

1 Introduction.

In this paper we show a duality which is related to rank two distributions of five variables: Let $D \subset TY$ be a subbundle of the tangent bundle of a five dimensional manifold Y with growth $(2, 3, 5)$ (see §2). It is known that for any point y of Y and for any direction of D_y, there exists uniquely a singular D-path (or an immersed abnormal extremal) through y with the given direction (see Proposition 4.1). Then the immersive singular D-paths form another five dimensional manifold X. Moreover we remark that X is endowed with a natural cone structure $C \subset TX$ induced from the Cartan prolongation E of D on a six dimensional manifold Z and a projection $Z \to X$.

Then we are naturally led to consider the corresponding objects for the cone structure $C \subset TX$ to singular paths for $D \subset TY$ and to obtain the "inverse" construction from the space X to the original space Y. To realise this, we regard the cone structure $C \subset TX$ as a control system $\mathbb{C} : E \to TX \to X$ on X defined by the differential map $E \hookrightarrow TZ \to TX$ of the projection $Z \to X$ and consider the space of singular $\mathbb{C}$-paths. Then we show that the original space Y is identified with the space of singular $\mathbb{C}$-paths on X, at least locally, while X is identified with the space of singular D-paths (Theorem 4.8). In fact, a singular $\mathbb{C}$-path on X consists of singular D-paths on Y which pass through a fixed point of Y.

Distributions with growth $(2, 3, 5)$ were studied by Cartan in the famous paper [7]. In fact they are called systems *of Cartan type* in [6]. In general, the space of singular paths

[*]This work was supported by KAKENHI No.22340030 and No.23654058.

(or abnormal extremals) of a given distribution plays an important role as it is an intrinsic invariant of the distribution. For spaces with $(2,3,5)$-distributions, the construction of the space of singular paths and the existence of natural double fibration was mentioned for the first time in unpublished lecture notes by Bryant [5]. On such spaces, indefinite conformal structures of signature $(3,2)$ were constructed by Nurowski [17]. Then, for the alternative constructions to Nurowski's, the exactly same cone structure as in this paper was explicitly used in [3]. In fact, in [3], the cone structure was given by a foliation on the space $P((D^2)^{\perp}) \subset P(T^*Y)$, while in this paper we describe it from an essentially same foliation in the space $P(D) \subset P(TY)$, which is canonically identified with $P((D^2)^{\perp})$ (see §4, in particular the proof of Proposition 4.1).

Note that, moreover in [8][9], it is shown that cone structures exist for rank 2 distributions on n-dimensional manifolds for arbitrary $n \geq 5$, and the cone structures were extensively used for constructions of the structures of absolute parallelism and invariants of such distributions. It is shown also that similar double fibrations exist for rank 2 distributions on manifolds of dimension $n \geq 5$ and also for distributions of higher rank in [8][9] implicitly and in [10] explicitly.

The constructions of double fibrations and cone structures are closely related to the general notion of pseudo-product structures in the sense of Tanaka ([19][20]). A *pseudo-product structure* on a manifold M is a distribution $E \subset TM$ on a manifold M with a decomposition $E = L \oplus K$ into integrable subbundles. Then locally we have a double fibration $P \xleftarrow{\pi_P} M \xrightarrow{\pi_N} N$ to the leaf space P of L and N of K respectively. Further we have the cone structures $K \hookrightarrow TM \xrightarrow{\pi_{P*}} TP$ on P and $L \hookrightarrow TM \xrightarrow{\pi_{N*}} TN$ on N respectively. Then we are naturally led to study the relation between singular paths on P, N and on M. Our duality theorem (Theorem 4.8) addresses to this natural general problem in the case of $(2,3,5)$-distributions. Moreover we observe an asymmetry on this duality in terms of singular paths (Proposition 4.3). It seems to be interesting problem to find other pseudo-product structures where the similar duality by singular paths holds. However we suppose the duality holds just in limited cases. For example, a different kind of duality was studied in [12] between a contact structure on a 3-dimensional space and an indefinite conformal structure of signature $(2,1)$ on another 3-dimensional space via an Engel structure. Nevertheless, the duality via singular paths does not hold in that case, since contact structures do not have any singular path, while the cone structure in that case actually has singular paths induced by the Engel line field. Note that distributions can be regarded as cone structures of special types. Then we hope the basic results given in this paper (Lemma 3.1, Lemma 3.3) can be helpful to treat singular paths for cone structures arising from pseudo-product structures.

We recall the basic definitions and terminologies needed in this paper in §2, and introduce a fundamental lemma for treating singular paths of cone structures in §3. In §4, we show our basic constructions and the main duality result mentioned above.

The abnormal extremals of a distribution D with growth $(2,3,5)$ with G_2-symmetry was calculated in [14]. On the other hand, from the view point of twistor theory, the double fibration $Y \leftarrow Z \rightarrow X$ with G_2-symmetry and the canonical geometric structures $E \subset TZ, D \subset TY$ and $C \subset TX$ have been explicitly constructed in [13]. Then we can determine singular paths, using the explicit representation, by direct calculations in G_2-case. In §5, we recall the explicit representations of the control systems $\mathbb{E} : E \hookrightarrow TZ \to Z$, $\mathbb{D} : D \hookrightarrow TY \to Y$ and $\mathbb{C} : E \to TX \to X$, and show partly the explicit calculations on their singular paths.

The $(2, 3, 5)$-distributions are studied in detail from various viewpoints by many authors (see for instance [3][6][22][23]), and even now we believe still there remain things to be uncovered on this subject.

All manifolds and mappings are assumed to be of class C^∞ unless otherwise stated. For an interval I, we say that an assertion holds for almost every $t \in I$ if it holds outside of some measure zero set of I.

The authors would like to thank the referee for the valuable comments.

2 Control systems and singular controls.

We need the following generalised notion of control systems ([1]). Let M be a manifold. A *control system* on M is given by a locally trivial fibration $\pi_\mathcal{U} : \mathcal{U} \to M$ over M and a fibered map $F : \mathcal{U} \to TM$ over the identity of M. In this paper, we assume that the fibre U of $\pi_\mathcal{U} : \mathcal{U} \to M$ is an open subset of $\mathbf{R}^r$. Note that in this paper we mainly treat a local case where $\pi_\mathcal{U}$ is trivial. However our results do not depend on any choice of trivialisation of $\pi_\mathcal{U}$.

For a given control system, an L^∞ (measurable, essentially bounded) map $c : [a, b] \to \mathcal{U}$ on an interval is called an *admissible control* if the curve $\gamma := \pi_\mathcal{U} \circ c : [a, b] \to M$ satisfies $\dot{\gamma}(t) = F(c(t))$, for almost every $t \in [a, b]$. Then the Lipschitz curve γ is called a *trajectory.* In this paper, we use the term "path" for a smooth (C^∞) immersive trajectory regarded up to parametrisation.

The *initial* (resp. *end*) *point* of γ is given by $\gamma(a) = \pi_\mathcal{U}(c(a))$ (resp. $\gamma(b) = \pi_\mathcal{U}(c(b))$. Note that, in our terminology, the term "control" contains the data on both the "control" in usual sense and (the initial point of) the trajectory. Therefore trajectory is determined uniquely by the control, while the control is not unique for the trajectory in general.

If we write locally $x(t) = \gamma(t)$ and $\widetilde{\gamma}(t) = (x(t), u(t))$, under a local triviality $\mathcal{U}|_V \cong V \times U$ over an open set $V \subset M$, then the control system is expressed by a family of vector fields, $\{f_u\}_{u\in U}$, $f_u(x) = F(x, u)$ over $V \subset M$ and by the equation $\dot{x}(t) = f_{u(t)}(x(t))$. In this local situation or in the case where the fibration $\pi_\mathcal{U}$ is globally trivial, for a given initial point, $u(t)$ is called a control defining the trajectory $x(t)$ as usual. Note that, for a given initial point x_0 and a given L^∞ control $u(t)$, the Cauchy problem $f_u(x) = F(x, u), x(0) = q_0$, has a unique Lipschitz solution (trajectory) $x(t)$ depending smoothly on q_0 by the classical Carathéodory theorem (see for example [2] 2.4.1).

Now, given a control system and given a point $q_0 \in M$, we denote by $\mathcal{C}_{ad}$ the set of all admissible controls $c : [a, b] \to \mathcal{U}$ with the initial point $\pi_\mathcal{U}(c(a)) = q_0$. Then it is known that $\mathcal{C}_{ad}$ is a Banach manifold ([2]). The *endpoint mapping* $\mathcal{E} : \mathcal{C}_{ad} \to M$ is defined by $\mathcal{E}(c) := \pi_\mathcal{U}(c(b))$. The control c with the initial point $\pi_\mathcal{U}(c(a)) = q_0$ is called *singular* or *abnormal*, if it is a singular point of $\mathcal{E}$, namely if the differential $\mathcal{E}_* : T_c\mathcal{C}_{ad} \to T_{\mathcal{E}(c)}M$ is not surjective. If c is a singular control, then the trajectory $\gamma = \pi_{\mathcal{U}} \circ c$ is called a *singular trajectory* or an *abnormal extremal* ([2][4]).

We see that if a control is singular then its restriction to any subinterval is singular. In fact, as a part of *Pontryagin maximal principle*, the general characterisation of singular controls is known. To state the characterisation, consider the fibre product of $\pi_{\mathcal{U}} : \mathcal{U} \to M$ and $\pi_{T^*M} : T^*M \to M$:

$$\mathcal{U} \times_M T^*M := \{(w, \varphi) \in \mathcal{U} \times T^*M \mid \pi_{\mathcal{U}}(w) = \pi_{T^*M}(\varphi)\},$$

(extended cotangent bundle), endowed with the natural projections $\Pi_{\mathcal{U}} : \mathcal{U} \times_M T^*M \to \mathcal{U}$ and $\Pi_{T^*M} : \mathcal{U} \times_M T^*M \to T^*M$. Regarding the local triviality $\mathcal{U}|_V \cong V \times U$ over a local coordinate neighbourhood $V \subset M$, $\mathcal{U} \times_M T^*M$ is identified with $(\mathcal{U}|_V) \times_V (T^*M|_V)$ and with $T^*M|_V \times U$.

We define the *Hamiltonian* function $H : \mathcal{U} \times_M T^*M \to \mathbf{R}$ of the control system $F : \mathcal{U} \to TM$ by

$$H(w, \varphi) := \langle \varphi, F(w) \rangle, \quad (w, \varphi) \in \mathcal{U} \times_M T^*M.$$

Note that the right-hand side is well defined, since $\pi_{T^*M}(\varphi) = \pi_{\mathcal{U}}(w) = \pi_{TM}(F(w))$. In particular, in the case where U is just one point, for a vector field ξ over M, we define $H_\xi : T^*M \to \mathbf{R}$ by $H_\xi(x, p) = \langle p, \xi(x) \rangle$.

For each point $\varphi \in T^*M$, we set $U_\varphi := \Pi_{T^*M}^{-1}(\varphi)$, the fibre of $\Pi_{T^*M} : \mathcal{U} \times_M T^*M \to T^*M$. Then U_φ is diffeomorphic to the fibre U of $\pi_{\mathcal{U}} : \mathcal{U} \to M$.

We define the *relative critical locus* $\Sigma(H) \subset \mathcal{U} \times_M T^*M$ of H by the set of critical points for the restrictions $H|_{U_\varphi} : U_\varphi \to \mathbf{R}$, $\varphi \in T^*M$.

Consider the canonical symplectic form ω on T^*M and the pull-back $\Omega := (\Pi_{T^*M})^*(\omega)$ on $\mathcal{U} \times_M T^*M$ by the mapping Π_{T^*M}. The kernel of the interior product $T_{(w,\varphi)}(\mathcal{U} \times_M T^*M) \to T^*_{(w,\varphi)}(\mathcal{U} \times_M T^*M)$, $\xi \mapsto i_\xi \Omega$ is equal to $T_{(w,\varphi)} U_\varphi$, for each point $(w, \varphi) \in \mathcal{U} \times_M T^*M$. Then consider the relative exterior differential $dH|_{\Sigma(H)} : \Sigma(H) \to \Pi^*_{T^*M}(T^*(T^*M))$, where $\Pi^*_{T^*M}(T^*(T^*M))$ is the pull-back of the vector bundle $T^*(T^*M)$ over T^*M by Π_{T^*M}. It defines a vector field $\overrightarrow{H} : \Sigma(H) \to \Pi^*_{T^*M}(T(T^*M))$ along $\Sigma(H)$, by

$$i_{\overrightarrow{H}} \Omega(w, \varphi) = -dH(w, \varphi), \quad (w, \varphi) \in \Sigma(H),$$

We call $\overrightarrow{H}$ the *Hamilton vector field*, which is just the family of Hamilton vector fields with parameter $u \in U$ restricted to $\Sigma(H)$, in local situation over M. Let $O \subset T^*M$ denote the zero-section of $\pi_{T^*M} : T^*M \to M$. Then we have:

Proposition 2.1 ([2]) *A control $c : [a, b] \to \mathcal{U}$ is a singular control if and only if there exists a lift $\beta : [a, b] \to (\mathcal{U} \times_M T^*M) \setminus \Pi_{T^*M}^{-1}(O)$ of c for $\Pi_{\mathcal{U}} : \mathcal{U} \times_M T^*M \to \mathcal{U}$ which satisfies that $\Pi_{T^*M} \circ \beta : [a, b] \to T^*M$ is Lipschitz and satisfies the constrained Hamiltonian equation:*

$$(\Pi_{T^*M} \circ \beta)'(t) = \overrightarrow{H}(\beta(t)), \quad \beta(t) \in \Sigma(H) (\subset \mathcal{U} \times_M T^*M),$$

for almost every $t \in [a, b]$.

The lift in the above Proposition 2.1 is called *a singular bi-extremal* or *an abnormal bi-extremal* for c.

In particular the notion of singular controls is a local notion:

Corollary 2.2 *A control $c : I \to \mathcal{U}$ is a singular control if and only if, for any subinterval $J \subset I$, the restriction $c : J \to \mathcal{U}$ of c is a singular control.*

The local description of the characterisation of singular control is given as follows:

Proposition 2.3 *Suppose* $\dim M = m, \dim U = r$. *Let*

$$(x; p; u) = (x_1, \dots, x_m; p_1, \dots, p_m; u_1, \dots, u_r)$$

be local coordinates of $\mathcal{U}|_V \times_V T^*M|_V \cong T^*M|_V \times U$ over coordinate neighbourhood $V \subset M$ and $H(x; p; u)$ the local expression of the Hamiltonian of a given control system. Then a control $c : [a, b] \to V \times U$ is singular if and only if there exists a lift $\beta : [a, b] \to T^*M|_V \times U$ of c, $\beta(t) = (x(t); p(t); u(t))$, which satisfies, for almost every $t \in [a, b]$,

$$\begin{cases} \dot{x}_i(t) = \dfrac{\partial H}{\partial p_i}(x(t); p(t); u(t)), \quad (1 \leq i \leq m) \\ \dot{p}_i(t) = -\dfrac{\partial H}{\partial x_i}(x(t); p(t); u(t)), \quad (1 \leq i \leq m) \\ \dfrac{\partial H}{\partial u_j}(x(t); p(t); u(t)) = 0, \quad (1 \leq j \leq r), \qquad p(t) \neq 0. \end{cases}$$

Consider two control systems $F, G : \mathcal{U} \to TM$ on M. Given a local trivialisation $\mathcal{U}|_V \cong V \times U$ over an open subset $V \subset M$, regard F, G as families of vector fields f_u, g_u respectively. Then we define a control system $[F, G] : \mathcal{U}|_V \to TV$ by $[F, G](x, u) = [f_u, g_u](x), x \in V, u \in U$. We denote by H_F the Hamiltonian function $H : T^*V \times U \to \mathbf{R}$ when we make stress on a control system F. In the next section, we will use the following fundamental formula (cf. [2], Chapter 11, for instance):

Lemma 2.4 *We have $dH_G(\overrightarrow{H}_F) = H_{[F,G]} : T^*V \times U \to \mathbf{R}$.*

In particular we have, in the case where $U = \mathrm{pt}$,

Lemma 2.5 *For vector fields ξ, η over M, we have $dH_\xi(\overrightarrow{H}_\eta) = H_{[\eta,\xi]} : T^*M \to \mathbf{R}$.*

A vector subbundle $D \subset TM$ is called a *distribution* or a *differential system* on M. Then the inclusion mapping $F : D \to TM$ with the bundle projection $\pi : D \to M$ defines a control system in the above sense. In this case, a control (resp. a trajectory, a path) is called a D-control (resp. a D-trajectory, a D-path). Then a D-control is uniquely determined from its D-trajectory. Let $\xi_1, \dots, \xi_r$ be a local frame of D. Then the Hamiltonian is given by

$$H_{\xi_i}(x;p) := \langle p, \xi_i(x) \rangle \quad \text{and} \quad H(x;p;u) := u_1 H_{\xi_1}(x;p) + \cdots + u_r H_{\xi_r}(x;p).$$

The local characterisation of singular D-controls is provided by the existence of $p(t) \neq 0$ satisfying

$$\begin{cases} \dot{x}(t) = \quad u_1\xi_1(x(t)) + \cdots + u_r\xi_r(x(t)), \\ \dot{p}_i(t) = -u_1 \dfrac{\partial H_{\xi_1}}{\partial x_i}(x(t);p(t)) - \cdots - u_r \dfrac{\partial H_{\xi_r}}{\partial x_i}(x(t);p(t)), \quad (1 \leq i \leq m), \\ H_{\xi_i}(x(t);p(t)) = 0, \quad (1 \leq j \leq m). \end{cases}$$

Let $D \subset TM$ be a distribution and $\mathcal{D}$ the sheaf of section-germs to D. The *small* (or *weak*) (resp. the *big* (or *strong*)) derived systems $\mathcal{D}^{(i)}$, (resp. $\mathcal{D}^i$), $i = 1, 2, 3, \dots$, are defined by $\mathcal{D}^{(1)} = \mathcal{D}^1 = \mathcal{D}$ and, inductively, by

$$\mathcal{D}^{(i+1)} := \mathcal{D}^{(i)} + [\mathcal{D}, \mathcal{D}^{(i)}], \quad (\text{resp.} \quad \mathcal{D}^{i+1} := \mathcal{D}^i + [\mathcal{D}^i, \mathcal{D}^i]), \ i \geq 1.$$

Note that $\mathcal{D}^{(2)} = \mathcal{D}^2$ and $\mathcal{D}^{(i)} \subset \mathcal{D}^i$, $i \geq 3$. The distribution D is called *bracket generating* if $\cup_{i=1}^{\infty} \mathcal{D}^{(i)} = \mathcal{T}M$, the total sheaf of vector fields over M. Suppose each $\mathcal{D}^{(i)}$ (resp. $\mathcal{D}^i$) coincides with the sheaf of section-germs to some subbundle of TM, written as $D^{(i)}$ (resp. D^i). In this situation, we call D has *small* (or *weak*) (resp. *big* (or *strong*)) growth $(n_1, n_2, \dots, n_i, \dots)$ if $\mathrm{rank}(D^{(i)}) = n_i$ (resp. $\mathrm{rank}(D^i) = n_i$), $i \geq 1$.

A singular control $c : I \to D$ of the control system $D \hookrightarrow TM \to M$ is called *regular* if there exists a singular bi-extremal $\beta : I \to D \times_M T^*M$ for c such that $\Pi_{T^*M} \circ \beta(t) \in D^{(2)\perp} \setminus D^{(3)\perp} \subset T^*M$ for any $t \in I$ ([16][11][15]). A singular control $c : I \to D$ is called *totally irregular* if any singular bi-extremal $\beta : I \to D \times_M T^*M$ for c satisfies that $\Pi_{T^*M} \circ \beta(t) \in D^{(3)\perp} \subset T^*M$ for any $t \in I$. Here, for a subbundle $E \subset TM$, we set $E^\perp = \{p \in T^*M \mid \langle p, v \rangle = 0, \text{ for any } \ v \in E_{\pi_{T^*M}(p)}\}$.

A singular trajectory (resp. a singular path) is called *regular* (resp. *totally irregular*) if so is its control. Note again that in the case of distribution, a control c is uniquely determined from the trajectory $\pi_D \circ c$.

3 Liftings of singular trajectories.

Let M, N be manifolds of dimension $m = n + k, n$ respectively, $E \subset TM$ a distribution on M of rank $r = \ell + k$, and $\pi : M \to N$ a fibration with $K = \mathrm{Ker}(\pi_*) \subset E$. We only treat the case where the π-fibre is an open subset of $\mathbf{R}^k$ and E is trivial over on each π-fibre.

Consider two control systems

$$\mathbb{E} : E \hookrightarrow TM \xrightarrow{\pi_{TM}} M, \text{ over } \ M, \qquad \text{and} \qquad \mathbb{E}/\pi : E \xrightarrow{\pi_*|_E} TN \xrightarrow{\pi_{TN}} N, \text{ over } \ N,$$

respectively. Note that the composition $\pi \circ \pi_E : E \to N$ of the bundle projection $\pi_E : E \to M$ and the fibration $\pi : M \to N$ is again a fibration over N, having the fibre $\pi_E^{-1}(W)$ for a typical fibre $W \subset M$ of π.

Moreover we suppose $E = L \oplus K$ for a complementary subbundle $L \subset E$ to K in E of rank ℓ and consider the third control system

$$\mathbb{L} : L \xrightarrow{\pi_*|_L} TN \xrightarrow{\pi_{TN}} N, \text{ over } \ N.$$

Then first we have

Lemma 3.1 *Let $\gamma : I \to N$ be a singular $(\mathbb{E}/\pi)$-trajectory. Suppose that there exists a Lipschitz abnormal bi-extremal $\beta : I \to E \times_N T^*N$ corresponding to γ. Then there exists a singular $\mathbb{E}$-trajectory $\widetilde{\gamma} : I \to M$ such that $\pi \circ \widetilde{\gamma} = \gamma$.*

Remark 3.2 In Lemma 3.1, we pose the Lipschitz condition on abnormal bi-extremals, because we have to regard some of control parameters as state variables.

Proof of Lemma 3.1: It is sufficient to deal with local situation on N: Take a system of local coordinates $x_1, \dots, x_n, w_1, \dots, w_k$ on M such that $\pi(x, w) = x$ and therefore $K = \langle \frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_k} \rangle$. Take a local frame $\xi_1, \dots, \xi_\ell$ of L, $\xi_i(x, w) = \sum_{j=1}^n c_{ij}(x, w) \frac{\partial}{\partial x_j}, (1 \leq i \leq \ell)$, so that $\xi_1, \dots, \xi_\ell, \frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_k}$ form a local frame of E. The system of canonical local coordinates of T^*M is given by $(x, w; p_1, \dots, p_n, \psi_1, \dots, \psi_k)$. For instance, $K^\perp \subset T^*M$ is defined locally by $\psi_1 = 0, \dots, \psi_k = 0$. For the control system $\mathbb{E}$, the Hamiltonian H is given by

$$H(x, w; p, \psi; \lambda, \mu) = \sum_{1 \leq i \leq \ell} \lambda_i H_{\xi_i}(x, w; p, \psi) + \sum_{1 \leq i \leq k} \mu_i H_{\partial/\partial w_i}(x, w; p, \psi),$$

where $H_{\xi_i}(x, w; p, \psi) = \sum_{1 \leq j \leq n} c_{ij}(x, w) p_j$, $H_{\partial/\partial w_i}(x, w; p, \psi) = \psi_i$, and $\lambda = (\lambda_1, \dots, \lambda_\ell)$ (resp, $\mu = (\mu_1, \dots, \mu_k)$) are the fibre coordinates of L (resp. K). The constrained Hamiltonian system is given by

$$\dot{x}_j = \sum_{1 \leq i \leq \ell} \lambda_i c_{ij}, (1 \leq j \leq n), \quad \dot{w}_j = \mu_j, (1 \leq j \leq k),$$

$$\dot{p}_\nu = -\sum_{1 \leq i \leq \ell, 1 \leq j \leq n} \lambda_i \frac{\partial c_{ij}}{\partial x_\nu} p_j, (1 \leq \nu \leq n), \quad \dot{\psi}_\kappa = -\sum_{1 \leq i \leq \ell, 1 \leq j \leq n} \lambda_i \frac{\partial c_{ij}}{\partial w_\kappa} p_j, (1 \leq \kappa \leq k),$$

$$\sum_{1 \leq j \leq n} c_{ij} p_j = 0, (1 \leq i \leq \ell), \quad \psi_i = 0, (1 \leq i \leq k).$$

The control system $\mathbb{E}/\pi$ over N is locally described as follows: The system of local coordinates of N (resp. T^*N, E) is given by $x = (x_1, \dots, x_n)$, (resp. $(x; p) = (x; p_1, \dots, p_n)$, $(x, w; \lambda, \mu) = (x, w; \lambda_1, \dots, \lambda_\ell, \mu_1, \dots, \mu_k)$). Then the system of local coordinates of $E \times_N T^*N$ is given by $(x, w; p; \lambda, \mu)$. Note that $\dim(E \times_N T^*N) = 2n + 2k + \ell$. In this case, the mapping $F : E \to TN$ is given locally by

$$F(x, w; \lambda, \mu) = \sum_{1 \leq i \leq \ell} \lambda_i \, \pi_{X*} \xi_i(x, w) + \sum_{1 \leq i \leq n} \mu_i \, \pi_{X*} \frac{\partial}{\partial w_i}(x, w) = \sum_{1 \leq i \leq \ell} \lambda_i \, \pi_{X*} \xi_i(x, w),$$

and the Hamiltonian function of the control system $\mathbb{E}/\pi$ is given by

$$H(x, w; p; \lambda, \mu) = \langle p, \sum_{1\leq i\leq \ell} \lambda_i\, \pi_{X*}\xi_i(x, w)\rangle = \sum_{1\leq i\leq \ell} \lambda_i \langle p, \pi_{X*}\xi_i(x, w)\rangle = \sum_{1\leq i\leq \ell, 1\leq j\leq n} \lambda_i c_{ij} p_j.$$

Here the control parameters are given by w, λ, μ. Note that H is independent of μ. The constrained Hamiltonian system is given by

$$\dot{x}_j = \sum_{1\leq i\leq \ell} \lambda_i c_{ij}, (1 \leq j \leq n), \quad \dot{p}_\nu = -\sum_{1\leq i\leq \ell, 1\leq j\leq n} \lambda_i \frac{\partial c_{ij}}{\partial x_\nu} p_j, (1 \leq \nu \leq n),$$
$$\sum_{1\leq j\leq n} c_{ij} p_j = 0, (1 \leq i \leq \ell), \qquad \sum_{1\leq i\leq \ell, 1\leq j\leq n} \lambda_i \frac{\partial c_{ij}}{\partial w_\kappa} p_j = 0, (1 \leq \kappa \leq k).$$

Let $\beta(t) = (x(t), w(t); p(t); \lambda(t), \mu(t))$ be an abnormal bi-extremal for $\mathbb{E}/\pi$. Note that $\mu(t)$ can be taken as arbitrary L^∞ function. Suppose $\beta(t)$ is Lipschitz. Then in particular the control $w(t)$ is Lipschitz. Replace $\mu(t)$ by $\dot{w}(t)$, which is of class L^∞, and take $\psi(t) = 0$. We set $\widetilde{\beta}(t) = (x(t), w(t); p(t), 0; \lambda(t), \dot{w}(t))$. Then $\widetilde{\beta}$ is an abnormal bi-extremal for $\mathbb{E}$ and $\widetilde{\gamma}(t) = (x(t), w(t))$ is a lift of $\gamma(t) = x(t), \pi \circ \widetilde{\gamma} = \gamma$. □

Second we remark that the projection $\pi_L : E = L \oplus K \to L$ induces the projection $\rho : E \times_N T^*N \to L \times_N T^*N$. Then we have

Lemma 3.3 *A curve $\beta : I \to E \times_N T^*N$ is an abnormal bi-extremal (resp. a Lipschitz abnormal bi-extremal) for $\mathbb{E}/\pi$ if and only if $\rho \circ \beta : I \to L \times_N T^*N$ is an abnormal bi-extremal (resp. a Lipschitz abnormal bi-extremal) for $\mathbb{L}$. Moreover any abnormal bi-extremal (resp. a Lipschitz abnormal bi-extremal) $\overline{\beta} : I \to L \times_N T^*N$ for $\mathbb{L}$ is written as $\rho \circ \beta$ by an abnormal bi-extremal (resp. a Lipschitz abnormal bi-extremal) $\beta : I \to E \times_N T^*N$ for $\mathbb{E}/\pi$.*

Proof: The system of local coordinates of $L \times_N T^*N$ is given $(x, w; p; \lambda)$, using the coordinates in the proof of Lemma 3.1. The constrained Hamiltonian system for $\mathbb{L}$ is of the same form for $\mathbb{E}/\pi$ if the ineffective component μ is deleted. Therefore $\beta(t) = (x(t), w(t); p(t); \lambda(t), \mu(t))$ is an abnormal bi-extremal (resp. a Lipschitz abnormal bi-extremal) for $\mathbb{E}/\pi$ if and only if $\rho \circ \beta(t) = (x(t), w(t); p(t); \lambda(t))$ is an abnormal bi-extremal (resp. a Lipschitz abnormal bi-extremal) for L. The second assertion is clear. □

Let $z \in M$. Let ξ be a section-germ at z to L and κ a section-germ at z to K. Then $[\xi, \kappa](z)$ modulo E_z depends only on the tangent vectors $\xi(z), \kappa(z) \in T_zM$. For $v \in L_z$, we define a linear map $\mathrm{ad}(v) : K_z \to T_zM/E_z$ by $\mathrm{ad}(v)(u) := [\xi, \kappa](z) \bmod E_z, \xi(z) = v, \kappa(z) = u$. We need the following result for the proof of our main result in the next section.

Lemma 3.4 *Let $(z; v) \in L, v \in L_z$. Then the differential map $\pi_{X*}|_L : L \to TX$ is an immersion at $(z; v)$ if and only if $\mathrm{ad}(v) : K_z \to T_zM/E_z$ is injective.*

Proof: We use the system of coordinates (x, w) of M and $(x, w; \lambda)$ of L as in Lemma 3.1 and, take a frame of L, $\xi_i(x, w) = \sum_{1 \leq j \leq n} c_{ij}(x, w)\frac{\partial}{\partial x_j}$, $(1 \leq i \leq \ell)$. Then $\pi_* : L \to TX$ is expressed by

$$(x, w; \lambda) \mapsto (x_1, \ldots, x_n, \sum_{1 \leq i \leq \ell} \lambda_i c_{i1}, \ldots, \sum_{1 \leq i \leq \ell} \lambda_i c_{in}).$$

The Jacobi matrix of the mapping π_* is given by

$$\begin{pmatrix} E_n & O & \cdots & O & O & \cdots & O \\ * & \sum_i \lambda_i \frac{\partial c_{i1}}{\partial w_1} & \cdots & \sum_i \lambda_i \frac{\partial c_{i1}}{\partial w_k} & c_{11} & \cdots & c_{\ell 1} \\ * & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \sum_i \lambda_i \frac{\partial c_{in}}{\partial w_1} & \cdots & \sum_i \lambda_i \frac{\partial c_{in}}{\partial w_k} & c_{1n} & \cdots & c_{\ell n} \end{pmatrix}.$$

Let $(z; v) \in L$ and $v = \sum_{1 \leq i \leq \ell} \lambda_i \xi_i(z)$. Then the Jacobi matrix at $(z; v) \in L$ is of rank $n + k + \ell$ if and only if

$$\left[\sum_{1 \leq i \leq \ell} \lambda_i \xi_i, \ \frac{\partial}{\partial w_1}\right], \ldots, \left[\sum_{1 \leq i \leq \ell} \lambda_i \xi_i, \ \frac{\partial}{\partial w_k}\right], \xi_1, \ldots, \xi_\ell$$

are linear independent at z, which is equivalent to that $\mathrm{ad}(v)$ is injective. $\Box$

4 Duality of singular paths.

Let Y be a 5-dimensional manifold and $D \subset TY$ a distribution on Y of rank two. Let Z be the set of tangential directions on D,

$$Z := \{z \mid z \text{ is an oriented one-dimensional linear subspace of } D_y \text{ for some } y \in Y\},$$

with the natural projection $\pi_Y : Z \to Y$. Define the Cartan prolongation $E \subset TZ$ of D, tautologically from D by $E_z := (\pi_{Y*})^{-1}(z)$ for any $z \in Z$ regarded as a line $z \subset T_{\pi_Y(z)}Y$ (see the definition in [6]). Suppose D has small growth $(2, 3, 5)$. Note that D has big growth $(2, 3, 5)$ as well. We give a detailed proof of the following known result, in order to use in the process of the proof on our main result.

Proposition 4.1 ([15][16]) *Let $D \subset TY$ be a distribution with growth $(2, 3, 5)$ on a 5-dimensional manifold Y. Then, given any point $y \in Y$ and any oriented $\ell \subset T_yY$, there exists an oriented singular D-trajectory through p with the direction ℓ, which is smooth and is unique up to parametrisation.*

Proof: Let D be a distribution with growth $(2, 3, 5)$ on a 5-dimensional manifold Y. Recall that D is of rank 2, D^2 is of rank 3 and $D^{(3)} = TY$. Let $\{\eta_1, \eta_2\}$ be a local frame of D. Then η_1, η_2 and $[\eta_1, \eta_2] = \eta_3$ form a local frame of D^2. Therefore, locally, say on an open set $U \subset Y$, there exists a frame $\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$ of TU such that

$$D = \langle \eta_1, \eta_2 \rangle, \quad [\eta_1, \eta_2] = \eta_3, \quad [\eta_1, \eta_3] = \eta_4, \quad [\eta_2, \eta_3] = \eta_5.$$

Then the Hamiltonian function H on $D \times_Y T^*Y = T^*U \times \mathbf{R}^2$ is given by

$$H(y, q; u) = u_1 H_{\eta_1}(y; q) + u_2 H_{\eta_2}(y; q),$$

where $H_{\eta_i}(y; q) = \langle q, \eta_i(y)\rangle,\ i = 1, 2$, and the constrained Hamiltonian system is given by

$$\begin{cases} \dot{y}(t) = u_1(t)\eta_1(y(t)) + u_2(t)\eta_2(y(t)), \\ \dot{q}_i(t) = -u_1(t)\dfrac{\partial H_{\eta_1}}{\partial y_i}(y(t); q(t)) - u_2(t)\dfrac{\partial H_{\eta_2}}{\partial y_i}(y(t); q(t)), \quad (1 \leq i \leq 5), \\ H_{\eta_1}(y(t); q(t)) = 0, \quad H_{\eta_2}(y(t); q(t)) = 0, \quad q(t) \neq 0. \end{cases}$$

Differentiating both sides of $H_{\eta_1}(y(t); q(t))(= \langle q(t), \eta_1(x(t))\rangle) = 0$ by t, we have, using Lemma 2.4,

$$\begin{aligned} 0 &= (dH_{\eta_1}(\overrightarrow{H}))(y(t); q(t); u(t)) = dH_{\eta_1}(u_1\overrightarrow{H}_{\eta_1} + u_2\overrightarrow{H}_{\eta_2})(y(t); q(t); u(t)) \\ &= u_2(t)H_{[\eta_2,\eta_1]}(y(t); q(t)). \end{aligned}$$

Similarly we have, from $H_{\eta_2}(y(t); q(t)) = 0$, that $u_1(t)H_{[\eta_1,\eta_2]}(y(t); q(t)) = 0$. Therefore if $(u_1(t), u_2(t)) \neq (0, 0)$, then $H_{\eta_3}(y(t); q(t)) = 0$. Thus we have that a Lipschitz curve $y(t)$ is a locally non-constant singular D-trajectory if and only if there exist $u(t)$ and $q(t) \in (D^2)^\perp \setminus 0$ satisfying the above constrained Hamiltonian system with $H_{\eta_3}(y(t); q(t)) = 0$ in addition. Moreover from $H_{\eta_3}(y(t); q(t)) = 0$, we have

$$u_1(t)H_{\eta_4}(y(t); q(t)) + u_2(t)H_{\eta_5}(y(t); q(t)) = 0.$$

Consider the vector field

$$Ab_{\eta_1,\eta_2} := H_{\eta_5}\overrightarrow{H}_{\eta_1} - H_{\eta_4}\overrightarrow{H}_{\eta_2}$$

over T^*Y (see [21]). Then Ab_{η_1,η_2} is tangent to $(D^2)^\perp$. In fact

$$Ab_{\eta_1,\eta_2}(H_{\eta_1}) = 0, Ab_{\eta_1,\eta_2}(H_{\eta_2}) = 0 \ \text{ and } \ Ab_{\eta_1,\eta_2}(H_{\eta_3}) = 0,$$

on $(D^2)^\perp \subset T^*Y$. Therefore any singular (oriented) path is obtained as the projection of an integral curve of Ab_{η_1,η_2} or $-Ab_{\eta_1,\eta_2}$ in $(D^2)^\perp$. $\square$

Let $[u_1, u_2]$ be a projective coordinate of the fibre of $\pi_Y : Z = PD \to Y$ and $z = u_2/u_1$ an affine coordinate of the fibre. Locally identify Z with $\mathbf{R} \times Y$. Set $\zeta = \partial/\partial z$. Then the Cartan prolongation E is generated by ζ and $\eta_1 + z\eta_2$.

Lemma 4.2 *The Cartan prolongation of a distribution with growth* $(2, 3, 5)$ *has small growth* $(2, 3, 4, 5, 6)$ *and big growth* $(2, 3, 4, 6)$.

Proof: Since $[\zeta, \eta_1 + z\eta_2] = [\zeta, \eta_1] + [\zeta, z\eta_2] = \eta_2$, we have $E^2 = \langle \zeta, \eta_1 + z\eta_2, \eta_2\rangle = \langle \zeta, \eta_1, \eta_2\rangle$, which is of rank 3. Since $[\eta_1 + z\eta_2, \eta_2] = \eta_3$, we have $E^{(3)} = E^3 = \langle \zeta, \eta_1, \eta_2, \eta_3\rangle$, which is of rank 4. Since $[\eta_1+z\eta_2, \eta_3] = \eta_4+z\eta_5$, we have $E^{(4)} = \langle \zeta, \eta_1, \eta_2, \eta_3, \eta_4+z\eta_5 \rangle$, that is of rank 5, while $E^4 = TZ$, that is of rank 6. Since $[\zeta, \eta_4+z\eta_5] = \eta_5$, we have $E^{(5)} = TZ$. $\Box$

Note that, by Proposition 4.1, any immersed D-trajectory lifts to a E-trajectory, and Z is foliated by lifted paths of singular D-paths. Thus we have a subbundle $K \subset E$ by this foliation. We set $L = \mathrm{Ker}(\pi_{Y*})$. Then $E = L \oplus K$. As we see below, this decomposition is intrinsically determined from (Z, E).

The notions of regular singular paths and totally irregular paths are defined at the end of §2. Then we have:

Proposition 4.3 *Let D be a distribution with growth $(2,3,5)$ on a five dimensional manifold Y, and (Z, E) the Cartan prolongation of D. Then any singular E-path (unparametrised immersed E-trajectory) is either a fibre of $\pi_Y : Z \to Y$ or the lift of a singular D-path. Any π_Y-fibre is a regular singular path of E. The lift of any singular D-path is a totally irregular singular path of E.*

Proof: Consider the Hamiltonian of E:

$$H(y, z; q, \varphi; \lambda, \mu) = \lambda\varphi + \mu H_{\eta_1+z\eta_2}(y, z; q, \varphi),$$

where $(y, z; q, \varphi)$ is the canonical coordinate system of T^*Z for a coordinate system (y, z) of Z and λ, μ are control parameters. Then the constrained Hamiltonian equation for an abnormal bi-extremal

$$\beta(t) = (y(t), z(t); q(t), \varphi(t); \lambda(t), \mu(t))$$

is given by

$$\begin{cases} \dot{z}(t) &= \lambda(t) \\ \dot{y}(t) &= \mu(t)(\eta_1 + z\eta_2)(y(t), z(t)) \\ \dot{\varphi}(t) &= -\mu(t)H_{\eta_2}(y(t); q(t)) \\ \dot{q}(t) &= -\mu(t)\frac{\partial(H_{\eta_1+z\eta_2})}{\partial y}(y(t), z(t); q(t), \varphi(t)) \end{cases}$$

with $\varphi(t) = 0, H_{\eta_1+z\eta_2}(y(t), z(t); q(t), \varphi(t)) = 0, (q(t), \varphi(t)) \neq 0$.

We are supposing $(y(t), z(t))$ is C^∞ and $(\dot{y}(t), \dot{z}(t)) \neq 0$ for all $t \in I$. Therefore $(\lambda(t), \mu(t))$ is C^∞ and $(\lambda(t), \mu(t)) \neq 0$. Suppose there is a non-empty sub-interval $J \subset I$ where $\mu(t) \neq 0$. Then, on J, we have

$$H_{\eta_1}(y(t); q(t)) = 0, \quad H_{\eta_2}(y(t); q(t)) = 0.$$

Then we have

$$0 = \lambda(t)H_{[\zeta,\eta_2]}(y(t); q(t)) + \mu(t)H_{[\eta_1+z\eta_2,\eta_2]}(y(t); q(t)) = \mu(t)H_{\eta_3}(y(t); q(t)).$$

Since $\mu(t) \neq 0$, we have $H_{\eta_3}(y(t); q(t)) = 0$. Thus $q(t) \in D^2$ and $(q(t), 0) \in E^{(3)\perp}$. Moreover we have

$$H_{\eta_4}(y(t); q(t)) + z(t)H_{\eta_5}(y(t); q(t)) = 0.$$

Therefore, on J, the immersive singular E-trajectory is the lift of an immersive singular D-trajectory and $(q(t), 0) \in E^{(4)\perp}$. Take maximal J. Then $J = I$. In fact, otherwise, take an boundary point t_0 of J which is an interior point of I. Then $\lambda(t_0) \neq 0$ and $\mu(t_0) = 0$, so $(\dot{y}(t_0), \dot{z}(t_0)) = (0, \lambda(t_0))$ must be tangent to the lift of an immersive singular D-trajectory, which is a contradiction. Thus, in this case, we have that the trajectory $(y(t), z(t))$ coincides with the lift of an immersive singular D-trajectory on I, which is totally irregular.

If $\mu(t) = 0$ on I, then, $y(t)$ is constant on I. Setting $y(t) = y_0$, we have that the trajectory $(y_0, z(t))$ gives a π_Y-fibre. In this case, for any $q_0 \neq 0$ with $H_{\eta_1}(y_0; q_0) = 0, H_{\eta_2}(y_0; q_0) = 0, H_{\eta_3}(y_0; q_0) \neq 0$, we have a singular bi-extremal $(y_0, z(t); q_0, 0; \dot{z}(t), 0)$ and $(q_0, 0) \in E^{2\perp} \setminus E^{(3)\perp}$. Therefore any π_Y-fibre is regular singular. $\square$

Remark 4.4 The lift of a singular D-path is irregular singular of "parabolic type" in the sense of [21].

Thus we have locally a fibration $\pi_X : Z \to X$ to another 5-dimensional manifold X and thus the double fibration

$$Y \xleftarrow{\pi_Y} Z \xrightarrow{\pi_X} X.$$

In fact the construction is local in Y: If Y is a germ at a point $y_0 \in Y$, then we can take Z as a germ along the simple closed curve $\pi_Y^{-1}(y_0) \cong S^1$ and X as a germ along the simple closed curve $\pi_X(\pi_Y^{-1}(y_0)) \cong S^1$. The Cartan prolongation $E \subset TZ$ is expressed as $\mathrm{Ker}(\pi_{Y*}) \oplus \mathrm{Ker}(\pi_{X*})$. Therefore, for each $z \in Z$, E_z projects to a line $\ell_z := \pi_{X*}(E_z) \subset T_{\pi_X(z)}X$ by $\pi_{X*} : T_zZ \to T_{\pi_X(z)}X$. Thus, for any $x \in X$, we have a family of lines $\{\ell_z \mid z \in \pi_X^{-1}(x)\} \subset T_xX$. We define a cone

$$C_x := \bigcup_{z \in \pi_X^{-1}(x)} \ell_z \subset T_xX,$$

and the cone field $C \subset TX$. This cone field is regarded as a control system as follows: Let $L = \mathrm{Ker}(\pi_{Y*}) \subset E$. Note that $\ell_z = \pi_{X*}(L_z)$. We set $F := \pi_{X*}|_L : L \to TX$, the composition of the inclusion to TZ and the differential $\pi_{X*} : TZ \to TX$ of the projection π_X. The projection is defined by $\pi_L = \pi_{TX} \circ F : L \to X$. We denote this control system by $\mathbb{C}$. Note that, in other word, $\mathbb{C}$ corresponds to the control system $\mathbb{L}$, using the notation in §3 applied to the projection $M = Z \to N = X$.

In what follows, we take a sufficiently small open neighbourhood of a point in Y such that the composition $\pi_X \circ \pi_E$ of the bundle projection $\pi_E : E \to Z$ and $\pi_X : Z \to X$ is a fibration with a fibre which diffeomorphic to $\mathbf{R}^3$ (the product of $\mathbf{R}^2$ and an open interval).

Lemma 4.5 *The differential map $\pi_{X*}|_{L\setminus 0} : L \setminus 0 \to TX$ restricted to L off the zero section is an immersion.*

Proof: We have two systems of local coordinates $y_1, \dots, y_5, z$ and $x_1, \dots, x_5, w$ of Z such that $\pi_Y(y, z) = y$ and that $\pi_X(x, w) = x$. The line bundle L is generated by

$\zeta = \frac{\partial}{\partial z}$ and the line bundle K is generated by $\varpi = \frac{\partial}{\partial w}$. By Lemma 4.2, we have that $[\zeta, \varpi] \neq 0$ mod.E. Therefore we have that $\mathrm{ad}(\zeta) : K \to TZ/K$ is injective. Therefore we have the result from Lemma 3.4. $\square$

Remark 4.6 The linear hull $H(C) \subset TX$ of the cone structure $C \subset TX$ defines a contact structure D' on X ([5]). In fact the contact structure is defined by $D'_x = \pi_{X*}(E^{(4)}_z)(\subset T_xX)$, for any $z \in Z$ with $\pi_X(z) = x$ (cf. Lemma 4.2). Moreover the tangent planes to the cones $C_{\gamma(t)}$, which is a Lagrangian plane in $D'_{\gamma(t)}$ along a singular trajectory $\gamma(t)$ of $\mathbb{C}$ is regarded as a curve in a Lagrange Grassmannian, which coincides with the (reduced) Jacobi curve introduced by Agrachev-Zelenko (cf.[22]).

Lemma 4.7 *Let $\gamma : [a, b] \to X$ be an immersive singular trajectory for the control system $\mathbb{C}$. Then there exists an immersive singular control $c : [a, b] \to L$ and a Lipschitz abnormal bi-extremal $\beta : [a, b] \to L \times_X T^*X$ for $\mathbb{C}$ corresponding to γ.*

Proof: We keep to take the systems of coordinates used in Lemma 3.1, Lemma 3.3 and Lemma 4.5. The Hamiltonian $H : L \times_X T^*X \to \mathbf{R}$ is given, in this case, by

$$H(x, w; p; \lambda) = \lambda\langle p, \pi_{X*}(\zeta(x, w))\rangle,$$

where (x, w, λ) (resp. (x, w), λ) is the system of coordinates of $L = \mathrm{Ker}(\pi_{Y*})$, (resp. of Z, of π_Y-fibres) and $\zeta = \frac{\partial}{\partial z}$ as in Lemma 4.5. The control parameters are given by w and λ. See the proofs of Lemma 3.1 and Lemma 3.3. Then the constrained Hamiltonian system is given by

$$\dot{x}(t) = \lambda(t)\pi_{X*}(\zeta(x(t), w(t))), \quad \dot{p}_i(t) = -\lambda(t)\langle p(t), \tfrac{\partial}{\partial x_i}\pi_{X*}(\zeta(x(t), w(t)))\rangle, (1 \leq i \leq 5),$$
$$\langle p(t), \pi_{X*}(\zeta(x(t), w(t)))\rangle = 0, \quad \langle p(t), \tfrac{\partial}{\partial w}\pi_{X*}(\zeta(x(t), w(t)))\rangle = 0.$$

Note that $\pi_{X*} \circ \zeta : Z \to TX$ is a smooth vector field along the projection $\pi_X : Z \to X$. Take $w_0 \in \mathbf{R}$ such that $(x(a), w_0)$ belongs to the coordinate neighbourhood in Z we are considering and $\dot{x}(a) \in (\pi_{X*})(L_{(x(a),w_0)})$. Since $\gamma(t) = x(t)$ is immersive, by Lemma 4.5, we have a unique smooth control $c : [a, b] \to L = \mathrm{Ker}(\pi_{Y*}) \subset E$, $c(t) = (x(t), w(t); \lambda(t))$ such that $\pi_L \circ c = \gamma$, $w(a) = w_0$. In fact, the coordinate $w(t)$ along π_X-fibre is determined by the velocity direction and the coordinate $\lambda(t)$ along π_Y-fibre is determined by the velocity of γ. Then $\beta(t) = (x(t), w(t); p(t); \lambda(t))$ is a Lipschitz abnormal bi-extremal. $\square$

Thus we have reached to the stage to show the following:

Theorem 4.8 *A singular $\mathbb{C}$-path is the π_X-image of a π_Y-fibre.*

Proof: Let $\gamma : I \to X$ be a singular $\mathbb{C}$-trajectory Then, by Lemma 4.7, γ lifts to a Lipschitz abnormal bi-extremal $\beta : I \to L \times_X T^*X$ for $\mathbb{C}$. Then, by Lemma 3.1, γ lifts to a singular $\mathbb{E}$-trajectory $\widetilde{\gamma}$ for the projection $\pi_X : Z \to X$, $\pi_X \circ \widetilde{\gamma} = \gamma$. Since γ is immersive, $\widetilde{\gamma}$ is also immersive. By Proposition 4.3, $\widetilde{\gamma}$ is either a π_Y-fibre or the lift of a singular D-trajectory. In this case, $\widetilde{\gamma}$ must be a π_Y-fibre, and we have that γ is the π_X-image of a π_Y-fibre up to parametrisation. $\square$

Remark 4.9 For any point x of X and for any direction of C_x, there exists uniquely a singular $\mathbb{C}$-path through x with the given direction. The space of singular $\mathbb{C}$-paths on X is identified with Y.

Remark 4.10 A singular $\mathbb{C}$-trajectory is not necessarily the image of a π_Y-fibre by π_X. For example a piecewise smooth $\mathbb{C}$-trajectory which consists of several images of π_Y-fibres by π_X is also a singular $\mathbb{C}$-trajectory. This remark applies to singular $\mathbb{D}$-trajectories of course.

Remark 4.11 The distribution $D \subset TY$ is also regarded as a cone structure defined by the control system $\mathbb{K} : K = \mathrm{Ker}(\pi_{X*}) \subset TZ \to TY \to Y$. Then, naturally, singular paths for $\mathbb{K}$ coincide with singular D-paths by Lemmas 3.1 and 3.3.

Remark 4.12 A singular bi-extremal for a control system is called *totally singular* if the Hessian matrix of the Hamiltonian with respect to the control parameter is zero along the bi-extremal and a trajectory is called *totally singular* if it possesses a totally singular bi-extremal ([2]). Then all singular trajectories on X for $\mathbb{C}$ and on Y for $\mathbb{D}$ are totally singular, and therefore the duality of singular paths on Y and X holds on the level of totally singular paths as well.

5 Example: G_2-case.

From the view point of twistor theory, the double fibration $Y \leftarrow Z \to X$ with G_2-symmetry and the canonical geometric structures, which are the G_2-Engel distribution $E \subset TZ$, the Cartan distribution $D \subset TY$ and the cone structure $C \subset TX$, have been explicitly constructed in [13]. In this section, we give the explicit representations of the associated control systems $\mathbb{E} : E \hookrightarrow TZ \to Z$, $\mathbb{D} : D \hookrightarrow TY \to Y$ and $\mathbb{C} : E \to TX \to X$, directly calculate the constrained Hamiltonian systems for the singular controls and determine the singular paths.

Example 5.1 (The control system $\mathbb{E}$ on Z.) On a space Z of dimension 6 with coordinates λ, x, y, z, u, v, consider the explicit double fibrations

$$(\lambda, x + \lambda y, y + \lambda z, v + \lambda x, u + \lambda(y^2 - xz)) \xleftarrow{\pi_Y} (\lambda, x, y, z, u, v) \xrightarrow{\pi_X} (x, y, z, u, v).$$

Then the G_2-Engel distribution $E = \mathrm{Ker}(\pi_{Y*}) \oplus \mathrm{Ker}(\pi_{X*})$ is generated by

$$\xi_1 = \frac{\partial}{\partial \lambda}, \quad \xi_2 = \frac{\partial}{\partial z} - \lambda\frac{\partial}{\partial y} + \lambda^2\frac{\partial}{\partial x} - \lambda^3\frac{\partial}{\partial v} + (\lambda^3 z + 2\lambda^2 y + \lambda x)\frac{\partial}{\partial u}.$$

For the coordinates $(\lambda, x, y, z, u, v; \kappa, p, q, r, \ell, m)$ of T^*Z, we have the Hamiltonian functions of ξ_1, ξ_2

$$H_{\xi_1} = \kappa, \quad H_{\xi_2} = r - \lambda q + \lambda^2 p - \lambda^3 m + (\lambda^3 z + 2\lambda^2 y + \lambda x)\ell,$$

and the total Hamiltonian function $H = aH_{\xi_1} + bH_{\xi_2}$, where (a, b) is the fibre coordinate of E. The constrained Hamiltonian system for singular controls is given by

$$\dot{\lambda} = a, \quad \dot{x} = b\lambda^2, \quad \dot{y} = -b\lambda, \quad \dot{z} = b, \quad \dot{u} = b(\lambda^3 z + 2\lambda^2 y + \lambda x), \quad \dot{v} = -b\lambda^3,$$
$$\dot{\kappa} = -b\{-q + 2\lambda p - 3\lambda^2 m + (3\lambda^2 z + 4\lambda y + x)\ell\},$$
$$\dot{p} = -b\lambda\ell, \quad \dot{q} = -b \cdot 2\lambda^2\ell, \quad \dot{r} = -b\lambda^3\ell, \quad \dot{\ell} = 0, \quad \dot{m} = 0,$$

and $H_{\xi_1} = 0, H_{\xi_2} = 0$. We set $\ell = \ell_0, m = m_0$. By differentiating both sides of $H_{\xi_2} = 0$, we have

$$a\{-q + 2\lambda p - 3\lambda^2 m_0 + (3\lambda^2 z + 4\lambda y + x)\ell_0\} = 0.$$

We consider two cases: one case where $a(t) \neq 0$ on a non-empty sub-interval and another case where $a(t)$ is identically zero.

In the first case, we have

$$-q + 2\lambda p - 3\lambda^2 m_0 + (3\lambda^2 z + 4\lambda y + x)\ell_0 = 0,$$

on the sub-interval. By differentiating both sides of the equation, we have

$$(2p - 6\lambda m_0) + (6\lambda z + 4y)\ell_0 = 0.$$

By differentiating both sides again, we have $z\ell_0 - m_0 = 0$. Assume $\ell_0 = 0$, then we have $p = q = r = 0$, which leads a contradiction with the condition on abnormal bi-extremals. Therefore $\ell_0 \neq 0$. We have that $\dot{z} = 0$ therefore $b = 0$, and that x, y, z, u, v are constant on the sub-interval. Moreover, if the singular trajectory is immersive, we have that x, y, z, u, v are constant on the whole interval, similarly to the proof of Proposition 4.3. Thus, in this case the singular control gives a π_X-fibre.

If a is identically zero, then λ is constant and

$$(x + \lambda y)' = 0, \quad (y + \lambda z)' = 0, \quad (v + \lambda x)' = 0, \quad (u + \lambda(y^2 - xz))' = 0.$$

Therefore, in the second case, the singular control gives a π_Y-fibre.

Example 5.2 (The control system $\mathbb{D}$ on Y.) For the Cartan distribution D on Y, let us consider the space with coordinates $\lambda, \mu, \nu, \tau, \sigma$ and the distribution $D \subset TY$ generated by

$$\eta_1 = \frac{\partial}{\partial\lambda} + \nu\frac{\partial}{\partial\mu} - (\lambda\nu - \mu)\frac{\partial}{\partial\tau} + \nu^2\frac{\partial}{\partial\sigma}, \quad \eta_2 = \frac{\partial}{\partial\nu} - \lambda\frac{\partial}{\partial\mu} + \lambda^2\frac{\partial}{\partial\tau} - (\lambda\nu + \mu)\frac{\partial}{\partial\sigma}.$$

For the coordinates $(\lambda, \mu, \nu, \tau, \sigma; p, q, r, \ell, m)$ of T^*Y, we set

$$H_{\eta_1} = p + \nu q - (\lambda\nu - \mu)\ell + \nu^2 m, \quad H_{\eta_2} = r - \lambda q + \lambda^2\ell - (\lambda\nu + \mu)m,$$

and $H = u_1 H_{\eta_1} + u_2 H_{\eta_2}$, where (u_1, u_2) is the fibre coordinate of D. Then we have the constrained Hamiltonian equation:

$$\dot{\lambda} = u_1, \quad \dot{\mu} = u_1\nu - u_2\lambda, \quad \dot{\nu} = u_2, \quad \dot{\tau} = -u_1(\lambda\nu - \mu) + u_2\lambda^2, \quad \dot{\sigma} = u_1\nu^2 - u_2(\lambda\nu + \mu),$$
$$\dot{p} = u_1\nu\ell - u_2(2\lambda\ell - \nu m), \quad \dot{q} = -u_1\ell + u_2 m, \quad \dot{r} = -u_1(q - \lambda\ell + 2\nu m) + u_2\lambda m,$$
$$\dot{\ell} = 0, \quad \dot{m} = 0, \quad p + \nu q - (\lambda\nu - \mu)\ell + \nu^2 m = 0, \quad r - \lambda q + \lambda^2\ell - (\lambda\nu + \mu)m = 0.$$

Then by direct calculations we can conclude that singular D-paths are π_Y-projections of π_X-fibres. Because this claim is known, we omit the details.

Example 5.3 (The control system $\mathbb{C}$ on X.) Let λ, x, y, z, u, v be local coordinates of Z. From the explicit form of the projection π_Y, we see that the line bundle $L = \mathrm{Ker}(\pi_{Y*})$ is generated by

$$\frac{\partial}{\partial z} + \lambda(-\frac{\partial}{\partial y} + x\frac{\partial}{\partial u}) + \lambda^2(\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial u}) + \lambda^3(-\frac{\partial}{\partial v} + z\frac{\partial}{\partial u}).$$

Then we can regard the cone structure $C \subset TX$ as the control system $\mathbb{C} : L \to TX \to X$ on X given by

$$\dot{\gamma} = \mu\left\{\frac{\partial}{\partial z} + \lambda(-\frac{\partial}{\partial y} + x\frac{\partial}{\partial u}) + \lambda^2(\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial u}) + \lambda^3(-\frac{\partial}{\partial v} + z\frac{\partial}{\partial u})\right\},$$

where $\gamma(t) = (x(t), y(t), z(t), u(t), v(t))$ and λ, μ are regarded as control parameters. The Hamiltonian function is given by

$$H = \mu\{r + \lambda(-q + x\ell) + \lambda^2(p + 2y\ell) + \lambda^3(-m + z\ell)\},$$

where $(x, y, z, u, v; p, q, r, \ell, m)$ is a coordinate system of T^*X. Then the constrained Hamiltonian system is given by

$$\begin{gathered}\dot{x} = \mu\lambda^2, \quad \dot{y} = -\mu\lambda, \quad \dot{z} = \mu, \quad \dot{u} = \mu(\lambda x + 2\lambda^2 y + \lambda^3 z), \quad \dot{v} = -\mu\lambda^3,\\ \dot{p} = -\mu\lambda\ell, \quad \dot{q} = -2\mu\lambda^2\ell, \quad \dot{r} = -\mu\lambda^3\ell, \quad \dot{\ell} = 0, \quad \dot{m} = 0,\\ r + \lambda(-q + x\ell) + \lambda^2(p + 2y\ell) + \lambda^3(-m + z\ell) = 0,\\ \mu\{-q + x\ell + 2\lambda(p + 2y\ell) + 3\lambda^2(-m + z\ell)\} = 0.\end{gathered}$$

Set $\ell = \ell_0, m = m_0$. Since our trajectory is immersive, we have $\mu \neq 0$. Therefore we have

$$-q + x\ell_0 + 2\lambda(p + 2y\ell_0) + 3\lambda^2(-m_0 + z\ell_0) = 0.$$

By differentiating both sides, we have

$$\dot{\lambda}\{2(p + 2y\ell_0) + 6\lambda(-m_0 + z\ell_0)\} = 0.$$

We assume $\dot{\lambda} \neq 0$. Then

$$2(p + 2y\ell_0) + 6\lambda(-m_0 + z\ell_0) = 0.$$

By differentiating both sides, we have $\dot{\lambda}(-m_0 + z\ell_0) = 0$, so $-m_0 + z\ell_0 = 0$. By differentiating both sides, we have $\mu\ell_0 = 0$, so $\ell = \ell_0 = 0$. Then we have $m = m_0 = 0$, and then $p = q = r = 0$. This contradicts with the condition on abnormal bi-extremals. Therefore we have $\dot{\lambda} = 0$, and λ is constant. Set $\lambda = \lambda_0$. Then we have, up to parametrisation and orientation,

$$x(t) = \lambda_0^2 t + x_0,\ y(t) = -\lambda_0 t + y_0,\ z(t) = t + z_0,\ v(t) = -\lambda_0^3 t + v_0,$$

and

$$u(t) = (\lambda_0 x_0 + 2\lambda_0^2 y_0 + \lambda_0^3 z_0)t + u_0,$$

where x_0 (resp. y_0, z_0, v_0, u_0) is the initial value of $x(t)$ (resp. $y(t), z(t), v(t), u(t)$). Thus we have shown that any singular $\mathbb{C}$-path is the π_X-projection of a π_Y-fibre in G_2-case directly.

Remark 5.4 In G_2-case [13], the π_X-image of a π_Y-fibre, namely a singular $\mathbb{C}$-path in X, is called a *Monge line*, while the π_Y-image of a π_X-fibre, namely a singular $\mathbb{D}$-path in Y, is called a *Cartan line.*

References

[1] A. A. Agrachev, *Geometry of Optimal Control Problems and Hamiltonian Systems,* Nonlinear and Optimal Control Theory, Springer-Verlag (2004), pp. 1–59.

[2] A.A. Agrachev, A.V. Sarychev, *Control Theory from Geometric Viewpoint,* Encyclopaedia of Mathematical Sciences **87**, Springer-Verlag, (2004).

[3] A. Agrachev, I. Zelenko, *Nurowski's conformal structures for* $(2,5)$*-distributions via dynamics of abnormal extremals,* Proceedings of RIMS Symposium on "Developments of Cartan Geometry and Related Mathematical Problems", RIMS Kokyuroku **1502**, 204-218, arxiv math.DG/0605059.

[4] B. Bonnard, M. Chyba, *Singular Trajectories and the Role in Control Theory,* Springer-Verlag (2003).

[5] R. L. Bryant, *Élie Cartan and geometric duality,* A lecture given at the Institut d'Élie Cartan on 19 June 1998.

[6] R.L. Bryant, L. Hsu, *Rigidity of integral curves of rank 2 distributions,* Inventiones Mathematicae, **114** (1993), 435–461.

[7] E. Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre,* Ann. Sci. Ecole Norm. Sup. (3), **27** (1910), 109–192.

[8] B. Doubrov, I. Zelenko, *On local geometry of nonholonomic rank* 2 *distributions*, Journal of London Mathematica Society, (2) **80–3** (2009), 545–566.

[9] B. Doubrov, I. Zelenko, *Equivalence of variational problems of higher order,* Differential Geometry and its Applications, **29–2**(2011), 255–270.

[10] B. Doubrov, I. Zelenko, *Prolongation of quasi-principal frame bundles and geometry of flag structures on manifolds,* preprint, arXiv math.DG/1210.7334v2.

[11] L. Hsu, *Calculus of variations via the Griffiths formalism,* Journal of Differential Geometry, **36** (1992), 551–589.

[12] G. Ishikawa, Y. Machida, M. Takahashi, *Asymmetry in singularities of tangent surfaces in contact-cone Legendre-null duality,* Journal of Singularities, **3** (2011), 126–143.

[13] G. Ishikawa, Y. Machida, M. Takahashi, *Singularities of tangent surfaces in Cartan's split* G_2*-geometry,* Hokkaido University Preprint Series in Mathematics #1020, (2012).

[14] Y. Kitagawa, *The Infinitesimal automorphisms of a homogeneous subriemannian contact manifold,* Thesis, Nara Women's University (2005).

[15] W. Liu, H.J. Sussman, *Shortest paths for sub-Riemannian metrics on rank-two distributions,* Memoirs of Amer. Math. Soc., **118–564**, Amer. Math. Soc., (1995).

[16] R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications,* Mathematical Surveys and Monographs, **91**, Amer. Math. Soc. (2002).

[17] P. Nurowski, *Differential equations and conformal structures,* Journal of Geometry and Physics, (2005) **55–1**, 19–49.

[18] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, *The Mathematical Theory of Optimal Processes,* Pergamon Press (1964).

[19] N. Tanaka, *On affine symmetric spaces and the automorphism groups of product manifolds,* Hokkaido Math. J. **14** (1985), 277–351.

[20] T. Yatsui, *On pseudo-product graded Lie algebras,* Hokkaido Math. J. **17** (1988), 333–343.

[21] I. Zelenko, *Nonregular abnormal extremals of* 2*-distribution: Existence, second variation, and rigidity.* Journal of Dynamical and Control Systems, **5–3** (1999), 347–283.

[22] I. Zelenko, *Fundamental form and the Cartan tensor of* $(2,5)$*-distributions coincide,* Journal of Dynamical and Control Systems, **12–2** (2006), 247–276.

[23] M. Zhitomirskii, *Exact normal form for* $(2,5)$ *distributions,* RIMS Kokyuroku **1502** (2006), 16–28.

Goo ISHIKAWA,
Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.
e-mail : ishikawa@math.sci.hokudai.ac.jp

Yumiko KITAGAWA,
Oita National College of Technology, Oita 870-0152, Japan.
e-mail : kitagawa@oita-ct.ac.jp

Wataru YUKUNO,
Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.
e-mail : yukuwata@math.sci.hokudai.ac.jp