博士学位論文

Studies on vector fields and differential forms on C^∞-schemes
(C^∞ スキーム上のベクトル場と微分形式に関する研究）

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Contents

1 Introduction

In this thesis, we study the theory of C^∞-rings and derivations on them. Based on these algebraic theory, we explain C^∞-ringed spaces and C^∞-schemes as generalized concepts of C^∞-manifolds. Moreover we study vector fields and differential forms on C^∞-ringed spaces and C^∞-schemes.

The one motivation of this article is to develop the theory of C^∞-manifolds, tangent vectors of C^∞-manifolds, vector fields of C^∞-manifolds and C^∞-functions on C^∞-manifolds in more general way. In the case of C^∞-manifolds, tangent vector spaces are finitely generated as $\mathbb{R}$-vector spaces, and tangent vector fields are locally expressed by finite C^∞-functions. Another motivation is to compare tangent vectors and tangent vector fields of C^∞-manifolds.

In §2, we explain and use the idea of *C^∞-rings* in [6] and [9]. C^∞-rings are $\mathbb{R}$-algebras which have operations by smooth functions $\mathbb{R}^n \to \mathbb{R}$. Let M be a C^∞-manifold and p a point of M. The set $C^\infty(M)$ of C^∞-functions on M, The set $C_p^\infty(M)$ of C^∞-function-germs at p on M, the set $\mathbb{R}[[x_1, \ldots, x_n]]$ of real formal power series on $\mathbb{R}^n$, and the set $\mathbb{R}$ of real numbers are C^∞-rings. Each of these C^∞-rings is expressed as a quotient C^∞-ring $C^\infty(\mathbb{R}^n)/I$ by an ideal I of $C^\infty(\mathbb{R}^n)$ as the $\mathbb{R}$-algebra. These are called *finitely generated C^∞-rings* which are main objects of our paper.

For a C^∞-ring $\mathfrak{C}$ and a homomorphism $p : \mathfrak{C} \to \mathbb{R}$, there exists a local C^∞-ring $\mathfrak{C}_p$ and its maximal ideal m_p of $\mathfrak{C}_p$. Then we define a k-jet $j_p^k(c) := c + m_p^{k+1}$ of p and a homomorphism $j_p^k : \mathfrak{C} \to \mathfrak{C}_p/m_p^{k+1}$. $\mathfrak{C}$ is called a *k-jet determined C^∞-ring* if it can be embedded into a power $\prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/m_p^{k+1}$ of $\mathfrak{C}_p/m_p^{k+1}$.

From [4], we have $\mathbb{R}$-*derivations* and $\mathbb{R}$-*cotangent modules* of C$^\infty$-rings. We have *C^∞-derivations* and *C^∞-cotangent modules* of C$^\infty$-rings by using the C$^\infty$-ring structure in a nontrivial way in [6]. For example, tangent vectors and exterior derivatives of differential forms on C^∞-manifolds can be regarded as $\mathbb{R}$-derivations. For a finitely generated C$^\infty$-ring $C^\infty(\mathbb{R}^n)/I$ and a zero point $p \in \mathbb{R}^n$ of I, we consider $\mathbb{R}$-derivations $C^\infty(\mathbb{R}^n)/I \to \mathbb{R}$ and a composition $e_p \circ V : C^\infty(\mathbb{R}^n)/I \to \mathbb{R}$ of an $\mathbb{R}$-derivation $V : C^\infty(\mathbb{R}^n)/I \to C^\infty(\mathbb{R}^n)/I$ and a homomorphism $e_p : C^\infty(\mathbb{R}^n)/I \to \mathbb{R}(e_p(f + I) := f(p))$. These $\mathbb{R}$-derivations are C$^\infty$-derivations.

In §3, we explain the idea of *C^∞-ringed spaces* and *C^∞-schemes* to generalize C^∞-manifolds. A C^∞-ringed space $\underline{X}$ is a topological space X with a sheaf $\mathcal{O}_X$ of C^∞-rings on X. For a C^∞-manifold M, we have a C^∞-ringed space which has the topological space M and a sheaf $\mathcal{O}_M$ defined by $\mathcal{O}_M(U) = C^\infty(U)$ for any open set $U \subset M$. We define a functor from the category of C^∞-rings to the category of C^∞-ringed spaces. We define a C^∞-scheme as a local C^∞-ringed space covered by affine C^∞-schemes.

Dominic Joyce uses C^∞-rings and C^∞-schemes to define d-manifolds and d-orbifolds, versions of manifolds and orbifolds related to Spivak's derived manifolds in [6].

In §4, we explain the main theorems of this thesis.

First, we consider a C^∞-ring $\mathfrak{C}$ or on a $\mathfrak{C}$-module $\mathfrak{M}$ which satisfies that any $\mathbb{R}$-derivation $d : \mathfrak{C} \to \mathfrak{M}$ is a C^∞-derivation. In the paper [14], we have proved for a C^∞-ring $\mathfrak{C}$ and a k-jet determined C^∞-ring $\mathfrak{D}$. Tangent vectors and tangent vector fields on C^∞-manifolds satisfy the above.

Second, we define tangent vectors and tangent vector fields on a C^∞-ringed space as $\mathbb{R}$-derivations of $\mathcal{O}_X$. This idea comes from tangent vectors are regarded as $\mathbb{R}$-derivations $C_p^\infty(M) \to \mathbb{R}$ and tangent vector fields are regarded as $\mathbb{R}$-derivations $C_p^\infty(M) \to C_p^\infty(M)$.

Third, we consider algebraic computability of differentials of C^∞-rings. We study about Nash functions and Leibniz complexities based on [5]. A Nash function is a function $f = f(x_1, \ldots, x_n) : \mathbb{R}^n \supset U \to \mathbb{R}$ which has a non-zero real polynomial $p(x, y) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}(x = (x_1, \ldots, x_n))$ such that $p(x, f(x)) = 0$ on U. Then, we can algebraically calculate the total differential df by the linearity and Leibniz rules of d.

The *Leibniz complexity* $LC(f)$ of f is defined as the minimal number of usages of Leibniz rules to compute df algebraically. Nash functions are characterized by the finiteness of Leibniz complexity (§4.5.2, Theorem 4.15). For a Nash function $f(x) : \mathbb{R}^n \supset U \to \mathbb{R}$ which has a non-zero real polynomial $p(x, y) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $p(x, f(x)) = 0$ on U, the Leibniz complexity $LC(f)$ of f is not exceeding the Leibniz complexity $LC(p)$ of p.

By Nash functions on $\mathbb{R}^n$ and a C^∞-ring, we define $\mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$ and $\mathfrak{L}\mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$. From the definition of Leibniz complexities in §4.5.2, we define two complexities $\mathrm{LComp}_{\mathfrak{C}}(f)$ and $\mathrm{Comp}_{\mathfrak{C}}(f)$.

2 C^∞-rings

In this section, we recall the notions of C^∞-rings and derivations of C^∞-rings.

2.1 C^∞-rings

Definition 2.1 ([3, 6]) *(1) A C^∞-**ring** is a set $\mathfrak{C}$ which is endowed with the following data;*
for any $l \in \{0\} \cup \mathbb{N}$ and any C^∞-map $f : \mathbb{R}^l \to \mathbb{R}$ (if $l = 0$, f is a constant value $c \in \mathbb{R}$), there are given an operation $\Phi_f : \mathfrak{C}^l \to \mathfrak{C}$ (if $l = 0$, an operation Φ_f is a constant on $\mathfrak{C}$) such that

- *for any* $k, l \in \{0\} \cup \mathbb{N}$, C^∞*-maps* $g : \mathbb{R}^k \to \mathbb{R}$ *and* $f_i : \mathbb{R}^l \to \mathbb{R}(i = 1, \ldots, k)$, *and* $c_1, \cdots, c_l \in \mathfrak{C}$,

  $\Phi_g(\Phi_{f_1}(c_1, \ldots, c_l), \ldots, \Phi_{f_k}(c_1, \ldots, c_l)) = \Phi_{g\circ(f_1,\ldots,f_k)}(c_1, \ldots, c_l)$ *and*

- *for any* $l \in \{0\}\cup\mathbb{N}$, *projections* $\pi_i : \mathbb{R}^l \to \mathbb{R}$ *defined as* $\pi_i(x_1, \ldots, x_l) = x_i (i = 1, \cdots, l)$ *and* $c_1, \ldots, c_l \in \mathfrak{C}$,

  $$\Phi_{\pi_i}(c_1, \ldots, c_l) = c_i.$$

(2) Let $\mathfrak{C}$ *and* $\mathfrak{D}$ *be* C^∞*-rings with operations* $\Phi_f : \mathfrak{C}^l \to \mathfrak{C}$ *and* $\Psi_f : \mathfrak{D}^l \to \mathfrak{D}$ *respectively for any* C^∞*-function* $f : \mathbb{R}^l \to \mathbb{R}$. *A homomorphism between* C^∞*-rings is a map* $\phi : \mathfrak{C} \to \mathfrak{D}$ *which satisfies*

$\phi\big(\Phi_f(c_1, \ldots, c_n)\big) = \Psi_f\big(\phi(c_1), \ldots, \phi(c_n)\big)$ *for any* $c_1, \cdots, c_l \in \mathfrak{C}, f : \mathbb{R}^l \to \mathbb{R}$.

(3) We will write ***C^∞Rings*** *for the category of* C^∞*-rings.*

We give examples of C^∞-rings and morphisms of C^∞-rings as followings.

Example 2.2 *The set* $\mathbb{R}$ *of real numbers has a structure of a* C^∞*-ring by the operation* $\Phi_f : \mathbb{R}^l \to \mathbb{R}$ *for* $f \in C^\infty(\mathbb{R}^l)$ *as* $\Phi_f(r_1, \ldots, r_l) := f(r_1, \ldots, r_l)$ *for any* $r_1, \ldots, r_l \in \mathbb{R}$.

Example 2.3 $C^\infty(M)$ *is a set of* C^∞*-functions on a* C^∞*-manifold* M. $C^\infty(M)$ *and the following operations* Φ_g *form a* C^∞*-ring.*

$$\Phi_g(f_1, \ldots, f_k)(x) := g(f_1(x), \ldots, f_k(x))$$

Example 2.4 *There exists following examples of morphisms between* C^∞*-rings.*

Suppose $f : X \to Y$ *is a smooth map between manifolds. We can define a morphism* $f^* : C^\infty(Y) \to C^\infty(X)$ *of* C^∞*-rings as* $f^*(c) := c \circ f \in C^\infty(X)$ *for* $c \in C^\infty(Y)$. *From the definition, we deduce the following properties.*

(1) For a smooth manifolds without boundary X, Y*,the map* $f \mapsto f^*$ *from smooth maps* $f : X \to Y$ *to morphisms of* C^∞*-rings* $\phi : C^\infty(Y) \to C^\infty(X)$ *is a 1-1 correspondence.*

(2) Let f^* *be a surjection. Assume that there exists two points* $x, x' \in X$ *such that* $x \neq x'$ *and* $f(x) = f(x')$. *There exists a* C^∞*-function* $\eta \in C^\infty(X)$ *such that* $\eta(x) = 0$ *and* $\eta(x') = 1$. *For the surjection* f^*, *there exists* $\iota \in C^\infty(Y)$ *such that* $\eta = f^*(\iota) = \iota \circ f$. *However* $\eta(x) = \iota(f(x)) = \iota(f(x')) = \eta(x')$ *contradicts* $\eta(x) = 0$ *and* $\eta(x') = 1$. *Therefore, we have* f *is an injection.*

(3) Let f be a surjection. Assume that there exists $c \in C^\infty(Y)$ such that $f^*(c) = c \circ f = 0$ on X. For any $y \in Y$, there exists $x \in X$ such that $y = f(x)$. Then $c(y) = c(f(x)) = 0$. We have $c = 0$ on X. Therefore, we have f is an injection.

Remark that any C^∞-ring has the natural $\mathbb{R}$-algebra structure. Define the addition on $\mathfrak{C}$ as $c + c' := \Phi_{(x,y)\mapsto x+y}(c, c')$, the multiplication on $\mathfrak{C}$ as $c \cdot c' := \Phi_{(x,y)\mapsto xy}(c, c')$, and the scalar multiplication by $\lambda \in \mathbb{R}$ as $\lambda c := \Phi_{x\mapsto \lambda x}(c)$. We see that elements 0 and 1 in $\mathfrak{C}$ are given by $0_{\mathfrak{C}} := \Phi_{\emptyset\mapsto 0}(\emptyset)$ and $1_{\mathfrak{C}} := \Phi_{\emptyset\mapsto 1}(\emptyset)$.

Let $\mathfrak{C}$ be a C^∞-ring with operations $\Phi_f : \mathfrak{C}^l \to \mathfrak{C}$ for any C^∞-function $f : \mathbb{R}^l \to \mathbb{R}$. Let $I \subset \mathfrak{C}$ be an ideal of the $\mathbb{R}$-algebra. We define a C^∞-ring structure on the quotient $\mathbb{R}$-algebra $\mathfrak{C}/I$ as follows. For any natural number $l \in \mathbb{N}$ and a C^∞-function $f \in C^\infty(\mathbb{R}^l)$, there exists C^∞-functions $g_1, \ldots, g_l \in C^\infty(\mathbb{R}^{2l})$ which satisfy

$$f(x_1 + y_1, \ldots, x_l + y_l) - f(x_1, \ldots, x_l) = \sum_{k=1}^{l} y_k g_k(x_1, \ldots, x_l, y_1, \ldots, y_l)$$

for any $(x_1, \ldots, x_l), (y_1, \ldots, y_l) \in \mathbb{R}^l$ by Hadamard's lemma (see [11] for the proof). Then we have

$$\Phi_f(c_1 + i_1, \ldots, c_l + i_l) - \Phi_f(c_1, \ldots, c_l) = \sum_{k=1}^{l} i_k \cdot \Phi_{g_k}(c_1, \ldots, c_l, i_1, \ldots, i_l)$$

for any $c_1, \ldots, c_l \in \mathfrak{C}$ and $i_1, \ldots, i_l \in I$. Therefore the quotient $\mathbb{R}$-algebra $\mathfrak{C}/I$ has a C^∞-ring structure with a following operation $\Phi^I_f : (\mathfrak{C}/I)^l \to \mathfrak{C}/I$ for any $f \in C^\infty(\mathbb{R}^l)$ defined as

$$\Phi^I_f(c_1 + I, \ldots, c_l + I) := \Phi_f(c_1, \ldots, c_l) + I \text{ for any } c_1, \ldots, c_l \in \mathfrak{C}.$$

Example 2.5 *The set $C^\infty_p(M)$ of C^∞-function-germs on M at a point p is a C^∞-ring, which has the unique maximal ideal m_p consisting of germs with zero values at p. We write $[f, U]_p$ as a C^∞-function-germ at p on M by a smooth function $f \in C^\infty(U)$ on an open set $U \subset M$.*

The set $C^\infty_p(M)/{m_p}^{k+1}$ of k-jets of C^∞-functions on M at a point p has a C^∞-ring structure.

2.2 The types of C^∞-rings

We proved that the set $C^\infty(\mathbb{R}^n)$ of smooth functions on n-dimensional real space $\mathbb{R}^n$ forms a C^∞-ring in Example 2.3.

Then we consider C^∞-rings which is expressed in terms of $C^\infty(\mathbb{R}^n)$ and its ideal I. Then we will classify C^∞-rings as to their ideals.

Definition 2.6 **([6])** *Let $\mathfrak{C}$ be a C^∞-ring.*

*(1) A C^∞-rings $\mathfrak{C}$ is called **finitely generated** if there exists $c_1, \ldots, c_n \in \mathfrak{C}$ such that for all $c \in \mathfrak{C}$ there exists $f \in C^\infty(\mathbb{R}^n)$ with $c = \Phi_f(c_1, \ldots, c_n)$.*

Thus there exists $n \in \mathbb{N}$ and an ideal $I \subset C^\infty(\mathbb{R}^n)$ which satisfy that $\mathfrak{C}$ is isomorphic to $C^\infty(\mathbb{R}^n)/I$ as a C^∞-ring.

Write $\boldsymbol{C^\infty Rings^{fg}}$ for the full subcategory of finitely generated C^∞-rings in $\boldsymbol{C^\infty Rings}$.

*(2) Let $\mathfrak{C}$ be a finitely generated C^∞-ring and $I \subset C^\infty(\mathbb{R}^n)$ an ideal of $C^\infty(\mathbb{R}^n)$ with $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$. We call $\mathfrak{C}$ **finitely presented** if I is a finitely generated $C^\infty(\mathbb{R}^n)$-module, i.e., there exists $i_1, \ldots, i_k \in C^\infty(\mathbb{R}^n)$ such that $I = \langle i_1, \ldots, i_k \rangle_{C^\infty(\mathbb{R}^n)}$.*

Write $\boldsymbol{C^\infty Rings^{fp}}$ for the full subcategory of finitely presented C^∞-rings in $\boldsymbol{C^\infty Rings}$.

*(3) An ideal $I \subset C^\infty(\mathbb{R}^n)$ is called **fair** if I is an ideal of elements $f \in C^\infty(\mathbb{R}^n)$ which satisfies $\pi_p(f) \in \pi_p(I)$ for any $p \in \mathbb{R}^n$. A C^∞-ring $\mathfrak{C}$ is called **fair** if it is isomorphic to $C^\infty(\mathbb{R}^n)/I$, where I is a fair ideal.*

Write $\boldsymbol{C^\infty Rings^{fa}}$ for the full subcategory of fair C^∞-rings in $\boldsymbol{C^\infty Rings}$.

(4) For a closed subset X of $\mathbb{R}^n$, define an ideal $m_X^\infty := \{c \in C^\infty(\mathbb{R}^n) | \partial^\alpha c = 0$ on X for all $k \geq 0$ and $|\alpha| \leq k\}$.

*We call an ideal $I \subset C^\infty(\mathbb{R}^n)$ **good** if there exists finite smooth functions $f_1, \ldots, f_k \in C^\infty(\mathbb{R}^n)$ and a closed subset X of $\mathbb{R}^n$ such that $I = \langle f_1, \ldots, f_k, m_X^\infty \rangle_{C^\infty(\mathbb{R}^n)}$.*

*A C^∞-ring $\mathfrak{C}$ is called **good** if it is isomorphic to $C^\infty(\mathbb{R}^n)/I$, where I is a good ideal.*

Write $\boldsymbol{C^\infty Rings^{go}}$ for the full subcategory of good C^∞-rings in $\boldsymbol{C^\infty Rings}$.

(5) Subcategories of $\boldsymbol{C^\infty Rings}$ satisfy

$$\boldsymbol{C^\infty Rings^{fp}} \subset \boldsymbol{C^\infty Rings^{go}} \subset \boldsymbol{C^\infty Rings^{fa}} \subset \boldsymbol{C^\infty Rings^{fg}} \subset \boldsymbol{C^\infty Rings}.$$

Definition 2.7 **([6])** *Let $\mathfrak{C}$ be a C^∞-ring.*

(1) *Let $\mathfrak{C}$ be a C^∞-ring and $S \subset \mathfrak{C}$ a subset of $\mathfrak{C}$. We call a C^∞-ring $\mathfrak{C}'$ with following properties a* ***localization*** *of $\mathfrak{C}$ at S and write $\mathfrak{C}[S^{-1}] := \mathfrak{C}'$.*

- *There exists a unique morphism $\pi : \mathfrak{C} \to \mathfrak{C}'$ such that $\pi(s)$ is invertible in $\mathfrak{C}'$ for all $s \in S$.*
- *If there exists a morphism $\phi : \mathfrak{C} \to \mathfrak{D}$ such that $\phi(s)$ is invertible in $\mathfrak{D}$ for all $s \in S$, there exists a unique morphism $\psi : \mathfrak{C}' \to \mathfrak{D}$ such that $\psi \circ \pi = \phi$, i.e., a following diagram is commutative.*

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "\pi"] \arrow[dr, "\phi"'] & \mathfrak{C}' \arrow[d, "\psi"] \\ & \mathfrak{D} \end{tikzcd}$$

(2) *A C^∞-ring $\mathfrak{C}$ is called a* **C^∞-local ring** *if $\mathfrak{C}$ has a unique maximal ideal $m_{\mathfrak{C}}$ which satisfies $\mathfrak{C}/m_{\mathfrak{C}} \cong \mathbb{R}$.*

Frankly speaking, finitely generated C^∞-rings are C^∞-rings which expressed in terms of smooth functions of $\mathbb{R}^n$, fair C^∞-rings are C^∞-rings which expressed in terms of germs of smooth functions at each $\mathbb{R}^n$.

Example 2.8 ([6]) *We have following examples of the above C^∞-rings.*

(1) *For a finitely generated C^∞-ring $C^\infty(\mathbb{R}^n)$, an open subset $U \subset \mathbb{R}^n$ and a smooth function $f \in C^\infty(\mathbb{R}^n)(U = f^{-1}(\mathbb{R}\backslash\{0\}))$, we have following isomorphisms.*

$$C^\infty(U) \cong C^\infty(\mathbb{R}^n)[f^{-1}] \cong C^\infty(\mathbb{R}^{n+1})/\langle yf(x_1, \ldots, x_n) - 1\rangle_{C^\infty(\mathbb{R}^{n+1})}.$$

(2) *For an $\mathbb{R}$-point $x : \mathfrak{C} \to \mathbb{R}$ and a subset $S := \{c \in \mathfrak{C} | x(c) \neq 0\}$, we have a localization $\mathfrak{C}_x := \mathfrak{C}[S^{-1}]$ with its projection $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$.*

(3) *Let $\eta \in C^\infty(\mathbb{R})$ be a smooth function with $\eta > 0$ on $(0, 1)$ and $\eta = 0$ on $\mathbb{R}\backslash(0, 1)$. Then the following ideal is not fair.*

$$I := \left\{ \sum_{a \in A} g_a(x)\eta(x - a) \middle| \begin{matrix} A \subset \mathbb{Z} \text{ is a finite set, and} \\ g_a \in C^\infty(\mathbb{R}) \text{ for all } a \in A \end{matrix} \right\}.$$

The subset of smooth functions $f \in C^\infty(\mathbb{R})$ which satisfies $[f, \mathbb{R}]_x \in \pi_x(I)$ for any $x \in \mathbb{R}$ is equals to

$$\overline{I} := \left\{ \sum_{a \in \mathbb{Z}} g_a(x)\eta(x - a) \middle| g_a \in C^\infty(\mathbb{R}) \text{ for all } a \in \mathbb{Z} \right\} \supset I.$$

$\overline{I}$ is not equal to I. Therefore, I is not a fair ideal.

From the following theorem, isomorphisms of C^∞-rings hold the property of fair, good and finitely presented of C^∞-rings.

Theorem 2.9 ([6]) *Suppose we have a following isomorphism of C^∞-rings*

$$C^\infty(\mathbb{R}^m)/I \cong C^\infty(\mathbb{R}^n)/J.$$

Then $C^\infty(\mathbb{R}^m)/I$ is fair(resp. good, finitely presented) if and only if $C^\infty(\mathbb{R}^n)/J$ is fair(resp. good, finitely presented).

2.3 Localizations and k-jets

Definition 2.10 ([6]) *Let $\mathfrak{C}$ be a C^∞-ring.*

(1) An $\mathbb{R}$**-point of** $\mathfrak{C}$ *is defined as a surjective homomorphism $p : \mathfrak{C} \to \mathbb{R}$ of C^∞-rings (see Definition 2.1 (2), Example 2.2).*

(2) Take any $\mathbb{R}$-point $p : \mathfrak{C} \to \mathbb{R}$. We call a C^∞-ring $\mathfrak{C}_p$ the **localization of** $\mathfrak{C}$ **at** *p if $\mathfrak{C}_p$ is a localization of $\mathfrak{C}$ at $p^{-1}(\mathbb{R}\backslash\{0\})$ with the projection homomorphism $\pi_p : \mathfrak{C} \to \mathfrak{C}_p$ in Definition 2.6. The localization $\mathfrak{C}_p$ always exists for any $\mathbb{R}$-point $p : \mathfrak{C} \to \mathbb{R}$. $\mathfrak{C}_p$ has the unique maximal ideal $m_p \subset \mathfrak{C}_p$ such that $\mathfrak{C}_p/m_p$ is isomorphic to $\mathbb{R}$ ([6]).*

(3) Let $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. For any $\mathbb{R}$-point $p : \mathfrak{C} \to \mathbb{R}$, define a natural homomorphism $j^k_p : \mathfrak{C} \to \mathfrak{C}_p/{m_p}^{k+1}$ by $j^k_p(c) := \pi_p(c) + m^{k+1}_p$ for $c \in \mathfrak{C}$ (If $k = \infty$, we mean ${m_p}^{k+1}$ by ${m_p}^\infty := \cap_{k\in\mathbb{N}} {m_p}^k$). We call $j^k_p(c)$ the k**-jet of** *c* **at** *p. Define a homomorphism $j^k : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/m^{k+1}_p$ as $j^k := (j^k_p)_{p:\mathfrak{C}\to\mathbb{R}}$.*

(4) For any $\mathbb{R}$-point $p : \mathfrak{C} \to \mathbb{R}$, define a homomorphism $j^g : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p$ as $j^g := (\pi_p)_{p:\mathfrak{C}\to\mathbb{R}}$.

Example 2.11 *Let M be a C^∞-manifold and $p \in M$. Define an $\mathbb{R}$-point $e_p : C^\infty(M) \to \mathbb{R}$ by $e_p(f) := f(p)$ for $f \in C^\infty(M)$.*

For the $\mathbb{R}$-point e_p, the localization $\big(C^\infty(M)\big)_{e_p}$ is isomorphic to the C^∞-ring $C^\infty_p(M)$ of C^∞-function-germs at p. Its unique maximal ideal is given by $m_{e_p} = \{[f,U]_p \in C^\infty_p(M) | f(p) = 0\}$.

Remark 2.12 *In I. Moerdijk and G.E. Reyes ([9] Definition 4.1), the point determined C^∞-rings, near-point determined C^∞-rings and germ determined C^∞-rings are introduced.*

Generalizing them, we define k-jet determined C^∞-rings by $\mathbb{R}$-points and their localizations in the next subsection §2.4.

2.4 k-jet determined C^∞-rings

We will define k-jet determined C^∞-rings. They will be used to prove our Theorem 4.1 and Lemma 4.2.

2.4.1 Point determined C^∞-rings

Let Λ be an index set and $\mathfrak{C}_\lambda$ a C^∞-ring with an operation $\Phi_f^\lambda : \mathfrak{C}_\lambda^n \to \mathfrak{C}_\lambda (f \in C^\infty(\mathbb{R}^n))$ for $\lambda \in \Lambda$. Its product $\mathfrak{C} := \prod_{\lambda\in\Lambda} \mathfrak{C}_\lambda$ is a C^∞-ring with operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for $f \in C^\infty(\mathbb{R}^n)$ as

$$\Phi_f(\{c_\lambda^1\}_{\lambda\in\Lambda}, \dots, \{c_\lambda^n\}_{\lambda\in\Lambda}) := \{\Phi_f^\lambda(c_\lambda^1, \dots, c_\lambda^n)\}_{\lambda\in\Lambda}.$$

Definition 2.13 *Let $\mathfrak{C}$ be a C^∞-ring. $\mathfrak{C}$ is* **point determined** *if it can be embedded into a power $\Pi_{i\in I}\mathbb{R}$ of $\mathbb{R}$.*

For an ideal $I \subset C^\infty(\mathbb{R}^n)$, we define a zero point set $Z(I)$ as a set of $p \in \mathbb{R}^n$ which satisfies $f(p) = 0$ for any $f \in I$.

Proposition 2.14 *Let $\mathfrak{C}$ be a C^∞-ring of the form $C^\infty(M)/I$, where M is a manifold. Then $\mathfrak{C}$ is point determined if and only if for all $f \in C^\infty(M)$ $f \in I$ if and only if $f|_{Z(I)} = 0$.*

We have a following lemma derived from Proposition 3.6 in [9].

Lemma 2.15 *Let $\mathfrak{C}$ be a C^∞-ring of the form $C^\infty(M)/I$, where M is a manifold. Let $\phi : \mathfrak{C} \to \mathbb{R}$ be an $\mathbb{R}$-point map.*

Then ϕ is of the form ev_x(the evaluation at x, $ev_x(f) = f(x)$) for a unique point x in the zero set $Z(I)$.

Proof) Suppose that there exists a C^∞-homomorphism $\phi : C^\infty(M) \to \mathbb{R}$ such that $\phi \neq ev_x$ for any $x \in M$.

There exists a property function $\rho \in C^\infty(M)$ of M. Define a property C^∞-function $p := \rho - \phi(\rho)$. p is a kernel of ϕ and its zero set $p^{-1}(0)$ is a compact set. For any $x \in p^{-1}(0)$ and $\phi \neq ev_x$, there exists a smooth function $g_x \in \ker\phi$ and an open neighborhood $U_x \subset M$ of $x \in M$ with $g_x(y) \neq 0$ for any $y \in U_x$. For the compactness of $p^{-1}(0)$, $p^{-1}(0)$ is covered by finite $U_{x_1}, \dots, U_{x_k}$. Then $g := p^2 + \sum_{i=1}^k (g_{x_i})^2$ satisfies that $g > 0$ on M and $\phi(g) = 0$. g is invertible in $C^\infty(M)$. Then $1 = \phi(g \cdot 1/g) = \phi(g)\phi(1/g) = 0$.

Suppose that there exists two points $x, x' \in M$ such that $ev_x(f) = f(x)$ and $ev'_x(f) = f(x')$ Assume that $x \neq x'$. Then there exists $\iota \in C^\infty(M)$ such that

$\iota(x) = 1$, $\iota(x') = 0$. However, it contradicts $\iota(x) = ev_x(\iota) = ev_{x'}(\iota) = \iota(x')$. We have $x = x'$.

Therefore, there exists a unique point $x \in M$ which satisfies $\phi = ev_x$. ■

Proof of Proposition 2.14) Suppose that $\mathfrak{C}$ is point determined. Take $f \in C^\infty(M)$ with $f \notin I$. Then $f \neq 0$. Therefore there is an $\mathbb{R}$-point map $\phi : \mathfrak{C} \to \mathbb{R}$ such that $\phi(f) \neq 0$. However for some $x \in Z(I)$, $\phi \equiv ev_x$ by Lemma 2.15, so $f(x) \neq 0$.

Suppose that a latter condition. Define a morphism $\phi : \mathfrak{C} \to \prod_{x \in Z(I)} \mathbb{R}$ as $\phi(f) := \{f(x)\}_{x \in Z(I)}$. Then $\phi(f) = 0$ if and only if $f \in I$ for the assumption of I. Therefore ϕ is an embedding to $\Pi_{x \in Z(I)}\mathbb{R}$. ■

Example 2.16 *Let $X \subset M$ be a subset of a manifold M. Define an ideal $m^0_X := \{f \in C^\infty(M) | f|_X \equiv 0\}$. For any $f \in m^0_X$, we have $f^{-1}(0) \supset X$. Therefore we have $X \subset \cap_{f \in m^0_X} f^{-1}(0) = Z(m^0_X) = \overline{X}$.*

Fix a smooth function $f \in C^\infty(M)$ with $f(x) = 0$ for all $x \in Z(m^0_X)$. For any point $y \in X \subset Z(m^0_X)$, $f(y) = 0$. Therefore $f \in m^0_X$.

Proposition 2.17 *For any ideal I of $C^\infty(M)$, if I is point determined, then I is also fair.*

For finite smooth functions $g_1, \ldots, g_n \in C^\infty(M)$, define an ideal $\langle g_1, \ldots, g_n \rangle_{C^\infty(M)}$ as a set of smooth functions generated by $g_1, \ldots, g_n$. From Lemma 2.18, $C^\infty(M)/\langle g_1, \ldots, g_n \rangle_{C^\infty(M)}$ is a point determined C^∞-rings for $g_1, \ldots, g_n \in C^\infty(M)$ which are independent.

Lemma 2.18 *Let M be a manifold. Suppose the smooth functions $g_1, \ldots, g_n \in C^\infty(M)$ are independent, i.e. for each smooth zero point $x \in Z(g_1, \ldots, g_n)$, the linear map $(d(g_1)_x, \ldots, d(g_n)_x) : T_xM \to \mathbb{R}^n$ is a surjection.*
Then $\langle g_1, \ldots, g_n \rangle_{C^\infty(M)} = m^0_{Z(g_1, \ldots, g_n)}$.

Proof) For $g_1, \ldots, g_n$, $g_i(y) = 0$ for any $i = 1, \ldots, n$ and $y \in Z(g_1, \ldots, g_n)$. Therefore, we have the inclusion $\langle g_1, \ldots, g_n \rangle \subset m^0_{Z(g_1, \ldots, g_n)}$.

Let us first note that for an arbitrary function $h \in C^\infty(M)$, $h \in \langle g_1, \ldots, g_n \rangle_{C^\infty(M)}$ if and only if for each $x \in M$ there is an open neighborhood U of x such that $h|_U \in \langle g_1|_U, \ldots, g_n|_U \rangle_{C^\infty(U)}$.

Take $h \in C^\infty(M)$ with $h|_{Z(g_1, \ldots, g_n)} = 0$, and choose $x \in M$.

- If $x \notin Z(g_1, \ldots, g_n)$, $g_i(x) \neq 0$ for some i. Then trivially there exists an open neighborhood U at x of M such that $g_i(y) \neq 0$ for any $y \in U$. Then we have $h|_U = (\frac{h|_U}{g_i|_U}) g_i|_U \in \langle g_1|_U, \ldots, g_n|_U \rangle_{C^\infty(U)}$.

- If $x \in Z(g_1, \ldots, g_n)$, then by hypothesis $g = (g_1, \ldots, g_n) : M \to \mathbb{R}^n$ is a submersion at x. By the local submersion theorem, there exist open neighborhoods U, V of origins and local coordinates $\phi : \mathbb{R}^m \supset U \to M$, $\psi : \mathbb{R}^n \supset V \to \mathbb{R}^n (\phi(0) = x, \psi(0) = 0)$ such that $g \circ \phi = \psi \circ p$

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d, "\phi"'] & V \arrow[d, "\psi"] \\ M \arrow[r, "g"] & \mathbb{R}^n \end{tikzcd}$$

  for a canonical projection $p(x_1, \ldots, x_m) = (x_1, \ldots, x_n)$.

  We claim that $h|_{\phi(U)} \in \langle g_1|_{\phi(U)}, \ldots, g_n|_{\phi(U)} \rangle_{C^\infty(\phi(U))}$. Indeed, replacing h by $h \circ \phi = k$, it suffices to note that if $k : U \to \mathbb{R}$ and $k|_{0 \times \mathbb{R}^{m-n}}$, then $k \in \langle \pi_1, \ldots, \pi_n \rangle$. But this is just Hadamard's lemma again: we can write

$$k(x_1, \ldots, x_m) = k(0, \ldots, 0, x_{n+1}, \ldots, x_m) + \sum_{i=1}^{n} x_i v_i(x_1, \ldots, x_m).$$

Therefore, we have that there is an open neighborhood U of x such that $h|_U \in \langle g_1|_U, \ldots, g_n|_U \rangle_{C^\infty(U)}$ for any $x \in M$ and $h \in \langle g_1, \ldots, g_n \rangle_{C^\infty(M)}$. ■

2.4.2 Weil algebras

Definition 2.19 ([9]) *A* **Weil algebra** *is a local $\mathbb{R}$-algebra W (a local ring with an $\mathbb{R}$-algebra structure) which is a finite dimensional $\mathbb{R}$-vector space and is written as $W = \mathbb{R} \oplus m$ (the first component is the $\mathbb{R}$-algebra structure and the second is the maximal ideal in W).*

For a k-jet determined finitely generated C^∞-ring $C^\infty(\mathbb{R}^n)/I$, its localization $C_x^\infty(\mathbb{R}^n)/I_x$ at $x \in Z(I)$ is a finitely generated $\mathbb{R}$-algebra.

Example 2.20 ([9]) *Let $k \in \{0\} \cup \mathbb{N}$, $n \in \mathbb{N}$ and $m \subset C_0^\infty(\mathbb{R}^n)$ be the maximal ideal of $C_0^\infty(\mathbb{R}^n)$.*

The ring

$$J_n^k := C_0^\infty(\mathbb{R}^n)/m^{k+1}$$

is called **the ring of jets of order k (in n-variables)**, *or* **the ring of k-jets**. *This J_n^k is a Weil algebra for any k.*

Lemma 2.21 ([9]) *Let W be a Weil algebra with a maximal ideal $m \subset W$ which satisfies $W = \mathbb{R} \oplus m$. Then $m^k = 0$ for some $k \in \mathbb{N}$.*

The converse of Lemma 2.21 does not hold in general, i.e. there exists a local $\mathbb{R}$-algebra which is an infinite dimensional $\mathbb{R}$-vector space and has a maximal ideal m with $m^k = 0$ for some $k \in \mathbb{N}$.

Example 2.22 *Let W be a formal algebra $\mathbb{R}[x_i | i \in \mathbb{N}] = \lim_{i\to\infty} \mathbb{R}[x_1, \dots, x_i]$, and an ideal $I := \langle x_i^k | i \in \mathbb{N}\rangle_{\mathbb{R}} = \lim_{i\to\infty}\langle x_1^k, x_2^k \dots, x_i^k\rangle_{\mathbb{R}}$ of W for $k \in \mathbb{N}$.*

The quotient $\mathbb{R}$-algebra

$$B := W/I = \lim_{i\to\infty} \mathbb{R}[x_1, \dots, x_i]/\langle x_1^k, x_2^k \dots, x_i^k\rangle_{\mathbb{R}}$$

is a local $\mathbb{R}$-algebra with a unique maximal ideal

$$m = \lim_{i\to\infty}\langle x_1, \dots, x_i\rangle_{\mathbb{R}}/\langle x_1^k, x_2^k, \dots, x_i^k\rangle_{\mathbb{R}}$$

which satisfies $m^k = 0$.

However B is the infinite dimensional $\mathbb{R}$-vector space.

From several results in [9], we have following results for an $\mathbb{R}$-algebra $C_0^\infty(\mathbb{R}^n)/I$.

Lemma 2.23 ([9]) *Let W be a Weil algebra. Then W is isomorphic to a ring $C_0^\infty(\mathbb{R}^n)/I$ for some ideal $I \subset C_0^\infty(\mathbb{R}^n)$ and W is finite dimensional as a real vector space.*

Proof) A Weil algebra W turns out to be isomorphic to a ring $C_0^\infty(\mathbb{R}^n)/I$ for some ideal $I \subset C_0^\infty(\mathbb{R}^n)$ from Theorem 3.17 of [9].

Moreover W is finite dimensional as a real vector space from Corollary 3.18 of [9]. ■

Moreover, homomorphisms of Weil algebras are homomorphisms of C^∞-rings.

Lemma 2.24 ([9]) *Let W be a Weil algebra and $\mathfrak{C}$ a C^∞-ring.*

Suppose that $\phi : W \to \mathfrak{C}$ (resp. $\psi : \mathfrak{C} \to W$) is a homomorphism of $\mathbb{R}$-algebras. Then ϕ (resp. ψ) is a homomorphism of C^∞-rings.

Proof) The homomorphism $\phi : W \to \mathfrak{C}$ of $\mathbb{R}$-algebras turns out to be a homomorphism of C^∞-rings from Corollary 3.19 of [9].

The homomorphism $\psi : \mathfrak{C} \to W$ of $\mathbb{R}$-algebras turns out to be a homomorphism of C^∞-rings from Proposition 3.20 of [9]. ■

From Lemma 2.23 and 2.24, we can regard Weil algebras as C^∞-rings.

2.4.3 Weakly nilpotent local $\mathbb{R}$-algebras

Definition 2.25 *An $\mathbb{R}$-algebra W is called a* **weakly nilpotent local $\mathbb{R}$-algebra** *if W has a unique maximal ideal $m \subset W$ and any $y \in m$ is a nilpotent element of W, i.e. $y^k = 0$ for some $k \in \mathbb{N}$.*

Example 2.26 *Let W be a formal algebra $\mathbb{R}[x_i | i \in \mathbb{N}] = \lim_{i\to\infty} \mathbb{R}[x_1, \ldots, x_i]$. Let I be an ideal defined by $\langle x_i^i | i \in \mathbb{N}\rangle_{\mathbb{R}} = \lim_{i\to\infty}\langle x_1, x_2^2 \ldots, x_i^i\rangle_{\mathbb{R}}$ of W.*

The quotient $\mathbb{R}$-algebra

$$B := W/I = \lim_{i\to\infty} \mathbb{R}[x_1, \ldots, x_i]/\langle x_1, x_2^2 \ldots, x_i^i\rangle_{\mathbb{R}}$$

is a weakly nilpotent local $\mathbb{R}$-algebra with a unique maximal ideal

$$m = \lim_{i\to\infty} \langle x_1, \ldots, x_i\rangle_{\mathbb{R}}/\langle x_1, x_2^2, \ldots, x_i^i\rangle_{\mathbb{R}}.$$

For any $k \in \mathbb{N}$, $m^k \neq 0$.

Proposition 2.27 *A Weil algebra is a weakly nilpotent local $\mathbb{R}$-algebra.*

Proposition 2.28 *Let W is a local $\mathbb{R}$-algebra with a maximal ideal $m \subset W$. Suppose that the maximal ideal m is generated by finite elements $a_1, \ldots, a_n \in m$ as the $\mathbb{R}$-algebra.*

If W is the weakly nilpotent local algebra, W is also the Weil algebra.

Proof) Clearly W is an $\mathbb{R}$-finite dimension vector space generated by $a_1, \ldots, a_n, 1$. There exists $i_k \in \mathbb{N}$ such that ${a_k}^{i_k} = 0$ for each $k = 1, \ldots, n$.

Suppose $p = \sum_{i=1}^n c_i a_i \in m (c_1, \ldots, c_n \in \mathbb{R})$. For any $j = 0, 1, \ldots, i_1 + i_2$, we have $j \geq i_1$ or $i_1 + i_2 - j \geq i_2$. Therefore, $a_1^j = 0$ or $a_2^{i_1+i_2-j} = 0$. For $c_1 a_1 + c_2 a_2 \in W (c_1, c_2 \in \mathbb{R})$, we have

$$(c_1 a_1 + c_2 a_2)^{i_1+i_2} = \sum_{j=0}^{i_1+i_2} \binom{i_1 + i_2}{j} {c_1}^j {c_2}^{i_1+i_2-j} {a_1}^j {a_2}^{i_1+i_2-j} = 0.$$

For $p \in m$, we have $p^{\sum_{k=1} i_k} = 0$. Therefore,

$$m^{\sum_{k=1} i_k} = 0.$$

We have that W is the Weil algebra. ■

For weakly nilpotent local $\mathbb{R}$-algebras, we regard them as C^∞-rings.

Corollary 2.29 *Let W be a weakly nilpotent local $\mathbb{R}$-algebra. W is a C^∞-ring.*

Proof) Suppose that W is finitely generated. Then W is a Weil algebra, and is also a finitely presented C^∞-ring from Lemma 2.23

Suppose that W is arbitrary weakly nilpotent local $\mathbb{R}$-algebra. There exists an index set $\{W_\lambda\}_{\lambda\in\Lambda}$ of finitely generated $\mathbb{R}$-subalgebras W_λ of W with an ordered set Λ of $\lambda < \lambda'$. W_λ is a Weil algebra for each $\lambda \in \Lambda$, i.e. a local C^∞-ring. Then $W = \lim_{\lambda\in\Lambda} W_\lambda$ is also a C^∞-ring since the limit of C^∞-rings exists([9]). ■

2.4.4 k-jet determined C^∞-rings

Definition 2.30 ([6]) *Let $\mathfrak{C}$ be a C^∞-ring.*

For any $\mathbb{R}$-point $p : \mathfrak{C} \to \mathbb{R}$, the localization $\mathfrak{C}_p$ by $\{s \in \mathfrak{C} | p(s) \neq 0\}$ always exists with the unique maximal ideal $m_p \subset \mathfrak{C}_p$ such that $\mathfrak{C}_p/m_p$ is isomorphic to $\mathbb{R}$.

Definition 2.31 *Let $\mathfrak{C}$ be a C^∞-ring and $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. $\mathfrak{C}$ is a* **k-jet determined C^∞-ring** *if the homomorphism $j^k : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/m_p^{k+1}$ defined in Definition 2.10 (3) is injective. Especially, a 0-jet determined C^∞-ring is a point determined C^∞-ring introduced in [9].*

$\mathfrak{C}$ is **near-point determined** *if the homomorphism $j^g : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p$ defined in Definition 2.10 (3) is injective (At [9], it is called* **near-point determined***).*

Example 2.32 *Suppose that M is a C^∞-manifold.*

(1) $C^\infty(M)$ is a point determined C^∞-ring.

For any $\mathbb{R}$-point $e_p : C^\infty(M) \to \mathbb{R}$, $j^0_p : C^\infty(M) \to C^\infty_p(M)/m_p$ is identified with e_p by the isomorphism $C^\infty_p(M)/m_p \cong \mathbb{R}$. Then, the homomorphism j^0 is identified with $e : C^\infty(M) \to \prod_{p\in M} \mathbb{R}$ defined by $e(f) := \{f(p)\}_{p\in M}$, which is clearly injective.

(2) Let $l, k \in \mathbb{N}$ and $0 \leq l < k \leq \infty$. Then $C^\infty_p(M)/{m_p}^{k+1}$ is not an l-jet determined C^∞-ring, but a k-jet determined C^∞-ring.

First, $C^\infty_p(M)/{m_p}^{k+1}$ has a unique $\mathbb{R}$-point $e_p : C^\infty_p(M)/{m_p}^{k+1} \to \mathbb{R}$ with $e_p(f + {m_p}^{k+1}) := f(p)$ for any $f + {m_p}^{k+1} \in C^\infty_p(M)/{m_p}^{k+1}$. For any $l' \leq k$, we have a homomorphism

$$j^{k,l'} : C^\infty_p(M)/{m_p}^{k+1} \to C^\infty_p(M)/{m_p}^{l'+1}$$

by $j^{k,l'}(f+{m_p}^{k+1}) := f+{m_p}^{l'+1}$. *Then the homomorphism*

$$j^{l'} : C_p^\infty(M)/{m_p}^{k+1} \to \big(C_p^\infty(M)/{m_p}^{k+1}\big)/\big(m_p/{m_p}^{k+1}\big)^{l'+1}$$

is identified with $j^{k,l'}$ *by the isomorphism*

$$\big(C_p^\infty(M)/{m_p}^{k+1}\big)/\big(m_p/{m_p}^{k+1}\big)^{l'+1} \cong C_p^\infty(M)/{m_p}^{l'+1}.$$

For $l<k$, $j^{k,l}$ *is not injective and* $C_p^\infty(M)/{m_p}^{k+1}$ *is not* l*-jet determined.*

$j^{k,k}$ *is an identity of* $C_p^\infty(M)/{m_p}^{k+1}$, *so* $C_p^\infty(M)/{m_p}^{k+1}$ *is* k*-jet determined.*

(3) *The* C^∞*-ring* $C_p^\infty(M)$ *of* C^∞*-function-germs on* (M,p) *is not* k*-jet determined for any* $k \in \{0\}\cup\mathbb{N}\cup\{\infty\}$.

Since there exists a nonzero C^∞*-function-germ* $f \in C_p^\infty(M)$ *such that* $f \in m_p^{k+1}$, *the homomorphism* $j^k : C_p^\infty(M) \to C_p^\infty(M)/m_p^{k+1}$ *is not injective.*

From the previous example, we have a following property of k-jet determined C^∞-rings.

Proposition 2.33 *Let* $\mathfrak{C}$ *be a* C^∞*-ring and* $k,l \in \{0\}\cup\mathbb{N}\cup\{\infty\}(k\le l)$.

If $\mathfrak{C}$ *is a* k*-jet determined* C^∞*-ring, then* $\mathfrak{C}$ *is also an* l*-jet determined* C^∞*-ring.*

We write $\mathbf{C^\infty Rings^{k-jet}}$, $\mathbf{C^\infty Rings^{pt}}$ *for the full subcategory of* k*-jet determined* C^∞*-rings and point determined* C^∞*-rings in* $\mathbf{C^\infty Rings}$. *Then, we have a following property.*

$$\begin{aligned}\mathbf{C^\infty Rings} &\supset \mathbf{C^\infty Rings^{\infty-jet}} \supset \cdots \\ &\supset \mathbf{C^\infty Rings^{(k+1)-jet}} \supset \mathbf{C^\infty Rings^{k-jet}} \supset \mathbf{C^\infty Rings^{(k-1)-jet}} \\ &\supset \cdots \\ &\supset \mathbf{C^\infty Rings^{2-jet}} \supset \mathbf{C^\infty Rings^{1-jet}} \supset \mathbf{C^\infty Rings^{0-jet}} = \mathbf{C^\infty Rings^{pt}}\end{aligned}$$

Proof) We have the homomorphism $j^{k,l} : \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/{m_p}^{l+1} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/{m_p}^{k+1}$ such that $j^k = j^{k,l}\circ j^l$.

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "j^l"] \arrow[dr, "j^k"'] & \prod_p \mathfrak{C}_p/{m_p}^{l+1} \arrow[d, "j^{k,l}"] \\ & \prod_p \mathfrak{C}_p/{m_p}^{k+1} \end{tikzcd}$$

If j^k is injective, then j^l is also injective. Therefore, we have Proposition 2.33. ■

For the definition of k-jet determined C^∞-rings, point determined C^∞-rings are 0-jet determined C^∞-rings.

2.5 $\mathbb{R}$-derivations and C^∞-derivations, $\mathbb{R}$-cotangent modules and C^∞-cotangent modules

Regarding operations by smooth functions on Euclidean spaces, which contain the addition, the multiplication and scalar multiplications by real values, we define two kinds of derivations on a C^∞-ring as follows.

Definition 2.34 ([6]) *(1) Suppose $\mathfrak{C}$ is a C^∞-ring, and $\mathfrak{M}$ a $\mathfrak{C}$-module.*

An $\mathbb{R}$**-derivation** *is an $\mathbb{R}$-linear map $d : \mathfrak{C} \to \mathfrak{M}$ such that*

$$d\big(c_1c_2\big) = c_2d(c_1) + c_1d(c_2) \text{ for any } c_1, c_2 \in \mathfrak{C}. \tag{1}$$

A C^∞**-derivation** *is an $\mathbb{R}$-linear map $d : \mathfrak{C} \to \mathfrak{M}$ such that*

$$\begin{aligned} d\big(\Phi_f(c_1, \ldots, c_n)\big) = \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot d(c_i) \\ \text{for all } n \in \mathbb{N}, f \in C^\infty(\mathbb{R}^n), c_1, \ldots, c_n \in \mathfrak{C}. \end{aligned} \tag{2}$$

(2) Let K be $\mathbb{R}$ or C^∞. Let $\mathfrak{C}$ be a C^∞-ring, $\mathfrak{M}$ a $\mathfrak{C}$-module and $d : \mathfrak{C} \to \mathfrak{M}$ a K-derivation. We call a pair $(\mathfrak{M}, d)$ a K**-cotangent module** *for $\mathfrak{C}$ if for any K-derivation $d' : \mathfrak{C} \to \mathfrak{M}'$ there exists a unique morphism $\phi : \mathfrak{M} \to \mathfrak{M}'$ of $\mathfrak{C}$-modules such that $\phi \circ d = d'$. Then we write $(\Omega_{\mathfrak{C},K}, d_{\mathfrak{C},K})$ for the K-cotangent module for $\mathfrak{C}$.*

(3) Let K be $\mathbb{R}$ or C^∞. Let $\phi : \mathfrak{C} \to \mathfrak{D}$ be a homomorphism of C^∞-rings. For the uniqueness of K-cotangent modules, there exists a unique morphism $\Omega_{\phi,K} : \Omega_{\mathfrak{C},K} \to \Omega_{\mathfrak{D},K}$ of $\mathfrak{C}$-modules with

$$d_{\mathfrak{D},K} \circ \phi = \Omega_{\phi,K} \circ d_{\mathfrak{C},K} : \mathfrak{C} \longrightarrow \Omega_{\mathfrak{D},K}. \tag{3}$$

We construct an $\mathbb{R}$-cotangent module and a C^∞-cotangent module for $\mathfrak{C}$ respectively. Let $\mathfrak{F}_{\mathfrak{C}}$ be a free $\mathfrak{C}$-module generated by $d(c)(c \in \mathfrak{C})$. Define elements of $\mathfrak{F}_{\mathfrak{C}}$ as

$$\begin{aligned}
Sum_{\mathfrak{C}}(c,c') :=& d(c+c') - d(c) - d(c'),\\
Prod_{\mathfrak{C}}(c,c') :=& d(c\cdot c') - cd(c') - c'd(c),\\
Func_{\mathfrak{C}}(f,c_1,\dots,c_m) :=& d(\Phi_f(c_1,\dots,c_m)) - \sum_{i=1}^{m}\Phi_{\frac{\partial f}{\partial x_i}}(c_1,\dots,c_m)d(c_i)\\
&\text{for any } m\in\{0\}\cup\mathbb{N}, c,c',c_1,\dots,c_m\in\mathfrak{C}, f\in C^\infty(\mathbb{R}^m)
\end{aligned} \tag{4}$$

and $\mathfrak{C}$-submodules of $\mathfrak{F}_{\mathfrak{C}}$ as

$$\begin{aligned}
\mathfrak{M}_{\mathfrak{C},\mathbb{R}} :=& \left\langle Sum_{\mathfrak{C}}(c,c'), Prod_{\mathfrak{C}}(c,c'), d(r1_{\mathfrak{C}}) \middle| c,c'\in\mathfrak{C}, r\in\mathbb{R}\right\rangle_{\mathfrak{C}} \text{ and}\\
\mathfrak{M}_{\mathfrak{C},C^\infty} :=& \left\langle \begin{matrix} Sum_{\mathfrak{C}}(c,c'),\\ Func_{\mathfrak{C}}(f,c_1,\dots,c_m),\\ d(r1_{\mathfrak{C}})\end{matrix} \middle| \begin{matrix} m\in\{0\}\cup\mathbb{N}, c,c',c_1,\dots,c_m\in\mathfrak{C},\\ f\in C^\infty(\mathbb{R}^m), r\in\mathbb{R}\end{matrix}\right\rangle_{\mathfrak{C}}.
\end{aligned} \tag{5}$$

Let $K=\mathbb{R},C^\infty$. Define a $\mathfrak{C}$-module $\Omega_{\mathfrak{C},K}$ by $\mathfrak{F}_{\mathfrak{C}}/\mathfrak{M}_{\mathfrak{C},K}$ and a K-derivation $d_{\mathfrak{C},K}:\mathfrak{C}\to\Omega_{\mathfrak{C},K}$ of $\mathfrak{C}$ as $d_{\mathfrak{C},K}(c):=d(c)+\mathfrak{M}_{\mathfrak{C},K}$.

For any K-derivation $V:\mathfrak{C}\to\mathfrak{M}$, we can define a homomorphism $\phi:\Omega_{\mathfrak{C},K}\to\mathfrak{M}$ of $\mathfrak{C}$-modules as $\phi(d(c)+\mathfrak{M}_{\mathfrak{C},K}):=V(c)$. ϕ satisfies that $\phi\circ d_{\mathfrak{C},K}=V$. Suppose that there exists another homomorphism $\phi':\Omega_{\mathfrak{C},K}\to\mathfrak{M}$ of $\mathfrak{C}$-modules which satisfies $\phi'\circ d_{\mathfrak{C},K}=V$. Since $\phi(d(c)+\mathfrak{M}_{\mathfrak{C},K})=V(c)=\phi'(d(c)+\mathfrak{M}_{\mathfrak{C},K})$ for any $c\in\mathfrak{C}$, we have $\phi=\phi'$. Therefore, ϕ is a unique homomorphism which satisfies $\phi\circ d_{\mathfrak{C},K}=V$.

Espencially for another K-cotangent module $(\mathfrak{M}',d')$ of $\mathfrak{C}$, there exists a unique homomorphism $\phi:\Omega_{\mathfrak{C},K}\to\mathfrak{M}'$ of $\mathfrak{C}$ which satisfies that $\phi\circ d_{\mathfrak{C},K}=d'$. For the property of the K-cotangent module $(\mathfrak{M}',d')$ of $\mathfrak{C}$, there exists a unique homomorphism $\psi:\mathfrak{M}'\to\Omega_{\mathfrak{C},K}$ of $\mathfrak{C}$ which satisfies that $\psi\circ d'=d_{\mathfrak{C},K}$. Then, $\phi\circ\psi\circ d'=d'$ and $\psi\circ\phi\circ d_{\mathfrak{C},K}=d_{\mathfrak{C},K}$. $\phi\circ\psi=id_{\mathfrak{M}'}$ and $\psi\circ\phi=id_{\Omega_{\mathfrak{C},K}}$ from the properties of $\mathfrak{M}'$ and $\Omega_{\mathfrak{C},K}$. Then $\mathfrak{M}'$ is isomorphic to $\Omega_{\mathfrak{C},K}$ as $\mathfrak{C}$-modules.

Therefore for any C^∞-ring $\mathfrak{C}$, there exists a K-cotangent module $(\Omega_{\mathfrak{C},K},d_{\mathfrak{C},K})$ of $\mathfrak{C}$ which is unique up to isomorphisms of $\mathfrak{C}$-modules.

Proposition 2.35 *Let $\mathfrak{C}$ be a C^∞-ring and $\mathfrak{F}_{\mathfrak{C}}$ a free $\mathfrak{C}$-module generated by $d(c)(c\in\mathfrak{C})$. For $\mathfrak{C}$-submodules $\mathfrak{M}_{\mathfrak{C},\mathbb{R}}$ and $\mathfrak{M}_{\mathfrak{C},C^\infty}$, we have following properties.*

(1) $\mathfrak{M}_{\mathfrak{C},\mathbb{R}}\subset\mathfrak{M}_{\mathfrak{C},C^\infty}$.

(2) If $\mathfrak{M}_{\mathfrak{C},\mathbb{R}}=\mathfrak{M}_{\mathfrak{C},C^\infty}$, an $\mathbb{R}$-cotangent module is a C^∞-cotangent module, i.e. $\Omega_{\mathfrak{C},\mathbb{R}}:=\mathfrak{F}_{\mathfrak{C}}/\mathfrak{M}_{\mathfrak{C},\mathbb{R}}$ is isomorphic to $\Omega_{\mathfrak{C},C^\infty}:=\mathfrak{F}_{\mathfrak{C}}/\mathfrak{M}_{\mathfrak{C},C^\infty}$ and $d_{\mathfrak{C},\mathbb{R}}$ is commutative to $d_{\mathfrak{C},C^\infty}$.

(3) $\mathfrak{M}_{\mathfrak{C},\mathbb{R}} = \mathfrak{M}_{\mathfrak{C},C^\infty}$ *if and only if any* $\mathbb{R}$*-derivation* $d : \mathfrak{C} \to \mathfrak{M}$ *become a* C^∞*-derivation.*

Proof)

(1) Take the C^∞-map $f(x_1, x_2) = x_1 \cdot x_2$.Then $Prod_{\mathfrak{C}}(c_1, c_2) = d(c_1c_2) - c_2d(c_1) - c_1d(c_2) = d\big(\Phi_f(c_1, c_2)\big) - \sum_{i=1}^{2} \Phi_{\frac{\partial f}{\partial x_i}}(c_1, c_2)d(c_i) = Func_{\mathfrak{C}}(f, c_1, c_2)$. Therefore, $\mathfrak{M}_{\mathfrak{C},\mathbb{R}} \subset \mathfrak{M}_{\mathfrak{C},C^\infty}$.

(2) Trivially.

(3) Take any $\mathfrak{C}$-module $\mathfrak{M}$ and $\mathbb{R}$-derivation $V : \mathfrak{C} \to \mathfrak{M}$. There exists a unique homomorphism $\phi_{\mathfrak{M}} : \Omega_{\mathfrak{C},\mathbb{R}} \to \mathfrak{M}$ such that $V = \phi_{\mathfrak{M}} \circ d_{\mathbb{R}}$.
For the assumption of $\mathfrak{M}_{\mathfrak{C},C^\infty} = \mathfrak{M}_{\mathfrak{C},\mathbb{R}}$, $d_{\mathbb{R}}$ is a C^∞-derivation.
Therefore, $V = \phi_{\mathfrak{M}} \circ d_{\mathfrak{C},\mathbb{R}}$ is a C^∞-derivation.

■

By Definition 2.34, we have that any C^∞-derivation is an $\mathbb{R}$-derivation.

Example 2.36 *Let M be a C^∞-manifold and $\Gamma(T^*M)$ the set of C^∞-sections to the cotangent bundle T^*M on M.*

(1) For any $f \in C^\infty(M)$, define a C^∞-section $df : M \to T^*M$ by $df(v) := v(f)$ for any $x \in M$ and $v \in T_xM$.

Define an $\mathbb{R}$-linear mapping $d : C^\infty(M) \to \Gamma(T^*M)$ by $d(f) := df$ for any $f \in C^\infty(M)$. This $\mathbb{R}$-mapping d is the C^∞-derivation.

(2) Let $V : M \to TM$ be a C^∞-vector field of M as V_x is a tangent vector at a point x on M.

Define $V(f) \in C^\infty(M)$ by $\big(V(f)\big)(x) := V_x(f)$ regarding the tangent vector V_x as a differential $C^\infty(M) \to \mathbb{R}$ at x. We regard $V : C^\infty(M) \to C^\infty(M)$ as an $\mathbb{R}$-derivation. Then V turns out to be a C^∞-derivation.

Remark 2.37 *From the definition of C^∞-rings and derivations in Definition 2.1 and Definition 2.34, we can define analytic rings and Nash rings, and analytic derivations and Nash derivations by using analytic functions and Nash functions.*

(1) An analytic ring (resp. a Nash ring) is a set $\mathfrak{A}$ which is endowed with the following data;
for any $l \in \{0\} \cup \mathbb{N}$ and any analytic-function (resp. Nash function)

$f : \mathbb{R}^l \to \mathbb{R}$ (if $l = 0$, f is a constant value $c \in \mathbb{R}$), there are given an operation $\Psi_f : \mathfrak{A}^l \to \mathfrak{A}$ (if $l = 0$, an operation Ψ_f is a constant on $\mathfrak{A}$) such that

- *for any $k \in \{0\} \cup \mathbb{N}$, analytic-maps (resp. Nash-maps) $g : \mathbb{R}^k \to \mathbb{R}$ and $f_i : \mathbb{R}^l \to \mathbb{R} (i = 1, \ldots, k)$, and $c_1, \cdots, c_l \in \mathfrak{A}$,*

  $\Psi_g(\Psi_{f_1}(c_1, \ldots, c_l), \ldots, \Psi_{f_k}(c_1, \ldots, c_l)) = \Psi_{g \circ (f_1, \ldots, f_k)}(c_1, \ldots, c_l)$ *and*

- *for all projections $\pi_i : \mathbb{R}^l \to \mathbb{R}$ defined as $\pi_i(x_1, \ldots, x_l) = x_i (i = 1, \cdots, l)$ and $c_1, \ldots, c_l \in \mathfrak{A}$,*

$$\Psi_{\pi_i}(c_1, \ldots, c_l) = c_i.$$

(2) An analytic-derivation (resp. a Nash-derivation) is an $\mathbb{R}$-linear map $d : \mathfrak{A} \to \mathfrak{M}$ such that

$$d\big(\Psi_f(c_1, \ldots, c_n)\big) = \sum_{i=1}^{n} \Psi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot d(c_i)$$

for any $c_1, \ldots, c_n \in \mathfrak{A}$ and an analytic function (resp. a Nash function) $f : \mathbb{R}^n \to \mathbb{R}$.

2.6 Derivations to $\mathbb{R}$ of a finitely generated C^∞-ring

For an open neighborhood of $p \in M$, a tangent space at p of U is equal to those of M. We can define tangent spaces of manifolds locally.

We will prove that tangent bundles of C^∞-rings

2.6.1 Local Property

Suppose U is an open subset of $\mathbb{R}^n$ and I an ideal of $C^\infty(\mathbb{R}^n)$. Define $\chi(U)$ as a set of $\mathbb{R}$-derivations $C^\infty(U) \to C^\infty(U)$.

Define an ideal I_U of $C^\infty(U)$ and a zero point set $Z(I_U)$ of I_U as

$$I_U := \big\langle \{ f \in C^\infty(U) | \text{There exists } g \in I \text{ such that } f \equiv g|_U . \} \big\rangle. \tag{6}$$

Its zero point set $Z(I_U)$ is equal to $Z(I) \cap U$.

For $p \in I_U$, define its Zariski tangent space and logarithmic tangent space as

$$\begin{aligned} T_p Z(I_U) &:= \{ v \in T_p\mathbb{R}^n | v : C^\infty(U)/I_U \to \mathbb{R} \text{ is an } \mathbb{R}\text{-derivation at } p \}, \\ T_p^l Z(I_U) &:= \left\{ v \in T_p\mathbb{R}^n \middle| \begin{array}{c} \text{There exists } X \in \chi(U) \\ \text{such that } X(I_U) \subset I_U \text{ and } X(p) = v. \end{array} \right\}. \end{aligned} \tag{7}$$

Remark 2.38 *$\{f \in C^\infty(U) | \text{There exists } g \in I \text{ such that } f \equiv g|_U.\}$ need not to be an ideal.*

Suppose $U = \mathbb{R}\backslash\{0\}$ and $I := \langle x \rangle$. Define $f \in C^\infty(U)$ such that

$$f(x) := \begin{cases} -1 & (x < 0) \\ 1 & (x > 0) \end{cases}.$$

xf is not a restriction of a smooth function $C^\infty(\mathbb{R})$.

So restrictions of ideals of $C^\infty(\mathbb{R}^n)$ needs not to be ideals.

Like $T_pZ(I)$, $T_pZ(I_U)$ is a finitely generated $\mathbb{R}$-module.

Theorem 2.39 *For any $v \in T_pZ(I_U)$, $v = \sum_{i=1}^n v(x_i + I_U)(\frac{\partial}{\partial x_i})_p$.*

2.6.2 The case of Zariski type

Lemma 2.40 *Let $I \subset C^\infty(\mathbb{R}^n)$ be an ideal and p a point of $Z(I)$. Take an open neighborhood $U \subset \mathbb{R}^n$ at p.*

(1) For any $v \in T_pZ(I)$, there exists a unique $\phi(v) \in T_pZ(I_U)$ such that

$$v(f + I) = \phi(v)(f|_U + I_U) \text{ for any } f \in C^\infty(\mathbb{R}^n). \tag{8}$$

(2) For any $w \in T_pZ(I_U)$, there exists a unique $\psi(w) \in T_pZ(I)$ such that

$$w(f|_U + I_U) = \psi(w)(f + I) \text{ for any } f \in C^\infty(\mathbb{R}^n). \tag{9}$$

Proof)

(1) If $v_i := v(x_i + I) \in \mathbb{R}$ for $i = 1, \ldots, n$, then $v = \sum_{i=1}^n v_i(\frac{\partial}{\partial x_i})_p$. As $v \in T_pZ(I)$, $\sum_{i=1}^n v_i \frac{\partial g|_U}{\partial x_i}(p) = 0$ for any $g \in I$. Then define $\phi(v) : C^\infty(U)/I_U \to \mathbb{R}$ as $\phi(v)(f + I_U) := \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}(p)$ with well-definedness. Suppose that $\phi(v), \phi'(v) \in T_pZ(I_U)$ satisfies (8). For each projection $x_i \in C^\infty(\mathbb{R}^n)$, $\phi(v)(x_i|_U + I_U) = v(x_i + I) = \phi(v)(x_i|_U + I_U)$. Therefore,

$$\begin{aligned} \phi(v) &= \sum_{i=1}^n \phi(v)(x_i|_U + I_U)\Big(\frac{\partial}{\partial x_i}\Big)_p \\ &= \sum_{i=1}^n \phi'(v)(x_i|_U + I_U)\Big(\frac{\partial}{\partial x_i}\Big)_p = \phi'(v). \end{aligned}$$

(2) From Theorem 2.39, we can write $w = \sum_{i=1}^n w(x_i|_U + I_U)(\frac{\partial}{\partial x_i})_p$ for $w \in T_pZ(I_U)$. Then define $w' := \sum_{i=1}^n w(x_i|_U + I_U)(\frac{\partial}{\partial x_i})_p$. For all $g \in I$, we have $w(g|_U + I_U) = 0$.

Then define $\psi(w) : C^\infty(\mathbb{R}^n)/I \to \mathbb{R}$ as $\psi(w)(f + I) := w(f|_U + I_U)$ with well-definedness. For the definition of w and $\psi(w)$,

$$\begin{aligned}\psi(w)(f+I) &= \sum_{i=1}^n w(x_i|_U + I_U)\frac{\partial f}{\partial x_i}(p) \\ &= \sum_{i=1}^n w(x_i|_U + I_U)\frac{\partial (f|_U)}{\partial x_i}(p) = w(f|_U + I_U)\end{aligned}$$

for any $f \in C^\infty(\mathbb{R}^n)$. Therefore $\psi(w) \in T_pZ(I)$ satisfies (9). We have $\psi(w)$ for w.

Suppose that $\psi(w), \psi'(w) \in T_pZ(I)$ satisfies (9). For all $f \in C^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\psi(w)(f+I) &= \sum_{i=1}^n \big(\psi(w)\big)(x_i + I)\frac{\partial f}{\partial x_i}(p) = \sum_{i=1}^n w(x_i|_U + I_U)\frac{\partial f}{\partial x_i}(p) \\ &= \sum_{i=1}^n \big(\psi'(w)\big)(x_i + I)\frac{\partial f}{\partial x_i}(p) = \psi'(w)(f+I).\end{aligned}$$

therefore, we obtain $\psi(w) = \psi'(w)$. ■

We have a morphism $\phi : T_pZ(I) \to T_pZ(I_U)$. $\psi : T_pZ(I_U) \to T_pZ(I)$ is an inverse of ϕ. Therefore we have a following theorem.

Theorem 2.41 *We have an isomorphism $\phi : T_pZ(I) \to T_pZ(I_U)$.*

2.6.3 The case of logarithmic type

Lemma 2.42 *Let $I \subset C^\infty(\mathbb{R}^n)$ be an ideal and p a point of $Z(I)$. Take an open neighborhood $U \subset \mathbb{R}^n$ of p.*

(1) For any vector field $X \in \chi(\mathbb{R}^n)$ with $X(I) \subset I$, there exists a vector field $Y \in \chi(U)$ such that

$$\begin{aligned}&Y(f|_U) \equiv X(f)|_U \text{ for any } f \in C^\infty(\mathbb{R}^n), \\ &Y(I_U) \subset I_U.\end{aligned} \tag{10}$$

(2) For any vector field $Y \in \chi(U)$ with $Y(I_U) \subset I_U$, there exists a vector field $X \in \chi(\mathbb{R}^n)$ and an open neighborhood $V \subset U$ of p such that

$$\begin{aligned} &X(f)|_V \equiv Y(f|_U)|_V \text{ for any } f \in C^\infty(\mathbb{R}^n), \\ &X(I) \subset I. \end{aligned} \tag{11}$$

Proof)

(1) Define $Y \in \chi(U)$ as $Y := X|_U$.

- Take $f \in C^\infty(\mathbb{R}^n)$. For any $x \in U$, $Y(f|_U)(x) = X(f)(x)$. Then we have $Y(h|_U) \equiv X(f)|_U$.
- Take $g \in I$. $Y(g|_U)(x) = X(g)(x)$ for any $x \in U$. We have $Y(g|_U) \in I_U$ for any $g \in I$. Then we have $Y(I_U) \subset I_U$.

Therefore $Y = X|_U$ satisfies (10).

(2) For open sets $V \subset U \subset \mathbb{R}^n$, there exists an open neighborhood W of p and a smooth function $\eta \in C^\infty(\mathbb{R}^n)$ such that $\overline{V} \subset W \subset \overline{W} \subset U$, $0 \leq \eta(x) \leq 1$ for any $x \in \mathbb{R}^n$ and

$$\eta(x) = \begin{cases} 1 & (x \in \overline{V}) \\ 0 & (x \in \mathbb{R}^n \backslash U) \end{cases}. \tag{12}$$

Define X as

$$X(x) := \begin{cases} \eta(x)Y(x) & (x \in U) \\ 0 & (x \in \mathbb{R}^n \backslash U) \end{cases}. \tag{13}$$

From the definition of η, X is a smooth vector section on $\mathbb{R}^n$.

- Take $f \in C^\infty(\mathbb{R}^n)$. For any $x \in V$,

$$Y(f|_U)(x) = \eta(x)Y(f|_U)(x) = X(f)(x). \tag{14}$$

  Then $Y(f|_U)|_V \equiv X(f)|_V$.
- For any $h \in I$, we have

$$X(h)(x) = \begin{cases} \eta(x)Y(h|_U)(x) & (x \in U) \\ 0 & (x \in \mathbb{R}^n \backslash U) \end{cases}. \tag{15}$$

  About $Y(h|_U) \in C^\infty(U)$, there exists $g_1, \ldots, g_k \in I$ such that $Y(h|_U) \equiv \sum_{i=1}^k c_i g_i|_U$.

For $c_i \in C^\infty(U)$, define

$$\hat{c}_i(x) := \begin{cases} \eta(x)c_i(x) & (x \in U) \\ 0 & (x \in \mathbb{R}^n \backslash U) \end{cases}. \tag{16}$$

Then we have $X(h) \equiv \sum_{i=1}^k \hat{c}_i g_i$.

■

Theorem 2.43 *We have an isomorphism $\phi^l : T_p^l Z(I) \to T_p^l Z(I_U)$.*

Proof) For $v = X_p \in T_p^l Z(I)(X : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n),\ X(I) \subset I)$, define $\phi(v) \in T_p^l Z(I_U)$ as $\phi(v) = Y_p$ by $Y : C^\infty(U) \to C^\infty(U)$ which satisfies Lemma 2.42 (1). The definition of $\phi : T_p^l Z(I) \to T_p^l Z(I_U)$ is well-defined and ϕ is a homomorphism of $\mathbb{R}$-vector spaces.

For $w = Y_p \in T_p^l Z(I_U)(Y : C^\infty(U) \to C^\infty(U),\ Y(I_U) \subset I_U)$, define $\psi(w) \in T_p^l Z(I)$ as $\psi(w) = X_p$ by $X : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ which satisfies Lemma 2.42 (2) for an open subset $V \subset U$. The definition of $\psi : T_p^l Z(I_U) \to T_p^l Z(I)$ is well-defined and ψ is a homomorphism of $\mathbb{R}$-vector spaces.

For $v = X_p \in T_p^l Z(I)$, $\psi(\phi(v)) = X'_p$ by $X' : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ which satisfies $X'(g)|_V = Y(g|_U)|_V = X(g)|_V$ for any $g \in C^\infty(\mathbb{R}^n)$ and $X'(I) \subset I$. Then $X_p = X'_p$ and $\psi \circ \phi = id_{T_p^l Z(I)}$.

For $w = Y_p \in T_p^l Z(I_U)$, $\phi(\psi(w)) = Y'_p$ by $Y' : C^\infty(U) \to C^\infty(U)$ which satisfies $Y'(f|_U)|_V = X(f)|_V = Y(f|_U)|_V$ for any $f \in C^\infty(\mathbb{R}^n)$ and $Y'(I_U) \subset I_U$. Then $Y_p = Y'_p$ and $\phi \circ \psi = id_{T_p^l Z(I_U)}$.

Therefore, we have an isomorphism $\psi^l : T_p^l Z(I) \to T_p^l Z(I_U)$. ■

2.7 Localizations of derivations

First, we want to check the well-definedness of localizations of derivations.

Lemma 2.44 *Let $\mathfrak{C}$ be a C^∞-ring and $S \subset \mathfrak{C}$ a subset of $\mathfrak{C}$ with a lozalization $\pi : \mathfrak{C} \to \mathfrak{C}[S^{-1}]$. Let K be $\mathbb{R}$ or C^∞.*

(1) $(\Omega_{\pi,K})_* : \Omega_{\mathfrak{C},K} \otimes_{\mathfrak{C}} \mathfrak{C}[S^{-1}] \to \Omega_{\mathfrak{C}[S^{-1}],K}$ is an isomorphism of $\mathfrak{C}[S^{-1}]$-modules.

(2) For any K-derivation $V : \mathfrak{C} \to \mathfrak{M}$, there exists a unique K-derivation $V' : \mathfrak{C}[S^{-1}] \to \mathfrak{M} \otimes_{\mathfrak{C}} \mathfrak{C}[S^{-1}]$ such that $V'(\pi(c)) = V(c) \otimes 1_{\mathfrak{C}[S^{-1}]}$.

(3) $(\Omega_{\pi,K})_*^{-1} \circ d_{\mathfrak{C}[S^{-1}],K} : \mathfrak{C}[S^{-1}] \to \Omega_{\mathfrak{C}[S^{-1}],K} \otimes_{\mathfrak{C}} \mathfrak{C}[S^{-1}]$ *is a* K*-cotangent module.*

Proof)

(1) For the case of $K = \mathbb{R}$, $(\Omega_{\pi,\mathbb{R}})_*$ is the isomorphism for Proposition 8.2A. in [4].

For the case of $K = C^\infty$, we prove that $(\Omega_{\pi,C^\infty})_*$ is the isomorphism. For a singleton $S = \{s\}$, $(\Omega_{\pi,C^\infty})_*$ is the isomorphism from Proposition 5.14. in [6]. For a finite set $S = \{s_1, \ldots, s_n\}$, we have $\mathfrak{C}[S^{-1}] \cong \mathfrak{C}[\{s_1 \cdot \ldots \cdot s_n\}^{-1}]$.

For a general subset $S \subset \mathfrak{C}$, we have a directed set Λ of λ which satisfies $\mathfrak{C}[S_\lambda^{-1}]$ is a localization of a finite set of $S_\lambda \subset S$ with a homomorphism $\pi_\lambda : \mathfrak{C} \to \mathfrak{C}[S_\lambda^{-1}]$. $(\Omega_{\pi,K})_* : \Omega_{\mathfrak{C},K} \otimes_{\mathfrak{C}} \mathfrak{C}[S_\lambda^{-1}] \to \Omega_{\mathfrak{C}[S_\lambda^{-1}],K}$ is an isomorphism for any λ.

Taking directed limits, we have $\Omega_{\mathfrak{C},K} \otimes_{\mathfrak{C}} \mathfrak{C}[S^{-1}] = \lim_\lambda \Omega_{\mathfrak{C},K} \otimes_{\mathfrak{C}} \mathfrak{C}[S_\lambda^{-1}] \cong \lim_\lambda \Omega_{\mathfrak{C}[S_\lambda^{-1}],K} = \Omega_{\mathfrak{C}[S^{-1}],K}$.

(2) Suppose that $V = d_{\mathfrak{C},K}$. Define $d'_{\mathfrak{C},K} := (\Omega_{\pi,K})_*^{-1} \circ d_{\mathfrak{C}[S^{-1}],K}$. Then $d'_{\mathfrak{C},K}(\pi(c)) = d_{\mathfrak{C},K}(c) \otimes 1_{\mathfrak{C}[S^{-1}]}$.

Suppose that $V : \mathfrak{C} \to \mathfrak{M}$ is a K-derivation. There exists a $\mathfrak{C}$-homomorphism $\phi : \Omega_{\mathfrak{C},K} \to \mathfrak{M}$ such that $V = \phi \circ d_{\mathfrak{C},K}$. Define $V' := (\phi \otimes_{\mathfrak{C}} 1_{\mathfrak{C}[S^{-1}]}) \circ d'_{\mathfrak{C},K}$. V' is a $\mathfrak{C}[S^{-1}]$-homomorphism and $V'(\pi(c)) = (\phi \otimes_{\mathfrak{C}} 1_{\mathfrak{C}[S^{-1}]})(d_{\mathfrak{C},K}(c) \otimes_{\mathfrak{C}} 1_{\mathfrak{C}[S^{-1}]}) = V(c) \otimes_{\mathfrak{C}} 1_{\mathfrak{C}[S^{-1}]}$.

(3) From (1) and (2), we have proved that $(\Omega_{\pi,K})_*^{-1} \circ d_{\mathfrak{C}[S^{-1}],K} : \mathfrak{C}[S^{-1}] \to \Omega_{\mathfrak{C}[S^{-1}],K} \otimes_{\mathfrak{C}} \mathfrak{C}[S^{-1}]$ is a K-cotangent module of $\mathfrak{C}[S^{-1}]$.

■

For a K-derivation $V : \mathfrak{C} \to \mathfrak{M}$, define a localization $V[S^{-1}] : \mathfrak{C}[S^{-1}] \to \mathfrak{M}[S^{-1}]$ of S, which satisfies $V[S^{-1}] \circ \pi_S = V \otimes 1_{\mathfrak{C}[S^{-1}]}$.

2.8 k-differential forms of a C^∞-ring

We define exterior derivations of C^∞-rings in this section.

As same as §2.5, define free $\mathfrak{C}$-modules and its elements.

Definition 2.45 *Let $\mathfrak{C}$ be a C^∞-ring with its cotangent bundles $(\Omega_{\mathfrak{C},\mathbb{R}}, d_{\mathfrak{C},\mathbb{R}})$ and $(\Omega_{\mathfrak{C},C^\infty}, d_{\mathfrak{C},C^\infty})$.*

For $k \in \mathbb{N}$, *define* $\mathfrak{F}^k_{\mathfrak{C}} := \langle (c_1, \ldots, c_k) | c_i \in \mathfrak{C}(i = 1, \ldots, k) \rangle_{\mathfrak{C}}$ *and write its element* $(c_1 \ldots, c_k)$ *and* $(c_0, \ldots, c_{j-1}, c_{j+1}, \ldots, c_k)$ *as* $(c_i)^k_{i=1}$ *and* $(c_i)^k_{i=0, i\neq j}$ *for* $j = 0, 1, \ldots, k$. *Define its elements as*

$$\begin{aligned}
Alt(\sigma; (c_i)^k_{i=1}) &:= (c_i)^k_{i=1} - sgn\sigma(c_{\sigma(i)})^k_{i=1} && (17)\\
Sum_{\mathfrak{C}}(c_0; (c_i)^k_{i=1}) &:= (c_0 + c_1, c_2, \ldots, c_k) - (c_i)^k_{i=0,i\neq 1} - (c_i)^k_{i=1}, && (18)\\
Sca_{\mathfrak{C}}(\lambda; (c_i)^k_{i=1}) &:= (\lambda c_1, c_2, \ldots, c_k) - \lambda 1_{\mathfrak{C}}(c_i)^k_{i=1} && (19)\\
Prod_{\mathfrak{C}}(c_0; (c_i)^k_{i=1}) &:= (c_0 c_1, c_2, \ldots, c_k) - c_0(c_i)^k_{i=1} - c_1(c_i)^k_{i=0,i\neq 1}, && (20)\\
Func_{\mathfrak{C}}(f, (c'_i)^m_{i=1}, (c_j)^{k-1}_{j=1}) &:= (\Phi_f((c'_i)^m_{i=1}), c_1, \ldots, c_{k-1}) && (21)\\
&\quad - \sum_{i=1}^m \Phi_{\frac{\partial f}{\partial x_i}}((c'_i)^m_{i=1}) \cdot (c'_i, c_1, \ldots, c_{k-1}).
\end{aligned}$$

$(c_1, \ldots, c_k, c'_1, \ldots, c'_m \in \mathfrak{C}, \lambda \in \mathbb{R}, \sigma$ *is a permutation of the set* $\{1, 2, \ldots, k\}$.)

Define $\mathfrak{M}^k_{\mathfrak{C},\mathbb{R}}$ *as a* $\mathfrak{C}$*-submodule of* $\mathfrak{F}^k_{\mathfrak{C}}$ *generated by (17), (18), (19), (20). Define* $\mathfrak{M}^k_{\mathfrak{C},C^\infty}$ *as a* $\mathfrak{C}$*-submodule of* $\mathfrak{F}^k_{\mathfrak{C}}$ *generated by (17), (18), (19), (21).*

From this property, we define exterior derivations by following mappings.

Proposition 2.46 *Define a mapping* $d^k : \mathfrak{F}^k_{\mathfrak{C}} \to \mathfrak{F}^{k+1}_{\mathfrak{C}}$ *as* $d^k\big(c \cdot (c_1, \ldots, c_k)\big) := (c, c_1, \ldots, c_k)$. *Then* $d^k(\mathfrak{M}^k_{\mathfrak{C},K}) \subset \mathfrak{M}^{k+1}_{\mathfrak{C},K}$ *for* $K = \mathbb{R}, C^\infty$.

Proof) For $c_0 \cdot Alt(\sigma; (c_i)^k_{i=1}) \in \mathfrak{M}^k_{\mathfrak{C},K}$,

$$\begin{aligned}
&d^k\big(c_0 \cdot Alt(\sigma; (c_i)^k_{i=1})\big)\\
&\qquad = (c_{i-1})^{k+1}_{i=1} - sgn\sigma(c_{\tau(i)-1})^{k+1}_{i=1}\\
&\qquad = (c_{i-1})^{k+1}_{i=1} - sgn\tau(c_{\tau(i)-1})^{k+1}_{i=1}\\
&\qquad = Alt(\tau; (c_i)^k_{i=1}).\\
&\qquad (\tau(1) := 1, \tau(i) := \sigma(i-1) + 1 (\forall i = 2, \ldots, k+1).)
\end{aligned}$$

For $c \cdot Sum_{\mathfrak{C}}(c_0; (c_i)^k_{i=1}) \in \mathfrak{M}^k_{\mathfrak{C},K}$,

$$\begin{aligned}
&d^k\big(c \cdot Sum_{\mathfrak{C}}(c_0; (c_i)^k_{i=1})\big) + \mathfrak{M}^{k+1}_{\mathfrak{C},K}\\
&\quad = (c, c_0 + c_1, c_2, \ldots, c_k) - (c, c_0, c_2, \ldots, c_k) - (c, c_1, c_2, \ldots, c_k) + \mathfrak{M}^{k+1}_{\mathfrak{C},K}\\
&\quad = -(c_0 + c_1, c, c_2, \ldots, c_k) + (c_0, c, c_2, \ldots, c_k) + (c_1, c, c_2, \ldots, c_k) + \mathfrak{M}^{k+1}_{\mathfrak{C},K}\\
&\quad = -Sum_{\mathfrak{C}}(c_0; (c_1, c, c_2, \ldots, c_k)) + \mathfrak{M}^{k+1}_{\mathfrak{C},K}.\\
&(\sigma(0) := 1, \sigma(1) = 0, \sigma(i) = i (\forall i = 2, \ldots, k).)
\end{aligned}$$

For $c \cdot Sca_{\mathfrak{C}}(\lambda; (c_i)_{i=1}^k) \in \mathfrak{M}_{\mathfrak{C},K}^k$,

$$
\begin{aligned}
d^k\big(c{\cdot}Sca_{\mathfrak{C}}(\lambda; (c_i)_{i=1}^k)\big) &+ \mathfrak{M}_{\mathfrak{C},K}^{k+1} \\
&= (c, \lambda c_1, c_2, \ldots, c_k) - (\lambda c, c_1, c_2, \ldots, c_k) + \mathfrak{M}_{\mathfrak{C},K}^{k+1} \\
&= (\lambda c_1, c, c_2, \ldots, c_k) - \lambda 1_{\mathfrak{C}}(c, c_1, c_2, \ldots, c_k) + \mathfrak{M}_{\mathfrak{C},K}^{k+1} \\
&= Sca_{\mathfrak{C}}(\lambda; (c_i)_{i=1}^k) + \mathfrak{M}_{\mathfrak{C},K}^{k+1}.
\end{aligned}
$$

$(\sigma(0) := 1, \sigma(1) = 0, \sigma(i) = i (\forall i = 2, \ldots, k).)$

For $c \cdot Func_{\mathfrak{C}}(f, (c'_i)_{i=1}^m, (c_j)_{j=1}^{k-1}) \in \mathfrak{M}_{\mathfrak{C},C^\infty}^k$,

$$
\begin{aligned}
&d^k\big(c \cdot Func_{\mathfrak{C}}(f, (c'_i)_{i=1}^m, (c_j)_{j=1}^{k-1})\big) + \mathfrak{M}_{\mathfrak{C},C^\infty}^{k+1} = (c, \Phi_f((c'_i)_{i=1}^m), c_1, \ldots, c_{k-1}) \\
&\quad - \sum_{i=1}^m (c \cdot \Phi_{\frac{\partial f}{\partial x_i}}\big((c'_i)_{i=1}^m\big), c'_i, c_1, \ldots, c_{k-1}) + \mathfrak{M}_{\mathfrak{C},C^\infty}^{k+1} \\
&= -(\Phi_f((c'_i)_{i=1,\ldots,m}), c, c_1, \ldots, c_{k-1}) - \sum_{i=1}^m \Big\{ c\big(\Phi_{\frac{\partial f}{\partial x_i}}((c'_i)_{i=1}^m), c'_i, c_1, \ldots, c_{k-1}\big) \\
&\quad + \Phi_{\frac{\partial f}{\partial x_i}}((c'_i)_{i=1}^m)(c, c'_i, c_1, \ldots, c_{k-1}) \Big\} + \mathfrak{M}_{\mathfrak{C},C^\infty}^{k+1} \\
&= -(\Phi_f((c'_i)_{i=1}^m), c, c_1, \ldots, c_{k-1}) - \sum_{i=1}^m \Big\{ c\big(\Phi_{\frac{\partial f}{\partial x_i}}((c'_i)_{i=1}^m), c'_i, c_1, \ldots, c_{k-1}\big) \\
&\quad - \Phi_{\frac{\partial f}{\partial x_i}}((c'_i)_{i=1}^m)(c'_i, c, c_1, \ldots, c_{k-1}) \Big\} + \mathfrak{M}_{\mathfrak{C},C^\infty}^{k+1} \\
&= -Func_{\mathfrak{C}}\big(f, (c'_i)_{i=1}^m, (c, c_1, \ldots, c_{k-1})\big) \\
&\quad - c \cdot \sum_{i=1}^m \big(\Phi_{\frac{\partial f}{\partial x_i}}((c'_i)_{i=1}^m), c'_i, c_1, \ldots, c_{k-1}\big) + \mathfrak{M}_{\mathfrak{C},C^\infty}^{k+1}.
\end{aligned}
\tag{22}
$$

For $\sum_{i=1}^{m}\big(\Phi_{\frac{\partial f}{\partial x_i}}((c'_i)_{i=1}^m), c'_i, c_1, \ldots, c_{k-1}\big) + \mathfrak{M}^{k+1}_{\mathfrak{C},C^\infty}$,

$$
\begin{aligned}
\sum_{i=1}^{m}&\Big(\Phi_{\frac{\partial f}{\partial x_i}}\big((c'_l)_{l=1}^m\big), c'_i, c_1, \ldots, c_{k-1}\Big) + \mathfrak{M}^{k+1}_{\mathfrak{C},C^\infty}\\
&=\sum_{i=1}^{m}\sum_{j=1}^{m}\Phi_{\frac{\partial^2 f}{\partial x_i \partial x_j}}\big((c'_l)_{l=1}^m\big)(c'_j, c'_i, c_1, \ldots, c_{k-1}) + \mathfrak{M}^{k+1}_{\mathfrak{C},C^\infty}\\
&=\sum_{1\le i<j\le m}\{\Phi_{\frac{\partial^2 f}{\partial x_i \partial x_j}}\big((c'_l)_{l=1}^m\big)\\
&\quad-\Phi_{\frac{\partial^2 f}{\partial x_i \partial x_j}}\big((c'_l)_{l=1}^m\big)\}\Big(c'_j, c'_i, c_1, \ldots, c_{k-1}\Big) + \mathfrak{M}^{k+1}_{\mathfrak{C},C^\infty}\\
&=0 + \mathfrak{M}^{k+1}_{\mathfrak{C},C^\infty}.
\end{aligned}
\tag{23}
$$

Then we have $d^k\big(c \cdot Func_{\mathfrak{C}}(f, (c'_i)_{i=1}^m, (c_j)_{j=1}^{k-1})\big) = 0$. We can prove $d^k\big(c \cdot Prod_{\mathfrak{C}}(c', c_1, c_2, \ldots, c_k)\big) \in \mathfrak{M}^{k+1}_{\mathfrak{C},\mathbb{R}}$ by $f(x_1, x_2) := x_1 \cdot x_2$ in equation (22) and (23).

Therefore, we have $d^k(\mathfrak{M}^k_{\mathfrak{C},K}) \subset \mathfrak{M}^{k+1}_{\mathfrak{C},K}$ for $k = 0, 1, 2, \ldots$ and $K = \mathbb{R}, C^\infty$. ■

From Proposition 2.46, we can define differential forms of C^∞-rings as follows.

Definition 2.47 *For $k = 0, 1, 2, \ldots$ and $K = \mathbb{R}, C^\infty$, define a $\mathfrak{C}$-module $\wedge^k\Omega_{\mathfrak{C},K}$ as*

$$
\wedge^k\Omega_{\mathfrak{C},K} := \begin{cases} \mathfrak{C} & (k = 0)\\ \Omega_{\mathfrak{C},K} & (k = 1)\\ \mathfrak{F}^k_{\mathfrak{C}}/\mathfrak{M}^k_{\mathfrak{C},K} & (k > 1)\end{cases}
\tag{24}
$$

and its element $cdc_1 \wedge \cdots \wedge dc_k := c(c_1, \ldots, c_k) + \mathfrak{M}^k_{\mathfrak{C},K}$. We can define $d^k_{\mathfrak{C},K} : \wedge^k\Omega_{\mathfrak{C},K} \to \wedge^{k+1}\Omega_{\mathfrak{C},K}$ as $d^k_{\mathfrak{C},K}(cdc_1 \wedge \cdots \wedge dc_k) := 1_{\mathfrak{C}}dc \wedge dc_1 \wedge \cdots \wedge dc_k$ for $k = 1, 2, \ldots$ and $d^k_{\mathfrak{C},K}(c) := 1_{\mathfrak{C}}dc$ for $k = 0$.

Proposition 2.48 *(1) The wedge product is skew-symmetric, i.e., for any $c_1, \ldots, c_k \in \mathfrak{C}$ and any permutation σ of $\{1, \ldots, k\}$,*

$$
dc_1 \wedge \ldots \wedge dc_k = sgn\sigma dc_{\sigma(1)} \wedge \ldots \wedge dc_{\sigma(k)}.
\tag{25}
$$

(2) The wedge products is multi-$\mathbb{R}$-linear, i.e., for any $i = 1, \ldots, k$, $c'_i, c_1, \ldots, c_k \in \mathfrak{C}$ and $\lambda, \lambda' \in \mathbb{R}$,

$$
\begin{aligned}
&dc_1 \wedge \ldots \wedge d(\lambda c_i + \lambda' c'_i) \wedge \ldots \wedge dc_k\\
&\quad= \lambda dc_1 \wedge \ldots \wedge dc_i \wedge \ldots \wedge dc_k + \lambda' dc_1 \wedge \ldots \wedge dc'_i \wedge \ldots \wedge dc_k.
\end{aligned}
\tag{26}
$$

Moreover for any $c_1, \ldots, c_k \in \mathfrak{C}$ and $f_1, \ldots, f_k \in C^\infty(\mathbb{R}^k)$, we have

$$\wedge_{i=1}^k d\big(\Phi_{f_i}(c_j)_{j=1}^k\big) = \det \left[\Phi_{\frac{\partial f_i}{\partial x_j}}(c_l)_{l=1}^k\right]_{i,j=1}^k dc_1 \wedge \ldots \wedge dc_k. \tag{27}$$

(3)

$$d^k_{\mathfrak{C},C^\infty}\big(\Phi_f(c_i)_{i=1}^l dc'_1 \wedge \ldots \wedge dc'_k\big) = \sum_{i=1}^l \Phi_{\frac{\partial f}{\partial x_i}}(c_i)_{i=1}^l dc_i \wedge dc'_1 \wedge \ldots \wedge dc'_k. \tag{28}$$

(4) $\{d^k_{\mathfrak{C},K}\}_{k=1}^\infty$ is coboundary operator, i.e.,

$$d^{k+1}_{\mathfrak{C},K} \circ d^k_{\mathfrak{C},K} = 0. \tag{29}$$

2.9 Limits and colimits of C^∞-rings

In [9], we have following properties about limits and colimits of C^∞-rings.

(1) Inverse limits of C^∞-rings are computed as inverse limits of their underlying sets.

(2) Directed colimits of C^∞-rings are computed as colimits of their underlying sets.

In this section, we compute finite colimits, inverse limits and directed colimits of C^∞-rings.

2.9.1 Finite colimits of C^∞-rings

For the following theorem, finitely presented C^∞-rings can be formed as pushouts.

Theorem 2.49 ([6]) *A C^∞-ring $\mathfrak{C}$ is finitely presented if and only if $\mathfrak{C}$ is a pushout of the following diagram with $n, k \in \mathbb{N}$ and morphisms $C^\infty(\mathbb{R}^k) \xrightarrow{\psi} C^\infty(\mathbb{R}^n) \xrightarrow{\phi} \mathfrak{C}$.*

$$\begin{tikzcd} C^\infty(\mathbb{R}^k) \arrow[r, "x_i \mapsto 0"] \arrow[d, "\psi"'] & \mathbb{R} \arrow[d, "1 \mapsto 1_{\mathfrak{C}}"] \\ C^\infty(\mathbb{R}^k) \arrow[r, "\phi"] & \mathfrak{C} \end{tikzcd}$$

From the following example, the category $\mathbf{C^\infty Rings}$ is closed under pushouts.

Example 2.50 ([6]) *For C^∞-rings $\mathfrak{C}$ and $\mathfrak{D}$, there exists morphisms $p_{\mathfrak{C}} : \mathbb{R} \to \mathfrak{C}$, $p_{\mathfrak{D}} : \mathbb{R} \to \mathfrak{D}$ with an initial object $\mathbb{R}$.*

Then there exists a unique C^∞-ring $\mathfrak{C}\hat{\otimes}\mathfrak{D}$ and unique morphisms $i_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}\hat{\otimes}\mathfrak{D}$, $i_{\mathfrak{D}} : \mathfrak{D} \to \mathfrak{C}\hat{\otimes}\mathfrak{D}$ such that;

(1) $i_{\mathfrak{C}} \circ p_{\mathfrak{C}} = i_{\mathfrak{D}} \circ p_{\mathfrak{D}}$, and

(2) if there exists morphisms $\phi_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{E}$, $\phi_{\mathfrak{D}} : \mathfrak{D} \to \mathfrak{E}$ with $\phi_{\mathfrak{C}} \circ p_{\mathfrak{C}} = \phi_{\mathfrak{D}} \circ p_{\mathfrak{D}}$, there exists a unique morphism $\phi_{\mathfrak{C}}\hat{\otimes}\phi_{\mathfrak{D}} : \mathfrak{C}\hat{\otimes}\mathfrak{D} \to \mathfrak{E}$ which satisfies a following equation.

$$\phi_{\mathfrak{C}} = \phi_{\mathfrak{C}}\hat{\otimes}\phi_{\mathfrak{D}} \circ i_{\mathfrak{C}},\ \phi_{\mathfrak{D}} = \phi_{\mathfrak{C}}\hat{\otimes}\phi_{\mathfrak{D}} \circ i_{\mathfrak{D}}.$$

We call $\mathfrak{C}\hat{\otimes}\mathfrak{D}$ a **coproduct** *of $\mathfrak{C}$ and $\mathfrak{D}$.*

We don't have to suppose that there exists morphism $p_{\mathfrak{C}}, p_{\mathfrak{D}}$ as $\mathbb{R}$ is an initial object of $\mathbf{C^\infty Rings}$.

As for the supposion of C^∞-rings $\mathfrak{C}, \mathfrak{D}$, there exists a C^∞-ring $\mathfrak{C}\hat{\otimes}\mathfrak{D}$ and morphisms $i_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}\hat{\otimes}\mathfrak{D}$, $i_{\mathfrak{D}} : \mathfrak{D} \to \mathfrak{C}\hat{\otimes}\mathfrak{D}$ such that they are unique up to isomorphisms.

Then we will write a pushout of C^∞-rings under finitely gerenated.

Example 2.51 ([6]) *By finitely generated C^∞-rings $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ and morphisms $\alpha : \mathfrak{C} \to \mathfrak{D}$, $\beta : \mathfrak{C} \to \mathfrak{E}$ of C^∞-rings, define a pushout $\mathfrak{F} := \mathfrak{D} \coprod_{\alpha,\mathfrak{C},\beta} \mathfrak{E}$ such that a following diagram is commutative.*

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \mathfrak{E} \arrow[d, "\delta"] \\ \mathfrak{D} \arrow[r, "\gamma"] & \mathfrak{F} \end{tikzcd} \tag{30}$$

For finitely generated C^∞-rings $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$, there exists natural numbers $l, m, n \in \mathbb{N}$ and ideals $I \subset C^\infty(\mathbb{R}^l), J \subset C^\infty(\mathbb{R}^m), K \subset C^\infty(\mathbb{R}^n)$ such that three following sequences are exact.

$$0 \to I \hookrightarrow C^\infty(\mathbb{R}^l) \xrightarrow{\phi} \mathfrak{C} \to 0,$$
$$0 \to J \hookrightarrow C^\infty(\mathbb{R}^m) \xrightarrow{\psi} \mathfrak{D} \to 0,$$
$$0 \to K \hookrightarrow C^\infty(\mathbb{R}^n) \xrightarrow{\chi} \mathfrak{E} \to 0.$$

Suppose that $C^\infty(\mathbb{R}^l)$ (resp. $C^\infty(\mathbb{R}^m)$, $C^\infty(\mathbb{R}^n)$) is generated by $\{x_1, \ldots, x_l\}$ (resp. $\{y_1, \ldots, y_m\}$, $\{z_1, \ldots, z_n\}$). We can make generators $\phi(x_1), \ldots, \phi(x_l)$ of $\mathfrak{C}$

by generators $x_1,\dots,x_n$.For $\alpha\big(\phi(x_i)\big) \in \mathfrak{D}$ and $\beta\big(\phi(x_i)\big) \in \mathfrak{E}(i = 1,\dots,l)$, there exists $f_i \in C^\infty(\mathbb{R}^m)$ and $g_i \in C^\infty(\mathbb{R}^n)$ such that $\alpha\big(\phi(x_i)\big) = \psi(f_i)$ and $\beta\big(\phi(x_i)\big) = \chi(g_i)$.

Suppose that $C^\infty(\mathbb{R}^{m+n})$ is generated by $\{w_1,\dots,w_m,w_{m+1},\dots,w_{m+n}\}$. Define an ideal L, a C^∞-ring $\mathfrak{F}'$, and a morphism ξ' as

$$L := \left\langle \begin{matrix} d(w_1,\dots,w_m), \\ e(w_{m+1},\dots,w_{m+n}), \\ f_i(w_1,\dots,w_m) - g_i(w_{m+1},\dots,w_{m+n}) \end{matrix} \middle| \begin{matrix} d \in J \subset C^\infty(\mathbb{R}^m) \\ e \in K \subset C^\infty(\mathbb{R}^n) \\ i = 1,\dots,l \end{matrix} \right\rangle_{C^\infty(\mathbb{R}^{m+n})},$$

$$\mathfrak{F}' := C^\infty(\mathbb{R}^{m+n})/L,\ \xi' : C^\infty(\mathbb{R}^{m+n}) \hookrightarrow \mathfrak{F}'.$$

We can define following morphisms $\gamma' : \mathfrak{D} \to \mathfrak{F}'$ and $\delta' : \mathfrak{E} \to \mathfrak{F}'$ as

$$\gamma'\Big(\phi\big(f(y_1,\dots,y_m)\big)\Big) := f(w_1,\dots,w_m) + L,$$
$$\delta'\Big(\chi\big(g(z_1,\dots,z_n)\big)\Big) := g(w_{m+1},\dots,w_{m+n}) + L$$

with well-defined by am ideal L and kernels of ϕ,ψ.

And for each generator $x_i \in C^\infty(\mathbb{R}^l)$, it satisfies

$$\begin{aligned} \delta' \circ \beta\big(\phi(x_i)\big) &= \delta' \circ \chi\big(g_i(z_1,\dots,z_n)\big) = g_i(w_{m+1},\dots,w_{m+n}) + L \\ &= f_i(w_1,\dots,w_m) + L = \gamma' \circ \psi\big(f_i(y_1,\dots,y_m)\big) = \gamma' \circ \alpha\big(\phi(x_i)\big). \end{aligned}$$

Then the diagram (30) is commutative, i.e. $\delta' \circ \beta = \gamma' \circ \alpha$.

So there exists a unique morphism $i' : \mathfrak{F} \to \mathfrak{F}'$ with $\delta' = i' \circ \delta, \gamma' = i' \circ \gamma$. The morphism i' satisfies

$$i'\big(\gamma \circ \psi(y_i)\big) = w_i + L,\ i'\big(\delta \circ \chi(z_j)\big) = w_{m+j} + L.$$

Therefore we have proved that $\mathfrak{F}$ is isomorphic to a finitely generated C^∞-ring $C^\infty(\mathbb{R}^{m+n})/L$ with a following ideal L.

$$L = \left\langle \begin{matrix} d(w_1,\dots,w_m), \\ e(w_{m+1},\dots,w_{m+n}), \\ f_i(w_1,\dots,w_m) - g_i(w_{m+1},\dots,w_{m+n}) \end{matrix} \middle| \begin{matrix} d \in J \subset C^\infty(\mathbb{R}^m) \\ e \in K \subset C^\infty(\mathbb{R}^n) \\ i = 1,\dots,l \end{matrix} \right\rangle.$$

Therefore we can write pushouts of fintiely generated C^∞-rings concretely, and we have a following property of pushouts of fintiely generated C^∞-rings.

Proposition 2.52 ([6]) *The full subcategories $\boldsymbol{C^\infty}$**Rings**$^{\boldsymbol{fg}}$, $\boldsymbol{C^\infty}$**Rings**$^{\boldsymbol{fp}}$, $\boldsymbol{C^\infty}$**Rings**$^{\boldsymbol{go}}$ are closed under pushouts and all finite colimits in $\boldsymbol{C^\infty}$**Rings**.*

The above property is not held for fair C^∞-rings. Its counterexample exists in Example 2.29 in [6].

2.9.2 Inverse limits of C^∞-rings

Theorem 2.53 ([9]) *Let I be an ordered set and $\{\mathfrak{C}_i\}_{i\in I}$ a set of C^∞-rings with C^∞-homomorphisms $u_{ij} : \mathfrak{C}_i \to \mathfrak{C}_j$ for any $i < j$. For $i < j < k$, $u_{ik} = u_{jk} \circ u_{ij}$ on $\mathfrak{C}_i$.*

There exists an **inverse limit** $\mathfrak{C}$ *of* $\{\mathfrak{C}_i\}$ *and a homomorphism $p_i : \mathfrak{C} \to \mathfrak{C}_i$ of C^∞-rings such that;*

(1) $u_{ij} \circ p_i = p_j$ for any $i < j$, and

(2) if there exists a C^∞-ring $\mathfrak{C}'$ and a homomorphism $q_i : \mathfrak{C}' \to \mathfrak{C}_i$ of C^∞-rings with $u_{ij} \circ q_i = q_j$ for any $i < j$, there exists a unique C^∞-homomorphism $q : \mathfrak{C}' \to \mathfrak{C}$ which satisfies $q_i = p_i \circ q$ for any i.

$$\begin{tikzcd} \mathfrak{C}' \arrow[d, "p_i"] \arrow[dr, "p_j"] & \\ \mathfrak{C}_i \arrow[r, "u_{ij}"] & \mathfrak{C}_j \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{C}' \arrow[r, "\exists 1 q"] \arrow[dr, "p_i"'] & \mathfrak{C} \arrow[d, "q_i"] \\ & \mathfrak{C}_i \end{tikzcd} \tag{31}$$

Proof) [**The existence of an inverse limit**] Define a C^∞-ring $\mathfrak{C}$ as

$$\mathfrak{C} := \left\{ (c_i)_{i\in I} \in \prod_{i\in I} \mathfrak{C}_i \middle| u_{ij}(c_i) = c_j \text{ for any } i < j \right\}. \tag{32}$$

This $\mathfrak{C}$ is an inverse limit of $\mathfrak{C}_i$. Define C^∞-homomorphisms $p_i : \mathfrak{C} \to \mathfrak{C}_i$ as $p_i\big((c_i)_i\big) = c_i$. Then $u_{ij} \circ p_i = p_j$ for any $i < j$.

[**The uniqueness of an inverse limit**] Suppose that there exists a C^∞-ring $\mathfrak{D}$ and C^∞-homomorphisms $q_i : \mathfrak{D} \to \mathfrak{C}_i$ such that $u_{ij} \circ q_i = q_j$ for any $i < j$. Define a C^∞-homomorphism $q : \mathfrak{D} \to \mathfrak{C}$ as $q(d) = \big(q_i(d)\big)_i$. q is well-defined by $u_{ij} \circ q_i = q_j$. Then $q_i = p_i \circ q$.

If there exists another C^∞-homomorphism $q' = (q'_i)_i : \mathfrak{D} \to \mathfrak{C}$ which satisfies $q_i = p_i \circ q'$ for any i, then $q'_i(d) = p_i\big(q'(d)\big) = q_i(d) = p_i\big(q(d)\big) = q_i(d)$ for any $d \in \mathfrak{D}$ and i. We have $q' = (q'_i)_i = (q_i)_i = q$.

$\mathfrak{C}$ is an inverse limit of $\mathfrak{C}_i$. ■

2.9.3 Directed colimits of C^∞-rings

Theorem 2.54 ([9]) *Let $\{\mathfrak{C}_i\}_{i\in I}$ be a family of C^∞-rings indexed by a directed set I (i.e. I is an ordered set which satisfies that there exists $k \geq i, j$ for any $i, j \in I$) with C^∞-homomorphisms $u_{ij} : \mathfrak{C}_i \to \mathfrak{C}_j$ for any $i \leq j$.*

There exists a **directed colimit** $\mathfrak{D}$ *of* $\{\mathfrak{C}_i\}$ *and a homomorphism $\iota_i : \mathfrak{C}_i \to \mathfrak{D}$ of C^∞-rings such that;*

(1) $\iota_j \circ u_{ij} = \iota_i$ *for any* $i \leq j$*, and*

(2) if there exists a C^∞*-ring* $\mathfrak{D}'$ *and a homomorphism* $\rho_i : \mathfrak{C}_i \to \mathfrak{D}'$ *of* C^∞*-rings with* $\rho_j \circ u_{ij} = \rho_i$ *for any* $i \leq j$*, there exists a unique* C^∞*-homomorphism* $\rho : \mathfrak{D} \to \mathfrak{D}'$ *which satisfies* $\rho_i = \rho \circ \iota_i$ *for any* i*.*

$$\begin{tikzcd} \mathfrak{C}_i \arrow[r, "u_{ij}"] \arrow[d, "\iota_i"'] & \mathfrak{C}_j \arrow[dl, "\iota_j"] \\ \mathfrak{D} & \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{C}_i \arrow[d, "\iota_i"'] \arrow[dr, "\rho_i"] & \\ \mathfrak{D} \arrow[r, "\exists 1\rho"] & \mathfrak{C}_i \end{tikzcd} \tag{33}$$

Lemma 2.55 *Let* $\{\mathfrak{C}_i\}_{i\in I}$ *be a family of* C^∞*-rings indexed by a directed set* I *with* C^∞*-homomorphisms* $u_{ij} : \mathfrak{C}_i \to \mathfrak{C}_j$ *for any* $i \leq j$*.*

Define a relation $\sim$ *as* $a_i \sim a_j$ *if and only if* $u_{ik}(a_i) = u_{jk}(a_j)$ *for some* $k \geq i, j$*. This is equivalence relation of* $\{\mathfrak{C}_i\}_{i\in I}$*.*

Proof) [**Reflexive**] Let $a \in \mathfrak{C}_i$ be any element of $\{\mathfrak{C}_i\}_{i\in I}$. Then $u_{ii}(a) = u_{ii}(a)$ and $a \sim a$.

[**Symmetric**] The symmetric law is trivial. Suppose that $a_{i_1} \in \mathfrak{C}_{i_1}, a_{i_2} \in \mathfrak{C}_{i_2}$ such that $a_{i_1} \sim a_{i_2}$. There exists $j \geq i_1, i_2$ such that $u_{i_1 j}(a_{i_1}) = u_{i_2 j}(a_{i_2})$. Then we have $u_{i_2 j}(a_{i_2}) = u_{i_1 j}(a_{i_1})$ and $a_{i_2} \sim a_{i_1}$.

[**Transivity**] Suppose that $a_{i_1} \in \mathfrak{C}_{i_1}, a_{i_2} \in \mathfrak{C}_{i_2}, a_{i_3} \in \mathfrak{C}_{i_3}$ such that $a_{i_1} \sim a_{i_2}$ and $a_{i_2} \sim a_{i_3}$. There exists $j \geq i_1, i_2$ and $k \geq i_2, i_3$ such that $u_{i_1 j}(a_{i_1}) = u_{i_2 j}(a_{i_2})$ and $u_{i_2 k}(a_{i_2}) = u_{i_3 k}(a_{i_3})$.

There exists $m \geq j, k$. Then we have $u_{i_1 m}(a_{i_1}) = u_{jm} \circ u_{i_1 j}(a_{i_1}) = u_{jm} \circ u_{i_2 j}(a_{i_2}) = u_{i_2 m}(a_{i_2}) = u_{km} \circ u_{i_2 k}(a_{i_2}) = u_{km} \circ u_{i_3 k}(a_{i_3}) = u_{i_3 m}(a_{i_3})$. We have that $a_{i_1} \sim a_{i_3}$. ■

Proof of Theorem 2.54) Define $\mathfrak{S}$ as a set of all elements of all $\mathfrak{C}_i$, i.e. $\mathfrak{S} := \coprod_i \mathfrak{C}_i = \{c | c \in \mathfrak{C}_i \text{ for some } i\}$. Define an equivalence relation $\sim$ of $\mathfrak{S}$ in Lemma 2.55.

Define a C^∞-ring $\mathfrak{D}$ as a quotient set of $\mathfrak{S}$ by $\sim$, that is, $\mathfrak{D} := \mathfrak{S}/\sim$. Write its element $[a_i]$ as $a_i/\sim$ for $a_i \in \mathfrak{C}_i$. This $\mathfrak{D}$ is a C^∞-ring, that is, for any $f \in C^\infty(\mathbb{R}^n)$, we define an operation $\Phi_f : \mathfrak{D}^n \to \mathfrak{D}$ as $\Phi_f^{\mathfrak{D}}([a_{i_1,1}], \ldots, [a_{i_n,n}]) := [\Phi_f^{\mathfrak{C}_k}\big(u_{i_1 k}(a_{i_1,1}), \ldots, u_{i_n k}(a_{i_n,n})\big)]$ for $a_{i_j,j} \in \mathfrak{C}_{i_j}$ and $k \geq i_1, \ldots, i_n$. For any i, define a homomorphism $\iota_i : \mathfrak{C}_i \to \mathfrak{D}$ as $\iota_i(a_i) = [a_i]$ for any $a_i \in \mathfrak{C}_i$. It satisfies $\iota_i = \iota_j \circ u_{ij}$ for any $i \leq j$ from the definition of $\mathfrak{D}$.

Suppose that there exists a C^∞-ring $\mathfrak{D}'$ and C^∞-homomorphisms $\rho_i : \mathfrak{C}_i \to \mathfrak{D}'$ such that $\rho_i = \rho_j \circ u_{ij}$. Define a C^∞-homomorphism $\rho : \mathfrak{D} \to \mathfrak{D}'$ as $\rho([a]) = \rho_i(a)$ for $a \in \mathfrak{C}_i$. It satisfies $\rho \circ \iota_i = \rho_i$ for any $i \in I$.

If there exists $\rho' : \mathfrak{D} \to \mathfrak{D}'$ such that $\rho' \circ \iota_i = \rho_i$, for $a \in \mathfrak{C}_i$, then $\rho([a]) = \rho \circ \iota_i(a) = \rho_i(a) = \rho' \circ \iota_i(a) = \rho'([a])$. We have proved that the existence and uniqueness of ρ.

We have that $\mathfrak{D}$ and $\iota : \mathfrak{C}_i \to \mathfrak{D}$ satisfy the condition(33). Therefore, $\mathfrak{D}$ is the directed colimit of $\mathfrak{C}_i$. ■

2.9.4 Limits of localizations

We check the existence of limits of localizations of C^∞-rings from [9].

Theorem 2.56 ([9]) *Let $\mathfrak{C}$ be a C^∞-ring and c an element of $\mathfrak{C}$.*

Then there exists a set $\Lambda = \{\mathfrak{C}_i\}$ of finitely generated C^∞-subrings of $\mathfrak{C}$ containing c such that $\mathfrak{C}[c^{-1}]$ is isomorphic to $\lim_i \mathfrak{C}_i[c^{-1}]$.

Lemma 2.57 *Let $\mathfrak{C}$ be a C^∞-ring and c an element of $\mathfrak{C}$. There exists a set $\Lambda = \{\mathfrak{C}_i\}$ of finitely generated C^∞-subrings of $\mathfrak{C}$ containing c.*

Define a set $\mathcal{P}_c$ of finite subsets of $\mathfrak{C}$ which can generate c as

$$\mathcal{P}_c := \left\{ S = \{c_1, \dots, c_n\} \middle| \begin{array}{c} \exists n \in \{0\} \cup \mathbb{N}, \exists c_1, \dots, c_n \in \mathfrak{C} \\ \exists f \in C^\infty(\mathbb{R}^n) \text{ such that } c = \Phi_f(c_1, \dots, c_n) \end{array} \right\}.$$

Lemma 2.58 *For $S = \{c_1, \dots, c_n\}, S' = \{c'_1, \dots, c'_{n'}\} \in \mathcal{P}_c$, define a relation $S \sim S'$ if there exists $f_1, \dots, f_n \in C^\infty(\mathbb{R}^{n'})$ and $g_1, \dots, g_{n'} \in C^\infty(\mathbb{R}^n)$ such that $c_i = \Phi_{f_i}(c'_1, \dots, c'_{n'})$ and $c'_j = \Phi_{g_j}(c_1, \dots, c_n)$ for $i = 1, \dots, n$ and $j = 1, \dots, n'$.*

This is the equivalence relation on $\mathcal{P}_c$.

Proof) [**Reflexive**] Suppose that $S = \{c_1, \dots, c_n\}$ is any element of $\mathcal{P}_c$. Define projections $\pi_i : \mathbb{R}^n \to \mathbb{R}$ as $\pi_i(x_1, \dots, x_n) = x_i$ for $i = 1, \dots, n$. Then $c_i = \Phi_{\pi_i}(c_1, \dots, c_n)$. We have $S \sim S$.

[**Symmetric**] The symmetric law is trivial.

[**Transivity**] Suppose that $S = \{c_1, \dots, c_n\}$, $S' = \{c'_1, \dots, c'_{n'}\}$, $S'' = \{c''_1, \dots, c''_{n''}\}$ such that $S \sim S'$ and $S' \sim S''$. There exists $f_1, \dots, f_n \in C^\infty(\mathbb{R}^{n'})$, $g_1, \dots, g_{n'} \in C^\infty(\mathbb{R}^n)$, $f'_1, \dots, f'_{n'} \in C^\infty(\mathbb{R}^{n''})$ and $g'_1, \dots, g'_{n''} \in C^\infty(\mathbb{R}^{n'})$ such that $c_i = \Phi_{f_i}(c'_1, \dots, c'_{n'})$, $c'_j = \Phi_{g_j}(c_1, \dots, c_n)$, $c'_{i'} = \Phi_{f'_{i'}}(c''_1, \dots, c''_{n''})$ and $c''_{j'} = \Phi_{g'_{j'}}(c'_1, \dots, c'_{n'})$ for $i = 1, \dots, n$, $j = 1, \dots, n'$, $i' = 1, \dots, n'$ and $j' = 1, \dots, n''$. By smooth functions $F_i := f_i \circ (f'_1, \dots, f'_{n'})$, $G_{j'} := g'_{j''} \circ (g_1, \dots, g_{n'})$, $c_i = \Phi_{F_i}(c''_1, \dots, c''_{n''})$ and $c''_{j'} = \Phi_{G_{j'}}(c_1, \dots, c_n)$ We have that $S \sim S''$. ■

Define $g(S)$ as the equivalence class of $S \in \mathcal{P}_c$ and G_c as the quotient set of $\mathcal{P}_c$ by $\sim$. Next, define a C^∞-subring $\mathfrak{C}(g) \subset \mathfrak{C}$ for $g \in G_c$ as a set which is generated by $c_1, \dots, c_n$ with $g = g(\{c_1, \dots, c_n\})$.

Lemma 2.59 *(1) This C^∞-subring $\mathfrak{C}(g)$ is well-defined.*

(2) $\mathfrak{C}(g)$ is finitely generated.

(3) For any fiinitely generated C^∞-subring $\mathfrak{C}'$ containing c, there exists $g \in G_c$ such that $\mathfrak{C}' = \mathfrak{C}(g)$.

Proof)

(1) Suppose that $S = \{c_1, \ldots, c_n\} \sim S' = \{c'_1, \ldots, c'_{n'}\}$, i.e. there exists C^∞-functions $f_1, \ldots, f_n \in C^\infty(\mathbb{R}^{n'})$ and $g_1, \ldots, g_{n'} \in C^\infty(\mathbb{R}^n)$ such that $c_i = \Phi_{f_i}(c'_1, \ldots, c'_{n'})$ and $c'_j = \Phi_{g_j}(c_1, \ldots, c_n)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n'$.

For any $b = \Phi_f(c_1, \ldots, c_n)$, $b = \Phi_{f\circ(f_1,\ldots,f_n)}(c'_1, \ldots, c'_{n'})$. Then, a C^∞-subring generated by $c_1, \ldots, c_n$ equals to a C^∞-subring generated by $c'_1, \ldots, c'_{n'}$.

Therefore, $\mathfrak{C}(g)$ is well-defined.

(2) Suppose that $g = g(S)$ by $S = \{c_1, \ldots, c_n\}$. Define a C^∞-homomorphism $\phi : C^\infty(\mathbb{R}^n) \to \mathfrak{C}(g)$ as $\phi(f(x_1, \ldots, x_n)) := \Phi_f(c_1, \ldots, c_n)$. For the previous property, ϕ is surjective. Then, $\mathfrak{C}(g)$ is isomorphic to $C^\infty(\mathbb{R}^n)/Ker\phi$ and $\mathfrak{C}(g)$ is finitely generated.

(3) For the definition of $\mathfrak{C}'$, there exists a C^∞-homomorphism $\phi : C^\infty(\mathbb{R}^n) \to \mathfrak{C}'$ such that ϕ is surjective. Define $g \in G_c$ as $g := g(S)$ by $S := \{\phi(x_1), \ldots, \phi(x_n)\}$. Then $\mathfrak{C}(g) = \phi(C^\infty(\mathbb{R}^n)) = \mathfrak{C}'$.

■

Proof of Lemma 2.57) Define a set Λ of $\mathfrak{C}(g)$ for any $g \in G_c$. From Lemma 2.59, Λ is a set of finitely generated C^∞-subrings of $\mathfrak{C}$ which containing c.

■

Define a set Λ of $\mathfrak{C}(g)$ for any $g \in G_c$. Λ is a set of finitely generated C^∞-subrings of $\mathfrak{C}$ which containing c.

Lemma 2.60 *A set Λ of finitely generated C^∞-subrings of $\mathfrak{C}$ containing c is a directed set.*

Proof) We have to prove that Λ is a directed set by $\subset$.

Take any $\mathfrak{C}_i, \mathfrak{C}_j \in \Lambda$ with $\mathfrak{C}_i \subset \mathfrak{C}_j$. There exists $S_i = \{c^i_1, \ldots, c^i_{n_i}\}, S_j = \{c^j_1, \ldots, c^j_{n_j}\}$ such that $\mathfrak{C}_i = g(S_i), \mathfrak{C}_j = g(S_j)$. Define $S_k := \{c^i_1, \ldots, c^i_{n_i}, c^j_1, \ldots, c^j_{n_j}\}$, $g_k := g(S_k)$ and $\mathfrak{C}_k := \mathfrak{C}(g_k)$. Then there exists inclusions $\mathfrak{C}_i \subset \mathfrak{C}_k$ and $\mathfrak{C}_j \subset \mathfrak{C}_k$.

Therefore, Λ is a directed set by $\subset$. ■

The proof of Theorem 2.56) From Lemma 2.60, there exists a directed set Λ_c is generated by finitely generated C^∞-subrings $\mathfrak{C}_i \subset \mathfrak{C}$ which contains $c \in \mathfrak{C}$ and inclusions $u_i : \mathfrak{C}_i \hookrightarrow \mathfrak{C}$ and $u_{ij} : \mathfrak{C}_i \hookrightarrow \mathfrak{C}_j$ which satisfies $u_i = u_j \circ u_{ij}$ for any $i \le j$.

By a localization of $\mathfrak{C}_i$ by c, we have finitely generated C^∞-rings $\mathfrak{C}_i[c^{-1}]$ and homomorphisms $u_{ij}[c^{-1}] : \mathfrak{C}_i[c^{-1}] \to \mathfrak{C}_j[c^{-1}]$ and homomorphisms $\pi_i : \mathfrak{C}_i \to \mathfrak{C}_i[c^{-1}]$ such that

(1) $u_{ij}[c^{-1}] \circ \pi_i = \pi_j \circ u_{ij}$ on $\mathfrak{C}_i$ for any $i \le j$ and

(2) $\pi_i(c)$ is invertible in $\mathfrak{C}_i[c^{-1}]$ for any i.

The set of $\mathfrak{C}_i[c^{-1}]$ is a directed set by $u_{ij}[c^{-1}] : \mathfrak{C}_i[c^{-1}] \to \mathfrak{C}_j[c^{-1}]$. Then there exists a directed colimit $\mathfrak{C}' = \lim_i \mathfrak{C}_i[c^{-1}]$ and homomorphisms $v_i : \mathfrak{C}_i[c^{-1}] \to \mathfrak{C}'$ with $v_j \circ u_{ij}[c^{-1}] = v_i$ for any $i \le j$.

Take any element $b = u_i(b_i) = u_j(b_j)$ for $b_i \in \mathfrak{C}_i$, $b_j \in \mathfrak{C}_j$. For the definitions of $\mathfrak{C}_i, \mathfrak{C}_j \subset \mathfrak{C}$ and inclusions, we have $b_i = b_j$ and $(\pi_k \circ u_{ik})(b_i) = (\pi_k \circ u_{jk})(b_j)$ for $k \ge i, j$.

$$\begin{aligned} v_i\big(\pi_i(b_i)\big) &= \big((v_k \circ u_{ik}[c^{-1}]) \circ \pi_i\big)(b_i) \\ &= \big(v_k \circ (\pi_k \circ u_{ik})\big)(b_i) \\ &= \big(v_k \circ (\pi_k \circ u_{jk})\big)(b_j) = v_j\big(\pi_j(b_j)\big). \end{aligned}$$

Therefore, we can define $\pi : \mathfrak{C} \to \mathfrak{C}'$ as $\pi(b) := v_i\big(\pi_i(b_i)\big)$ for any $b = u_i(b_i) \in \mathfrak{C}(b_i \in \mathfrak{C}_i)$. For $\pi(c) = v_i\big(\pi_i(c)\big)$, we have that $v_i\big(\pi_i(c)^{-1}\big)$ is an invertible element.

Suppose that $\pi' : \mathfrak{C} \to \mathfrak{D}$ is another homomorphism which satisfies that $\pi'(c)$ is invertible in $\mathfrak{D}$. For any $\mathfrak{C}'$, $\pi' \circ u_i : \mathfrak{C}_i \hookrightarrow \mathfrak{C} \to \mathfrak{D}$ is a homomorphism which satisfies that $\pi'\big(u_i(c)\big)$ is invertible in $\mathfrak{D}$. Then we have a unique homomorphism $\phi_{\pi' \circ u_i} : \mathfrak{C}_i[c^{-1}] \to \mathfrak{D}$ such that $\pi' \circ u_i = \phi_{\pi' \circ u_i} \circ \pi_i$.

Define a homomorphism $\phi_{\mathfrak{D}} : \mathfrak{C}' \to \mathfrak{D}$ as $\phi_{\mathfrak{D}}(d) := \phi_{\pi' \circ u_i}(d_i)$ for $d = v_i(d_i)(d_i \in \mathfrak{C}_i[c^{-1}])$. Then

$$\big(\phi_{\mathfrak{D}} \circ \pi\big)(u_i(b_i)) = \big(\phi_{\mathfrak{D}} \circ (v_i \circ \pi_i)\big)(b_i) = \big(\phi_{\pi' \circ u_i} \circ \pi_i\big)(b_i) = \big(\pi' \circ u_i\big)(b_i) = \pi'\big(u_i(b_i)\big).$$

We have $\phi_{\mathfrak{D}} \circ \pi = \pi'$.

Suppose that there exists another homomorphism $\phi'_{\mathfrak{D}} : \mathfrak{C}' \to \mathfrak{D}$ which satisfies that $\phi'_{\mathfrak{D}} \circ \pi = \pi'$. For $d = v_i(\pi_i(d_i))(d_i \in \mathfrak{C}_i)$,

$$\phi'_{\mathfrak{D}}\big(v_i(\pi_i[c^{-1}](d_i))\big) = \phi'_{\mathfrak{D}}(\pi(d_i)) = \pi(d_i) = \phi_{\mathfrak{D}}(\pi(d_i)) = \phi_{\mathfrak{D}}\big(v_i(\pi_i[c^{-1}](d_i))\big).$$

For $d = v_i\big((\pi_i[c^{-1}](c))^{-1}\big)$,

$$\begin{aligned}\phi'_{\mathfrak{D}}\Big(v_i\big(\pi_i(c)^{-1}\big)\Big) &= \phi'_{\mathfrak{D}}\Big(v_i\big(\pi_i(c)\big)\Big)^{-1} \\ &= \phi_{\mathfrak{D}}\Big(v_i\big(\pi_i(c)\big)\Big)^{-1} = \phi_{\mathfrak{D}}\Big(v_i\big(\pi_i(c)^{-1}\big)\Big).\end{aligned}$$

We have $\phi'_{\mathfrak{D}} = \phi_{\mathfrak{D}}$. Therefore, $\phi_{\mathfrak{D}}$ is a unique homomorhism which satisfies $\phi_{\mathfrak{D}} \circ \pi = \pi'$.

$\mathfrak{C}' = \lim_i \mathfrak{C}_i[c^{-1}]$ is a localization of $\mathfrak{C}$ at $\{c\}$. ■

3 C^∞-ringed spaces

3.1 C^∞-ringed spaces

A C^∞-ringed space is a pair of a topological space and a sheaf of C^∞-rings over it. We consider sheaves with the construction of C^∞-rings.

Definition 3.1 ([6]) *(1) A* C^∞**-ringed space** $\underline{X} := (X, \mathcal{O}_X)$ *is a topological space X with a sheaf $\mathcal{O}_X$ of C^∞-rings on X.*

(2) Let $\underline{X} = (X, \mathcal{O}_X)$, $\underline{Y} = (Y, \mathcal{O}_Y)$ be C^∞-ringed spaces. A **morphism** $\underline{f} = (f, f^\#) : \underline{X} \to \underline{Y}$ **of** C^∞**-ringed spaces** *is a continuous map $f : X \to Y$ with a morphism $f^\# : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ of sheaves of C^∞-rings on X.*

(3) Write $\mathbf{C^\infty RS}$ *for the category of C^∞-ringed spaces.*

(4) A **local** C^∞**-ringed space** $\underline{X} = (X, \mathcal{O}_X)$ *is a C^∞-ringed space for which a directed colimit $\mathcal{O}_{X,x} := \lim_{U \ni x} \mathcal{O}_X(U)$ is a local ring for all $x \in X$.*

(5) Write $\mathbf{LC^\infty RS}$ *for the full subcategory of* $\mathbf{C^\infty RS}$ *of local C^∞-ringed spaces.*

We will explain compositions of morphisms between C^∞-ringed spaces later.

Example 3.2 *(1) For a C^∞-manifold X, we define the C^∞-ringed space $(X, \mathcal{O}_X)$ as the topological space X with the sheaf $\mathcal{O}_X$ of C^∞-rings on X defined as $\mathcal{O}_X(U) := C^\infty(U)$ for each open set $U \subset X$.*

(2) Let $f : X \to Y$ be a smooth map between manifolds.

We can define a morphism $f^\# : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ of sheaves of C^∞-rings on X with a morphism $f^\#(U) : f^{-1}(\mathcal{O}_Y)(U) \to \mathcal{O}_X(U)$ of C^∞-rings defined as $f^\#(U)\big([c]_{V \supset f(U)}\big) := (c \circ f)|_U$ for each open set $U \subset X$.

Then we defined a morphism $\underline{f} := (f, f^\#) : \underline{X} \to \underline{Y}$.

Thus the following functor $F_{\boldsymbol{Man}}^{C^\infty \boldsymbol{RS}} : \boldsymbol{Man} \to C^\infty \boldsymbol{RS}$ is defined as

$$F_{\boldsymbol{Man}}^{C^\infty \boldsymbol{RS}}(X) := (X, \mathcal{O}_X),$$
$$F_{\boldsymbol{Man}}^{C^\infty \boldsymbol{RS}}(f : X \to Y) := (f, f^\#) : F_{\boldsymbol{Man}}^{C^\infty \boldsymbol{RS}}(X) \to F_{\boldsymbol{Man}}^{C^\infty \boldsymbol{RS}}(Y).$$

3.1.1 Pushouts and pullbacks

Let X and Y be topological spaces. Take $f : X \to Y$ as a continuous mapping of topological spaces.

We can define $f^{\#}$ as a morphism $f^{\#} : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ of sheaves on X or a morphism $f^{\#} : \mathcal{O}_Y \to f^*(\mathcal{O}_X)$ of sheaves on Y. However either will do from [7].

First, we define pushforwards of sheaves on X by f in Definition A.14. of [7] and pullbacks of sheaves on X by f in Definition A.15. of [7].

Definition 3.3 ([7]) *Let X, Y be topological spaces and $f : X \to Y$ a continuous mapping.*

(1) Let $\mathcal{O}_X$ be a sheaf of C^∞-rings on X with a restriction $\rho_{UU'} : \mathcal{O}_X(U) \to \mathcal{O}_X(U')$ for any open subsets $U' \subset U \subset X$.

For an open subset $V \subset Y$, define a C^∞-ring $f^*(\mathcal{O}_X)(V) := \mathcal{O}_X(f^{-1}(V))$ with an operation $\Psi_g(c_1, \ldots, c_n) := \Phi_g(c_1, \ldots, c_n)$ for any $g \in C^\infty(\mathbb{R}^n)$.

For open subsets $V' \subset V \subset Y$, define a homomorphism $\rho'_{VV'} : f^*(\mathcal{O}_X)(V) \to f^*(\mathcal{O}_X)(V')$ of C^∞-rings as $\rho'_{VV'}(c) := \rho_{f^{-1}(V)f^{-1}(V')}(c)$.

Define a sheaf $f^*(\mathcal{O}_X)$ with a restriction $\rho'_{VV'}$.

(2) Let $\phi : \mathcal{O}_X \to \mathcal{O}'_X$ be a morphism of sheaves of C^∞-rings on X.

For an open subset $V \subset Y$, define $f^*(\phi)(V) : f^*(\mathcal{O}_X)(V) \to f^*(\mathcal{O}'_X)(V)$ as $f^*(\phi)(V)(c) := \phi(f^{-1}(V))(c)$.

Define $f^*(\phi) : f^*(\mathcal{O}_X) \to f^*(\mathcal{O}'_X)$ is a homomorphism of sheaves of C^∞-rings on Y.

Definition 3.4 ([7]) *Let X, Y be topological spaces and $f : X \to Y$ a continuous mapping.*

(1) Let $\mathcal{O}_Y$ be a sheaf of C^∞-rings on Y.

For an open subset $U \subset X$, define a C^∞-ring $f^{-1}(\mathcal{O}_Y)(U) := \lim_{V \supset f(U)} \mathcal{O}_Y(V)$ with an operation $\Psi_g([c_1]_{V_1 \supset f(U)}, \ldots, [c_n]_{V_n \supset f(U)}) := [\Phi_g(c_1, \ldots, c_n)]_{V_1 \cap \cdots \cap V_n \supset f(U)}$ for any $g \in C^\infty(\mathbb{R}^n)$.

For open subsets $U' \subset U \subset X$, define a morphism $\rho''_{UU'} : f^{-1}(\mathcal{O}_Y)(U) \to f^{-1}(\mathcal{O}_Y)(U')$ of C^∞-rings as $\rho''_{UU'}\big([c]_{V \supset f(U)}\big) := [c]_{V \supset f(U')}$.

Define a sheaf $f^{-1}(\mathcal{O}_Y)$ with a restriction $\rho''_{UU'}$.

(2) *Let $\psi : \mathcal{O}_Y \to \mathcal{O}'_Y$ be a morphism of sheaves of C^∞-rings on Y.*

For an open subset $U \subset X$, define $f^{-1}(\psi)(U) : f^{-1}(\mathcal{O}_Y)(U) \to f^{-1}(\mathcal{O}'_Y)(U)$ as $f^{-1}(\psi)(U)([c]_{V \supset f(U)}) := [\psi(V)(c)]_{V \supset f(U)}$.

Define $f^{-1}(\psi) : f^{-1}(\mathcal{O}_Y) \to f^{-1}(\mathcal{O}'_Y)$ is a morphism of sheaves of C^∞-rings on X.

Proposition 3.5 *Let $\underline{X}$ and $\underline{Y}$ be C^∞-ringed spaces and $f : X \to Y$ a continuous mapping.*

For any open set $V \subset Y$, define a morphism $\Psi'(V) : \mathcal{O}_Y(V) \to f^*\big(f^{-1}(\mathcal{O}_Y)\big)(V)$ of C^∞-rings as $\big(\Psi'(V)\big)(c) := [c]_{V \supset f(f^{-1}(V))}$. Define a morphism $\Psi' : \mathcal{O}_Y \to f^*\big(f^{-1}(\mathcal{O}_Y)\big)$ of sheaves on Y.

For any open set $U \subset X$, define a morphism $\Phi'(U) : f^{-1}\big(f^*(\mathcal{O}_X)\big)(U) \to \mathcal{O}_X(U)$ of C^∞-rings as $\Phi'(U)\big([c_V]_{V \supset f(U)}\big) := \rho_{f^{-1}(V)U}(c_V)$. Define a morphism $\Phi' : f^{-1}\big(f^*(\mathcal{O}_X)\big) \to \mathcal{O}_X$ of sheaves on X.

For the above morphisms, we have a following bijection $Hom\big(f^{-1}(\mathcal{O}_Y), \mathcal{O}_X\big) \cong Hom\big(\mathcal{O}_Y, f^*(\mathcal{O}_X)\big)$.

Therefore if we have a morphism $f^\# : f^{-1}\big(\mathcal{O}_Y\big) \to \mathcal{O}_X$, we can have a morphism $f^\# : \mathcal{O}_Y \to f^*(\mathcal{O}_X)$. The converse is also true.

Next, we will explain natural isomorphisms about pullback sheaves from Remark A.16. in [7].

Proposition 3.6 ([7]) *We consider continuous mappings $f : X \to Y$, $g : Y \to Z$ of topological spaces.*

(1) *For a sheaf $\mathcal{O}_Z$ of C^∞-rings on Z, $f^{-1}\big(g^{-1}(\mathcal{O}_Z)\big)$ and $(g \circ f)^{-1}(\mathcal{O}_Z)$ have a following isomorphism*

$$I_{f,g}(\mathcal{O}_Z) : (g \circ f)^{-1}(\mathcal{O}_Z) \xrightarrow{\sim} f^{-1}\big(g^{-1}(\mathcal{O}_Z)\big).$$

(2) *For a sheaf $\mathcal{O}_X$ of C^∞-rings on X, ${id_X}^{-1}(\mathcal{O}_X)$ and $\mathcal{O}_X$ have a following isomorphism*

$$\delta_X(\mathcal{O}_X) : {id_X}^{-1}(\mathcal{O}_X) \xrightarrow{\sim} \mathcal{O}_X.$$

Then, we define compositions of morphisms between C^∞-ringed spaces from Remark A.16. in [7].

Definition 3.7 ([7]) *For morphisms $\underline{f} : \underline{X} \to \underline{Y}$, $\underline{g} : \underline{Y} \to \underline{Z}$ ($\underline{f} = (f, f^\#)$, $\underline{g} = (g, g^\#)$) of C^∞-ringed spaces, define a composition $\underline{g \circ f} := \big(g \circ f, f^\# \circ f^{-1}(g^\#) \circ I_{f,g}(\mathcal{O}_Z)\big)$.*

3.2 Spectra

Definition 3.8 ([6]) *Let $\mathfrak{C}$ be a C^∞-ring and $\mathfrak{M}$ a $\mathfrak{C}$-module. By a functor $Spec : C^\infty Rings^{op} \to LC^\infty RS$, a C^∞-ringed space $\underline{X} := Spec\mathfrak{C}$ is*

(1) Define a topological space $X_{\mathfrak{C}}$ as followings by C^∞-ring $\mathfrak{C}$.

- *Define a set $X_{\mathfrak{C}} := \{x : \mathfrak{C} \to \mathbb{R} | x \text{ is an } \mathbb{R}\text{-point of } \mathfrak{C}.\}$.*
- *For each $c \in \mathfrak{C}$, define a mapping $c_* : X_{\mathfrak{C}} \to \mathbb{R}$ as $c_*(x) := x(c)$. Set a topology $\mathcal{T}_{\mathfrak{C}}$ of $X_{\mathfrak{C}}$ as a smallest topology such that c_* is continuous for all $c \in \mathfrak{C}$.*

(2) For an open subset $U \subset X_{\mathfrak{C}}$, define $\mathcal{O}_{X_{\mathfrak{C}}}(U)$ as a set of functions $s : U \to \coprod_{x \in U} \mathfrak{C}_x$ with following properties.

- *For each $x \in U$, $s(x) \in \mathfrak{C}_x$ is satisfied.*
- *U is covered by open sets V for which there exists $c, d \in \mathfrak{C}$ with $x(d) \neq 0$ and $\pi_x(c)\pi_x(d)^{-1} = s(x)$ for all $x \in V$.*

(3) Therefore define the following C^∞-ringed space

$$Spec\mathfrak{C} := (X_{\mathfrak{C}}, \mathcal{O}_{X_{\mathfrak{C}}}). \tag{34}$$

The definition of Spec : $\mathbf{C^\infty Rings} \to \mathbf{LC^\infty RS}$ is complex. However we have following examples.

Example 3.9 ([6]) *(1) Let X, Y be C^∞-manifolds and $\phi : C^\infty(Y) \to C^\infty(X)$ a morphism of C^∞-rings.*

(a) From a proof about real spaces in [9] and the existence of property functions on manifolds, the map $f \mapsto f^*$ from points x of X to $\mathbb{R}$-points $x : C^\infty(X) \to \mathbb{R}$ of C^∞-rings $C^\infty(X)$ is a 1-1 correspondence. Then $X_{C^\infty(X)}$ is homeomorphic to the topological space X. Therefore $Spec(C^\infty(X)) = (X, \mathcal{O}_X)(\mathcal{O}_X(U) := C^\infty(U))$.

(b) Morphisms $\phi : C^\infty(Y) \to C^\infty(X)$ of C^∞-rings correspond to smooth maps $f : X \to Y$ of manifolds with $\phi(c) = c \circ f$ for any $c \in C^\infty(Y)$. Then $Spec\phi = (f, f^\#)$.

(2) For $\mathbb{R}$ as a C^∞-ring, we have $Spec\mathbb{R} = \underline{*}$.

Example 3.10 *Let $\phi : \mathfrak{C} \to \mathfrak{D}$ be a morphism of finitely generated C^∞-rings with $\mathfrak{C} = C^\infty(\mathbb{R}^n)/I$ and $\mathfrak{D} = C^\infty(\mathbb{R}^m)/J$.*

Take a smooth mapping $f = (f_1, \dots, f_n) : \mathbb{R}^m \to \mathbb{R}^n$ which satisfy $\phi(c + I) = c \circ f + J$ for all $c \in C^\infty(\mathbb{R}^n)$. Define $\underline{f_\phi} = (f_\phi, f_\phi^\#) = Spec\phi : Spec\mathfrak{D} \to Spec\mathfrak{C}$.

For all $x \in Z(J) \subset \mathbb{R}^m$, define $f_\phi(x) = x \circ \phi$. And for $y_i + I \in C^\infty(\mathbb{R}^n)(i = 1, \dots, n)$, $x \circ \phi(y_i + I) = x(f_i + J) = f_i(x)$. Therefore $f_\phi \equiv f|_{Z(J)}$.

For each $x \in Z(J)$, we have local C^∞-rings $\mathfrak{C}_{f(x)} = C^\infty_{f(x)}(\mathbb{R}^n)/I \cdot C^\infty_{f(x)}(\mathbb{R}^n)$, $\mathfrak{D}_x (= C^\infty_x(\mathbb{R}^m)/J \cdot C^\infty_x(\mathbb{R}^m)$ and a homomorphism

$$\phi_x : \mathfrak{C}_{f(x)} \to \mathfrak{D}_x$$

is $\phi_x\big([g, \mathbb{R}^n]_{f(x)} + I \cdot C^\infty_{f(x)}(\mathbb{R}^n)\big) = [g \circ f, \mathbb{R}^m]_x + J \cdot C^\infty_x(\mathbb{R}^m))$.

The morphism $f^\#(U) : \mathcal{O}_{X_\mathfrak{C}}(U) \to \mathcal{O}_{X_\mathfrak{D}}(f^{-1}(U))$ is

$$\begin{aligned}\big(f^\#(U)s\big)(v) &= \phi_x(s(f(v))) \\ &= \phi_x([g_u, \mathbb{R}^n]_{f(v)} + I \cdot C^\infty_{f(v)}(\mathbb{R}^n)) \\ &= [g_{f(v)} \circ f, \mathbb{R}^m]_v + I \cdot C^\infty_v(\mathbb{R}^m).\end{aligned} \tag{35}$$

for any $s \in \mathcal{O}_{X_\mathfrak{C}}(U)(s(u) = [g_u(x), \mathbb{R}^n]_u + I \cdot C^\infty_u(\mathbb{R}^n) \in C^\infty_u(\mathbb{R}^n)/I \cdot C^\infty_u(\mathbb{R}^n))$,

3.3 Finite limits of C^∞-ringed spaces

As for the existence of finite colimits of C^∞-rings and finite limits of topological spaces, we have a following property.

Proposition 3.11 ([6]) *All finite limits exist in the category* $\mathbf{C^\infty RS}$.

Proof) We can form finite limits by forming fibre products. Then we have only to prove that we can form fibre products of $\mathbf{C^\infty RS}$

Suppose that $\underline{X}, \underline{Y}, \underline{Z}$ are C^∞-ringed spaces and $\underline{f} : \underline{X} \to \underline{Z}$, $\underline{g} : \underline{Y} \to \underline{Z}$ morphisms of C^∞-ringed spaces.

[**Topological spaces**] For continuous mappings $f : X \to Z$, $g : Y \to Z$ of topological spaces, we can give a fibre product of topological spaces $X \overset{\pi_X}{\leftarrow} W \overset{\pi_Y}{\to} Y$ with $f \circ \pi_X = g \circ \pi_Y$ as follows.

$$\begin{aligned}W &:= \{(x, y) \in X \times Y | f(x) = g(y)\}, \\ &\pi_X(x, y) := x, \ \pi_Y(x, y) := y.\end{aligned}$$

[**Sheaves of C^∞-rings**] For morphisms $f^\# : f^{-1}(\mathcal{O}_Z) \to \mathcal{O}_X$, $g^\# : g^{-1}(\mathcal{O}_Z) \to \mathcal{O}_Y$ of sheaves of C^∞-rings, we can give morphisms of sheaves on W as

$$\pi_X^{-1}(f^{\#}) \circ I_{\pi_X,f}(\mathcal{O}_Z) : (f \circ \pi_X)^{-1}(\mathcal{O}_Z) \to \pi_X^{-1}(\mathcal{O}_X),$$
$$\pi_Y^{-1}(g^{\#}) \circ I_{\pi_Y,g}(\mathcal{O}_Z) : (g \circ \pi_Y)^{-1}(\mathcal{O}_Z) \to \pi_Y^{-1}(\mathcal{O}_Y).$$

Then as for colimits of C^∞-rings, we can define a colimit of $\pi_X^{-1}(\mathcal{O}_X) \overset{\pi_X^{\#}}{\to} \mathcal{O}_W \overset{\pi_X^{\#}}{\leftarrow} \pi_X^{-1}(\mathcal{O}_Y)$ as

$$\begin{gathered} \mathcal{O}_W := \pi_X^{-1}(\mathcal{O}_X) \coprod \pi_Y^{-1}(\mathcal{O}_Y), \\ \pi_X^{\#} : \pi_X^{-1}(\mathcal{O}_X) \to \mathcal{O}_W,\ \pi_Y^{\#} : \pi_Y^{-1}(\mathcal{O}_Y) \to \mathcal{O}_W \end{gathered} \tag{36}$$

with $\pi_X^{\#} \circ \big(\pi_X^{-1}(f^{\#}) \circ I_{\pi_X,f}(\mathcal{O}_Z)\big) = \pi_Y^{\#} \circ \big(\pi_Y^{-1}(g^{\#}) \circ I_{\pi_Y,g}(\mathcal{O}_Z)\big)$.

From the above construction, we have $\underline{f} \circ \underline{\pi_X} = \underline{g} \circ \underline{\pi_Y}$.

Suppose that there exists a C^∞-ringed space $\underline{W'}$ and morphisms of $\underline{\pi'_X} : \underline{W'} \to \underline{X},\ \underline{\pi'_Y} : \underline{W'} \to \underline{Y}$ C^∞-ringed spaces with $\underline{f} \circ \underline{\pi'_X} = \underline{g} \circ \underline{\pi'_Y}$.

- For continuous mappings $h = (\pi'_X, \pi'_Y) : W' \to W$ of topological spaces, $\pi_X \circ h = \pi'_X,\ \pi_Y \circ h = \pi'_Y$ is satisfied.
- For morphisms $(\pi'_X)^{\#} : \pi'_X(\mathcal{O}_X) \to \mathcal{O}_{W'},\ (\pi'_Y)^{\#} : \pi'_Y(\mathcal{O}_Y) \to \mathcal{O}_{W'}$ and $h^{-1}(\mathcal{O}_W) = h^{-1}\big(\pi_X^{-1}(\mathcal{O}_X)\big) \coprod h^{-1}\big(\pi_Y^{-1}(\mathcal{O}_Y)\big)$, we can define $h^{\#} : h^{-1}(\mathcal{O}_W) \to \mathcal{O}_{W'}$ as

$$h^{\#} := \big((\pi'_X)^{\#} \circ I_{h,\pi_X}(\mathcal{O}_X)\big) \coprod \big((\pi'_Y)^{\#} \circ I_{h,\pi_Y}(\mathcal{O}_Y)\big). \tag{37}$$

Thus we had defined $\underline{W}$ which satisfies the universality. ■

Then we have the following property about finite limits of C^∞-ringed spaces.

Proposition 3.12 ([6]) *The full subcategory* $\mathbf{LC^\infty RS}$ *of local C^∞-ringed spaces in* $\mathbf{C^\infty RS}$ *is closed under finite limits in* $\mathbf{C^\infty RS}$.

3.4 Definition of C^∞-schemes

We define a C^∞-scheme (resp. an affine C^∞-scheme) which is locally(resp. globally)represented by an image of Spec from C^∞-rings. And classify C^∞-schemes as to C^∞-rings.

Definition 3.13 ([6]) *Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-ringed space.*

(1) *A C^∞-ringed space $\underline{X}$ is called* **affine C^∞-scheme** *if it is isomorphic to* $Spec\mathfrak{C}$ *for some C^∞-ring $\mathfrak{C}$. And the C^∞-ringed space $\underline{X}$ is called* **finitely presented***(resp.* **good***,* **fair***) if the C^∞-ring $\mathfrak{C}$ is finitely generated(resp. good, fair).*

Write $\boldsymbol{C^\infty Sch}$, $\boldsymbol{C^\infty Sch^{fp}}$, $\boldsymbol{C^\infty Sch^{go}}$, $\boldsymbol{C^\infty Sch^{fa}}$ *for the full subcategory of all, finitely presented, good, fair, C^∞-schemes, respectively.*

(2) *If there exists an open cover $\{\underline{U_\lambda}\}_{\lambda\in\Lambda}$ such that $\underline{U_\lambda}$ is an affine C^∞-scheme for each λ, we call $\underline{X}$* ***C^∞-scheme****. Furthermore, if there exists an open cover $\{\underline{U_\lambda}\}_{\lambda\in\Lambda}$ such that $\underline{U_\lambda}$ is an* **affine finitely presented***(resp.* **good***,* **fair***)* **C^∞-scheme** *for each λ, we call $\underline{X}$* **locally finitely presented** *(***locally good***,* **locally fair***)* **C^∞-scheme***.*

Write $\boldsymbol{C^\infty Sch}$, $\boldsymbol{C^\infty Sch^{fp}}$, $\boldsymbol{C^\infty Sch^{go}}$, $\boldsymbol{C^\infty Sch^{fa}}$ *for the full subcategory of all, locally finitely presented, locally good, locally fair, C^∞-schemes, respectively.*

(3) *We call a C^∞-ringed space $\underline{X}$* **separated***(resp.* **second countable***,* **compact***,* **paracompact***) if the underlying topological space X is Hausdorff(resp. second countable, compact, paracompact).*

(4) *Define a functor* $\Gamma : \boldsymbol{LC^\infty RS} \to \boldsymbol{C^\infty Rings}^{op}$ *as:*

$$\begin{aligned}\Gamma\big((X,\mathcal{O}_X)\big) &:= \mathcal{O}_X(X)\\ &\quad \textit{for any local } C^\infty\textit{-ringed space } (X,\mathcal{O}_X),\\ \Gamma\big((f,f^\#)\big) &:= f^\#(Y) : \mathcal{O}_Y(Y) \to f^*(\mathcal{O}_X)(Y) = \mathcal{O}_X(X)\\ &\quad \textit{for any homomorphism } (f,f^\#) : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y).\end{aligned}$$

(5) *For all C^∞-ring $\mathfrak{C}$ and local C^∞-ringed space $\underline{X}$, there are a functorial isomorphism* $Hom_{\boldsymbol{C^\infty Rings}}(\mathfrak{C},\Gamma(\underline{X})) \cong Hom_{\boldsymbol{LC^\infty RS}}(\underline{X}, Spec\mathfrak{C})$.

From Theorem 4.12 and Theorem 4.26 of [6], the following theorems are proved.

Theorem 3.14 ([6]) *The full subcategories* $\mathbf{AC^\infty Sch^{fp}}$, $\mathbf{AC^\infty Sch^{go}}$, $\mathbf{AC^\infty Sch^{fa}}$ *and* $\mathbf{AC^\infty Sch}$ *are closed under all finite limits and fibre product in* $\mathbf{L}C^\infty\mathbf{RS}$.

Theorem 3.15 ([6]) *The full subcategories* $C^\infty\mathbf{Sch^{fp}}$, $C^\infty\mathbf{Sch^{go}}$, $C^\infty\mathbf{Sch^{fa}}$ *and* $C^\infty\mathbf{Sch}$ *are closed under all finite limits and fibre product in* $\mathbf{L}C^\infty\mathbf{RS}$.

3.5 MSpectra

Suppose that $\mathfrak{C}$ is a C^∞-ring and $(X, \mathcal{O}_X) := \mathbf{Spec}\mathfrak{C}$ is an affine C^∞-scheme. For any $c \in \mathfrak{C}$, define $\Phi_{\mathfrak{C}}(c) : X \to \coprod_{x\in X} \mathfrak{C}_x$ as $\big(\Phi_{\mathfrak{C}}(c)\big)(y) := c_y = \pi_y(c) \in \mathfrak{C}_y$ for any $y \in X$. $\Phi_{\mathfrak{C}}(c)$ is an element of $\mathcal{O}_X(X)$. Define $\Phi_{\mathfrak{C}} : \mathfrak{C} \to \Gamma(\mathbf{Spec}\mathfrak{C}) = \mathcal{O}_X(X)$.

Definition 3.16 ([6]) *Suppose that $\mathfrak{C}$ is a C^∞-ring and $(X, \mathcal{O}_X) := \boldsymbol{Spec}\mathfrak{C}$ is an affine C^∞-scheme.*

(1) Let $\mathfrak{M}$ be a $\mathfrak{C}$-module.

- *We have a $\mathfrak{C}$-module $\mathcal{O}_X(U)$ by a homomorphism $\rho_{XU} \circ \Phi_{\mathfrak{C}} : \mathfrak{C} \to \mathcal{O}_X(U)$ for each $U \subset X$, thus we have a tensor $\mathfrak{M} \otimes_{\mathfrak{C}} \mathcal{O}_X(U)$.*
- *For each open sets $V \subset U \subset X$, define a restriction as*

$$id_{\mathfrak{M}} \otimes \rho_{UV} : \mathfrak{M} \otimes_{\mathfrak{C}} \mathcal{O}_X(U) \to \mathfrak{M} \otimes_{\mathfrak{C}} \mathcal{O}_X(V).$$

- *We can give a presheaf $\mathfrak{M} \otimes_{\mathfrak{C}} \mathcal{O}_X$ by $\mathfrak{M} \otimes_{\mathfrak{C}} \mathcal{O}_X(U), id_M \otimes \rho_{UV}$.*

Then define MSpec$\mathfrak{M}$ as a sheafification of the presheaf $\mathfrak{M} \otimes_{\mathfrak{C}} \mathcal{O}_X$.

(2) For a morphism $\alpha : \mathfrak{M} \to \mathfrak{M}'$ of $\mathfrak{C}$-module, define a morphism MSpec α : MSpec $\mathfrak{M} \to$ MSpec $\mathfrak{M}'$ of sheaves as a following pullback

$$\alpha \otimes_{\mathfrak{C}} id_{\mathcal{O}_X} : \mathfrak{M} \otimes_{\mathfrak{C}} \mathcal{O}_X \to \mathfrak{M}' \otimes_{\mathfrak{C}} \mathcal{O}_X.$$

(3) From the above, define a following functor of categories

$$MSpec : \mathfrak{C}\textbf{-mod} \to \mathcal{O}_X\textbf{-mod}.$$

3.6 Locally finite sums on C^∞-ringed spaces

For a C^∞-ring $\mathfrak{C}$, we define locally finite sums and a topology of $\mathfrak{C}$ by a topological space of Spec $\mathfrak{C}$. Its idea is a generalization of an idea about topology of function space.

Definition 3.17 ([6]) *Let $\mathfrak{C}$ be a C^∞-ring and $(X, \mathcal{O}_X) :=$ Spec$\mathfrak{C}$ an affine C^∞-scheme.*

(1) Consider a formal expression $\sum_{a\in A} c_a (\forall a \in A, c_a \in \mathfrak{C})$. We say that $\sum_{a\in A} c_a$ is a **locally finite sum** *if X can be covered by open sets $U \subset X$ such that*

$$\#\{a \in A | \pi_x(c_a) \neq 0_{\mathfrak{C}_x}\} < \infty \text{ for any } x \in U.$$

(2) *For a locally finite sum $\sum_{a\in A} c_a$, we say that $c \in \mathfrak{C}$ is a* **limit** *of $\sum_{a\in A} c_a$ written $c = \sum_{a\in A} c_a$ if c satisfies*

$$\pi_x(c) = \sum_{a\in A} \pi_x(c_a) \text{ for any } x \in X.$$

(3) *Suppose the topological space X is locally compact. (This is automatic if $\mathfrak{C}$ is finitely generated.)*

For each element $c \in \mathfrak{C}$ and compact set $S \subset X$, define an open neighborhood $\mathcal{U}_{c,S}$ of $c \in \mathfrak{C}$ as

$$\mathcal{U}_{c,S} := \left\{ c' \in \mathfrak{C} \middle| \pi_x(c') = \pi_x(c) \text{ for all } x \in S \right\}.$$

Then we have the topology of $\mathfrak{C}$ with basis $\mathcal{U}_{c,S}$ for all c and S.

As for locally finite sums and limits of C^∞-rings, we can say fair C^∞-rings in a following property.

Proposition 3.18 ([6]) *An ideal $I \subset C^\infty(\mathbb{R}^n)$ is fair if and only if I is closed under locally finite sums.*

Proof) Suppose that I is fair. Take a locally finite sum $\sum_{a\in A} c_a (\forall a \in A, c_a \in I)$ of I. For each point $x \in \mathbb{R}^n$, there exists an open neighborhood $U_x \subset \mathbb{R}^n$ and $c \in C^\infty(\mathbb{R}^n)$ such that for all but finitely many $a \in A$ we have $\pi_y(c_a) = 0$ for all $y \in U_s$, and it satisfies $c = \sum_{a\in A} c_a$.

For each $x \in \mathbb{R}^n$, $\pi_x(I)$ is an ideal and $\sum_{a\in A} \pi_x(c_a)$ is a finite sum. Then $\pi_x(c) \in \pi_x(I)$. Since I is fair, c is in I.

Suppose I is closed under locally finite sums. Take $c \in C^\infty(\mathbb{R}^n)$ and suppose that there exists an element $c_x \in I$ with $\pi_x(c) = \pi_x(c_x)$ for each point $x \in \mathbb{R}^n$. We have an open neighborhood U_x of x with $c|_{U_x} = c_x|_{U_x}$ by the definition of mapping germs. As $\{U_x\}_{x\in\mathbb{R}^n}$ is an open cover of $\mathbb{R}^n$, there exists an open cover $\{U_b\}_{b\in B}$ of $\mathbb{R}^n$ such that

- the open cover $\{U_b\}_{b\in B}$ is locally finite,
- there exists a partition of unity $\{\eta_b\}_{b\in B} \subset C^\infty(\mathbb{R}^n)$ subordinate to $\{U_b\}_{b\in B}$, and
- there exists $x_b \in \mathbb{R}^n$ with $U_b \subset U_{x_b}$ for each element $b \in B$.

As for the supposition of locally finite sums, we have $\sum_{b\in B} \eta_b c_{x_b} = c \in I$. ■

As well as manifolds, we can define supports and partitions of unity of C^∞-schemes.

Definition 3.19 ([6]) *Suppose $\underline{X} = (X, \mathcal{O}_X)$ is a C^∞-scheme.*

(1) We call a formal expression $\sum_{a\in A} c_a (A$: an index set, $c_a \in \mathcal{O}_X(X))$ a **locally finite sum** *if there exists an open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X such that*

$$\#\{a \in A | \rho_{XU_\lambda}(c_a) \neq 0\} < \infty \text{ for all } \lambda \in \Lambda.$$

Then $\sum_{a\in A} \pi_x(c_a)$ is a finite sum for each $x \in X$ and it become meaningful.

Furthermore for a locally finite sum $\sum_{a\in A} c_a$, if there exists $c \in \mathcal{O}_X(X)$ such that $\pi_x(c) := \sum_{a\in A} \pi_x(c_a)$ for each $x \in X$, we call c a **limit** *of $\sum_{a\in A} c_a$.*

(2) Take $c \in \mathcal{O}_X(X)$. As $\mathcal{O}_X$ is a sheaf, if there exists open sets $V_i \subset X(i \in I)$ such that $\rho_{XV_i}(c) = 0_{\mathcal{O}_X(V_i)}$, $\rho_{XV}(c) = 0_{\mathcal{O}_X(V)}$ for $V = \cup V_i$.

Define the **support** *suppc of c as*

$$suppc := X \backslash \sup\big\{V \subset X \big| V \text{ is open}, \rho_{XV}(c) = 0\big\}. \tag{38}$$

(3) For an open cover $\{U_a\}_{a\in A}$ of X and each $a \in A$, suppose that we have $\eta_a \in \mathcal{O}_X(X)$ with $supp\eta_a \subset U_a$.

$\{\eta_a\}_{a\in A}$ is a **partition of unity subordinate to** *$\{U_a\}_{a\in A}$ if the following conditions are satisfied.*

(a) $\sum_{a\in A} \eta_a$ is a locally finite sum on $\underline{X}$,

(b) and $\sum_{a\in A} \eta_a = 1$.

As well as the existence of partition of unity for any open covers of manifolds, there exists partitions of unity for any open covers of C^∞-schemes.

Proposition 3.20 ([6]) *Suppose that a C^∞-scheme $\underline{X}$ is separated, paracompact and locally fair.*

For an open cover $\{\underline{U_a}\}_{a\in A}$ of $\underline{X}$, there exists a partition of unity $\{\eta_a\}_{a\in A}$ $\big(\forall a \in A,\ \eta_a \in \mathcal{O}_X(X)\big)$ subordinate to the open cover $\{\underline{U_a}\}_{a\in A}$.

3.7 Derivations on $\mathcal{O}_X$-modules

Definition 3.21 *Let $\mathfrak{C}$ be a C^∞-ring, $\mathfrak{M}$ a $\mathfrak{C}$-module. Let $\underline{X} = (X, \mathcal{O}_X) :=$ Spec$\mathfrak{C}$ and $\mathcal{E}_X :=$ MSpec$\mathfrak{M}$. Take an $\mathbb{R}$-derivation(resp. a C^∞-derivation) $d : \mathfrak{C} \to \mathfrak{M}$.*

From Lemma 2.44, we can localize $d : \mathfrak{C} \to \mathfrak{M}$ *as* $d_x := d[S_x^{-1}] : \mathfrak{C}_x \to \mathfrak{M} \otimes_{\mathfrak{C}} \mathfrak{C}_x$ *for any* $x : \mathfrak{C} \to \mathbb{R}$ *and a subset* $S_x := x^{-1}(\mathbb{R} \backslash \{0\})$*. For an open subset* $U \subset X$*, define an* $\mathbb{R}$*-derivation(resp. a* C^∞*-derivation)*

$$d(U) : \mathcal{O}_X(U) \to MSpec\mathfrak{M}(U) \tag{39}$$

as $d(U)(c) : U \to \coprod_{x \in U} \mathfrak{M} \otimes_{\mathfrak{C}} \mathcal{O}_{X,x}$ *with* $\big(d(U)(c)\big)(x) := d_x(c(x)) \in \mathfrak{M} \otimes_{\mathfrak{C}} \mathfrak{C}_x$ *for any* $c \in \mathcal{O}_X(U)$*.*

Define $MSpecd : \mathcal{O}_X \to \mathcal{E}_X$ *as* $MSpecd(U) := d(U) : \mathcal{O}_X(U) \to \mathcal{E}_X(U)$ *for any open set* $U \subset X$*.*

4 Derivations, differentials, tangent vectors, and tangent vector fields

4.1 Derivation of a k-jet determined C^∞-ring

Suppose that $\mathfrak{C}$, $\mathfrak{D}$ are C^∞-rings and $\phi : \mathfrak{C} \to \mathfrak{D}$ a homomorphism of C^∞-rings. The aim is to show that any $\mathbb{R}$-derivation $V : \mathfrak{C} \to \mathfrak{D}$ along ϕ becomes a C^∞-derivation under certain conditions of $\mathfrak{C}$, $\mathfrak{D}$ and ϕ.

Theorem 4.1 ([15]) *Let $\mathfrak{C}, \mathfrak{D}$ be C^∞-rings, $\phi : \mathfrak{C} \to \mathfrak{D}$ a homomorphism of C^∞-rings and $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. Suppose that $\mathfrak{D}$ is k-jet determined.*
Then any $\mathbb{R}$-derivation $V : \mathfrak{C} \to \mathfrak{D}$ over ϕ is a C^∞-derivation.

We have to prove that

$$V(\Phi_f(c_1, \dots, c_n)) = \sum_{i=1}^{n} \phi(\Phi_{\frac{\partial f}{\partial x_i}}(c_1, \dots, c_n))V(c_i) \tag{40}$$

for any $f \in C^\infty(\mathbb{R}^n)$ and $c_1, \dots, c_n \in \mathfrak{C}$.

For a smooth function $f \in C^\infty(\mathbb{R}^n)$, we write $f(x)$, $f(r)$ and $\Phi_f(c)$ as $f(x_1, \dots, x_n)$, $f(r_1, \dots, r_n)$ and $\Phi_f(c_1, \dots, c_n)$ for the coordinate $(x_1, \dots, x_n)$ of $\mathbb{R}^n$, $r := (r_1, \dots, r_n) \in \mathbb{R}^n$ and $c := (c_1, \dots, c_n)$ by $c_1, \dots, c_n \in \mathfrak{C}$.

For $\alpha = (\alpha_1, \dots, \alpha_n) \in (\{0\} \cup \mathbb{N})^n$, we define $e_i := (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$, $|\alpha| := \sum_{i=1}^{n} \alpha_i \in \{0\} \cup \mathbb{N}$, $\alpha! := \prod_{i=1}^{n} \alpha_i!$, $c^\alpha := \prod_{i=1}^{n} c_i^{\alpha_i}$ for $c_1, \dots, c_n \in \mathfrak{C}$ and $\frac{\partial^\alpha f}{\partial x^\alpha} := \frac{\partial^{|\alpha|} f}{\partial {x_1}^{\alpha_1} \cdots \partial {x_n}^{\alpha_n}}$.

For the proof of Theorem 4.1, we will prepare Lemma 4.2.

Lemma 4.2 ([15]) *Let $\mathfrak{C}, \mathfrak{D}$ be C^∞-rings, $\phi : \mathfrak{C} \to \mathfrak{D}$ a homomorphism of C^∞-rings. Suppose that $\mathfrak{D}$ is a local C^∞-ring which is a weakly nilpotent local $\mathbb{R}$-algebra with a unique maximal ideal $m \subset \mathfrak{D}$.*
Then any $\mathbb{R}$-derivation $V : \mathfrak{C} \to \mathfrak{D}$ along ϕ is a C^∞-derivation.

Proof) Suppose $c_1, \dots, c_n \in \mathfrak{C}$ and $f \in C^\infty(\mathbb{R}^n)$. From the locality of $\mathfrak{D}$, take a real number $p_i \in \mathbb{R}$ and a natural number $k_i \in \mathbb{N}$ such that $\phi(c_i) - p_i = \phi(c_i - p_i) \in m$ and $(\phi(c_i) - p_i)^{k_i} = 0$ for each $i = 1, \dots, n$. Let $k \in \mathbb{N}$ be the sum of k_i. Then

$$\phi\big((c - p)^\alpha\big) = 0 \text{ for any } |\alpha| \geq k + 1. \tag{41}$$

Using Hadamard's lemma by induction, there exists smooth functions $G_{\alpha,p}, G^i_{\beta,p} \in C^\infty(\mathbb{R}^n)(|\alpha| = k+2, |\beta| = k+1, i = 1, \dots, n)$ such that

$$f(x) = f(p) + \sum_{1 \le |\alpha| \le k+1} \frac{(x-p)^\alpha}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(p) + \sum_{|\alpha|=k+2} (x-p)^\alpha G_{\alpha,p}(x), \quad (42)$$

$$\frac{\partial f}{\partial x_i}(x) = \sum_{0 \le |\beta| \le k} \frac{(x-p)^\beta}{\beta!} \frac{\partial^{\beta+e_i} f}{\partial x^{\beta+e_i}}(p) + \sum_{|\beta|=k+1} (x-p)^\beta G^i_{\beta,p}(x). \quad (43)$$

For each $i = 1, \dots, n$, we can describe $\phi\big(\Phi_{\frac{\partial f}{\partial x_i}}(c)\big)$ as followings by the equations (41) and (43).

$$\begin{aligned}
\phi\big(\Phi_{\frac{\partial f}{\partial x_i}}(c)\big) =&\phi\Big(\sum_{0 \le |\beta| \le k} \frac{(c-p)^\beta}{\beta!} \frac{\partial^{\beta+e_i} f}{\partial x^{\beta+e_i}}(p) + \sum_{|\beta|=k+1} (c-p)^\beta \Phi_{G^i_{\beta,p}}(c)\Big) \\
=&\phi\Big(\sum_{0 \le |\beta| \le k} (\beta+e_i)_i \frac{(c-p)^\beta}{(\beta+e_i)!} \frac{\partial^{\beta+e_i} f}{\partial x^{\beta+e_i}}(p)\Big) \\
=&\phi\Big(\sum_{1 \le |\alpha| \le k+1} \alpha_i \frac{(c-p)^{\alpha-e_i}}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(p)\Big)
\end{aligned} \quad (44)$$

($\{\beta \in (\{0\} \cup \mathbb{N})^n | 0 \le |\beta| \le k\}$ and $\{\alpha \in (\{0\} \cup \mathbb{N})^n | 1 \le |\alpha| \le k+1, \alpha_i \ge 1\}$ are one-to-one relation with $\alpha \leftrightarrow \alpha + e_i$).
Then we can describe $V(\Phi_f(c))$ as followings by the equations (41) and (44).

$$\begin{aligned}
V(\Phi_f(c)) =& \sum_{1 \le |\alpha| \le k+1} \sum_{i=1}^n \frac{\partial^\alpha f}{\partial x^\alpha}(p) \phi\Big(\frac{\alpha_i (c-p)^{\alpha-e_i}}{\alpha!}\Big) V(c_i) \\
&+ \sum_{|\alpha|=k+2} \big\{V((c-p)^\alpha)\phi(\Phi_{G_{\alpha,p}}(c)) + \phi\left((c-p)^\alpha\right) V(\Phi_{G_{\alpha,p}}(c))\big\} \\
=& \sum_{i=1}^n \phi\Big(\sum_{1 \le |\alpha| \le k+1} \alpha_i \frac{(c-p)^{\alpha-e_i}}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(p)\Big) V(c_i) = \sum_{i=1}^n \phi\big(\Phi_{\frac{\partial f}{\partial x_i}}(c)\big) V(c_i).
\end{aligned}$$

Therefore, we have the equation (40). ■

We can prove Theorem 4.1 in the same way as the proof of Lemma 4.2.

Proof of Theorem 4.1) Suppose $c_1, \dots, c_n \in \mathfrak{C}$ and $f \in C^\infty(\mathbb{R}^n)$.
Take any $\mathbb{R}$-point $q : \mathfrak{D} \to \mathbb{R}$ and $l \in \{0\} \cup \mathbb{N}$. $\mathfrak{D}_q/{m_q}^{l+1}$ is a local C^∞-ring with a unique maximal ideal $m_q/{m_q}^{l+1}$. $\mathfrak{D}_q/{m_q}^{l+1}$ satisfies that

$(\mathfrak{D}_q/{m_q}^{l+1})/(m_q/{m_q}^{l+1})$ is isomorphic to $\mathfrak{D}_q/m_q$, or to $\mathbb{R}$ and $(m_q/{m_q}^{l+1})^{l+1} = 0$. Therefore, $\mathfrak{D}_q/{m_q}^{l+1}$ is a weakly nilpotent local $\mathbb{R}$-algebra. For $j^l_q : \mathfrak{D} \to \mathfrak{D}_q/{m_q}^{l+1}$, $j^l_q \circ V : \mathfrak{C} \to \mathfrak{D}_q/{m_q}^{l+1}$ is an $\mathbb{R}$-derivation along $j^l_q \circ \phi : \mathfrak{C} \to \mathfrak{D}_q/{m_q}^{l+1}$. From Lemma 4.2, $j^l_q \circ V$ is the C^∞-derivation. Therefore, we have

$$j^l_q\Big(\sum_{i=1}^n \phi\big(\Phi_{\frac{\partial f}{\partial x_i}}(c_1, \dots, c_n)\big)V(c_i)\Big) = j^l_q\Big(V\big(\Phi_f(c_1, \dots, c_n)\big)\Big). \tag{45}$$

Suppose that $\mathfrak{D}$ is k-jet determined for $k \in \{0\} \cup \mathbb{N}$. For any $\mathbb{R}$-point $q : \mathfrak{D} \to \mathbb{R}$, we have the equation (45) for $l = k$. From the assumption of k-jet determined, we have the equation (40). Then, we have that V is a C^∞-derivation.

Suppose that $\mathfrak{D}$ is ∞-jet determined. For any $\mathbb{R}$-point $q : \mathfrak{D} \to \mathbb{R}$ and $l \in \{0\} \cup \mathbb{N}$, we have the equation (45). From the assumption of ∞-jet determined, we have the equation (40). Therefore, we have that V is C^∞-derivation. ■

4.1.1 Examples of k-jet determined C^∞-rings

We have following properties of Theorem 4.1.

Proposition 4.3 *Let $\mathfrak{C}$ be a point determined C^∞-ring of the form $C^\infty(\mathbb{R}^n)/I$.*

For any $\mathbb{R}$-derivation $V : \mathfrak{C} \to \mathfrak{C}$, there exists smooth functions $v_1, \dots, v_n \in C^\infty(\mathbb{R}^n)$ such that

$$V(f + I) = \Big(\sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}\Big) + I \text{ for any } f \in C^\infty(\mathbb{R}^n).$$

Remark 4.4 *Let N, M be C^∞-manifolds, and $f : N \to M$ a C^∞-mapping.*

Vector fields $V : N \to f^*(TM)$ can be regarded as C^∞-derivations $d : C^\infty(M) \to C^\infty(N)$ as followings.

For $g \in C^\infty(M)$, define a function $d(g) : N \to \mathbb{R}$ as $\big(d(g)\big)(p) := V_p g$ for $p \in N$ and $V_p \in T_{f(p)}M$. $d(g)$ is an element of $C^\infty(N)$.

Take coordinates $(U_p, \phi = (x_1, \dots, x_n))$, $(U_{f(p)}, \psi = (y_1, \dots, y_m))$ ($\phi : N \supset U_p \tilde{\to} V_p \subset \mathbb{R}^n$, $\psi : M \supset U_{f(p)} \tilde{\to} V_{f(p)} \subset \mathbb{R}^m$). Then we can write $V(q) = \sum_{i=1}^m v_{i,p}(q)(\frac{\partial}{\partial y_i})_{f(q)}$ by $v_{1,p}, \dots, v_{m,p} \in C^\infty(U_p)$ for any $q \in U_p$. Then we have

$$\big(d(g)\big)(q) = \sum_{i=1}^m v_{i,p}(q) \frac{\partial g}{\partial y_i}(f(q)) \text{ for any } q \in U_p.$$

Therefore, $d(g) \in C^\infty(N)$.

Take smooth functions $g_1, \dots, g_l \in C^\infty(M)$ and $h = C^\infty(\mathbb{R}^l)$. For any $q \in U_p$,

$$\begin{aligned}
d\big(\Phi_h(g_1, \dots, g_l)\big)(q) &= d\big(h \circ (g_1, \dots, g_l)\big)(q) \\
&= \sum_{i=1}^{m} v_{i,p}(q) \frac{\partial\big(h \circ (g_1, \dots, g_l)\big)}{\partial y_i}(f(q)) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{l} v_{i,p}(q) \frac{\partial h}{\partial z_j}\Big(g_1\big(f(q)\big), \dots, g_l\big(f(q)\big)\Big) \frac{\partial g_j}{\partial y_i}(f(q)) \\
&= \sum_{j=1}^{l} \frac{\partial h}{\partial z_j}\Big(g_1\big(f(q)\big), \dots, g_l\big(f(q)\big)\Big) \Big\{ \sum_{i=1}^{m} v_{i,p}(q) \Big(\frac{\partial g_j}{\partial y_i}(f(q)) \Big) \Big\} \\
&= \Big\{ \sum_{j=1}^{l} \big(\Phi_{\frac{\partial h}{\partial z_j}}(g_1, \dots, g_l) \circ f\big) d(g_j) \Big\}(q).
\end{aligned}$$

We have $d\big(\Phi_h(g_1, \dots, g_l)\big) = \sum_{j=1}^{l} f^*\big(\Phi_{\frac{\partial h}{\partial z_j}}(g_1, \dots, g_l)\big) d(g_j)$. Therefore, $d : C^\infty(M) \to C^\infty(N)$ is the C^∞-derivation over $f^* : C^\infty(M) \to C^\infty(N)$.

From D. Joyce [6] Remark 5.12., we have the following example showing that the assumption in our Theorems is essential.

Example 4.5 ([6]) *Let $\mathfrak{C}$ be a C^∞-ring $C^\infty(\mathbb{R})$ of C^∞-functions on $\mathbb{R}$. For the $\mathbb{R}$-cotangent module $(\Omega_{\mathfrak{C},\mathbb{R}}, d_{\mathfrak{C},\mathbb{R}})$, $d_{\mathfrak{C},\mathbb{R}} : \mathfrak{C} \to \Omega_{\mathfrak{C},\mathbb{R}}$ is an $\mathbb{R}$-derivation but not a C^∞-derivation.*

In fact, for the exponential $e^x \in C^\infty(\mathbb{R})$, $e^x d_{\mathfrak{C},\mathbb{R}}(x) - d_{\mathfrak{C},\mathbb{R}}(e^x) \neq 0$ in $\Omega_{\mathfrak{C},\mathbb{R}}$. Note that the assumptions of Theorem 4.1 are not satisfied in this example.

4.2 Tangent spaces of C^∞-ringed spaces

We have defined Zariski and logarithmic tangent spaces of C^∞-rings. In this section, we define Zariski or logarithmic tangent spaces of C^∞-rings by its derivations at each point.

4.2.1 Tangent vectors and tangent vector fields of C^∞-ringed spaces

Definition 4.6 *Let $\underline{X} := (X, \mathcal{O}_X)$ be a local C^∞-ringed space. Take a point p of X. Then, we define an $\mathbb{R}$-point $e_p : \mathcal{O}_{X,p} \to \mathbb{R}$.*

(1) *We call* $v : \mathcal{O}_X(X) \to \mathbb{R}$ *a* **tangent vector of** $\underline{X}$ **at** p *if* v *is an* $\mathbb{R}$*-derivation, i.e.*

$$v(fg) = v(f)e_p([g,X]_p) + e_p([f,X]_p)v(g) \quad \text{for all } f,g \in \mathcal{O}_X(X). \tag{46}$$

And define $T_p\underline{X}$ *for the set of tangent vectors of* $\underline{X}$ *at* p*.*

(2) *A* **tangent vector field** ξ *on* $\underline{X}$ *assigns an* $\mathbb{R}$*-derivation* $\xi(U) : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$*, such that* $\xi(V) \circ (\cdot|_V) = (\cdot|_V) \circ \xi(U)$ *for each inclusion of open sets* $V \subset U \subset X$*.*

For a tangent vector field ξ *on* $\underline{X}$*, define* $\xi_p : \mathcal{O}_{X,p} \to \mathcal{O}_{X,p}$ *as*

$$\xi_p\big([f,U]_p\big) := [\xi(U)(f), U]_p \text{ for any } [f,U]_p \in \mathcal{O}_{X,p}. \tag{47}$$

And we can define a tangent vector $v : \mathcal{O}_{X,p} \to \mathbb{R}$ *of* $\underline{X}$ *at* p *as*

$$v\big([f,U]_p\big) := e_p\big(\xi_p([f,U]_p)\big) \text{ for any } [f,U]_p \in \mathcal{O}_{X,p}. \tag{48}$$

We call v *a logarithmic tangent vector of* $\underline{X}$ *at* p*.*

We have define Zariski and logarithmic tangent spaces of C^∞-rings. In this section, we define Zariski or logarithmic tangent spaces of C^∞-rings by its derivations at each point.

4.2.2 C^∞-ringed spaces in the case of finitely generated C^∞-rings

For C^∞-schemes, tangent vectors and tangent vector fields are locally represented as derivations of C^∞-rings. Then, we show an example of finitely generated C^∞-schemes.

Example 4.7 *Take a* C^∞*-ring* $\mathfrak{C} := C^\infty(\mathbb{R}^n)/I$*. Define a fair ideal*

$$\overline{I} := \{f \in C^\infty(\mathbb{R}^n) | [f,\mathbb{R}^n]_u \in I \cdot C^\infty_u(\mathbb{R}^n) \text{ for all } u \in \mathbb{R}^n\}. \tag{49}$$

$\overline{I}$ *is a smallest fair ideal of* $C^\infty(\mathbb{R}^n)$ *which contains* I*.*

Define a C^∞*-scheme* $(X, \mathcal{O}_X) = Spec\mathfrak{C}$ *as the topological space* $X := \{x : \mathfrak{C} \to \mathbb{R} | x \text{ is an } \mathbb{R}\text{-point of } \mathfrak{C}.\}$*, and* $(C^\infty(\mathbb{R}^n)/I)_u = C^\infty_u(\mathbb{R}^n)/I \cdot C^\infty_u(\mathbb{R}^n) = C^\infty_u(\mathbb{R}^n)/\overline{I} \cdot C^\infty_u(\mathbb{R}^n)$ *for each* $u \in Z(I)$*.*

Proof) *Define a projective* $\pi_u : C^\infty(\mathbb{R}^n)/I \to C^\infty_u(\mathbb{R}^n)/I \cdot C^\infty_u(\mathbb{R}^n)$ *as* $\pi_u(f + I) := [f,\mathbb{R}^n]_u + I$*.*

[Localization] *For each $s+I \in C^\infty(\mathbb{R}^n)/I$ which satisfies $s(u) \neq 0$, take a smooth function $t \in C^\infty(\mathbb{R}^n)$ such that $s \cdot t \equiv 1$ on an open neighborhood of $u \in \mathbb{R}^n$. Then*

$$\begin{aligned}[s, \mathbb{R}^n]_u \cdot [t, \mathbb{R}^n]_u + I \cdot C_u^\infty(\mathbb{R}^n) &= [s \cdot t, \mathbb{R}^n]_u + I \cdot C_u^\infty(\mathbb{R}^n) \\ &= [1, \mathbb{R}^n]_u + I \cdot C_u^\infty(\mathbb{R}^n) \\ &= 1_{C_u^\infty(\mathbb{R}^n)/I \cdot C_u^\infty(\mathbb{R}^n)}.\end{aligned}$$

[Universality] *Take a projective $\pi'_u : C^\infty(\mathbb{R}^n)/I \to \mathfrak{D}$ such that $\pi'_u(s+I)$ is invertible for all $s \in C^\infty(\mathbb{R}^n)(s(u) \neq 0)$.*

For all $f \in C^\infty(\mathbb{R}^n)$ which satisfies $[f, \mathbb{R}^n] \in I \cdot C_u^\infty(\mathbb{R}^n)$, there exists $\iota \in I$ and $\eta \in C^\infty(\mathbb{R}^n)(\eta \equiv 1$ on an open neighborhood of $u)$ such that $\eta f = \eta \iota$.

$$\begin{aligned}\pi'_u(f+I) &= \big(\pi'_u(\eta+I)\big)^{-1} \pi'_u(\eta f + I) \\ &= \big(\pi'_u(\eta+I)\big)^{-1} \pi'_u(\eta \iota + I) = 0.\end{aligned}$$

Then we can define $\phi : C_u^\infty(\mathbb{R}^n)/I \cdot C_u^\infty(\mathbb{R}^n) \to \mathfrak{D}$ such that $\pi'_u = \phi \circ \pi_u$. ■

For each open set $U \subset Z(I)$, define $\mathcal{O}_X(U)$ as the set of functions $s : U \to \coprod_{u \in U} C_u^\infty(\mathbb{R}^n)/I \cdot C_u^\infty(\mathbb{R}^n)$ which satisfies

- *$s(u) \in C_u^\infty(\mathbb{R}^n)/I \cdot C_u^\infty(\mathbb{R}^n)$ and*
- *for each $u \in U$, there exists an open neighborhood $V_u \subset U$ of u and a smooth function $f_u \in C^\infty(\mathbb{R}^n)$ which satisfy $s(v) = [f_u, \mathbb{R}^n]_v + I \cdot C_u^\infty(\mathbb{R}^n)$ for all $v \in V_u$.*

4.2.3 Differentials on local C^∞-ringed spaces

Take a morphism $\underline{f} = (f, f^\#) : \underline{X} \to \underline{Y}$ of local C^∞-rings. For each $x \in X$, we will define $d\underline{f}_x = (\underline{f}_*)_x : T_x\underline{X} \to T_{f(x)}\underline{Y}$.

$f^\# : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ is a morphism of sheaves of C^∞-rings on X such that $f^\#(U) : f^{-1}(\mathcal{O}_Y)(U) \to \mathcal{O}_X(U)$ is a homomorphism of C^∞-rings for each open set $U \subset X$. And for each $x \in X$, we have a homomorphism $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ of local C^∞-rings.

Lemma 4.8 *Let $\underline{X}$ and $\underline{Y}$ be local C^∞-ringed spaces.*

Let $v : \mathcal{O}_{X,x} \to \mathbb{R}$ be a tangent vector of $\underline{X}$ at x. Define $w := v \circ f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathbb{R}$. Then w is a tangent vector of $\underline{Y}$ at $f(x)$.

Proof) [w **is** $\mathbb{R}$**-linear**] For all $c, c' \in \mathcal{O}_{Y,f(x)}$ and $r, r' \in \mathbb{R}$,

$$\begin{aligned}(v \circ f_x^{\#})(rc + r'c') &= v\big(r \cdot f_x^{\#}(c) + r' \cdot f_x^{\#}(c')\big) \\ &= r \cdot \big(v \circ f_x^{\#}\big)(c) + r' \cdot \big(v \circ f_x^{\#}\big)(c').\end{aligned}$$

We proved that w is an $\mathbb{R}$-linear mapping.

[w **is the derivation**] For all $c, c' \in \mathcal{O}_{Y,f(x)}$,

$$\begin{aligned}\big(v \circ f_x^{\#}\big)(c \cdot c') &= v\big(f_x^{\#}(c) \cdot f_x^{\#}(c')\big) \\ &= v\big(f_x^{\#}(c)\big)e_p\big([f_x^{\#}(c')]_p\big) + e_p\big([f_x^{\#}(c)]_p\big)v\big(f_x^{\#}(c')\big).\end{aligned}$$

As $\underline{X}, \underline{Y}$ are local C^∞-ringed spaces, we have $e_p(f_x^{\#}([c_p])) = e_{f(p)}([c]_{f(p)})$.

$$\big(v \circ f_x^{\#}\big)(c \cdot c') = v\big(f_x^{\#}(c)\big)e_{f(p)}\big([c']_{f(p)}\big) + e_{f(p)}\big([c]_{f(p)}\big)v\big(f_x^{\#}(c')\big).$$

Then, $w = v \circ f_y^{\#}$ is the $\mathbb{R}$-derivation. ■

Therefore, we can define an $\mathbb{R}$-linear $d\underline{f}_x (= (\underline{f}_*)_x) : T_x\underline{X} \to T_{f(x)}\underline{Y}$ as

$$d\underline{f}_x(v) := v \circ f_x^{\#} : \mathcal{O}_{Y,f(x)} \to \mathbb{R} \text{ for any } v : \mathcal{O}_{X,x} \to \mathbb{R}.$$

4.3 Differential forms on C^∞-schemes

4.3.1 Differential forms of C^∞-schemes

Definition 4.9 ([6]) *(1) For a C^∞-scheme $\underline{X}$, we define a presheaf $PT^*\underline{X}$ on $\underline{X}$ as followings.*

$$\begin{aligned}PT^*\underline{X}(U) &:= \Omega_{\mathcal{O}_X(U)} \text{ for any } U \subset X, \\ \Omega_{\rho_{UU'}} : \Omega_{\mathcal{O}_X(U)} &\to \Omega_{\mathcal{O}_X(V)} \text{ for any } U' \subset U \subset X.\end{aligned} \tag{50}$$

Then define $T^*\underline{X}$ as the sheafication of $\mathcal{O}_X$-modules.

(2) For a morphism between C^∞-schemes $\underline{f} : \underline{X} \to \underline{Y}$, we define a presheaf $\underline{f}^*(PT^*\underline{Y})$ on $\underline{X}$ as followings

$$\begin{aligned}\underline{f}^*(PT^*\underline{Y})(U) &= \lim_{V \supset f(U)} \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U) \\ &\quad \text{for any } U \subset X, \\ \lim_{V \supset f(U)} id_{\Omega_{\mathcal{O}_Y(V)}} \otimes_{\mathcal{O}_Y(V)} \rho_{UU'} &: \underline{f}^*(PT^*\underline{Y})(U) \to \underline{f}^*(PT^*\underline{Y})(U') \\ &\quad \text{for any } U' \subset U \subset X.\end{aligned} \tag{51}$$

Then define $\underline{f}^*(T^*\underline{Y})$ as the sheafication of $\mathcal{O}_X$-modules.

(3) For a morphism $\underline{f} : \underline{X} \to \underline{Y}$ of C^∞-schemes, define a morphism $P\Omega_{\underline{f}} : \underline{f}^*(PT^*\underline{Y}) \to PT^*\underline{X}$ of presheaves on X

$$\big(P\Omega_{\underline{f}}\big)(U) := \lim_{V \supset f(U)} \big(\Omega_{\rho_{f^{-1}(V)U} \circ f^{\#}(V)}\big)_* \text{ for any } U \subset X. \tag{52}$$

Then define $\Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \to T^*\underline{X}$ as the sheafication of $\mathcal{O}_X$-modules.

4.3.2 Dual spaces of tangent spaces

We want to define a bilinear mapping $T^*\underline{X} \times T_x\underline{X} \to \mathbb{R}$.

Proposition 4.10 *Let $\underline{X} = (X, \mathcal{O}_X)$ be a locally C^∞-ringed space. Take $x \in X$, an open set $U \subset X$ and $v \in T_x\underline{X}$.*

Tangent vectors at $x \in X$ are C^∞-derivations, i.e. for any tangent vector $v : \mathcal{O}_{X,x} \to \mathbb{R}$, we have

$$v\big(\Phi_f(c_1, \ldots, c_n)\big) = \sum_{i=1}^{n} e_x\big([\Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n), U]_x\big) v(c_i) \tag{53}$$

for any $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathcal{O}_X(U)$ and smooth function $f : \mathbb{R}^n \to \mathbb{R}$.

Proof) This proposition satisfies the condition of Lemma 4.2.
Then we have Proposition 4.10. ■

4.3.3 Dual spaces of tangent vector spaces

Next, we want to define a bilinear mapping $T^*\underline{X} \times \chi_{\underline{X}} \to \mathcal{O}_X$.

Suppose that $\underline{X}$ is a separated, paracompact and locally fair C^∞-scheme and $V : \mathcal{O}_X \to \mathcal{O}_X$ is a derivation on $\underline{X}$. Take a point $x \in \underline{X}$.Then there exists a fair C^∞-ring $\mathfrak{C}_x = C^\infty(\mathbb{R}^{n_x})/I_x$ and an open neighborhood $\underline{U_x} \subset \underline{X}$ of x with an isomorphism $\underline{\phi_x} : \underline{U_x} \tilde{\to} \mathrm{Spec} C^\infty(\mathbb{R}^{n_x})/I_x =: \underline{V_x}$.

Consider

$$V_x : \mathcal{O}_{V_x} \to \mathcal{O}_{U_x} \xrightarrow{V} \mathcal{O}_{U_x} \to \mathcal{O}_{V_x}. \tag{54}$$

For each $y' \in V_x \subset \mathbb{R}^{n_x}(y' = \phi_x(x'))$, $\mathcal{O}_{V_x,y'} = C^\infty_{y'}(\mathbb{R}^{n_x})/I_x C^\infty_{y'}(\mathbb{R}^{n_x})$.

4.3.4 The partition of unity in a separated, paracompact and locally fair C^∞-scheme

Proposition 4.11 *Let $\underline{X} := (X, \mathcal{O}_X)$ be a separated, paracompact and locally fair C^∞-scheme.*

Suppose that $\{\underline{U_a}\}_{a\in A}$ is an open cover of $\underline{X}$. Then there exists a partition of unity $\{\eta_a \in \mathcal{O}_X(X)\}_{a\in A}$ on $\underline{X}$ subordinated to $\{\underline{U_a}\}_{a\in A}$, such that

- *For any $a \in A$, $supp\eta_a = \{x \in X | [\eta_a, X]_x \neq 0\} \subset U_a$,*
- *$\sum_{a\in A} \eta_a$ is a locally finite sum,*
- *$\sum_{a\in A} \eta_a \equiv 1$.*

Lemma 4.12 *Let $\underline{X} := (X, \mathcal{O}_X)$ be a separated, paracompact and locally fair C^∞-scheme.*

Take $x \in X$ and its open neighborhood U. Then there exists $\eta \in \mathcal{O}_X(X)$ such that $supp\eta \subset U$ and $[\eta, X]_x = 1$.

4.3.5 Tangent vectors of a local C^∞-ringed space

Proposition 4.13 *Let $\underline{X} := (X, \mathcal{O}_X)$ be a separated, paracompact and locally fair C^∞-scheme.*

Suppose $D : \mathcal{O}_X(X) \to \mathcal{O}_X(X)$ be an $\mathbb{R}$-derivation of $\mathcal{O}_X(X)$. Then there is a unique vector field $\xi : \mathcal{O}_X \to \mathcal{O}_X$ such that $\xi(X) = D$.

Proof) First, we prove that: for any $f \in \mathcal{O}_X(X)$ and an open set $U \subset X$ such that $f|_U \equiv 0$, we have $(Df)|_U \equiv 0$.

Take any point $x \in U$. There exists open sets U_x, W_x and $\eta_x \in \mathcal{O}_X(X)$ such that $x \in U_x \subset W_x \subset \overline{W_x} \subset U \subset X$, $\eta_x|_{U_x} \equiv 1$ and $\eta_x|_{X\backslash\overline{W_x}} \equiv 0$.

For any $y \in U$, $[(1-\eta_x)f, X]_y = 0_{\mathcal{O}_{X,y}} = [f, X]_y$ and for any $y \in X\backslash\overline{W_x}$, we have $[(1-\eta_x)f, X]_y = [f, X]_y)$. Therefore we got $(1-\eta_x)f \equiv f$.

For $x \in U_x \subset W_x \subset U$, we have $[f, X]_x = 0, [\eta_x, X]_x = 0$. Therefore,

$$[Df, X]_x = [D(1-\eta_x), X]_x[f, X]_x + [1-\eta_x, X]_x[Df, X]_x = 0$$

Then we have $(Df)|_U \equiv 0$.

Suppose that x is any point of X and V is an any open neighborhood at x in X.

Take $h \in \mathcal{O}_X(V)$. There exists $h' \in \mathcal{O}_X(X)$ and an open neighborhood U at X of X such that $h|_U \equiv h'|_U$. If there exists $h'' \in \mathcal{O}_X(X)$ such that $h|_U \equiv h''|_U$, we have $(Dh')|_U \equiv (Dh'')|_U$. Then we can define $\xi_x([h, V]_x) := [Dh', X]_x$ with well-definedness.

We define $\xi_x : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ for any $x \in X$. For an $\mathbb{R}$-derivation D, ξ_x is an $\mathbb{R}$-derivation and $Df \equiv \xi(X)(f)$ for any $f \in \mathcal{O}_X(X)$. ■

4.4 Differential forms on C^∞-ringed spaces

Let $\underline{X} := (X, \mathcal{O}_X)$ be a C^∞-ringed space. Like cotangent sheaves of C^∞-schemes, we want to define sheaves of differential forms on C^∞-schemes.

- $\omega : \chi_{\underline{X}} \times \ldots \times \chi_{\underline{X}} \to \mathcal{O}_X$ is multi-linear and skew-symmetric.
- $\Omega^0_{\underline{X}} = \mathcal{O}_X$, $\Omega^1_{\underline{X}} = T^*\underline{X}$.
- For $\omega \in \Omega^p_{\underline{X}}$,$\eta \in \Omega^q_{\underline{X}}$, we have a wedge product

$$\begin{aligned}&(\omega \wedge \eta)(X_1, \ldots, X_{p+q})\\ &= \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q}} (sgn\sigma)\omega(X_{\sigma(1)}, \ldots, X_{\sigma(p)})\eta(X_{\sigma(p+1)}, \ldots, X_{\sigma p+q}).\end{aligned}$$

- For a nonnegative integer k, we have derivations $d_k : \Omega^k_{\underline{X}} \to \Omega^{k+1}_{\underline{X}}$.

In §4.3.1, we defined $\mathcal{P}T^*\underline{X}$ to associate to each open set $U \subset X$ the cotangent module $\Omega_{\mathcal{O}_X(U)}$, and cotangent sheaf $T^*\underline{X}$ of $\underline{X}$ to be the sheaf of $\mathcal{O}_X$-modules associated to $\mathcal{P}T^*\underline{X}$.

4.4.1 Differential forms on C^∞-ringed spaces

For any integer $n \geq 0$, define $\mathcal{P}\Omega^n_{\underline{X}}$ to associate to each open set $U \subset X$ the n-th exterior algebra

$$\mathcal{P}\Omega^n_{\underline{X}}(U) := \wedge^n \Omega_{\mathcal{O}_X(U)}$$

by Definition 2.47.

For each inclusion of open sets $V \subset U \subset X$, the morphism $\mathcal{P}\Omega^n_{\rho_{UV}}$: $\mathcal{P}\Omega^n_{\underline{X}}(U) \to \mathcal{P}\Omega^n_{\underline{X}}(V)$ of $\mathcal{O}_X(U)$-module as

$$\mathcal{P}\Omega^n_{\rho_{UV}}\Big(\sum_{i=1}^k c_i dc_{i,1} \wedge \ldots \wedge dc_{i,n}\Big) := \sum_{i=1}^k \rho_{UV}(c_i) d\rho(c_{i,1}) \wedge \ldots \wedge d\rho(c_{i,n}).$$

Then we have $\wedge^n\Omega_{\rho_{UV}} \circ \mu_{\mathcal{O}_X(U)} = \mu_{\mathcal{O}_X(U)} \circ (\rho_{UV} \times \wedge^n\Omega_{\rho_{UV}})$ such that

$$\begin{tikzcd} \mathcal{P}\Omega^n_{\underline{X}}(U) \arrow[r, "\mathcal{P}\Omega^n_{\rho_{UV}}"] \arrow[d, "(\cdot)|_V"] & \mathcal{P}\Omega^n_{\underline{X}}(U), \arrow[d, "(\cdot)|_V"] \\ \mathcal{P}\Omega^n_{\underline{X}}(V) \arrow[r, "\mathcal{P}\Omega^n_{\rho_{UV}}"] & \mathcal{P}\Omega^n_{\underline{X}}(V) \end{tikzcd} \tag{55}$$

and $\mathcal{P}\Omega^n_{\underline{X}}$ is a presheaf of $\mathcal{O}_X$-modules on $\underline{X}$.

Define the n-th differential complex $\Omega^n_{\underline{X}}$ of $\underline{X}$ to be the sheaf of $\mathcal{O}_X$-modules associated to $\mathcal{P}\Omega^n_{\underline{X}}$.

4.4.2 Wedge products on C^∞-ringed spaces

Next we define wedge product $\bigwedge : \Omega^p_{\underline{X}} \times \Omega^q_{\underline{X}} \to \Omega^{p+q}_{\underline{X}}$ as a morphism of $\underline{X}$ for $p, q \geq 0$.

The wedge product $\mathcal{P}\bigwedge$ of two differential forms of presheafs $\mathcal{P}\Omega^n_{\underline{X}}$ is defined as follows: for each open $U \subset X$, if $\tau = \sum c_i dc_{i1} \wedge \ldots \wedge dc_{ip} \in \mathcal{P}\Omega^p_{\underline{X}}(U)$ and $\omega = \sum c'_j dc'_{j1} \wedge \ldots \wedge dc'_{jq} \in \mathcal{P}\Omega^q_{\underline{X}}(U)$, then

$$\mathcal{P}\bigwedge(\tau,\omega) := \sum_{i,j} c_i c'_j dc_{i1} \wedge \ldots \wedge dc_{ip} \wedge dc'_{j1} \wedge \ldots \wedge dc'_{jq} \mathcal{P}\Omega^{p+q}_{\underline{X}}(U). \tag{56}$$

We have a following commutative diagram

$$\begin{tikzcd} \mathcal{P}\Omega^p_{\underline{X}}(U) \times \mathcal{P}\Omega^q_{\underline{X}}(U) \arrow[r, "\mathcal{P}\bigwedge"] \arrow[d, "(\cdot)|_V \times (\cdot)|_V"] & \mathcal{P}\Omega^{p+q}_{\underline{X}}(U), \arrow[d, "(\cdot)|_V"] \\ \mathcal{P}\Omega^p_{\underline{X}}(V) \times \mathcal{P}\Omega^q_{\underline{X}}(V) \arrow[r, "\mathcal{P}\bigwedge"] & \mathcal{P}\Omega^{p+q}_{\underline{X}}(V) \end{tikzcd} \tag{57}$$

Therefore, we have defined $\mathcal{P}\wedge : \mathcal{P}\Omega^p_{\underline{X}} \times \mathcal{P}\Omega^q_{\underline{X}} \to \mathcal{P}\Omega^{p+q}_{\underline{X}}$.

Define the wedge product $\wedge : \Omega^p_{\underline{X}} \times \Omega^q_{\underline{X}} \to \Omega^{p+q}_{\underline{X}}$ of $\underline{X}$ to be the sheaf of $\mathcal{O}_X$-modules associated to $\mathcal{P}\wedge : \mathcal{P}\Omega^p_{\underline{X}} \times \mathcal{P}\Omega^q_{\underline{X}} \to \mathcal{P}\Omega^{p+q}_{\underline{X}}$. We write $\tau \wedge \omega := \wedge(\tau, \omega)$.

4.4.3 Derivations on C^∞-ringed spaces

We will define a derivation $d^n_{\underline{X}} : \Omega^n_{\underline{X}} \to \Omega^{n+1}_{\underline{X}}$ for any integer $n \geq 0$.

For any integer $n \geq 0$, define $\mathcal{P}d^n_{\underline{X}}(U) : \mathcal{P}\Omega^n_{\underline{X}}(U) \to \mathcal{P}\Omega^{n+1}_{\underline{X}}(U)$ to associate to each open set $U \subset X$ as $\mathcal{P}d^n_{\underline{X}}(U)(\sum_{i=1}^k c_i dc_{i,1} \wedge \ldots \wedge dc_{i,n}) := \sum_{i=1}^k dc_i \wedge$

$dc_{i,1} \wedge \ldots \wedge dc_{i,n}$ by Definition 2.47. For $n \geq 0$, we have

$$\begin{tikzcd} \mathcal{P}\Omega^n_{\underline{X}}(U) \arrow[r, "\mathcal{P}d^n_{\underline{X}}(U)"] \arrow[d, "(\cdot)|_V"] & \mathcal{P}\Omega^{n+1}_{\underline{X}}(U), \arrow[d, "(\cdot)|_V"] \\ \mathcal{P}\Omega^n_{\underline{X}}(V) \arrow[r, "\mathcal{P}d^n_{\underline{X}}(V)"] & \mathcal{P}\Omega^{n+1}_{\underline{X}}(V) \end{tikzcd} \tag{58}$$

Therefore, we have defined $\mathcal{P}d^n_{\underline{X}} : \mathcal{P}\Omega^n_{\underline{X}} \to \mathcal{P}\Omega^{n+1}_{\underline{X}}$.

Define the derivations $d^n_{\underline{X}} : \Omega^n_{\underline{X}} \to \Omega^{n+1}_{\underline{X}}$ to be the sheaf of $\mathcal{O}_X$-modules associated to $\mathcal{P}d^n_{\underline{X}} : \mathcal{P}\Omega^n_{\underline{X}} \to \mathcal{P}\Omega^{n+1}_{\underline{X}}$ for any integer $n \geq 0$.

4.5 Leibniz complexity of Nash functions

In this section, we give several results in [5] and show additional remarks.

Let $f = f(x_1, \ldots, x_n)$ be a C^∞ function on an open subset $U \subset \mathbb{R}^n$. Then f is called a **Nash function** on U if f is analytic-algebraic on U, i.e. if f is analytic on U and there exists a non-zero polynomial $P(x, y) \in \mathbb{R}[x, y]$, $x = (x_1, \ldots, x_n)$, such that $P(x, f(x)) = 0$ for any $x \in U$ ([10][13][1]). If U is semi-algebraic, then, f is a Nash function if and only if f is analytic and the graph of f in $U \times \mathbb{R} \subset \mathbb{R}^{n+1}$ is a semi-algebraic set ([1]). For a further significant progress on global study of Nash functions, see [2].

4.5.1 Leibniz complexities of Nash functions

Suppose that a smooth function $f(x_1, \ldots, x_n) \in C^\infty(\mathbb{R}^n)$ has a nonzero real polynomial $p(x, y) := \sum_{|a|, b < \infty} c_{a,b} x^a y^b \in \mathbb{R}[x_1, \ldots, x_n, y]$ such that

$p(x, f(x)) \equiv 0$. On $\Omega_{C^\infty(\mathbb{R}^n),\mathbb{R}} = \mathfrak{F}_{C^\infty(\mathbb{R}^n)}/\mathfrak{M}_{C^\infty(\mathbb{R}^n),\mathbb{R}}$,

$$
\begin{aligned}
d(p(x, f(x))) &= \sum_{|a|,b<\infty} c_{a,b}\Big\{\sum_{i=1}^n a_i x^{a-e_i} f(x)^b d(x_i) + b x^a f(x)^{b-1} d(f)\Big\} \\
&= \sum_{|a|,b<\infty} c_{a,b}\Big[\sum_{i=1}^n \Big\{a_i x^{a-e_i} f(x)^b d(x_i) + b x^a f(x)^{b-1} \frac{\partial f}{\partial x_i}(x) d(x_i)\Big\} \\
&\quad + b x^a f(x)^{b-1}\Big\{d(f) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) d(x_i)\Big\}\Big] \\
&= \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(\sum_{|a|,b<\infty} c_{a,b} x^a f(x)^b\Big) d(x_i) \\
&\quad + \sum_{|a|,b<\infty}\Big\{c_{a,b} b x^a f(x)^{b-1}\Big(d(f) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) d(x_i)\Big)\Big\} \\
&= \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(p(x, f(x))\Big) d(x_i) + \frac{\partial p}{\partial y}(x, f(x))\Big(d(f) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) d(x_i)\Big).
\end{aligned}
$$

For $p(x, f(x)) \equiv 0$, we have $\frac{\partial}{\partial x_i}\big(p(x, f(x))\big) = 0$ for $i = 1, \ldots, n$ and $d\big(p(x, f(x))\big) = 0$.

Therefore, if a smooth function $f(x)$ is the Nash function with a non-zero real polynomial $p(x, y) \in \mathbb{R}[x, y]$ such that $p(x, f(x)) = 0$, we have

$$\frac{\partial p}{\partial y}(x, f(x))\Big(d(f) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) d(x_i)\Big) = 0$$

on $\Omega_{C^\infty(\mathbb{R}^n),\mathbb{R}}$. If $\frac{\partial p}{\partial y}(x, f(x)) \neq 0$ for any $x \in \mathbb{R}^k$, we have $d(f) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) d(x_i) = 0$.

We can write Theorem 4.15 as a following.

Theorem 4.14 *For $f(x) \in C^\infty(\mathbb{R}^m)$, $LComp_{C^\infty(\mathbb{R}^m)}(f(x)) < \infty$, i.e. $g(x)d(f(x)) - \sum_{i=1}^m g(x)\frac{\partial f}{\partial x_i}(x)d(x_i)$ is generated by $Prod_{C^\infty(\mathbb{R}^m)}(\cdot,\cdot)$, $Sum_{C^\infty(\mathbb{R}^m)}(\cdot,\cdot)$, $Sca_{C^\infty(\mathbb{R}^m)}(\cdot,\cdot)$ for a non-zero smooth function $g(x) \in C^\infty(\mathbb{R}^m)$ if and only if f is a Nash function, i.e. there exists a non zero polynomial $p(x, y) \in \mathbb{R}[x_1, \ldots, x_m, y]$ such that $p(x, f(x)) \equiv 0$.*

4.5.2 Algebraic computability of differentials

Let $C^\infty(U)$ (resp. $C^\omega(U)$, $N^\infty(U)$) denote the set of all C^∞ functions (resp. analytic functions, Nash functions) on an open subset $U \subset \mathbb{R}^n$. We take the space $\Omega_{C^\infty(U),\mathbb{R}}$ and the universal $\mathbb{R}$-derivation $d_{C^\infty(U),\mathbb{R}} : C^\infty(U) \to \Omega_{C^\infty(U),\mathbb{R}}$ from §2.5.

If $\mathfrak{M}$ is any $C^\infty(U)$-module and $V : \mathfrak{C} \to \mathfrak{M}$ is any derivation, then there exists a unique $C^\infty(U)$-homomorphism $\phi : \Omega_{C^\infty(U),\mathbb{R}} \to \mathfrak{M}$ such that $V = \phi \circ d_{C^\infty(U),\mathbb{R}}$ from §2.5.

Suppose U is connected. Consider the set $S \subset N^\infty(U)$ of non-zero Nash functions i.e. Nash functions which are not identically zero on U. Then S is closed under the multiplication. Let $\widetilde{C^\infty}(U) = C^\infty(U)[S^{-1}]$ denote the localization of $C^\infty(U)$ by S. We consider the space $\Omega_{\widetilde{C^\infty}(U),\mathbb{R}}$ of Kähler differentials of the $\mathbb{R}$-algebra $\widetilde{C^\infty}(U)$.

Then we have:

Theorem 4.15 ([5]) *Let U be a semi-algebraic connected open subset of $\mathbb{R}^n$. Then the following conditions on an analytic function $f \in C^\omega(U)$ are equivalent to each other:*
(1) *f is a Nash function on U.*
(2) *There exists a non-zero Nash function $g \in N^\infty(U)$ such that*

$$g\left(d_{C^\infty(U),\mathbb{R}}(f) \;-\; \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d_{C^\infty(U),\mathbb{R}}(x_i)\right) = 0,$$

in the space $\Omega_{C^\infty(U),\mathbb{R}}$ of Kähler differentials of $C^\infty(U)$.
(3)

$$d_{\widetilde{C^\infty}(U),\mathbb{R}}(f) \;=\; \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d_{\widetilde{C^\infty}(U),\mathbb{R}}(x_i),$$

in the space $\Omega_{\widetilde{C^\infty}(U),\mathbb{R}}$ of Kähler differentials of $\widetilde{C^\infty}(U)$.
(4) *There exist $f_1, \ldots, f_n \in \widetilde{C^\infty}(U)$ such that*

$$d_{\widetilde{C^\infty}(U),\mathbb{R}}(f) \;=\; \sum_{i=1}^{n} f_i d_{\widetilde{C^\infty}(U),\mathbb{R}}(x_i),$$

in the space $\Omega_{\widetilde{C^\infty}(U),\mathbb{R}}$ of Kähler differentials of $\widetilde{C^\infty}(U)$.

Proof) [**(1) $\Rightarrow$ (2)**] Let $f \in C^\infty(U)$ be a Nash function and $P(x, y)$ be a non-zero polynomial satisfying $P(x, f) = 0$ and $\frac{\partial P}{\partial y}(x, f) \neq 0$. Then, by taking

$\mathbb{R}$-derivation $d_{C^\infty(U),\mathbb{R}}$ on both sides of $P(x,f)=0$, we have in $\Omega_{C^\infty(U),\mathbb{R}}$,

$$\begin{aligned}
0 &= d_{C^\infty(U),\mathbb{R}}(P(x,f)) = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(x,f) d_{C^\infty(U),\mathbb{R}}(x_i) + \frac{\partial P}{\partial y}(x,f) d_{C^\infty(U),\mathbb{R}}(f) \\
&= \sum_{i=1}^{n} \left(-\frac{\partial P}{\partial y}(x,f) \frac{\partial f}{\partial x_i} \right) d_{C^\infty(U),\mathbb{R}}(x_i) + \frac{\partial P}{\partial y}(x,f) d_{C^\infty(U),\mathbb{R}}(f) \\
&= \frac{\partial P}{\partial y}(x,f) \left(d_{C^\infty(U),\mathbb{R}}(f) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d_{C^\infty(U),\mathbb{R}}(x_i) \right),
\end{aligned}$$

and that $\frac{\partial P}{\partial y}(x,f)$ is a non-zero Nash function on U.
The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear.
[**(4) $\Rightarrow$ (1)**] Suppose f is not a Nash function on U and $d_{\widetilde{C^\infty}(U),\mathbb{R}}(f) - \sum_{i=1}^n f_i d_{\widetilde{C^\infty}(U),\mathbb{R}}(x_i) = 0$ in $\Omega_{\widetilde{C^\infty}(U),\mathbb{R}}$. Since f is not a Nash function, there exists a point $a \in U$ such that $f \in \mathcal{F}_{\mathbb{R}^n,a} \subset Q(\mathcal{F}_{\mathbb{R}^n,a})$ is not algebraic. Here $\mathcal{F}_{\mathbb{R}^n,a} = C_a^\infty(\mathbb{R}^n)/m_a^\infty$ is the $\mathbb{R}$-algebra of formal series, $\mathfrak{M} = Q(\mathcal{F}_{\mathbb{R}^n,a})$ is its quotient field and the Taylor series of f at a is written also by the same symbol f. Moreover, we have $d(f) - \sum_{i=1}^n f_i d(x_i) = 0$ in the $\mathbb{R}$-cotangent module $\Omega_{\mathfrak{M},\mathbb{R}}$ of $\mathfrak{M}$, via the homomorphism $\widetilde{C^\infty}(U) \to \mathfrak{M}$ defined by taking the Taylor series. Then, in the free $\mathfrak{M}$-module $\mathcal{F}_{\mathfrak{M}}$ generated by elements $\{d(h) \mid h \in \mathfrak{M}\}$, $d(f) - \sum_{i=1}^n f_i d(x_i)$ is a finite sum of elements of type

$$\begin{aligned}
aSum_{\mathfrak{M}}(h,k) &= a(d(h+k) - d(h) - d(k)), \\
bSca_{\mathfrak{M}}(\lambda,\ell) &= b(d(\lambda\ell) - \lambda d(\ell)), \\
cProd_{\mathfrak{M}}(p,q) &= c(d(pq) - p(q) - qd(p)).
\end{aligned}$$

Here $a,h,k,b,\ell,c,p,q \in \mathfrak{M}$, $\lambda \in \mathbb{R}$. Now we take the subfield $\mathfrak{L} \subset \mathfrak{M}$ generated over the rational function field $\mathfrak{K} = \mathbb{R}(x)$ by $f, f_i (1 \le i \le n)$ and those a,h,k,b,ℓ,c,p,q which appear in the above expression of $d(f) - \sum_{i=1}^n f_i d(x_i)$: $\mathfrak{L} = K(f, h_1, \ldots, h_m)$, which is a finitely generated field over K by f and for some $h_1, \ldots, h_m \in \mathfrak{M}$. Then we have $d(f) - \sum_{i=1}^n f_i d(x_i) = 0$ also in $\Omega_{\mathfrak{L},\mathbb{R}}$.

Take any non-zero element $u \in \mathfrak{L}$ and fix it. Since f is transcendental over $\mathfrak{K}$ in the usual sense, we can define an $\mathbb{R}$-derivation $D_0 : \mathfrak{K}(f) \to \mathfrak{L}$ on the extension field $\mathfrak{K}(f)$ over $\mathfrak{K}$ by f, by $D_0(x_j) = 0, 1 \le j \le n, D_0(f) = u$.
Then we define a derivation $D_1 : \mathfrak{K}(f, h_1) \to \mathfrak{L}$, $D_1|_{\mathfrak{K}(f)} = D_0$ as follows: If h_1 is transcendental over $\mathfrak{K}(f)$, then we set $D_1(h_1) = 0$. If h_1 is algebraic over $\mathfrak{K}(f)$, then we set $D_1(h_1)$ as the element in $\mathfrak{K}(f, h_1)$ which is determined by the algebraic relation of h_1 over $\mathfrak{K}(f)$ and D_0. In fact, if $\sum_{k=0}^m a_k h_1^{m-k} = 0,\ a_k \in$

$\mathfrak{K}(f)$, is a minimal algebraic relation of h_1 over $\mathfrak{K}(f)$, then we would have

$$\sum_{k=0}^{m} D_0(a_k)h_1^{m-k} + \left(\sum_{k=0}^{m-1}(m-k)a_k h_1^{m-k-1}\right) D_1(h_1) = 0.$$

Since $\sum_{k=0}^{m-1}(m-k)a_k h_1^{m-k-1} \neq 0$ by the minimality assumption, $D_1(h_1)$ is uniquely determined by

$$D_1(h_1) = -\left(\sum_{k=0}^{m} D_0(a_k)h_1^{m-k}\right) / \left(\sum_{k=0}^{m-1}(m-k)a_k h_1^{m-k-1}\right).$$

Here it is essential that we discuss $\mathbb{R}$-derivations over a field. Thus we extend D_1 into an $\mathbb{R}$-derivation $D = D_m : \mathfrak{L} \to \mathfrak{L}$ by a finitely number of steps. Note that we need not to use Zorn's lemma to show the existence of extension of derivation. Then by the universality of the $\mathbb{R}$-derivations, there exists an $\mathfrak{L}$-linear map $\rho : \Omega_{\mathfrak{L},\mathbb{R}} \to \mathfrak{L}$ such that $\rho \circ d_{\mathfrak{L},\mathbb{R}} = D : \mathfrak{L} \to \mathfrak{L}$. Here $d_{\mathfrak{L},\mathbb{R}} : \mathfrak{L} \to \Omega_{\mathfrak{L},\mathbb{R}}$ is the universal $\mathbb{R}$-derivation of $\mathfrak{L}$. Then we have

$$0 = \rho\left(d_{\mathfrak{L},\mathbb{R}}(f) - \sum_{i=1}^{n} f_i d_{\mathfrak{L},\mathbb{R}}(x_i)\right) = D(f) = u.$$

This leads to contradiction with the assumption $u \neq 0$. Thus we have that f is a Nash function.

$\blacksquare$

Remark 4.16 *If U is not connected, then Theorem 4.15 does not hold. In fact, let $U = \mathbb{R} \setminus \{0\}$ and set $f(x) = e^x$ if $x > 0$ and $f(x) = 1$ if $x < 0$. Then $f \in C^\omega(U)$ and $f \notin N^\infty(U)$. However the condition (2) is satisfied if we take as g the non-zero Nash function on U defined by $g(x) = 0 (x > 0), g(x) = 1 (x < 0)$.*

4.5.3 Estimates on Leibniz complexity

Let $U \subset \mathbb{R}^n$ be a connected open subset. For a Nash function $f \in N^\infty(U)$, we define the Leibniz complexity of f by the minimal number of terms corresponding to Leibniz rule for $g(d(f) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d(x_i))$ in the free $C^\infty(U)$-module $\mathfrak{F}_{C^\infty(U)}$ among all expressions for all non-zero $g \in N^\infty(U)$. The definition is based on the statement (2) of Theorem 4.15. We do not care about the number of terms corresponding to linearity of the differential. Moreover we will do not count the term generated by the relation $d(1 \cdot 1) - 1d(1) - 1d(1)$. Therefore we use the relation $d(c) = 0$ for $c \in \mathbb{R}$ freely.

Let $\mathrm{LC}(f)$ denote the Leibniz complexity of f.

First we show general basic inequalities:

Lemma 4.17 *For $f(x), g(x) \in N^\infty(U)$, we have*
(1) $\mathrm{LC}(f+g) \leq \mathrm{LC}(f) + \mathrm{LC}(g)$,
(2) $\mathrm{LC}(fg) \leq \mathrm{LC}(f) + \mathrm{LC}(g) + 1$.

Proof) Let $h(x)d(f(x)) \in \mathfrak{M}_{C^\infty(U),\mathbb{R}}$ (resp. $k(x)d(g(x)) \in \mathfrak{M}_{C^\infty(U),\mathbb{R}}$) be expressed using the terms of Leibniz rule minimally i.e. $\mathrm{LC}(f)$-times (resp. $\mathrm{LC}(g)$-times), for a non-zero $h(x) \in N^\infty(U)$ (resp. a non-zero $k(x) \in N^\infty(U)$), except for the term $d(1 \cdot 1) - 1d(1) - 1d(1)$. Then $h(x)k(x)d\big((f+g)(x)\big) = k(x)\big\{h(x)d(f(x))) + h(x)(k(x)d(g(x))\big\} \in \mathfrak{M}_{C^\infty(U),\mathbb{R}}$ is expressed using Leibniz rule at most $\mathrm{LC}(f) + \mathrm{LC}(g)$ times. Therefore we have (1). Moreover, by using Leibniz rule once, we have

$$h(x)k(x)d\big((fg)(x)\big) = hk(gd(f) + fd(g)) = kg(hd(f)) + hf(kd(g))$$

in $\Omega_{\mathcal{E}(U)}$. Then, using Leibniz rule $\mathrm{LC}(f) + \mathrm{LC}(g)$ times, we compute $d(f)$ and $d(g)$, and thus $d(fg)$. Therefore we have (2). ■

4.5.4 Leibniz complexity of C^∞-rings

In §4.5.3, we have defined Leibniz complexities of Nash functions. In this section, we define Leibniz complexities in C^∞-rings by using the definition of Leibniz complexity of Nash functions.

Let $\mathfrak{C}$ be a general C^∞-ring and $\mathfrak{F}_{\mathfrak{C}}$ a free $\mathfrak{C}$-module generated by $d(c)$ for $c \in \mathfrak{C}$. For $h_1, h_2 \in \mathfrak{C}$ and $c \in \mathbb{R}$. Define elements of $\mathfrak{F}_{\mathfrak{C}}$ as

$$\begin{aligned}
Prod_{\mathfrak{C}}(c_1, c_2) &:= d(c_1c_2) - c_1d(c_2) - c_2d(c_1),\\
Sum_{\mathfrak{C}}(c_1, c_2) &:= d(c_1 + c_2) - d(c_1) - d(c_2),\\
Sca_{\mathfrak{C}}(\lambda, c_1) &:= d(\lambda c_1) - \lambda d(c_1),\\
Func_{\mathfrak{C}}(f, c_1, \ldots, c_m) &:= d(\Phi_f(c_1, \ldots, c_m)) - \sum_{i=1}^{m} \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_m)d(c_i).
\end{aligned}$$

Define a subset $\mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$ of a function $f \in C^\infty(\mathbb{R}^n)$ which satisfies that $Func_{\mathfrak{C}}(f, c_1, \ldots, c_m)$ is generated by a finite sum of $Prod_{\mathfrak{C}}(h_1, h_2)$, $Sum_{\mathfrak{C}}(h_1, h_2)$, $Sca_{\mathfrak{C}}(c, h_1)$ for any $c_1, \ldots, c_m \in \mathfrak{C}$.

Define a subset $\mathfrak{L}\mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$ of a function $f \in C^\infty(\mathbb{R}^n)$ which satisfies that there exists a non-zero smooth function $g \in C^\infty(\mathbb{R}^m)$ such that $\Phi_g(c)Func_{\mathfrak{C}}(f, c_1, \ldots, c_m)$ is generated by a finite sum of $Prod_{\mathfrak{C}}(h_1, h_2)$, $Sum_{\mathfrak{C}}(h_1, h_2)$, $Sca_{\mathfrak{C}}(c, h_1)$ for any $c_1, \ldots, c_m \in \mathfrak{C}$.

For $f \in C^\infty(\mathbb{R}^n)$, define two type complexities of $\mathfrak{C}$.

Definition 4.18 *Let $\mathfrak{C}$ is a C^∞-ring and $f \in C^\infty(\mathbb{R}^m)$ for $m \in \{0\} \cup \mathbb{N}$.*

(1) If $Func_{\mathfrak{C}}(f, c_1, \ldots, c_m)$ is generated by $Prod_{\mathfrak{C}}(\cdot,\cdot)$,$Sum_{\mathfrak{C}}(\cdot,\cdot)$ and $Sca_{\mathfrak{C}}(\cdot,\cdot)$, i.e.,

$$\begin{aligned} Func_{\mathfrak{C}}(f, c_1, \ldots, c_m) = &\sum_{i=1}^{p} c_i \cdot Prod_{\mathfrak{C}}(c_{1,i}, c_{2,i}) \\ &+ \sum_{j=1}^{q} c'_j \cdot Sum_{\mathfrak{C}}(c'_{1,j}, c'_{2,j}) + \sum_{k=1}^{r} c''_k \cdot Sca_{\mathfrak{C}}(\lambda_k, c''_{1,k}) \end{aligned} \tag{59}$$

for any $c_1, \ldots, c_m \in \mathfrak{C}$, define $Comp_{\mathfrak{C}}(f)$ as a minimal number p of (59).

(2) If $\Phi_g(c_1, \ldots, c_m)\{Func_{\mathfrak{C}}(f, c_1, \ldots, c_m)\}$ is generated by $Prod_{\mathfrak{C}}(\cdot,\cdot)$,$Sum_{\mathfrak{C}}(\cdot,\cdot)$ and $Sca_{\mathfrak{C}}(\cdot,\cdot)$, i.e.,

$$\begin{aligned} \Phi_g(c_1, \ldots, c_m)\{Func_{\mathfrak{C}}(f, c_1, \ldots, c_m)\} = &\sum_{i=1}^{p} c_i \cdot Prod_{\mathfrak{C}}(c_{1,i}, c_{2,i}) \\ &+ \sum_{j=1}^{q} c'_j \cdot Sum_{\mathfrak{C}}(c'_{1,j}, c'_{2,j}) + \sum_{k=1}^{r} c''_k \cdot Sca_{\mathfrak{C}}(\lambda_k, c''_{1,k}) \end{aligned} \tag{60}$$

for any $c_1, \ldots, c_m \in \mathfrak{C}$ and some non-zero action Φ_g for $g \in C^\infty(\mathbb{R}^m)$, define $LComp_{\mathfrak{C}}(f)$ as a minimal number p of (60).

We have $\mathrm{Comp}_{\mathfrak{C}}(f) \geq \mathrm{LComp}_{\mathfrak{C}}(f)$. From the definition of Leibniz complexity of Nash functions in §4.5.3, $LC(f) = \mathrm{LComp}_{C^\infty(U)}(f)$ for an open set $U \subset \mathbb{R}^n$ and $f \in C^\infty(U)$.

For any C^∞-ring $\mathfrak{C}$, we have $\mathrm{Comp}_{\mathfrak{C}}(f) \leq \mathrm{Comp}_{C^\infty(\mathbb{R}^n)}(f)$ for any $f \in C^\infty(\mathbb{R}^n)$.

As same as Lemma 4.17, we have following properties of complexities.

Proposition 4.19 *Suppose that $f, g \in C^\infty(\mathbb{R}^n)$.*

(1) For a real value $\lambda \in \mathbb{R}$, $Comp_{\mathfrak{C}}(\lambda f) = \begin{cases} Comp_{\mathfrak{C}}(f) & (\lambda \neq 0) \\ 0 & (\lambda = 0) \end{cases}$.

(2) $Comp_{\mathfrak{C}}(f+g) \leq Comp_{\mathfrak{C}}(f) + Comp_{\mathfrak{C}}(g)$.

(3) $Comp_{\mathfrak{C}}(fg) \leq Comp_{\mathfrak{C}}(f) + Comp_{\mathfrak{C}}(g) + 1$.

(4) For a projection $x_i : \mathbb{R}^n \to \mathbb{R}$, $Comp_{\mathfrak{C}}(x_i^m) \leq \begin{cases} m-1 & (m \geq 1) \\ k & (m = 2^k, k \geq 0) \end{cases}$.

These properties hold also for $LComp_{\mathfrak{C}}(f)$.

Proof)

(1) Suppose that $\lambda \neq 0$. For $Func_{\mathfrak{C}}(\lambda f, c_1, \ldots, c_m) = d(\Phi_{\lambda f}(c_1, \ldots, c_m)) - \sum_{i=1}^{m} \Phi_{\frac{\partial(\lambda f)}{\partial x_i}}(c_1, \ldots, c_m)d(c_i)$,

$$
\begin{aligned}
&d(\Phi_{\lambda f}(c)) - \sum_{i=1}^{m} \Phi_{\frac{\partial(\lambda f)}{\partial x_i}}(c)d(c_i) \\
&\quad = \{d(\Phi_{\lambda f}(c)) - \lambda d(\Phi_f(c))\} + \lambda d(\Phi_f(c)) - \lambda \sum_{i=1}^{m} \Phi_{\frac{\partial f}{\partial x_i}}(c)d(c_i) \\
&\quad = Sca_{\mathfrak{C}}(\lambda, \Phi_f(c)) + \lambda \cdot \underline{Func_{\mathfrak{C}}(f, c_1, \ldots, c_m)}
\end{aligned}
$$

Then we have $\mathrm{Comp}_{\mathfrak{C}}(\lambda f) \leq \mathrm{Comp}_{\mathfrak{C}}(f)$. Replacing f and λ by λf and $\frac{1}{\lambda}$, we have $\mathrm{Comp}_{\mathfrak{C}}(\lambda f) \geq \mathrm{Comp}_{\mathfrak{C}}(f)$.

Therefore, we have $\mathrm{Comp}_{\mathfrak{C}}(\lambda f) = \mathrm{Comp}_{\mathfrak{C}}(f)$.

(2) For $Func_{\mathfrak{C}}(f+g, c_1, \ldots, c_n) = d(\Phi_{f+g}(c)) - \sum_{i=1}^{n} \Phi_{\frac{\partial(f+g)}{\partial x_i}}(c)d(c_i)$,

$$
\begin{aligned}
&d(\Phi_{f+g}(c)) - \sum_{i=1}^{n} \Phi_{\frac{\partial(f+g)}{\partial x_i}}(c)d(c_i) \\
&=\big(d(\Phi_f(c) + \Phi_g(c)) - d(\Phi_f(c)) - d(\Phi_g(c))\big) \\
&\quad + \big(d(\Phi_f(c)) + d(\Phi_g(c))\big) - \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(c)d(c_i) - \sum_{i=1}^{n} \Phi_{\frac{\partial g}{\partial x_i}}(c)d(c_i) \\
&=Sum_{\mathfrak{C}}(\Phi_f(c), \Phi_g(c)) \\
&\quad + \{d(\Phi_f(c)) - \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(c)d(c_i)\} + \{(d(\Phi_g(c)) - \sum_{i=1}^{n} \Phi_{\frac{\partial g}{\partial x_i}}(c)d(c_i))\} \\
&=Sum_{\mathfrak{C}}(\Phi_f(c), \Phi_g(c)) + \underline{Func_{\mathfrak{C}}(f, c_1, \ldots, c_n)} + \underline{Func_{\mathfrak{C}}(g, c_1, \ldots, c_n)}
\end{aligned}
$$

We have that $\mathrm{Comp}_{\mathfrak{C}}(f+g) \leq \mathrm{Comp}_{\mathfrak{C}}(f) + \mathrm{Comp}_{\mathfrak{C}}(g)$.

(3) For $Func_{\mathfrak{C}}(fg, c_1, \ldots, c_n) = d(\Phi_{fg}(c)) - \sum_{i=1}^{n} \Phi_{\frac{\partial(fg)}{\partial x_i}}(c)d(c_i)$,

$$\begin{aligned}
d(\Phi_{fg}(c)) - \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}g + f\frac{\partial g}{\partial x_i}}(c)d(c_i)& \\
=&\big\{d(\Phi_f(c)\Phi_g(c)) - \Phi_f(c)d(\Phi_g(c)) - \Phi_g(c)d(\Phi_f(c))\big\} \\
&+ \{\Phi_f(c)d(\Phi_g(c)) + \Phi_g(c)d(\Phi_f(c))\} \\
&- \sum_{i=1}^{n}\{\Phi_{\frac{\partial f}{\partial x_i}}(c)\Phi_g(c)d(c_i) + \Phi_f(c)\Phi_{\frac{\partial g}{\partial x_i}}(c)d(c_i)\} \\
=&Prod_{\mathfrak{C}}(\Phi_f(c), \Phi_g(c)) + \Phi_f(c)\underline{\Big(d(\Phi_g(c)) - \sum_{i=1}^{n} \Phi_{\frac{\partial g}{\partial x_i}}(c)d(c_i)\Big)} \\
&+ \Phi_g(c)\underline{\Big(d(\Phi_f(c)) - \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(c)d(c_i)\Big)} \\
=&Prod_{\mathfrak{C}}(\Phi_f(c), \Phi_g(c)) + \Phi_f(c) \cdot \underline{Func_{\mathfrak{C}}(g, c_1, \ldots, c_n)} \\
&+ \Phi_g(c) \cdot \underline{Func_{\mathfrak{C}}(f, c_1, \ldots, c_n)}
\end{aligned}$$

We have that $\mathrm{Comp}_{\mathfrak{C}}(fg) \leq \mathrm{Comp}_{\mathfrak{C}}(f) + \mathrm{Comp}_{\mathfrak{C}}(g) + 1$.

(4) Suppose that $m = 1$. For $Func_{\mathfrak{C}}(x_i, c_1, \ldots, c_n) = d\big(\Phi_{x_i}(c_1, \ldots, c_n)\big) - 1_{\mathfrak{C}}d(c_i)$,

$$d\big(\Phi_{x_i}(c_1, \ldots, c_n)\big) - 1_{\mathfrak{C}}d(c_i) = d(c_i) - 1d(c_i) = (1-1)d(c_i) = 0.$$

We have that $\mathrm{Comp}_{\mathfrak{C}}(x_i) \leq 0$, i.e. $\mathrm{Comp}_{\mathfrak{C}}(x_i) = 0$

Suppose that m is an arbitrary natural number. For $Func_{\mathfrak{C}}({x_i}^m, c_1, \ldots, c_n) = d\big(\Phi_{{x_i}^m}(c_1, \ldots, c_n)\big) - \Phi_{m{x_i}^{m-1}}(c_1, \ldots, c_n)d(c_i) = d(c_i^m) - mc_i^{m-1}d(c_i)$,

$$\begin{aligned}
d(c_i^m) - (mc_i^{m-1})d(c_i)& \\
=&\{d(c_i^m) - c_id(c_i^{m-1}) - c_i^{m-1}d(c_i)\} \\
&+ \{c_id(c_i^{m-1}) + c_i^{m-1}d(c_i)\} - (nc_i^{m-1})d(c_i) \\
=&Prod_{\mathfrak{C}}(c_i, c_i^{n-1}) + c_i\{d(c_i^{m-1}) + (n-1)c_i^{m-1}d(c_i)\} \\
=&Prod_{\mathfrak{C}}(c_i, c_i^{m-1}) + c_i\{Func_{\mathfrak{C}}(x_i^{m-1}, c_1, \ldots, c_n)\}.
\end{aligned}$$

Then we have $\mathrm{Comp}_{\mathfrak{C}}(x_i^n) \leq 1 + \mathrm{Comp}_{\mathfrak{C}}(x_i^{n-1})$.

Suppose that $m = 2^k$. For k is arbitrary natural number,

$$\begin{aligned}
&d(c_i^{2^k}) - 2^k c_i^{2^k-1} d(c_i) \\
&\quad = \{d(c_i^{2^k}) - 2c_i^{2^{(k-1)}} d(c_i^{2^{(k-1)}})\} + \{2c_i^{2^{(k-1)}} d(c_i^{2^{(k-1)}}) - (2^k c_i^{2^k-1}) d(c_i)\} \\
&\quad = Prod_{\mathfrak{C}}(c_i^{2^{(k-1)}}, c_i^{2^{(k-1)}}) + 2c_i^{2^{(k-1)}} \{d(c_i^{2^{(k-1)}}) - (2^{(k-1)} c_i^{2^{(k-1)}-1}) d(c_i)\} \\
&\quad = Prod_{\mathfrak{C}}(c_i^{2^{(k-1)}}, c_i^{2^{(k-1)}}) + 2c_i^{2^{(k-1)}} \{Func_{\mathfrak{C}}(x_i^{2^{(k-1)}}, c_1, \ldots, c_n)\}.
\end{aligned}$$

We have $\mathrm{Comp}_{\mathfrak{C}}(x_i^{2^k}) \leq 1 + \mathrm{Comp}_{\mathfrak{C}}(x_i^{2^{k-1}})$. $\mathrm{Comp}_{\mathfrak{C}}(x_i^{2^k}) \leq (k-1) + \mathrm{Comp}_{\mathfrak{C}}(x_i^{2^0}) = k-1$.

Therefore, we have $\mathrm{Comp}_{\mathfrak{C}}(x_i^{2^k}) \leq k-1$ for arbitrary natural number k.

■

4.5.5 Leibniz complexities for real polynomials

By the definition of Leibniz complexity, we have the affine invariance:

Lemma 4.20 ([5]) *Let $f \in N^\infty(U)$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be an affine isomorphism. Then $f \circ \varphi \in N^\infty(\varphi^{-1}(U))$ satisfies* $\mathrm{LC}(f \circ \varphi) = \mathrm{LC}(f)$.

Lemma 4.21 *Let $f \in \mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$ and $\phi : \mathbb{R}^m \to \mathbb{R}^n$ be an affine morphism. $f \circ \phi \in \mathfrak{N}_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^m)$ and $Comp_{\mathfrak{C}}(f \circ \phi) \leq Comp_{\mathfrak{C}}(f)$.*

Proof) In $\Omega_{\mathfrak{C},\mathbb{R}}$, $d(\Phi_{f\circ\phi}(c))$ is equals to $\sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(\Phi_\phi(c)) d(\phi_i(c))$ by using Leibniz rule $\mathrm{Comp}_{\mathfrak{C}}(f)$ times.

For the affine morphism ϕ, $\sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(\Phi_\phi(c)) d(\phi_i(c))$ is equals to

$$\sum_{i=1}^n \sum_{j=1}^m \Phi_{\frac{\partial f}{\partial x_i}\circ\phi}(c) \Phi_{\frac{\partial \phi_i}{\partial y_j}}(c) d(c_j) = \sum_{j=1}^m \Phi_{\frac{\partial (f\circ\phi)}{\partial y_j}}(c) d(c_j).$$

Therefore, we have $\mathrm{Comp}_{\mathfrak{C}}(f \circ \phi) \leq \mathrm{Comp}_{\mathfrak{C}}(f)$. ■

Lemma 4.22 *Let $f \in \mathfrak{L}\mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be an affine isomorphism. $f \circ \phi \in \mathfrak{L}\mathfrak{N}_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$ and $LComp_{\mathfrak{C}}(f \circ \phi) = LComp_{\mathfrak{C}}(f)$.*

Proof) Suppose that there exists a smooth function $g \in C^\infty(\mathbb{R}^n)$ which is a non-zero as an action $\Phi_g : \mathfrak{C}^n \to \mathfrak{C}$ and $\Phi_g(c)d(\Phi_f(c))$ is equals to $\Phi_g(c)\sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c)d(c_i)$ by using Leibniz rules $\mathrm{LComp}_{\mathfrak{C}}(f)$ times.

In $\Omega_{\mathfrak{C},\mathbb{R}}$, $\Phi_{g\circ\phi}(c)d(\Phi_{f\circ\phi}(c))$ is equals to $\Phi_{g\circ\phi}(c)\sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(\Phi_\phi(c))d(\phi_i(c))$ by using Leibniz rule $\mathrm{LComp}_{\mathfrak{C}}(f)$ times.

For the affine isomorphism $\phi = (\phi_1, \ldots, \phi_n)$, $\Phi_{g\circ\phi} : \mathfrak{C}^n \to \mathfrak{C}$ is a non-zero action. $\Phi_{g\circ\phi}(c)\sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(\Phi_\phi(c))d(\phi_i(c))$ is equals to

$$\Phi_{g\circ\phi}(c)\sum_{i=1}^n\sum_{j=1}^n \Phi_{\frac{\partial f}{\partial x_i}\circ\phi}(c)\Phi_{\frac{\partial \phi_i}{\partial y_j}}(c)d(c_j) = \Phi_{g\circ\phi}(c)\sum_{j=1}^n \Phi_{\frac{\partial (f\circ\phi)}{\partial y_j}}(c)d(c_j).$$

We have $\mathrm{LComp}_{\mathfrak{C}}(f\circ\phi) \leq \mathrm{LComp}_{\mathfrak{C}}(f)$.

From the inverse mapping ϕ^{-1}, we have $\mathrm{LComp}_{\mathfrak{C}}(f\circ\phi) = \mathrm{LComp}_{\mathfrak{C}}(f)$. ■

In general it is a difficult problem to determine the exact value of the Leibniz complexity even for an polynomial function.

Example 4.23 ([5]) Let $n = 1$. $\mathrm{LC}(x + c) = 0$. $\mathrm{LC}(x^2 + bx + c) = 1$. $\mathrm{LC}(\sqrt{x^2+1}) = 2$. Let $n = 2$. For $\lambda \in \mathbb{R}$, we have

$$\mathrm{LC}(x_1^2 + x_2^2 + \lambda x_1 x_2) = \begin{cases} 1 & \text{if} \quad |\lambda| \geq 2 \\ 2 & \text{if} \quad |\lambda| < 2. \end{cases}$$

In fact, $x_1^2+x_2^2+\lambda x_1x_2 = (x_1+\frac{\lambda}{2}x_2)^2+(1-\frac{\lambda^2}{4})x_2^2$. Moreover $x_1^2+x_2^2+\lambda x_1x_2 = (x_1+\alpha x_2)(x_1+\beta x_2)$ for some $\alpha, \beta \in \mathbb{R}$ if and only if $|\lambda| \geq 2$.

Let $n = 1$ and write $x = x_1$. Then we have $\mathrm{LC}(x^0) = \mathrm{LC}(1) = 0, \mathrm{LC}(x) = 0, \mathrm{LC}(x^2) = 1, \mathrm{LC}(x^3) = 2, \mathrm{LC}(x^4) = 2$. For example we calculate $d(x^4) = 2x^2d(x^2) = 4x^3d(x)$ by using Leibniz rule twice, and we can check that it is impossible to calculate $d(x^4)$ by using Leibniz rule just once.

To observe the essence of the problem to estimate the Leibniz complexity, let us digress to consider "the problem of strips". Let k be a positive integer. Suppose we have a sheet of paper having width k and, using a pair of scissors, we make k-strips of width 1. We may cut several sheets of the same width at once by piling them. Then the problem is to minimize the total number of cuts. Clearly it is at most $k - 1$. Now we show one strategy for the problem. Consider the binary expansion of k:

$$k = 2^{\mu_r} + 2^{\mu_{r-1}} + \cdots + 2^{\mu_1},$$

for some integers $\mu_r > \mu_{r-1} > \cdots > \mu_1 \geq 0$. We set $\mu = \mu_r$. Then *the number of digits* ('1' or '0') is given by $\mu + 1$, while r is *the number of units*, '1', appearing in the binary expansion. Then first we cut the sheet into r sheets of width $2^{\mu}, 2^{\mu_{r-1}}, \ldots, 2^{\mu_1}$ by $(r-1)$-cuts. Second divide the sheet of width 2^{μ} into sheets of width $2^{\mu_{r-1}}$ by $\mu - \mu_{r-1}$-cuts. Third divide the piled sheets of width $2^{\mu_{r-1}}$ into sheets of width $2^{\mu_{r-2}}$ by $\mu_{r-1} - \mu_{r-2}$-cuts, and so on. Iterating the process, we have sheets of width 2^{μ_1}, which we divide into strips of width 1 by μ_1-cuts finally. The total number of cuts by this method is given by $\mu + r - 1$.

Lemma 4.24 ([5]) *For a positive integer k, we have*

$$\mathrm{LC}(x^k) \leq \mu + r - 1.$$

Proof) In general, the process of cutting of a pile of sheets of width ℓ into those of width ℓ' and $\ell'' = \ell - \ell'$ respectively is translated into, for a constant λ,

$$d(\lambda x^{\ell}) = \lambda d(x^{\ell}) = \lambda x^{\ell''} d(x^{\ell'}) + \lambda x^{\ell'} d(x^{\ell''}).$$

Here we use Leibniz rule just once and the linearity of the derivation. A piling is realized by just the distributive law of the module structure. Therefore the method described above in the strip problem implies the estimate of Leibniz complexity. ■

Lemma 4.25 *Take μ as a number of digits of k and r as a number of units of k (For $k = \sum_{j=1}^{r} 2^{p_i}$ $(0 \leq p_i \leq \ldots \leq p_r,\ \mu = r)$).*

$$LComp_{C^\infty(\mathbb{R})}(x^k) \leq \mu + r - 1 \tag{61}$$

Proof) We have $d(x^{2^{\mu}}) = 2^{\mu} x^{2^{\mu}-1} d(x)$, in $\Omega_{C^\infty(\mathbb{R}),\mathbb{R}}$ for using μ Leibniz rules $Prod_{C^\infty(\mathbb{R})}(x^{2^{\mu'-1}}, x^{2^{\mu'-1}}) = d(x^{2^{\mu'}}) - 2x^{2^{\mu'-1}} d(x^{2^{\mu'-1}}) (\mu' = 1, \ldots, \mu)$.

We have

$$d(x^{\sum_{j=1}^{r} 2^{p_j}}) = \sum_{j=1}^{r} x^{\sum_{j=1, j \neq i}^{r} 2^{p_j}} d(x^{2^{p_j}})$$

in $\Omega_{C^\infty(\mathbb{R}),\mathbb{R}}$ for using $r-1$ Leibniz rules $Prod_{C^\infty(\mathbb{R})}(x^{2^{p_s}}, x^{\sum_{j=1}^{s-1} 2^{p_j}}) = d(x^{\sum_{j=1}^{s} 2^{p_j}}) - x^{\sum_{j=1}^{s-1} 2^{p_j}} d(x^{2^{p_s}}) - x^{2^{p_s}} d(x^{\sum_{j=1}^{s-1} 2^{p_j}}) (s = 2, \ldots, r)$.

Therefore, we have $\mathrm{LComp}_{C^\infty(\mathbb{R})}(x^k) \leq \mu + r - 1$. ■

Lemma 4.26 ([5]) *For $f \in N^\infty(U)$ and a natural number $k \geq 1$, we have* $\mathrm{LC}(f^k) \leq \mathrm{LC}(f) + \mathrm{LC}(x^k)$.

Proof) If f is a constant function, then $\mathrm{LC}(f^k) = 0$, so the inequality holds trivially. We suppose f is not a constant function. By definition, for some non-zero $g \in N^\infty(\mathbb{R})$, $gd(x^k)$ is deformed into $g\, kx^{k-1}d(x)$ in $\Omega_{C^\infty(\mathbb{R}),\mathbb{R}}$ using Leibniz rules $\mathrm{LC}(x^k)$-times. Using the same procedure, $(g \circ f)d(f^k(x))$ is deformed into $(g \circ f)kf^{k-1}d(f(x))$ in $\Omega_{C^\infty(U)}$ using Leibniz rules $\mathrm{LC}(x^k)$-times. Note that $g \circ f$ is non-zero in $N^\infty(U)$. Moreover, using Leibniz rules $\mathrm{LC}(f)$ times, $h(g \circ f)kf^{k-1}d(f(x))$ is deformed into $h(g \circ f)\sum_{i=1}^n kf^{k-1}(\frac{\partial f}{\partial x_i})d(x_i)$ for some non-zero $h \in N^\infty(U)$. Since $g \circ f$ is non-zero, $h(g \circ f)$ is non-zero. ■

Lemma 4.27 *For $f \in \mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$ and $k \in \mathbb{N}$, we have $Comp_{\mathfrak{C}}(f^k) \leq Comp_{\mathfrak{C}}(f) + Comp_{\mathfrak{C}}(x^k)$.*

Proof) $d\big(\Phi_{f^k}(c)\big) = d\big((\Phi_f(c))^k\big)$ is equals to $k\Phi_f(c)^k d\big(\Phi_f(c)\big)$ by using Leibniz rules $\mathrm{Comp}_{\mathfrak{C}}(x^k)$ times. It equals to $k\Phi_f(c)^k \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c)d(c_i)$ by using Leibniz rules $\mathrm{Comp}_{\mathfrak{C}}(f)$ times.

Therefore, we have $\mathrm{Comp}_{\mathfrak{C}}(f^k) \leq \mathrm{Comp}_{\mathfrak{C}}(f) + \mathrm{Comp}_{\mathfrak{C}}(x^k)$. ■

Lemma 4.28 *For $f \in \mathfrak{L}\mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$ and $k \in \mathbb{N}$, we have $LComp_{\mathfrak{C}}(f^k) \leq LComp_{\mathfrak{C}}(f) + LComp_{\mathfrak{C}}(x^k)$.*

Proof) There exists non-zero actions $\Phi_g : \mathfrak{C}^n \to \mathfrak{C}$, $\Phi_h : \mathfrak{C} \to \mathfrak{C}$ such that $\Phi_g(c_1, \ldots, c_n)d\big(\Phi_f(c_1, \ldots, c_n)\big)$, $\Phi_h(c')d(c'^k)$ are equal to $\Phi_g(c_1, \ldots, c_n)\sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n)d\big(c_i\big)$, $\Phi_h(c')kc'^{k-1}d(c')$ by using Leibniz rules $\mathrm{LComp}_{\mathfrak{C}}(f)$, $\mathrm{LComp}_{\mathfrak{C}}(x^k)$ times for any $c_1, \ldots, c_n, c' \in \mathfrak{C}$.

$\Phi_{(h \circ f)g}(c) = \Phi_h(\Phi_f(c))\Phi_g(c) : \mathfrak{C}^n \to \mathfrak{C}$ is a non-zero action.
Then $\Phi_h(\Phi_f(c))\Phi_g(c)d\big(\Phi_{f^k}(c)\big)$ is equals to $k\Phi_h(\Phi_f(c))\Phi_g(c)\Phi_f(c)^{k-1}d\big(\Phi_f(c)\big)$ by using Leibniz rules $\mathrm{LComp}_{\mathfrak{C}}(f)$. It equals to $k\Phi_h(\Phi_f(c))\Phi_g(c)\Phi_f(c)^{k-1}\sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c)d(c_i)$ by using Leibniz rules $\mathrm{LComp}_{\mathfrak{C}}(x^k)$.

Therefore, we have $\mathrm{LComp}_{\mathfrak{C}}(f^k) \leq \mathrm{LComp}_{\mathfrak{C}}(f) + \mathrm{LComp}_{\mathfrak{C}}(x^k)$. ■

Remark 4.29 ([5]) The estimate in Lemma 4.24 is, by no means, best possible. For example, let $k = 31$. Then $31 = 2^4 + 2^3 + 2^2 + 2^1 + 2^0$. Therefore $r = 5$ and $\mu = 4$. So the estimate gives us that $\mathrm{LC}(x^{31}) \leq 8$. However $\mathrm{LC}(x^{31}) \leq 6$. In fact, since $32 = 2^5$, we have by Lemma 4.24,

$$xd(x^{31}) = d(x^{32}) - x^{31}d(x) = 32x^{31}d(x) - x^{31}d(x) = 31x^{31}d(x),$$

by using Leibniz rule 6 times. Then algebraically we have $d(x^{31}) = 31x^{30}d(x)$.

As above, we consider "the problem of strips" starting from several number of sheets, say, s, having width k_s, k_{s-1}, and k_1 respectively. Then we have

Lemma 4.30 $p(x) := \sum_{i=1}^{s} a_i x^{k_i} (a_i \in \mathbb{R}\backslash\{0\}, 0 \leq k_1 < \ldots < k_s)$.

Take μ_i as a number of digits of k_i and r_i as a number of units of k_j (For $k_i = \sum_{j=1}^{r_i} 2^{p_{i,j}} (0 \leq p_{i,1} \leq \ldots \leq p_{i,r_i} = \mu_i)$). Then we have

$$LComp_{C^\infty(\mathbb{R})}(p(x)) \leq \mu + \sum_{i=1}^{s} (r_i - 1). \tag{62}$$

Proof) We have

$$d(x^{\sum_{j=1}^{r_i} 2^{p_{i,j}}}) = \sum_{j=1}^{r_i} x^{\sum_{j=1, j\neq i}^{r_i} 2^{p_{i,j}}} d(x^{2^{p_{i,j}}})$$

in $\Omega_{C^\infty(\mathbb{R}),\mathbb{R}}$ for using $r_i - 1$ Leibniz rules $d(x^{\sum_{j=1}^{s} 2^{p_{i,j}}}) = x^{\sum_{j=1}^{s-1} 2^{p_{i,j}}} d(x^{2^{p_{i,s}}}) + x^{2^{p_{i,s}}} d(x^{\sum_{j=1}^{s-1} 2^{p_{i,j}}}) (s = 2, \ldots, r_i)$. We have the sum of $d(x^{2^p}) (p = 1, \ldots, \mu)$.

We have $d(x^{2^\mu}) = 2^\mu x^{2^\mu - 1} d(x)$ in $\Omega_{C^\infty(\mathbb{R}),\mathbb{R}}$ for using μ Leibniz rules $d(x^{2^{\mu'}}) = 2x^{2^{\mu'-1}} d(x^{2^{\mu'-1}}) (\mu' = 1, \ldots, \mu)$.

Therefore, we have $\mathrm{LComp}_{C^\infty(\mathbb{R})}(p(x)) \leq \mu + \sum_{i=1}^{s} (r_i - 1)$. ■

We estimate the Leibniz complexity for a polynomial of n-variables. Let $p(x) = p(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$. We set $p(x) = \sum b_\alpha x^\alpha, b_\alpha \in \mathbb{R}$, by using multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers. It is trivial that $\mathrm{LC}(p)$ is at most the total number of multiplications of variables:

$$\sum_{b_\alpha \neq 0} \max\{|\alpha| - 1,\ 0\}.$$

Instead we consider the number

$$\sigma(p) := \sum_{b_\alpha \neq 0} \max\{\#\{i \mid 1 \leq i \leq n, \alpha_i > 0\} - 1,\ 0\},$$

which is needed just to separate the variables on differentiation, and we try to save the additional usage of Leibniz rule.

Suppose that, by arranging terms with respect to x_i for each $i, 1 \leq i \leq n$,

$$p(x) = a_{i,s(i)} x_i^{k_{i,s(i)}} + a_{i,s(i)-1} x_i^{k_{i,s(i)-1}} + \cdots + a_{i,1} x_i^{k_{i,1}},$$

where $a_{i,j}$ is a non-zero polynomial of $x_1, \dots, x_n$ without x_i, $(1 \leq j \leq s(i))$, and $k_{i,s(i)} > k_{i,s(i)-1} > \cdots > k_{i,1} \geq 0$. The maximal exponent $k_{i,s(i)}$ is written as $\deg_{x_i} p$, the degree of p in the variable x_i. For the binary expansion of $\deg_{x_i} p$, let μ_i denote (the number of digits of $\deg_{x_i} p$) -1. Moreover let $r_{ij}, 1 \leq j \leq s(i)$ denote the number of units of the exponent k_{ij} for the binary expansion. Then we have

Lemma 4.31 ([5]) By using the linearly, $\boldsymbol{d}(c) = 0, c \in \mathbb{R}$, and Leibniz rule $\sigma(p) + \sum_{i=1}^{n} \Big(\mu_i + \sum_{j=1}^{s(i)} (r_{ij} - 1)\Big)$-times, we have $\boldsymbol{d}(p) = \sum_{i=1}^{n} (\frac{\partial p}{\partial x_i}(x)) \boldsymbol{d}(x_i)$ in $\Omega_{\mathcal{E}(U)}$. In particular we have the estimate

$$\mathrm{LC}(p) \leq \sigma(p) + \sum_{i=1}^{n} \left(\mu_i + \sum_{j=1}^{s(i)} (r_{ij} - 1) \right).$$

Lemma 4.32 *$p(x) := \sum_{\alpha \neq 0} b_\alpha x^\alpha (b_\alpha \in \mathbb{R} \backslash \{0\}, \alpha = (\alpha_1, \dots, \alpha_n) \in (\{0\} \cup \mathbb{N})^n)$.*

We can write $p(x) = \sum_{j=1}^{s(i)} a_{i,j} x_i^{k_{i,j}}$ for $i = 1, \dots, n$ ($a_{i,j}$ is a non-zero real polynomial of $x_1, \dots, \hat{x}_i, \dots, x_n$ for $1 \leq j \leq s(i)$, $k_{i,s(i)} > k_{i,s(i)-1} > \dots > k_{i,1} \geq 0$).

Let $\mu_{i,j}$ be a number of digits of $k_{i,j}$ and $r_{i,j}$ as a number of units of $k_{i,j}$ (For $k_{i,j} = \sum_{k=1}^{r_{i,j}} 2^{p_{i,j,k}} (0 \leq p_{i,j,1} \leq \dots \leq p_{i,j,r_{i,j}} = \mu_{i,j})$). Define

$$\sigma(p(x)) := \sum_{b_\alpha \neq 0} \max\big\{ \#\{i | 1 \leq i \leq n, \alpha_i > 0\} - 1, 0 \big\}. \tag{63}$$

Then we have

$$LComp_{C^\infty(\mathbb{R}^n)}(p(x)) \leq \sigma(p(x)) + \sum_{i=1}^{n} (\mu_{i,s(i)} + \sum_{j=1}^{s(i)} (r_{i,j} - 1)). \tag{64}$$

Proof) For any $\alpha \neq 0 (b_\alpha \neq 0)$, we have $d(x^\alpha) = \sum_{i=1}^{n} x^{\alpha - \alpha_i e_i} d(x_i^{\alpha_i})$ in $\Omega_{C^\infty(\mathbb{R}^n), \mathbb{R}}$ for using $\#\{i | 1 \leq i \leq n, \alpha_i > 0\} - 1$ Leibniz rules. Then, we can calculate from $d(p(x))$ to a sum of $d(x_i^{k_{i,j}})$ for using $\sigma(p(x))$ Leibniz rules.

For $d(x_i^{k_{i,j}})$, we have

$$d(x^{\sum_{j=1}^{r_i} 2^{p_{i,j}}}) = \sum_{j=1}^{r_i} x^{\sum_{j=1, j \neq i}^{r_i} 2^{p_{i,j}}} d(x^{2^{p_{i,j}}})$$

in $\Omega_{C^\infty(\mathbb{R}),\mathbb{R}}$ for using $r_{i,j}-1$ Leibniz rules $d(x^{\sum_{j=1}^{s} 2^{p_{i,j}}}) = x^{\sum_{j=1}^{s-1} 2^{p_{i,j}}} d(x^{2^{p_{i,s}}}) + x^{2^{p_{i,s}}} d(x^{\sum_{j=1}^{s-1} 2^{p_{i,j}}})(s = 2,\ldots,r_{i,j})$. Then, we can calculate from the sum of $d(x_i^{k_{i,j}})$ to a sum of $d(x_i^{2^{p_{i,j,k}}})$ for using $\sum_{i=1}^{n}\sum_{j=1}^{s(i)}(r_{i,j}-1)$ Leibniz rules.

We have the sum of $d(x^{2^p})(p = 1,\ldots,\mu)$.

We have $d(x^{2^\mu}) = 2^\mu x^{2^\mu - 1} d(x)$ in $\Omega_{C^\infty(\mathbb{R}),\mathbb{R}}$ for using μ Leibniz rules $d(x^{2^{\mu'}}) = 2x^{2^{\mu'-1}} d(x^{2^{\mu'-1}})(\mu' = 1,\ldots,\mu)$.

Therefore, we have $\mathrm{LComp}_{C^\infty(\mathbb{R}^n)}(p(x)) \leq \sigma(p(x)) + \sum_{i=1}^{n}(\mu_{i,s(i)} + \sum_{j=1}^{s(i)}(r_{i,j} - 1))$. ■

Proposition 4.33 ([5]) *Under the above notations, we have the estimate*

$$\mathrm{LC}(f) \leq \sigma(P) + \sum_{i=0}^{n}\left(\mu_i + \sum_{j=1}^{s(i)}(r_{ij}-1)\right).$$

In particular we have

$$\mathrm{LC}(f) \leq \sigma(P) + \sum_{i=0}^{n}\{(\deg_{x_i} P + 2)(\log_2(\deg_{x_i} P) - 1)\} + n + 1.$$

4.5.6 Leibniz complexities for Nash functions

Lemma 4.34 *$f(x) : U \to \mathbb{R}$ is a Nash function on U.*

There exists a non zero real polynomial $p(x) := \sum_{\alpha\neq 0} b_\alpha x^\alpha (b_\alpha \in \mathbb{R}\backslash\{0\}, \alpha = (\alpha_1,\ldots,\alpha_n) \in (\{0\}\cup\mathbb{N})^n)$. We have

$$LComp_{C^\infty(U)}(f(x)) \leq LComp_{C^\infty(\mathbb{R}^n)}(p(x)). \tag{65}$$

We can write $p(x) = \sum_{j=1}^{s(i)} a_{i,j} x_i^{k_{i,j}}$ for $i = 1,\ldots,n$ ($a_{i,j}$ is a non-zero real polynomial of $x_1,\ldots,\hat{x}_i,\ldots,x_n$ for $1 \leq j \leq s(i)$, $k_{i,s(i)} > k_{i,s(i)-1} > \ldots > k_{i,1} \geq 0$).

Let $\mu_{i,j}$ be a number of digits of $k_{i,j}$ and $r_{i,j}$ as a number of units of $k_{i,j}$ (For $k_{i,j} = \sum_{k=1}^{r_{i,j}} 2^{p_{i,j,k}} (0 \leq p_{i,j,1} \leq \ldots \leq p_{i,j,r_{i,j}} = \mu_{i,j})$). Define

$$\sigma(p(x)) := \sum_{b_\alpha\neq 0} \max\bigl\{\#\{i | 1 \leq i \leq n, \alpha_i > 0\} - 1, 0\bigr\}.$$

Then we have

$$LComp_{C^\infty(\mathbb{R}^n)}(p(x)) \leq \sigma(p(x)) + \sum_{i=1}^{n}\bigl(\mu_{i,s(i)} + \sum_{j=1}^{s(i)}(r_{i,j}-1)\bigr). \tag{66}$$

Example 4.35 ([5]) *Let* $n = 1, f = \frac{1}{\sqrt{x^2+1}}$ *and* $P(x, y) = y^2 - x^2 - 1$. *Then* $\sigma(P) = 0, \mu_0 = \mu_1 = 1$ *and* $r_{ij} = 1$. *Therefore the first inequality gives us that* $\mathrm{LC}(f) \leq 2$ *as is seen in Introduction.*

Proof of Proposition 4.33.
We write the right hand side by ψ of the first inequality. By Lemma 4.31, we have, by using Leibniz rule ψ-times,

$$\boldsymbol{d}(P(x,y)) = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(x,y)\boldsymbol{d}x_i + \frac{\partial P}{\partial y}(x,y)\boldsymbol{d}y,$$

modulo several linearity relations and $\boldsymbol{d}c, c \in \mathbb{R}$ in $\Omega_{\mathcal{E}(U\times\mathbb{R})}$. Then, substituting y by f, we have that

$$0 = \boldsymbol{d}(P(x,f)) = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(x,f)\boldsymbol{d}x_i + \frac{\partial P}{\partial y}(x,f)\boldsymbol{d}f,$$

in $\Omega_{\mathcal{E}(U)}$, therefore that

$$\frac{\partial P}{\partial y}(x,f)\left(\boldsymbol{d}f - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\boldsymbol{d}x_i\right) = 0,$$

in $\Omega_{\mathcal{E}(U)}$, by using Leibniz rule at most ψ-times. Thus we have the first inequality. The second equality is obtained from the first equality combined with the inequalities derived by the definitions:

$$2^{\mu_i} \leq \deg_{x_i} P < 2^{\mu_i+1},\ \ s(i) \leq \deg_{x_i} P + 1,\ \text{and}\ \ r_{ij} \leq \mu_i,$$

$(1 \leq j \leq s(i), 0 \leq i \leq n)$. ■

In [12], the complexity $\mathrm{C}(f)$ of a Nash function f is defined as the minimum the total degree $\deg P$ of non-zero polynomials $P(x,y)$ with $P(x,f) = 0$. Moreover we define

$$\mathrm{S}(f) := \min\left\{\sigma(P \circ \psi)\Big|_{\psi \text{ is an affine isomorphism on } \mathbb{R}^{n+1}}^{P(x,f)=0, \deg P = \mathrm{C}(f),}\right\},$$

i.e. the minimum of the number σ for any defining polynomial P of f with minimal total degree under any choice of affine coordinates. We can regard $\mathrm{S}(f)$ a complexity for the separation of variables in differentiation of f. Then we have the following result:

Corollary 4.36 ([5]) *Let $f \in \mathcal{N}(U)$ be a Nash function on a connected open set $U \subset \mathbb{R}^n$. Then we have an estimate on the Leibniz complexity* $\mathrm{LC}(f)$ *by the Ramanakoraisina's complexity* $\mathrm{C}(f)$ *and another complexity* $\mathrm{S}(f)$*,*

$$\mathrm{LC}(f) \leq \mathrm{S}(f) + (n+1)(\mathrm{C}(f)+2)(\log_2 \mathrm{C}(f) - 1) + n + 1.$$

Proof) Since $\deg_{x_i} P \leq \mathrm{C}(f)\,(0 \leq i \leq n)$ we have the above estimate by Proposition 4.33 and Lemma 4.20. ∎

Naturally we would like to pose a problem to obtain any lower estimate of Leibniz complexity.

4.5.7 Leibniz complexities

Let $\mathfrak{C} := C^\infty(\mathbb{R}^n)$ and $\mathfrak{F}_\mathfrak{C}$ be a free $\mathfrak{C}$-module generated by $d(c)$ for $c \in \mathfrak{C}$.

Define an $\mathbb{R}$-cotangent module $\Omega_{\mathfrak{C},\mathbb{R}} := \mathfrak{F}_\mathfrak{C}/\mathfrak{M}_{\mathfrak{C},\mathbb{R}}$ and its $\mathbb{R}$-derivationas $d_\mathfrak{C} : \mathfrak{C} \to \Omega_{\mathfrak{C},\mathbb{R}}$ as $d_\mathfrak{C}(c) := d(c) + \mathfrak{M}_{\mathfrak{C},\mathbb{R}}$.

They satisfy $\mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n) \subset \mathfrak{L}\mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$.

Proposition 4.37 *(1) $f(x) := \frac{1}{p(x)}$ ($p(x) \in \mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$ such that $p(x) \neq 0$ for any $x \in \mathbb{R}^n$). We have $f(x) \in \mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$.*

(2) For $p(x), q(x) \in \mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$, we have $f(x) := p(x)q(x) \in \mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$.

(3) For $p(y_1, \ldots, y_m) \in \mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^m)$ and $q_1(x), \ldots, q_m(x) \in \mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$, $f(x) = p(q_1(x), \ldots, q_m(x)) \in \mathfrak{N}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$.

Proof)

(1) We have $f(x)p(x) \equiv 1$. On $\Omega_{C^\infty(\mathbb{R}^n),\mathbb{R}}$,

$$\begin{aligned}
0 = d(f(x)p(x)) &= f(x)d(p(x)) + p(x)d(f(x)) \\
&= \sum_{i=1}^n f(x)\frac{\partial p}{\partial x_i} d(x_i) + p(x)d(f(x)), \\
p(x)d(f(x)) &= -\sum_{i=1}^n f(x)\frac{\partial p}{\partial x_i} d(x_i), \\
d(f(x)) &= -\sum_{i=1}^n \left(\frac{f(x)}{p(x)}\frac{\partial p}{\partial x_i}\right) d(x_i) \\
&= \sum_{i=1}^n \left(-\frac{1}{p(x)^2}\frac{\partial p}{\partial x_i}\right) d(x_i) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} d(x_i).
\end{aligned}$$

Therefore, $Func_{\mathfrak{C}}(\frac{1}{p}(x), c_1, \dots, c_n)$ is generated by $Prod_{\mathfrak{C}}(\cdot,\cdot)$, $Sum_{\mathfrak{C}}(\cdot,\cdot)$, $Sca_{\mathfrak{C}}(\cdot,\cdot)$ and $\frac{1}{p}(x)$ is in $\mathfrak{N}^{\infty}_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$.

(2) We have $d(p(x)) = \sum_{i=1}^{n} \frac{\partial p}{\partial x_i}(x) d(x_i)$ and $d(q(x)) = \sum_{i=1}^{n} \frac{\partial q}{\partial x_i}(x) d(x_i)$ in $\Omega_{C^\infty(\mathbb{R}^n),\mathbb{R}}$. Then we have

$$\begin{aligned} d\big(p(x)q(x)\big) &= p(x) d\big(q(x)\big) + q(x) d\big(p(x)\big) \\ &= p(x)\sum_{i=1}^{n} \frac{\partial q}{\partial x_i}(x) d(x_i) + q(x) \sum_{i=1}^{n} \frac{\partial p}{\partial x_i}(x) d(x_i) \\ &= \sum_{i=1}^{n} \Big\{ p(x) \frac{\partial q}{\partial x_i}(x) + q(x) \frac{\partial p}{\partial x_i}(x) \Big\} d(x_i). \end{aligned}$$

(3) For $p(y_1, \dots, y_m) \in \mathfrak{N}^{\infty}_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^m)$, $q_1(x), \dots, q_m(x) \in \mathfrak{N}^{\infty}_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$, we have $d(p(y)) = \sum_{i=1}^{m} \frac{\partial p}{\partial y_i}(y) d(y_i)$ in $\Omega_{C^\infty(\mathbb{R}^m),\mathbb{R}}$. and $d(q_k(x)) = \sum_{j=1}^{n} \frac{\partial q_k}{\partial x_j}(x) d(x_j)$ in $\Omega_{C^\infty(\mathbb{R}^n),\mathbb{R}}$.

$$\begin{aligned} d(p(q_1(x), \dots, q_m(x))) &= \sum_{i=1}^{m} \frac{\partial p}{\partial y_i}((q_1(x), \dots, q_m(x)) d(q_i) \\ &= \sum_{i=1}^{m} \frac{\partial p}{\partial y_i}((q_1(x), \dots, q_m(x)) \sum_{j=1}^{n} \frac{\partial q_i}{\partial x_j}(x) d(x_j) \\ &= \sum_{j=1}^{n} \{ \sum_{i=1}^{m} \frac{\partial p}{\partial y_i}((q_1(x), \dots, q_m(x)) \frac{\partial q_i}{\partial x_j}(x) \} d(x_j) \\ &= \sum_{j=1}^{n} \{ \frac{\partial p(q_1, \dots, q_m)}{\partial x_j}(x) \} d(x_j). \end{aligned}$$

■

Proposition 4.38 *(1) For a Nash function $f(x) \in C^\infty(\mathbb{R}^n)$, we have $f(x) \in \mathfrak{LN}^{\infty}_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$.*

(2) $f(x) := \frac{1}{p(x)}$ ($p(x) \in \mathfrak{LN}^{\infty}_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$ such that $p(x) \neq 0$ for some $x \in \mathbb{R}^n$). We have $f(x) \in \mathfrak{LN}^{\infty}_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$.

(3) For $p(x), q(x) \in \mathfrak{LN}^{\infty}_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$, we have $f(x) := p(x)q(x) \in \mathfrak{LN}^{\infty}_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$.

Proof)

(1) From Theorem 4.15, we have a non-zero real polynomial $p(x, y) \in \mathbb{R}[x, y]$ such that $p(x, f(x)) = 0$ and $\frac{\partial p}{\partial y}(x, , f(x)) \neq 0$. And we have

$$\frac{\partial p}{\partial y}(x, f(x))\Big(d(f) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)d(x_i)\Big) = 0$$

on $\Omega_{C^\infty(\mathbb{R}^n),\mathbb{R}}$. Therefore, we have $f \in \mathfrak{LN}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$

(2) Suppose that $g \in C^\infty(\mathbb{R}^n)$ is a smooth function which satisfies that $\Phi_g(c)Func_{\mathfrak{C}}(p(x), c_1, \ldots, c_n)$ is generated by $Prod_{\mathfrak{C}}(\cdot,\cdot)$, $Sum_{\mathfrak{C}}(\cdot,\cdot)$, $Sca_{\mathfrak{C}}(\cdot,\cdot)$. We have $f(x)p(x) \equiv 1$. On $\Omega_{C^\infty(\mathbb{R}^n),\mathbb{R}}$,

$$\begin{aligned} 0 = d(f(x)p(x)) &= f(x)d(p(x)) + p(x)d(f(x)) \\ p(x)d(f(x)) &= -\frac{1}{p(x)}d(p(x)) \\ p(x)^2 d(f(x)) &= -d(p(x)). \end{aligned}$$

Therefore, $\Phi_{g(x)p(x)^2}(c)Func_{\mathfrak{C}}(\frac{1}{p}(x), c_1, \ldots, c_n)$ is generated by $Prod_{\mathfrak{C}}(\cdot,\cdot)$, $Sum_{\mathfrak{C}}(\cdot,\cdot)$, $Sca_{\mathfrak{C}}(\cdot,\cdot)$ and $\frac{1}{p}(x)$ is in $\mathfrak{LN}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$.

(3) For any $c_1, \ldots, c_n \in \mathfrak{C}$, there exists non-zero functions $g, h \in C^\infty(\mathbb{R}^n)$ such that $\Phi_g(c)d(\Phi_p(c)) = \sum_{i=1}^n \Phi_{g\cdot\frac{\partial p}{\partial x_i}}(c)d(c_i)$
and $\Phi_h(c)d(\Phi_q(c)) = \sum_{i=1}^n \Phi_{h\cdot\frac{\partial q}{\partial x_i}}(c)d(c_i)$ in $\Omega_{\mathfrak{C},\mathbb{R}}$. Then we have

$$\begin{aligned} \Phi_{gh}(c)d\big(\Phi_{pq}(c)\big) &= \Phi_{hp}(c)\Phi_g(c)d\big(\Phi_q(c)\big) + \Phi_{gq}(c)\Phi_h(c)d\big(\Phi_p(c)\big) \\ &= \Phi_{hp}(c)\Phi_g(c)\sum_{i=1}^{n}\Phi_{\frac{\partial q}{\partial x_i}}(c)d(c_i) + \Phi_{gq}(c)\Phi_h(c)\sum_{i=1}^{n}\Phi_{\frac{\partial p}{\partial x_i}}(c)d(c_i) \\ &= \Phi_{gh}(c)\sum_{i=1}^{n}\Big\{\Phi_{p\frac{\partial q}{\partial x_i}+q\frac{\partial p}{\partial x_i}}(c)\Big\}d(c_i). \end{aligned}$$

in $\Omega_{\mathfrak{C},\mathbb{R}}$. Therefore, we have that $pq \in \mathfrak{LN}^\infty_{\mathfrak{C},\mathbb{R}}(\mathbb{R}^n)$.

■

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