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Research on Statistical Mechanics of Labor Markets Based on Simulation Substantiated by Actual Data

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Abstract

This thesis provides theoretical studies on labor markets incorporating econophysics through statistical mechanical approaches.

Physical science researchers have long had a reputation of meddling in areas outside the restricted domain of physical science. In some cases, the non-traditional approaches that researchers introduce can result in new insights, and in due course, this new inter-disciplinary area of research can become a recognized part of science. However, econophysics, the use of physical theories and methods developed by physicists to understand economic phenomena and solve economic problems, leads to a development of interdisciplinary research field. It is rooted in statistical mechanics, and some of its applications to the study of economics are usually including uncertainty or stochastic processes and nonlinear dynamics.

Based on the statistical mechanics, we introduce a probabilistic agent-based model of labor markets for university graduates, in particular, in Japan. To make an effective model of the market, with several hypotheses, we construct the local energy function of each company and describe the probability that an arbitrary company gathers students at each business year by a Boltzmann-Gibbs distribution.

Inspired by the unsupervised learning or self-organization in the machine learning context, we attempt to draw a ‘learning curve’ for the collective behavior of job-seeking ‘non-intelligent’ labors in successive job-hunting processes. In our model, the diversity of students’ behavior is built-in by means of the Jaynes-Shannon’s Max-Ent(Maximum Entropy) principle. We discuss the speed of convergence for the error-measurement, where we consider a scenario in which the students do not use any information about the result of job-hunting in a previous process. Our approach enables us to examine the existence of the condition on which macroscopic quantity, say, ‘stage-wise unemployment rate’ becomes ‘scale-invariant’ in the sense that it does not depend on the job-hunting stage. From the macroscopic view point, the problem could be regarded as a human resource allocation.

We provide a mathematical model to investigate the human resource allocation problem for students. The basic model is described by the Potts spin glass which is well known in the research field of statistical physics. In the model, each Potts spin represents the action of one student, and it takes a discrete variable corresponding to the company he/she applies for. We construct the energy to include the distinct effects on students’ behavior. The correlations (the adjacent matrix) between students are taken into account through the pairwise spin-spin interactions. We carry out computer simulations to examine the efficiency of the model. We also show that some chiral representation of the Potts spin enables us to obtain some analytical

insights into our labor markets.

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Chapter 1

Introduction

In recent Japan, deterioration of the employment rate is now one of the most serious problems [1, 2]. Actually, various attempts to overcome these difficulties have been done by central or local governments. Apparently, labor (work) is important not only for each of us to earn our daily bread, but also for our state to keep the revenues by collecting the taxes from labors. Especially, the employment rate is getting worse and the government has over-issued quite a lot of government bonds to compensate a lack of the tax revenues and the national debt is now becoming a serious risk to cause a national-wide bankruptcy.

Furthermore, humans (or labors) are the most important resources in our society. This is because they can produce not only various products and services in the society, but also they contribute to the society by paying their taxes. For this reason, in each scale of society, for example, from nation to companies or much smaller communities such as laboratory (or research group) of university, allocation of human resources is one of the essential problems. Needless to say, such appropriate allocation of human resource is regarded as a ‘matching problem’ between individuals and some ‘groups’ such as companies, and the difference among individuals in their abilities or preference makes the problem difficult. A typical example of the human resource allocation is found in simultaneous recruiting of new graduates in Japan. Students who are looking for their jobs might research several candidates of companies to enter and send the applications through the web site. However, the students incline to apply to

well-established companies, whereas they do not like to get a job in relatively small companies. This fact enhances the so-called ‘mismatch’ between labors (students) and companies. We can easily see the situation of job searching process in Japan. At the job fair, we find that some booths could collect a lot of students. On the other hand, some other booths could not attract the students’s attentions. Therefore, the job-matching itself is apparently governed by some ‘collective behavior’ of students. Namely, each student seems to behave by looking at their ‘neighbors’ and adapting to the ‘mood’ in their community, or they sometimes can share the useful information (of course, such information is sometimes extremely ‘biased’) about the market via Internet or social networking service.

To consider the effective policy and to carry out it for sweeping away the unemployment uncertainty, it seems that we should investigate the labor markets scientifically and if possible, one should simulate artificial labor markets in computer to reveal the essential features of the problem. In fact, in macroeconomics (labor science), there exist a lot of effective attempts to discuss the macroscopic properties [3, 4, 5, 6, 7, 8] including so-called search theory [9, 10, 11, 12]. However, apparently, the macroscopic approaches lack of their microscopic view points, in their arguments, the behavior of microscopic heterogeneous agents such as labors or companies are neglected.

Motivated by the above background and requirement, in this thesis, we shall propose a probabilistic model based on the concept of statistical mechanics for stochastic labor markets, in particular, Japanese labor markets for university (college) graduates and investigate the behavior analytically.

This thesis is organized as follows. In the next chapter, we introduce several important concepts used in this paper. In particular, we account for two distinct quantities, namely, ‘cumulative’ and ‘stage-wise’ quantities. The generic properties of them would be explained. In the same section, we also show several important empirical findings. In Chap. 3, we introduce our probabilistic labor market which is constructed by means of Jaynes-Shannon’s MaxEnt principle. The numerical results are given in the next Chap. 4. In Chap. 5, we introduce the so-called Potts model, and explain several macroscopic quantities. Our preliminary results for several

job-searching and selection scenarios by students and companies The last chapter is devoted to summary.

Chapter 2

Empirical evidence in labor markets

2.1 The Philips curve

Let us first mention the relationship between the unemployment and the inflation rates in recent Japan (from 1970s to 2000s) in empirical evidence of the labor market. This relationship is generally called as *Philips curve*. In the original paper of Philips [13], he found the relation for the empirical data set from the middle of 19th century to the beginning of 20th century (1861-1913) in UK. However, up to now, a lot of verifications have been done for various data sets in various countries. Therefore, we are confirmed that the Philips curve should be regarded as one of the ‘universal properties’ in labor markets.

In Fig. 2-1, we plot the Philips curves of Japan in 1980s, 90s, 2000s and the age-dependence of the curve (the lower-right panel) [14]. In these panels, the fitting curve (the solid curve): $\pi + b \propto U^{-c}$ is actually obtained by the least square estimation for the parameter b and c from the n -data points $(X_i, Y_i) = (\log U_i, \log(\pi_i + b))$, $i = 1, \dots, n$. From the lower-right panel in Fig. 2-1, we clearly find that the curve is slowly getting ‘flat’.

From these empirical findings, we easily notice that the curve changes (evolves) in time and the behavior might be dependent on the situation of economy in the country.

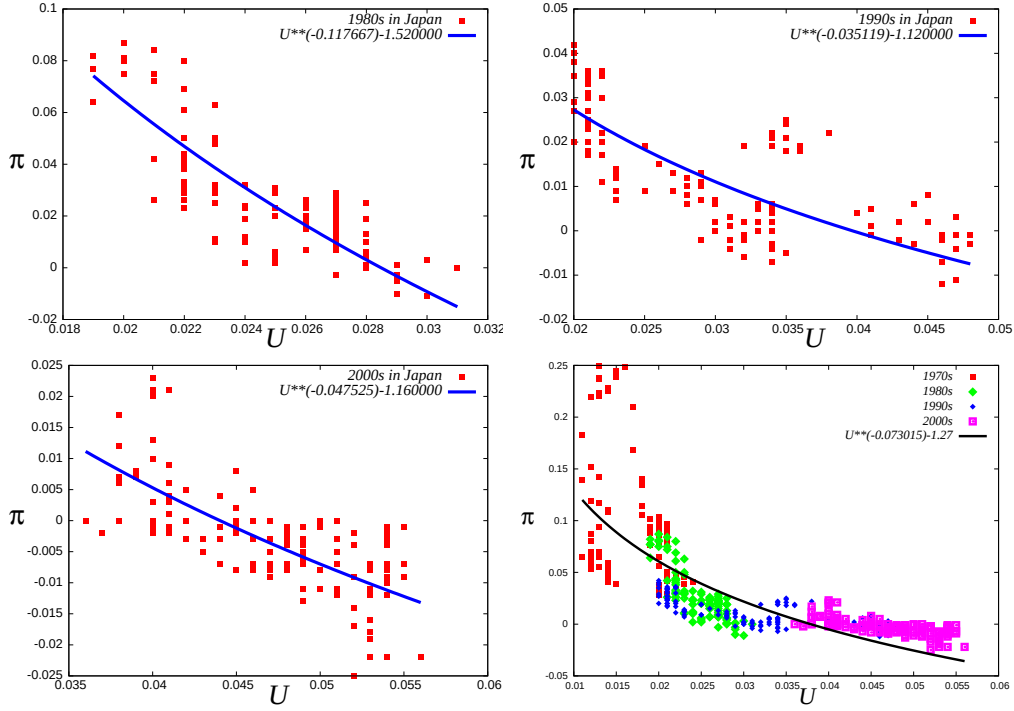


Figure 2-1: The Philips curves in Japan. From the upper left to the lower right, the curves in 1980s, 90s, and 2000s are plotted. In the lower-right panel, Philips curves in Japan (from 1970s to 2000s) is shown by a single plot. We clearly find that the Philips curve is getting ‘flat’.

Therefore, it is important for us to reveal the dependence from the microscopic point of view and it is our motivation to deal with the problem in our study.

2.2 ‘Stage-wise’ unemployment rate and ‘cumulative’ employment rate

As we already mentioned in the previous section, employment uncertainty in young generation, especially in university graduates has been one of the serious social problems in Japan. Actually, the employment rate (the informal acceptance rate) reported in four times (October, December and February, April in the next year) within a business year is getting worse. In Fig. 2-2 (left), we plot the empirical trends of employment rate $1 - U$ changing in year for the four employment opportunities (let us specify the stage by n as $n = 0, 1, 2$ and 3) in the last seventeen years. From this

figure, we clearly observe that in each year the employment rate increases monotonically as the stage n goes on (see the left panel). This means that the students and companies make up these four labor markets having the different (reduced) system sizes. Namely, after the first $n = 0$ stage, several excellent students go away from the market and several popular companies also close recruiting students because they might satisfy their quota immediately after they start to advertise for the candidates for the posts. Hence, the size of the market (the number of students and positions) is definitely getting small as the job-hunting stage goes on as $n = 0, 1, 2, \dots$.

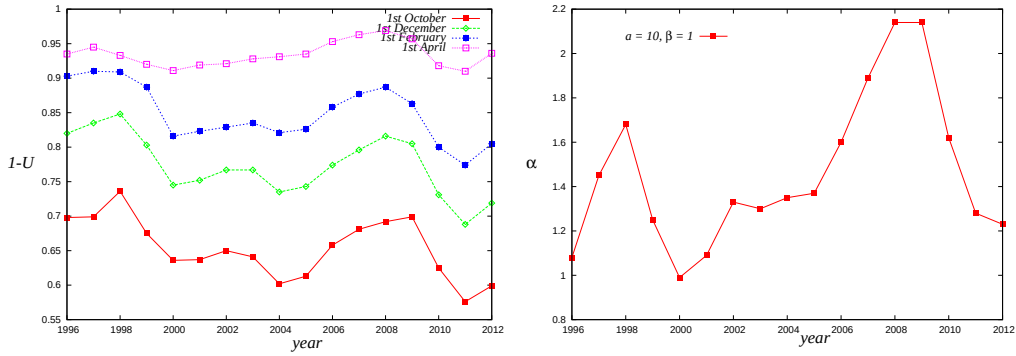


Figure 2-2: The left panel shows the changing trends in cumulative employment rate for each employment opportunities, based on employment statistical data of university graduates in 1996 to 2012 by *MEXT* (Ministry of Education, Culture, Sports, Science and Technology) [15] & *MHLW* (Ministry of Health, Labor and Welfare) [16]. The right panel exhibits the annual job-offer ratio at October, that is, $\alpha(0) \equiv \alpha$. We find that the α is greater than unity except for in 2000.

In the right panel of Fig. 2-2, we plot the annual job-offer ratio α which is usually released to the public in every October by *MEXT* (Ministry of Education, Culture, Sports, Science and Technology) [15] & *MHLW* (Ministry of Health, Labor and Welfare) [16]. From this figure, we observe that the α is greater than 1 except for the statistics in 2000. In this sense, Japanese labor markets for university graduates are almost ‘seller’s market’ in which the number of opened positions is larger than that of job-seeking students.

We found that the employment rate $1 - U$ is getting better as the term goes on as October, December, February, and April. This fact is naturally accepted because the employment rate $1 - U$ is ‘cumulative quantity’. In this sense, the $1 - U$ is a

monotonically increasing function of the term. However, when we assume that the market might shrink gradually as the term goes on, namely, the size of the market such as the number of students, positions might decrease, one could define the ‘size-reduced market’ which is induced from the market at the previous stage (term) after. In this sense, there might exist successive opportunities for students who failed in the previous stages to apply to the positions in companies in the size-reduced market after each term is over.

Then, an interesting query might arise in our mind. Namely, is it possible for us to conclude that the unemployment rate at the stage n (what we call ‘stage-wise’ unemployment rate) should be the size (scale)-independent as $U^{(n)} = U (= \text{constant})$, where we defined the ‘stage-wise’ unemployment rate as

$$U^{(n)} = \frac{\text{\# of students who do not get any job even after the } n\text{-stage is over}}{\text{\# of students who are looking for the job after the } (n-1)\text{-stage is over}}. \quad (2.1)$$

To answer this highly relevant question, let us define the ‘cumulative’ employment rate at each stage $n = 1, 2, \dots$ by

$$(1 - U)_n = \frac{\text{total \# of students who got the position until the stage } n}{\text{total \# of students}}. \quad (2.2)$$

Obviously, unemployment rate shown in Fig. 2-2 (left) is the above cumulative employment rate and it increases monotonically up to $(1 - U)_n$ as the stage n goes on. On the other hand, the behavior of stage-wise unemployment rate $U^{(n)}$ might be more complicated and it contains useful information about the difficulties of matching between companies and students in each job-hunting process. However, unfortunately, there is no empirical data conferring the stage-wise unemployment rate opened to the public and we should estimate the quantities from the available data such as $(1 - U)_n$.

2.3 Job-worker mismatch and its measurement

We should notice that in our model system, one has three relevant quantities, namely, unemployment rate $U^{(n)}$, labor shortage ratio $\Omega^{(n)}$ and job-offer ratio $\alpha^{(n)}$. To obtain a

useful relationship between these quantities, let us define the total number of students and companies by N and K , respectively. Each student and company are specified by labels $i(= 1, \dots, N)$, $k(= 1, \dots, K)$, and let v_k^* be the quota of the company k and we define the number of students whom the company k obtains by m_k . Then, we have

$$\Omega = \frac{1}{V} \sum_{k=1}^K (v_k^* - m_k) = 1 - \frac{1}{V} \sum_{k=1}^K m_k = 1 - \frac{1 - U}{\alpha}, \quad (2.3)$$

where we used the definitions of total vacancy

$$V = \sum_{k=1}^K v_k^* \quad (2.4)$$

and the job-offer ratio

$$\alpha = \frac{V}{N}. \quad (2.5)$$

Apparently, for $\alpha = V/N > 1$, the labor market behaves as a ‘seller’s market’, whereas for $\alpha < 1$, the market becomes a ‘buyer’s market’. Therefore, one confirms that U , Ω and α should satisfy the following single equation [17, 18].

$$U = \alpha\Omega + 1 - \alpha. \quad (2.6)$$

By solving the above equation with respect to Ω , we have the alternative form

$$\Omega = \frac{\alpha - (1 - U)}{\alpha}. \quad (2.7)$$

By using (2.7), we calculate the labor shortage ratio Ω by means of the observables $1 - U$ and α as we showed in Fig. 2-2. Here it should be kept in mind that the above relation is satisfied for cumulative quantities although we did not mention it clearly. Namely, (2.7) (or (2.6)) should be written as

$$\Omega_n = \frac{\alpha - (1 - U)_n}{\alpha} \quad (2.8)$$

where Ω_n stands for the cumulative labor shortage ratio. It could be defined explicitly as

$$\Omega_n = \frac{\# \text{ of positions which are opened for students until the stage } n}{\text{total } \# \text{ of positions in society}}, \quad (2.9)$$

which is different from the stage-wise labor shortage ratio

$$\Omega^{(n)} = \frac{\# \text{ of positions which are still opened for students even after } n\text{-stage is over}}{\# \text{ of positions which are opened for students after } (n-1)\text{-stage is over}}. \quad (2.10)$$

In Fig. 2-3, we plot the cumulative employment rate $(1 - U)_n$ (left) and labor shortage ratio Ω_n (right) evaluated by employment statistical data of university graduates in 1996 to 2012 by *MEXT* & *MHLW* [15, 16]. From the left panel, we are confirmed that the $(1 - U)_n$ increases monotonically as the stage n gains. On the other hand, the cumulative labor shortage ratio Ω_n is apparently a monotonically decreasing function of n .

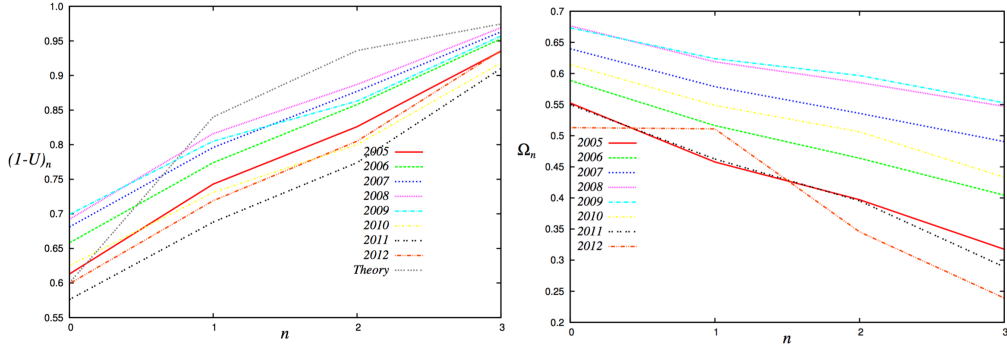


Figure 2-3: The cumulative employment rate $(1 - U)_n$ (left) and labor shortage ratio Ω_n (right) evaluated by employment statistical data of university graduates in 1996 to 2012 by *MEXT* & *MHLW* [15, 16]. Ω_n is calculated by $\Omega_n = (\alpha - (1 - U)_n)/\alpha$. A line caption ‘Theory’ in the left panel denotes $(1 - U)_n = 1 - U^n$ which is obtained from the assumption $U^n = U = 0.4$. From this figure, we find that the empirical data does not follow the theoretical law: $1 - U^n$.

In following, we write $\Omega_n = \Omega$, $(1 - U)_n = 1 - U$ for simplicity and draw the $U - \Omega$ curve using the empirical data for the cumulative employment rate $1 - U$.

In Fig. 2-4, we show the $U - \Omega$ curves for seventeen years in October, December, February and April, which are calculated by using the data [15, 16]. From these four panels, we find that the degree of job-worker mismatch between the students

and companies in 2010 is the worst in the past seventeen years. However, in 2011 and 2012, the mismatch is even improved by compromise between companies and students.

As we have shown above, the $U - \Omega$ curve tells us useful information about the job-worker mismatch. Especially, the trajectory as shown in Fig. 2-4 is reflected by changing students' tendencies for the choice of companies, government's policy, or business cycle in the society.

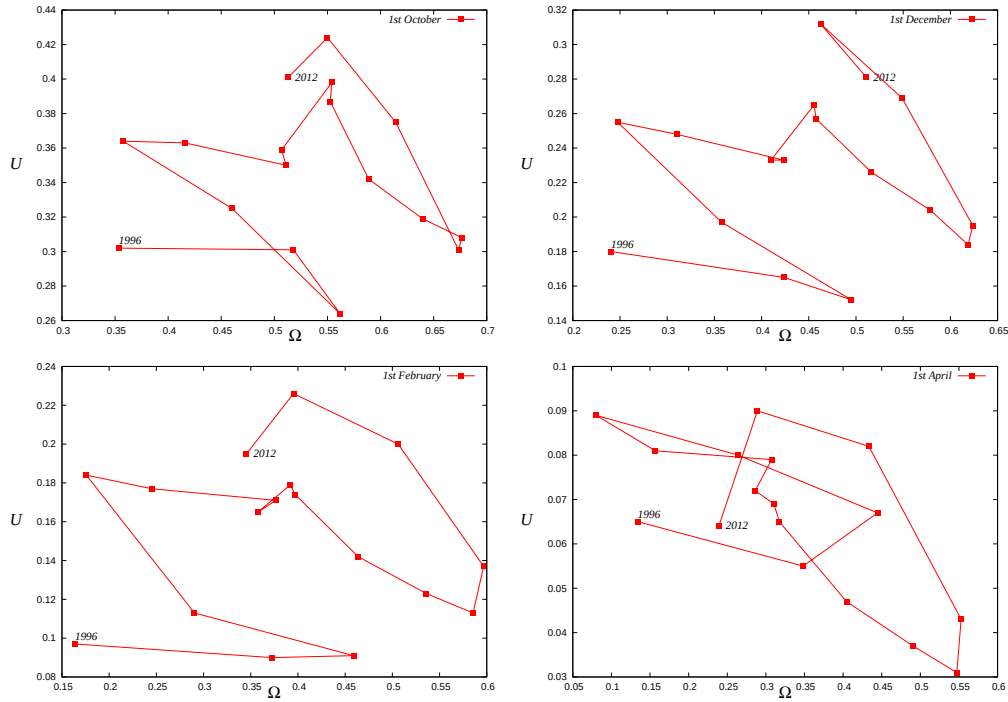


Figure 2-4: The $U - \Omega$ curves for October, December, February and April, which are calculated by empirical data of $1 - U$ [15, 16] and (2.7). We find that the degree of job-worker mismatch between the students and companies in 2010 is the worst in last seventeen years. However, in 2011 and 2012, the mismatch is even improved by compromise between companies and students.

2.4 Representation of $(\alpha^{(n)}, U^{(n)}, \Omega^{(n)})$ in terms of α , $(1 - U)_n$, $(1 - U)_{n-1}$

In the previous section, we showed the $U - \Omega$ curve for the cumulative quantities $(1 - U)_n$ and Ω_n . From the result, we were confirmed that job-worker mismatch between the students and companies was actually enhanced around 2010. However, the information about the $U - \Omega$ curve does not tell us to what extent the market becomes difficult or easy for students to obtain the jobs, or to what extent the market is sufficient or insufficient for companies to fill their quota as the stage n goes on. In order to investigate this kind of macroscopic properties of the market, we should draw the $U - \Omega$ curve for ‘stage-wise’ quantities instead of the cumulative ones.

Then, we should notice that the relationship (2.7) (or (2.6)) should be satisfied for the ‘stage-wise’ quantities $(\alpha^{(n)}, U^{(n)}, \Omega^{(n)})$ as well. Hence, by setting $\Omega = \Omega^{(n)}$, $U = U^{(n)}$ and $\alpha = \alpha^{(n)}$, we obtain a single relationship:

$$U^{(n)} = \alpha^{(n)}\Omega^{(n)} + 1 - \alpha^{(n)}. \quad (2.11)$$

We should keep in mind that $\alpha^{(0)} = \alpha$, $U^{(0)} = U_0 = U$, $\Omega^{(0)} = \Omega_0 = \Omega$, and the stage-wise job-offer ratio $\alpha^{(n)}$ is defined as

$$\alpha^{(n)} = \frac{\# \text{ of positions opened for students after } n\text{-stage is over}}{\# \text{ of students who are looking for the position after } n\text{-stage is over}}. \quad (2.12)$$

From the definition, $\alpha \equiv \alpha^{(0)}$ appearing in (2.7) for the cumulative quantities should be $\alpha^{(0)} = V/N$ and the counter empirical data is opened to the public in the October’s statistics.

Here we attempt to rewrite the set of relevant quantities $(\alpha^{(n)}, U^{(n)}, \Omega^{(n)})$ in the stage-wise sense at the n -th job-hunting stage in terms of the cumulative employment rates $(1 - U)_n, (1 - U)_{n-1}$ and job-offer ratio $\alpha \equiv \alpha^{(0)}$. For this purpose, we first examine the expression for $\alpha^{(1)}$.

The total positions (vacancies) after the first stage is over, namely, $V^{(1)}$ is given

by

$$V^{(1)} = V^{(0)} - (1 - U^{(0)})N = \{\alpha - 1 + U^{(0)}\} N \quad (2.13)$$

where we used $V \equiv V^{(0)} = \alpha N$. On the other hand, the number of students who could not get any position at $n = 1$ is given as $N^{(1)} \equiv NU^{(0)}$. Hence, we have

$$\alpha^{(1)} = \frac{V^{(1)}}{N^{(1)}} = \frac{\{\alpha - 1 + U^{(0)}\} N}{U^{(0)}N} = \frac{\alpha - 1 + U^{(0)}}{U^{(0)}}. \quad (2.14)$$

Using the same way as the above $V^{(1)}$, we obtain the number of positions still remaining after the second stage as

$$\begin{aligned} V^{(2)} &= V^{(1)} - (1 - U^{(1)})U^{(0)}N \\ &= V - (1 - U^{(0)})N - (1 - U^{(1)})U^{(0)}N = \{\alpha - 1 + U^{(1)}U^{(0)}\} N. \end{aligned} \quad (2.15)$$

Therefore, when we notice that $N^{(2)} = U^{(1)}U^{(0)}N$, one can rewrite the $\alpha^{(2)}$ in terms of $U^{(0)}, U^{(1)}$ and α as

$$\alpha^{(2)} = \frac{V^{(2)}}{N^{(2)}} = \frac{\alpha - 1 + U^{(1)}U^{(0)}}{U^{(1)}U^{(0)}}. \quad (2.16)$$

Repeating this procedure up to n -stage, we easily obtain

$$\frac{V^{(n)}}{N} = \alpha - 1 + \prod_{k=0}^{n-1} U^{(k)}, \quad \frac{N^{(n)}}{N} = \prod_{k=0}^{n-1} U^{(k)} \quad (2.17)$$

and this simply reads

$$\alpha^{(n)} = \frac{V^{(n)}}{N^{(n)}} = \frac{\alpha - 1 + \prod_{k=0}^{n-1} U^{(k)}}{\prod_{k=0}^{n-1} U^{(k)}} (n \geq 1), \alpha^{(0)} = \alpha. \quad (2.18)$$

We next rewrite the above $\alpha^{(n)}$ by means of the cumulative employment rate $(1 - U)_{n-1}$ which is available as an empirical data set shown in Fig. 2-2. Obviously, $V^{(n)}$ is given in terms of $(1 - U)_{n-1}$ as $V^{(n)} = \{\alpha - (1 - U)_{n-1}\} N$. On the other hand, the number of students who could not get any job until n -stage is evaluated by

$N^{(n)} = N - (1 - U)_{n-1}N = \{1 - (1 - U)_{n-1}\}N$. Thus, we have

$$\alpha^{(n)} = \frac{V^{(n)}}{N^{(n)}} = \frac{\alpha - (1 - U)_{n-1}}{1 - (1 - U)_{n-1}}. \quad (2.19)$$

By comparing (2.18) and (2.19), we immediately obtain

$$(1 - U)_{n-1} = 1 - \prod_{k=0}^{n-1} U^{(k)}. \quad (2.20)$$

In order to derive the expression of ‘stage-wise’ unemployment rate $U^{(k)}$ in terms of the cumulative employment rate $(1 - U)_k$, we shall write down (2.20) for $n = 2, 1$ and $n = 0$. This procedure immediately leads to $(1 - U)_2 = 1 - U^{(0)}U^{(1)}U^{(2)}$, $(1 - U)_1 = 1 - U^{(0)}U^{(1)}$ and $(1 - U)_0 = 1 - U^{(0)}$. Solving these equations with respect to $U^{(k)}$, $k = 0, 1, 2$, we obtain

$$U^{(0)} = 1 - (1 - U)_0, \quad U^{(1)} = \frac{1 - (1 - U)_1}{1 - (1 - U)_0}, \quad U^{(2)} = \frac{1 - (1 - U)_2}{1 - (1 - U)_1}. \quad (2.21)$$

The above result is easily generalized to $U^{(n)}$ as

$$U^{(n)} = \frac{1 - (1 - U)_n}{1 - (1 - U)_{n-1}}. \quad (2.22)$$

It is quite easy for us to prove that $0 \leq U^{(n)} \leq 1$ holds. When we notice that the cumulative employment is a monotonically increasing function of n , one has $(1 - U)_{n-1} \leq (1 - U)_n$, that is, $1 - (1 - U)_n \leq 1 - (1 - U)_{n-1}$. Hence, we conclude

$$0 \leq \frac{1 - (1 - U)_n}{1 - (1 - U)_{n-1}} \equiv U^{(n)} \leq 1. \quad (2.23)$$

Using the relation (2.11), we immediately obtain

$$\Omega^{(n)} = \frac{U^{(n)} + \alpha^{(n)} - 1}{\alpha^{(n)}} = \frac{\alpha - (1 - U)_n}{\alpha - (1 - U)_{n-1}}. \quad (2.24)$$

From the fact that $(1 - U)_n$ decreases monotonically as n increases, we can easily show $0 \leq \Omega^{(n)} \leq 1$ using the same argument as in $U^{(n)}$ (2.23).

Let us summarize the result,

$$(\alpha^{(n)}, U^{(n)}, \Omega^{(n)}) = \left(\frac{\alpha - (1 - U)_{n-1}}{1 - (1 - U)_{n-1}}, \frac{1 - (1 - U)_n}{1 - (1 - U)_{n-1}}, \frac{\alpha - (1 - U)_n}{\alpha - (1 - U)_{n-1}} \right). \quad (2.25)$$

In following, we shall investigate the generic properties of these stage-wise quantities.

2.4.1 On the monotonic behavior of stage-wise job-offer ratio

It should bear in mind that the behavior of the stage-wise job-offer ratio $\alpha^{(n)}$ is strongly dependent on the initial value $\alpha^{(0)} = \alpha$. Actually, one can evaluate the difference (gap) $\alpha^{(n)} - \alpha^{(n-1)}$ as

$$\alpha^{(n)} - \alpha^{(n-1)} = \frac{(\alpha - 1) \{(1 - U)_{n-1} - (1 - U)_{n-2}\}}{\{1 - (1 - U)_{n-1}\} \{1 - (1 - U)_{n-2}\}}. \quad (2.26)$$

Taking into account the the fact that the cumulative employment rate $(1 - U)_n$ is a monotonically increasing function of stage n , namely, $(1 - U)_{n-1} \geq (1 - U)_{n-2}$, and it is lower than unity, that is, $(1 - U)_{n-1}, (1 - U)_{n-2} \leq 1$, we conclude that

$$\begin{aligned} \alpha^{(n)} &\geq \alpha^{(n-1)} \quad (\alpha > 1 : \text{monotonically increasing}) \\ \alpha^{(n)} &\leq \alpha^{(n-1)} \quad (\alpha < 1 : \text{monotonically decreasing}). \end{aligned} \quad (2.27)$$

Apparently, the stage-wise job-offer ratio is one of the measurement to quantify the difficulties of matching process between companies and students on the condition that one does not consider the extreme ‘biased-behavior’ of the students’ preferences. Therefore, the above result tells us that the job-hunting becomes more difficult as stage n goes on when the job-offer ratio released to the public in October $\alpha^{(0)} = \alpha$ is lower than 1, whereas it becomes easier as the stage n increases when $\alpha > 1$. Actually in Fig. 2-5, we plot the $\alpha^{(n)}$ by making use of the empirical data. The left panel in Fig. 2-5 shows the results for 1996, 2005 and 2009 having $\alpha > 1$, whereas the right panel is the result for the market in 2000 having $\alpha = 0.99 < 1$. From this figure, we clearly find that the behavior of $\alpha^{(0)}$ is categorized into distinct two classes by

$\alpha > 1$ or $\alpha < 1$. It should be stressed that this result is independent on the choice of concrete matching algorithm for collective intelligence of students in job-hunting process. We should mention that no similar monotonic behavior is observed in $\Omega^{(n)}$ or $U^{(n)}$.

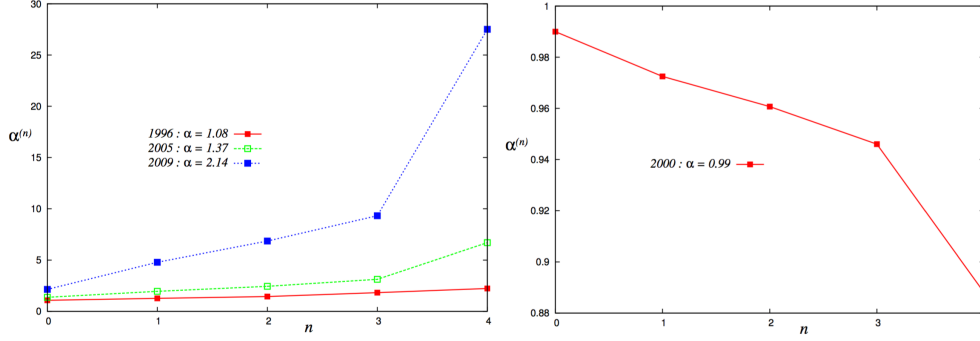


Figure 2-5: The stage-wise job-offer ratio $\alpha^{(n)} = \{\alpha - (1 - U)_{n-1}\} / \{1 - (1 - U)_{n-1}\}$. The left panel shows several typical examples for the case of $\alpha > 1$ (for the statistics in 1996, 2005 and 2009), whereas the right panel is shown as a result for $\alpha < 1$ (for the statistics in 2000).

From the above general result, we obtain several insights into the behavior of recent Japanese labor market for university graduates. Very recently, in April 2013, a Japanese think tank, *Recruit Works Institute* [20] reported that the job-offer ratio in 2012 for the market at the first stage which includes all undergraduate students and all companies was $\alpha^{(0)} \equiv \alpha = 1.28(> 1)$. Hence, from the above discussion, the stage-wise job-offer ratio $\alpha^{(n)}$ increases as n increases. Thus, the students should not rush themselves to get the positions because it becomes much easier for them to obtain the jobs as n goes on. However, the think tank also reported that the ‘effective’ job-offer ratio was $\alpha = 0.54(< 1)$ when we restrict ourselves to the market of established and large-scale companies having more than 5000 employees. This fact implies that the students who are looking for the positions in such established companies should hurry to obtain the jobs at as early-stage as possible because the stage-wise $\alpha^{(n)}$ decreases monotonically to zero for $\alpha < 1$. On the other hand, the effective job-offer ratio for the market in which there are only small enterprises having less than 300 employees was 3.26 in 2012. This fact might be a reason why a job-worker mismatch is difficult to be overcome in recent Japanese labor markets for university graduates.

2.4.2 The asymptotic limit

As we already explained in the introduction, in Japan, the statistics for the cumulative employment rate is opened to the public in four seasons, namely, in October ($n = 0$), December ($n = 1$), February ($n = 2$) and April ($n = 3$). However, it is possible for us to imagine the case of quite large n , especially, the asymptotic limit of stage-wise quantities $(\alpha^{(n)}, U^{(n)}, \Omega^{(n)})$ in the limit of $n \rightarrow \infty$. Hence, here we shall consider the limit $n \rightarrow \infty$ in the results (2.25) obtained in the previous sections.

We first consider the case of $\alpha < 1$. To consider the limit $n \rightarrow \infty$, we should notice that the number of students who get their jobs becomes identical to the total vacancy V in the limit of $n \rightarrow \infty$ for $\alpha < 1$. Namely,

$$\lim_{n \rightarrow \infty} (1 - U)_n = \frac{V}{N} = \alpha. \quad (2.28)$$

In this sense, the cumulative employment rate could not increase up to unity and it is apparently bounded from the above by the job-offer ratio opened in October ($n = 0$), namely, $\alpha^{(0)} = \alpha$. Therefore, we easily confirm at the end

$$\lim_{n \rightarrow \infty} \alpha^{(n)} = 0, \quad \lim_{n \rightarrow \infty} U^{(n)} = 1, \quad \lim_{n \rightarrow \infty} \Omega^{(n)} = 0 \quad (2.29)$$

where we used the fact

$$\lim_{n \rightarrow \infty} \{(1 - U)_n - (1 - U)_{n-1}\} = \epsilon \quad (2.30)$$

for an infinitesimal constant $\epsilon (> 0)$ accompanying with (2.28) to derive the above second and third limits as

$$\lim_{n \rightarrow \infty} U^{(n)} = \lim_{n \rightarrow \infty} \frac{1 - (1 - U)_n}{1 - (1 - U)_{n-1}} = \lim_{n \rightarrow \infty} \frac{1 - (1 - U)_n}{1 - (1 - U)_n + \epsilon} = \frac{1 - \alpha}{1 - \alpha + \epsilon} = 1, \quad (2.31)$$

$$\lim_{n \rightarrow \infty} \Omega^{(n)} = \lim_{n \rightarrow \infty} \frac{\alpha - (1 - U)_n}{\alpha - (1 - U)_{n-1}} = \lim_{n \rightarrow \infty} \frac{\alpha - (1 - U)_n}{\alpha - (1 - U)_n + \epsilon} = \frac{0}{\epsilon} = 0. \quad (2.32)$$

We next consider the case of $\alpha > 1$. For this case, all students can get the job in the limit of $n \rightarrow \infty$. Hence, we have

$$\lim_{n \rightarrow \infty} (1 - U)_n = 1. \quad (2.33)$$

By taking into account this fact, we immediately obtain

$$\lim_{n \rightarrow \infty} \alpha^{(n)} = \infty, \quad \lim_{n \rightarrow \infty} U^{(n)} = 0, \quad \lim_{n \rightarrow \infty} \Omega^{(n)} = 1 \quad (2.34)$$

where we used the relation (2.30) and (2.33) to get the second and third limits as

$$\lim_{n \rightarrow \infty} U^{(n)} = \lim_{n \rightarrow \infty} \frac{1 - (1 - U)_n}{1 - (1 - U)_n + \epsilon} = \frac{0}{\epsilon} = 0, \quad (2.35)$$

$$\lim_{n \rightarrow \infty} \Omega^{(n)} = \lim_{n \rightarrow \infty} \frac{\alpha - (1 - U)_n}{\alpha - (1 - U)_n + \epsilon} = \frac{\alpha - 1}{\alpha - 1 + \epsilon} = 1. \quad (2.36)$$

Finally, we would like to stress that the marginal case $\alpha = 1$ is rather special in the sense that $\alpha^{(n)}$ becomes independent of n as

$$\alpha^{(n)} = 1. \quad (2.37)$$

This result is intuitively understood as follows. Let us define $\Delta_V^{(n)}$ and $\Delta_N^{(n)}$ as the number of positions which are occupied and the number of students who obtain the position at n -stage, respectively. Then, $\alpha^{(n)}$ is given as

$$\alpha^{(n)} \equiv \frac{V^{(n)}}{N^{(n)}} = \frac{V - \Delta_V^{(1)} - \Delta_V^{(2)} - \dots - \Delta_V^{(n-1)}}{N - \Delta_N^{(1)} - \Delta_N^{(2)} - \dots - \Delta_N^{(n-1)}} = 1 \quad (2.38)$$

where we used $V = \alpha N = N$. We also utilized the fact that the number of students who obtain the positions at each stage is identical to the number of positions occupied by those students, namely, $\Delta_V^{(k)} = \Delta_N^{(k)}, k = 1, 2, \dots, n-1$. From the result (2.25), we also confirm

$$U^{(n)} = \Omega^{(n)} \quad (2.39)$$

for $\alpha = 1$. Thus, the performance of students to find the positions is identical to that of companies to fill the quota at each stage n .

In Fig. 2-6, we plot the $U - \Omega$ curves evaluated by means of stage-wise quantities. Namely, $U^{(1)} - \Omega^{(1)}$ curve (left) and $U^{(3)} - \Omega^{(3)}$ curve (right). From the right panel accompanying with the upper left panel in Fig. 2-4, which is the $U^{(0)} - \Omega^{(0)}$ curve from the definition, we clearly find that in 1996, the lowest job-worker mismatch in $U^{(0)} - \Omega^{(0)}$ curve is getting worse as the stage n gains up to $n = 3$ as shown in the right panel of Fig. 2-6. From the result, we are confirmed that the job-worker mismatch could be changed as the stage n goes on and it implies that students might change their strategies (‘softening’ their policies in some sense) during their job-hunting processes to get the positions.

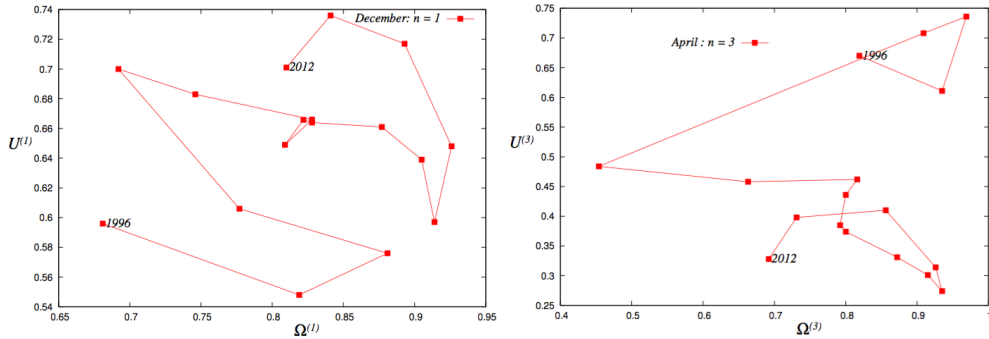


Figure 2-6: The $U - \Omega$ curves evaluated by means of stage-wise quantities. Namely, $U^{(1)} - \Omega^{(1)}$ curve (left) and $U^{(3)} - \Omega^{(3)}$ curve (right). The trajectories $(\Omega_{year}^{(n)}, U_{year}^{(n)})$, year = 1996, 1997, $\dots$, 2012 are calculated in terms of the cumulative employment rate $(1 - U)_n$ by (2.25).

2.5 ‘Learning curve’ for the collective intelligence

In the previous sections, we derived the stage-wise quantities $(\alpha^{(n)}, U^{(n)}, \Omega^{(n)})$ in terms of the cumulative employment rate $(1 - U)_n$, $(1 - U)_{n-1}$ and α . We also discussed several generic properties concerning these stage-wise quantities. However, we need a concrete ‘learning algorithm’ of collective intelligence which consists of job-seeking students. If these job-seeking students are ‘intelligent’ enough to obtain their positions as quick as possible. In other words, if the students’ strategies are effective enough to fill the total vacancies at as early stage as possible, the ‘learning algorithm’ should

be regarded as the best one. As the result, the quantity:

$$\varepsilon_n \equiv \begin{cases} 1 - (1 - U)_n & (\alpha \geq 1) \\ \alpha - (1 - U)_n & (\alpha < 1) \end{cases} \quad (2.40)$$

which is regarded as an error-measurement for the collective intelligence, converges to zero at the earliest stage of n . Therefore, the n -dependence of the error-measurement ε_n should be referred to as ‘learning curve’ [21, 22], which is usually used in the literature of machine learning, for the behavior of collective intelligence of students.

Here we should mention that from (2.8), the cumulative labor shortage ratio Ω_n is a monotonically decreasing function of n because $(1 - U)_n$ increases monotonically. Therefore, the asymptotic limit is given by

$$\lim_{n \rightarrow \infty} \Omega_n = \begin{cases} 0 & (\alpha \geq 1) \\ \frac{\alpha-1}{\alpha} & (\alpha < 1) \end{cases} \quad (2.41)$$

Obviously, we need to provide a specific learning algorithm to evaluate the learning curve. In following, we examine the learning curve for a simplest scenario, namely, the stage-wise unemployment rate $U^{(n)}$ or labor shortage ratio $\Omega^{(n)}$ is independent of n , namely, $U^{(n)} = U$ or $\Omega^{(n)} = \Omega$.

From (2.11), one can extract useful information about the n -dependence of the job-offer ratio on the condition that either U or Ω is n -independent. We first consider the case in which the labor shortage ratio Ω is independent of n . By substituting (2.11) with a constant Ω , that is, $U^{(k)} = \alpha^{(k)}\Omega + 1 - \alpha^{(k)}$ into (2.18), we have

$$\alpha^{(n)} = \frac{\alpha - (1 - \Omega^n) \prod_{k=0}^{n-1} \alpha^{(k)}}{1 - (1 - \Omega^n) \prod_{k=0}^{n-1} \alpha^{(k)}}. \quad (2.42)$$

By solving the above equation with respect to $\alpha^{(n)}$, we have the solution as a function of constant Ω as

$$\alpha^{(n)} = \frac{\alpha \Omega^n}{1 - \alpha(1 - \Omega^n)}, \quad \alpha < 1. \quad (2.43)$$

where we should remember that we defined $\alpha \equiv \alpha^{(0)}$. Then, it should be stressed

that the scale-invariant Ω is achieved if and only if the job-offer ratio α at the initial stage is smaller than unity. In other words, the scale-invariant Ω is never achieved for $\alpha > 1$. From the general result we mentioned in the previous section, for the case of $\alpha < 1$, the job-offer ratio $\alpha^{(n)}$ is a monotonically decreasing function of n . Therefore, we ultimately confirm that $\lim_{n \rightarrow \infty} \alpha^{(n)} = 0$ and $\lim_{n \rightarrow \infty} U^{(n)} = 1$. Namely, ‘perfect unemployment state’ could be realized at the end of our job-hunting process $n \rightarrow \infty$. In Fig. 2-7 (left), we plot the $\alpha^{(n)}$ as a function of n for the initial job-offer ratio $\alpha = 0.5$ and 0.9 . The learning curve for this case is easily obtained as follows. Substituting (2.43) into (2.7), we have $U^{(k)} = \{1 - \alpha(1 - \Omega^{k+1})\} / \{1 - \alpha(1 - \Omega^k)\}$. Hence, accompanying this result with $(1 - U)_n = 1 - \prod_{k=0}^{n-1} U^{(k)}$, the error-measurement ε_n defined by (2.40) for the case of $\alpha < 1$ is now written as

$$\begin{aligned} \varepsilon_n &= \alpha - (1 - U)_n \\ &= \alpha - \left\{ 1 - \prod_{k=0}^{n-1} \frac{1 - \alpha(1 - \Omega^{k+1})}{1 - \alpha(1 - \Omega^k)} \right\} = \alpha \Omega^n = \alpha \exp[-n \log(1/\Omega)]. \end{aligned} \quad (2.44)$$

Therefore, the error-measurement ε_n (‘learning curve’) decays exponentially to zero. In other words, the cumulative employment rate $(1 - U)_n$ converges to the theoretical upper bound $\alpha (< 1)$ with an exponential speed.

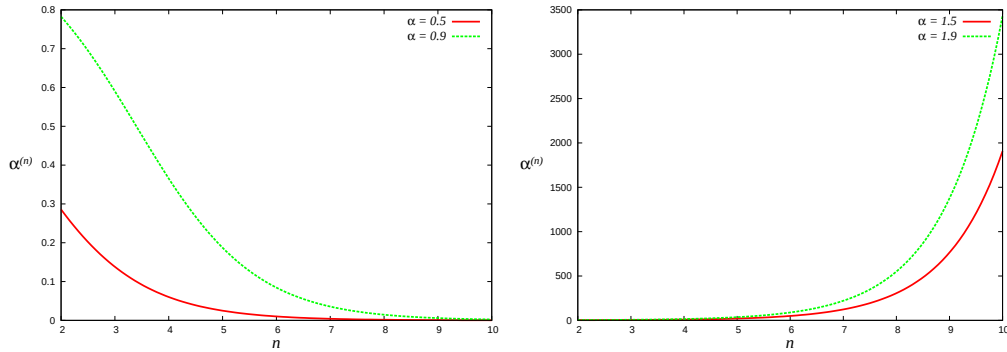


Figure 2-7: The n -dependence of job-offer ratio $\alpha^{(n)}$ for the case of n -invariant $\Omega = 0.4$ (left) and n -invariant $U = 0.4$ (right). The behavior of $\alpha^{(n)}$ is completely determined by the sign of $\alpha - 1$ ($\alpha = 1$ is the marginal).

On the other hand, for the case of n -independent U , by setting $U^{(k)} = U$ in (2.18),

we immediately obtain

$$\alpha^{(n)} = \frac{\alpha - 1 + \prod_{k=0}^{n-1} U^{(k)}}{\prod_{k=0}^{n-1} U^{(k)}} = \frac{\alpha - 1 + U^n}{U^n}, \quad \alpha > 1. \quad (2.45)$$

Here we should notice that the scale-invariant U is achieved if and only if α satisfies $\alpha > 1$. Then, the $\alpha^{(n)}$ is a monotonically increasing function of n and it reads $\lim_{n \rightarrow \infty} \alpha^{(n)} = \infty$ and $\lim_{n \rightarrow \infty} \Omega^{(n)} = 1$. This means that ‘perfect labor shortage state’ appears at the final stage of our job-hunting process. When we use the error measurement in the collective behavior of students from (2.40) as

$$\varepsilon_n \equiv 1 - (1 - U)_n \quad (2.46)$$

for $\alpha > 1$, the ε_n as a function of n could be regarded as a learning curve for this case. For the scale-invariant U , the curve decays to zero exponentially as

$$\varepsilon_n = U^n = \exp[-n \log(1/U)]. \quad (2.47)$$

Therefore, the error-measurement decays exponentially to zero, in other words, the cumulative employment rate $(1-U)_n$ converges to unity, which means that all students can obtain the position asymptotically.

The above two limiting cases seem to be rather special. Thus, one can naturally assume that both Ω and U could be dependent on the job-hunting stage n . In order to consider such general case somehow or other, we shall revisit our probabilistic labor market [17, 18]. Of course, in macroeconomics (labor science), there exist a lot of effective attempts to discuss the macroscopic properties [5, 6, 8] including so-called search theory [9, 10, 11, 12]. However, it should be stressed that the macroscopic approaches lack of their microscopic viewpoint, namely, in their arguments, the behavior of microscopic (heterogeneous) agents such as job-seekers or companies are neglected.

Chapter 3

Model system

3.1 Hypotheses in modeling

We first introduce several basic properties which should be satisfied in our probabilistic labor market. As we already mentioned, in our model system, the labors(the job seekers) are restricted to university graduates and the other persons on the job searching are neglected for simplicity. Strictly speaking, this assumption should not be justified, however, as is well known, Japanese society is still somewhat conservative and once labors get their jobs, they stay in the company by their retirement age. In this sense, we omit the contribution from the on the job searching persons to the labor market because the fraction at the present time might be negligibly small. In our modeling, we shall assume that the following four simple points (i)–(iv) should be taken into account to construct the labor markets.

- (i) Each company recruits constant numbers of newcomers in each year.
- (ii) If the company takes too much or too less applications which are far beyond or far below the quota, the ability of the company to gather applicants in the next business year decreases.
- (iii) Each company is apparently ranked according to various perspectives, and it is available for all students.

- (iv) A diversity of making the decision of labor should be taken into account by means of maximization of Shannon's entropy under some relevant constraints.

In following, we shall attempt to construct the labor markets by considering the above four essential points. Our model system is a variant of urn models [3, 26] or the so-called *Kolkata Paise Restaurant Problem (KPRP)* [27, 28].

3.2 System parameters

First of all, we define the system parameters and their sizes such as the number of companies, open positions (vacancies) and applicants. Let us define the total number of companies as K and each of them is distinguished by the label $k = 1, 2, \dots, K$. Then, the number of the quota of the company k is specified by v_k^* . In real labor markets, the quota v_k^* itself fluctuates in time (business year) and it changes according to the gross margin in the previous year and some companies in financial difficulties might decrease the quota. However, in our modeling, we fix the value and regard the quota as a 'uniform' and 'time-independent' variable (in that sense, we can regard it as 'quenched variable' in the literature of disordered physics).

Hence, the total quota (total job vacancy in society) in each business year V is given by

$$V = \sum_{k=1}^K v_k^*. \quad (3.1)$$

On the other hand, we define the number of students (the number of new university graduates) by N and each of the students is specified by the index i as $i = 1, 2, \dots, N$. Then, we introduce

$$\alpha \equiv \frac{V}{N} \quad (3.2)$$

as job-offer ratio and it is independent of system size of V and N . Apparently, for $\alpha = V/N > 1$, that is $V > N$, the labor market behaves as a 'seller's market', whereas for $\alpha < 1$, the market becomes a 'buyer's market'. For this model system, we might assume that each student posts his/her single application to the company.

As is well-known in statistical physics, critical phenomena such as ‘order-disorder phase transition’ could be taken place when the system size goes to infinity (what we call *thermodynamic limit*). On the analogy of this fact, our artificial labor market described by a probabilistic model also could show similar critical phenomena in the limit of $V, N \rightarrow \infty$. Of course, it is impossible for us to take the limit in computer simulations, hence we should set these system parameters to large but finite values, namely, $V, N \gg 1$. Therefore, in our analysis given below, the relevant system parameter is job-offer ratio α rather than V or N because obviously the $\alpha = V/N$ is a system-size-independent parameter for $\mathcal{O}(V) = \mathcal{O}(N)$. For this reason, in our modeling, we shall investigate the macroscopic behavior of the market through the job-offer ratio α instead of the number of vacancy V , nor the number of students N .

3.3 Energy function: Student’s preference of market history or ranking

Considering the condition (ii) cited above, we shall define the following local mismatch measurement:

$$h_k(t) = \frac{1}{V} |v_k^* - v_k(t)| = \frac{1}{\alpha N} |v_k^* - v_k(t)| \quad (3.3)$$

where $v_k(t)$ denotes the number of students who seek for the position in the company k at the business year t . Hence, the above local mismatch measurement $h_k(t)$ stands for the difference (gap) between the number of applicants $v_k(t)$ and the quota v_k^* for each company k .

We should keep in mind that from the condition (i) mentioned before, the v_k^* is a business year t -independent constant. Some analysts reported that the increase of unemployment rate in recent Japanese labor market is due to the mismatch between the university graduates and companies. Namely, most of students look for the positions in famous and already established large companies and they do not want to work for a small business such like fostering venture businesses. Such a sort of ‘mismatch’ could be quantified by $h_k(t)$.

On the other hand, based on the condition (iii), we define the ranking of the company k by $\epsilon_k(> 1)$ which is independent of the business year t . Here we assume that the ranking of the company k is higher if the value of ϵ_k is larger. In our agent-based modeling, we simply set the value as

$$\epsilon_k = 1 + \frac{k}{K}. \quad (3.4)$$

In such a situation, the company $k = K$ is the highest ranking company and the company $k = 1$ is the lowest. We should mention that in the second term in the right hand side of (3.4), we divided the k by the number of companies K so as to make the maximum ranking factor $\max_k \epsilon_k = \epsilon_K = 2$ a size-independent quantity.

From the above set-up and on the analogy of the Boltzmann-Gibbs distribution in conventional statistical mechanics, we define the probability $P_k(t)$ that the company k gathers their applicants at time t as

$$P_k(t) = \frac{\epsilon_k}{Z} \exp[-H_k(\beta_k, h_k)] \equiv \frac{\exp[-E(\epsilon_k, H_k(\beta_k, h_k))]}{Z}, \quad (3.5)$$

$$Z \equiv \sum_{k=1}^K \exp[-E(\epsilon_k, H_k(\beta_k, h_k))] \quad (3.6)$$

where we defined Z as the normalization constant for probability (a sort of partition function in statistical physics).

We also defined two τ -dimensional *market history vectors* with the length τ : $\beta_k \equiv (\beta_k(t-1), \dots, \beta_k(t-\tau))$ and $h_k \equiv (h_k(t-1), \dots, h_k(t-\tau))$. Then, H_k appearing in the probability (3.5) is defined by the inner product of these two vectors as

$$H_k(\beta_k, h_k) \equiv \beta_k \cdot h_k \quad (3.7)$$

Here we should notice that the above inner product choice for the expression of H_k is just an example and one can easily extend (modify) the functional form to much more generalized one: $H_k = f(\beta_k, h_k)$ including (3.7) as a special case.

With the above definitions, the energy function of our probabilistic labor market for each company k is written explicitly by

$$E(\epsilon_k, h_k; t) \equiv -\gamma \log \epsilon_k + \sum_{l=1}^{\tau} \beta_k(t-l) h_k(t-l). \quad (3.8)$$

We should notice that the ‘Markovian nature’ is built in the system through the second term appearing in the right hand side of (3.8). In our modeling, we shall simply choose a particular market history vector as $\beta_k = (\beta, 0, \dots, 0)$ for all k . Namely, a simple Markovian nature is built in the dynamics of labor markets. However, it should be noticed here that the other meaningful choices such as $\beta_k = (e^{-|t-1|}, e^{-|t-2|}, \dots, e^{-|t-\tau|})$, for which the history effect on the labor market is exponentially dumped according to the length from the present, are rather straightforward.

Thus, the energy function (3.8) is now simplified as

$$E(\epsilon_k, h_k; t) \equiv -\gamma \log \epsilon_k + \sum_{l=1}^{\tau} \beta_l h_k(t-l). \quad (3.9)$$

The parameters $\gamma > 0$ and β specify the probability from the macroscopic point of view. Namely, the company k having relatively small $h_k(t)$ can gather a lot of applicants in the next business year and the ability is controlled by the parameter β (we used the condition (ii) which was mentioned in the previous section). On the other hand, the high ranked company can gather lots of applicants and the degree of the ability is specified by the parameter γ (we used the condition (iii) which was mentioned in the previous section).

Based on this energy function, the behavior of students can be considered as follows. From the first term of the energy function, students tend to post their applications to companies with relatively high ranking. However, from the second term appearing in the energy function, we should notice that even if the company k gathers a lot of applicants as over the quota at some year t , the probability that the company k gathers the applicants at the next business year $t+1$ decreases. Therefore, the second term depends on the history of the labor market as a ‘negative feedback’

on the first term. The information about the market in the previous business year $h_k(t)$ is much more important for the students for $\gamma > \beta$, whereas the ranking of companies becomes more essential for $\gamma < \beta$ to decrease the energy function.

In Fig. 3-1, we plot $P_{k=K}$ and $P_{k=1}$ as a function of γ for $\beta = 0$. We clearly find that the company $k = K$ starts to occupy macroscopic order of students when γ increases. The behavior becomes steep when the total number of companies K is large.

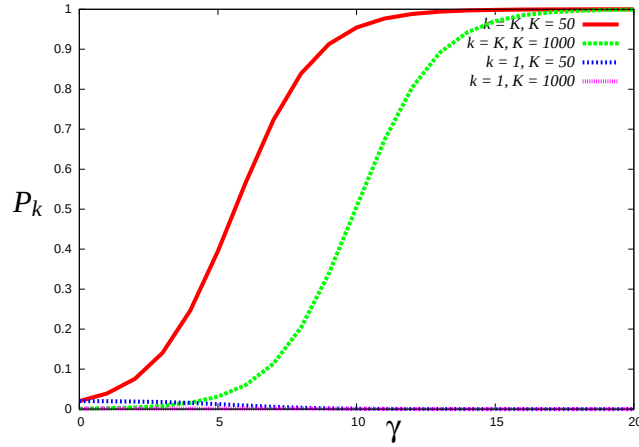


Figure 3-1: $P_{k=K}$ and $P_{k=1}$ as a function of γ for $\beta = 0$. We find that the company $k = K$ starts to occupy macroscopic order of students when γ increases. The behavior becomes steep when the total number of companies K is quite large.

Here we would like to mention the relationship between these parameter set (γ, β) and heterogeneity in our labor market. Obviously, the preferences of companies' ranking and market history depend on a private individual because each student possesses his/her own strategy to obtain the suitable position. In this sense, these parameters should be selected as (γ_i, β_i) for $i = 1, \dots, N$. However, this type of choice might cause tremendous difficulties in our modeling because we should calibrate $2N$ parameters from the empirical evidence. Of course, one can assume that γ and β are generated by some parametric distributions $P(\gamma)$ and $P(\beta)$ which contain a finite number of parameters. However, for this choice also, we need to determine the parameters from limited empirical data and it is not so easy for us to carry out. Therefore, in our study, we simply assume that the parameters are independent of

student. As we see in the next section, heterogeneous properties (diversities) in our model system are induced by means of the Jaynes-Shannon's MaxEnt principle.

3.4 Jaynes-Shannon's MaxEnt principle

In the previous sections, we introduced the energy function (3.8) to be minimized for each student to choose the company to apply for. However, minimizing the energy function leads to a unique solution without any diversity. The lack of diversity for the students' choices sometimes causes the tremendous global mismatch. In order to introduce the diversity into the system, we utilize the so-called *Jaynes-Shannon's MaxEnt principle* [23]. In following, we derive the aggregation probability for the students by means of the maximum entropy principle.

As we mentioned, the decision-makings of students that make up the labor market should be 'diverse' from the condition (iv). In order to quantify the diversity, we maximize the following Shannon's entropy:

$$H = - \sum_{k=1}^K P_k(t) \log P_k(t) \quad (3.10)$$

under the normalization condition of the probability $P_k(t)$, namely,

$$\sum_{k=1}^K P_k(t) = 1 \quad (3.11)$$

and the conservation of energy is

$$E = \sum_{k=1}^K P_k(t) E(\epsilon_k, h_k; t) \quad (3.12)$$

in each time (annual). Then, we consider the following variational problem described

by the functional concerning [23]:

$$f[P_k(t)] = -\sum_{k=1}^K P_k(t) \log P_k(t) + \lambda_1 \left\{ 1 - \sum_{k=1}^K P_k(t) \right\} + \lambda_2 \left\{ E - \sum_{k=1}^K P_k(t) E(\epsilon_k, h_k; t) \right\} \quad (3.13)$$

where λ_1 and λ_2 are Lagrange multipliers. By solving this variational problem with respect to $P_k(t)$ so as to satisfy

$$\frac{\delta f[P_k(t)]}{\delta P_k(t)} = \frac{\delta f[P_k(t)]}{\delta \lambda_1} = \frac{\delta f[P_k(t)]}{\delta \lambda_2} = 0, \quad (3.14)$$

the probability $P_k(t)$ that company k gathers the students at time t is given by

$$\begin{aligned} P_k(t) &= \frac{\exp[-E(\epsilon_k, h_k; t)]}{\sum_{k=1}^K \exp[-E(\epsilon_k, h_k; t)]} \\ &= \frac{\exp[\gamma \log(1 + \frac{k}{K}) - \beta |v_k^* - v_k(t-1)|]}{\sum_{k=1}^K \exp[\gamma \log(1 + \frac{k}{K}) - \beta |v_k^* - v_k(t-1)|]}, \end{aligned} \quad (3.15)$$

where we chose $e^{-1-\lambda_1} = \left\{ \sum_{k=1}^K \exp[\gamma \log(1 + \frac{k}{K}) - \beta |v_k^* - v_k(t-1)|] \right\}^{-1}$ and set $\lambda_2 = 1$ (unit ‘temperature’ to control the diversity of the system) for simplicity. Therefore, for each time step t , each student posts his/her application to the company k according to the above aggregation probability (3.15). Apparently, under the condition of $\gamma = 0, \beta = 0$, the probability $P_k(t)$ is consistent with the same probability of $P_k(t) = 1/K$.

Then, it is worth while for us to mention that the probability described by (3.15) has a quite similar formula to the so-called *multinomial (non-linear) logit model or softmax regression* in the literature of statistics [24, 25]. Actually, when we define the so-called softmax function as

$$\text{softmax}(k, x_1, x_2, \dots, x_K) = \frac{\exp(x_k)}{\sum_{i=1}^K \exp(x_i)}, \quad (3.16)$$

the aggregation probability $P_k(t)$ is written by

$$P_k(t) = \text{softmax}(k, -E_1(t), -E_2(t), \dots, -E_K(t)). \quad (3.17)$$

Henceforth, the actions of students to post their applications are governed by the aggregation probability (3.15). To monitor their ‘microscopic’ actions, we construct the job matching process and introduce several microscopic and macroscopic quantities in the following sections.

3.5 Job matching process

We should notice that for the probability $P_k(t)$, each student i decides to post their applications to the company k at time t as

$$a_{ik}(t) = \begin{cases} 1 & (\text{with prob. } P_k(t)) \\ 0 & (\text{with prob. } 1 - P_k(t)) \end{cases} \quad (3.18)$$

where $a_{ik}(t) = 1$ means that the student $i(= 1, \dots, N)$ posts their applications to the company k , and $a_{ik}(t) = 0$ means that he/she does not. In our modeling, we assume that each student posts their applications a -times on average. In other words, the company k takes $aNP_k(t)$ -applications on average.

We can now evaluate how many acceptances are obtained by a student and let us define the number by s_i for each student $i(= 1, \dots, N)$. where $s_{ik}(t)=1$ means that the student i receives an acceptance from the company k in the business year, and $s_{ik}(t)=0$ means that he/she does not. Then, we should notice that the total number of acceptances for the student i is defined by

$$s_i(t) = \sum_{k=1}^K s_{ik}(t). \quad (3.19)$$

Then, we discuss the number of applications that companies could get from students. Firstly, we consider the simplest case that each student i posts his/her single application to the company k with probability $P_k(t)$. And as previously stated, because the definition that $P_k(t)$ is independent of student $i(= 1, \dots, N)$, the company k takes the number of applications by $NP_k(t)$ on average, and the social total applications is given by $N \sum_{k=1}^K P_k(t) = N$. This means that each student i only gets one

chance to post their applications to one of the company k . However the reality is not so simple, in our modeling, we extend this situation by assuming that each student i posts the applications a -times on average, and as a result, each company k could take $aNP_k(t)$ applications averagely.

In following, we introduce some selection methods that we have defined at present stage study.

3.5.1 Random selection

As the most simple selection method, the so-called ‘Random selection’ is defined as each student i has the same attributes for each company k . In other words, all the students have the equal chance to obtain acceptances from the companies. And there are two possibilities about the number of applications $v_k(t)$ that the companies gathered, over the quota v_k^* or not. Hence, each student i obtains their acceptances from the company k at time t is given by

$$s_{ik}(t) = \begin{cases} \Theta(v_k^* - v_k(t))\delta_{a_{ik}(t),1} & (\text{with prob. } 1) \\ \Theta(v_k(t) - v_k^*)\delta_{a_{ik}(t),1} & (\text{with prob. } v_k^*/v_k(t)) \end{cases} \quad (3.20)$$

where $\Theta(\dots)$ denotes the conventional step function defined by

$$\Theta(x) = \begin{cases} 1 & (x \geq 0) \\ 0 & (x < 0) \end{cases} \quad (3.21)$$

and $\delta_{x,y}$ stands for the Kronecker’s delta defined by

$$\delta_{a_{ik}(t),1} = \begin{cases} 1 & (\text{if } a_{ik}(t) = 1) \\ 0 & (\text{if } a_{ik}(t) \neq 1) \end{cases} \quad (3.22)$$

Thus, the first term of (3.20) means that the $s_{ik}(t)$ takes 1 with unit probability when the student i posts the application to the company k and the total number of applications the company k gathered does not exceed the quota v_k^* . On the other hand, the second term means that the $s_{ik}(t)$ also takes 1 with probability $v_k^*/v_k(t)$

even if $v_k(t) > v_k^*$ holds. In other words, for $v_k(t) > v_k^*$, the informally accepted v_k^* students are randomly selected from $v_k(t)$ students.

Therefore, each student i has an equal chance to get the acceptance from company k , that means the ‘random selection’ is the fairest method for all the students. However, this is also a simple simulation, and has failed to show the mismatch between companies and students in job matching process. In following, we consider a scenario in which each company that obtained applications over the quota selects the students according to their rankings.

3.5.2 Ranking selection based on scores of students

As a quantitative distinguishing criterion, we defined the ranking of the company k by ϵ_k to denote their ability in recruiting students, which could be much more effective when there is a larger ranking preference parameter γ . In a similar way, we define the ranking of each student i by their scores as

$$\epsilon_i = 1 + \frac{i}{N}. \quad (3.23)$$

Namely, the student ‘ $i = N$ ’ is the highest ranking student and the student ‘ $i = 1$ ’ is the lowest. Be different than the random selection, in this method, when the company k gathers the total number of applications exceed their quota v_k^* , they send their acceptances to the students that have the higher ranking.

This makes the higher ranking students easier to monopolize the positions of the companies. As compared to ‘random selection’, this simplex ranking method is unfair for students. In reality, be different to the university entrance exam, when companies choose their employees, they are not limited to consider the students’ undergraduate transcripts, but also test their capability such as communication skills, team spirit and so on. In order to conform to this complex selection process, in following, we try to use the concept of ‘valuation basis’ of companies to describe the personal attributes of students.

3.5.3 Complex selection based on valuation basis of companies

In modern society, an increasing number of companies is turning to find and develop diverse talents, which forces the companies to choose their employees from various angles. However, in our model definition, we did not take into account the details of valuation process by companies so far. This is because the valuation basis is unfortunately not opened for the public and it is somewhat ‘black box’ for students [29]. However, recently, several web sites [30, 31] for supporting job hunting might collect a huge number of information about students as their ‘scores’ of aptitude test.

Then, based on the proposed method which is called ‘skill-weightings’ [32], the students skills are quantified in a general standard. And the companies combine their actual needs, and use these skill sets in different weightings to evaluate the employee job performance. For example, an employee with a skill set (A, B) has output at company k is given by

$$\mathbf{Y}^{(k)} = \lambda^k A + (1 - \lambda^k)B \quad (3.24)$$

with $0 \leq \lambda^k \leq 1$, and λ^k reflects the condition that each company k may weight the skills differently from another company k' . Similarly, in our model, to reflect this kind of ‘diversity’ more objectively, the concept of ‘feature vector’ [33, 34] is imported into our model system.

Hence, we might have a M -dimensional vector which based on $e_1 \equiv (1, 0, \dots, 0)$, $e_2 \equiv (0, 1, \dots, 0)$, $e_M \equiv (0, 0, \dots, 1)$, each of whose component represents a score for a given question, for each student $i = 1, \dots, N$ as

$$\begin{aligned} \mathbf{x}^{(i)} &= x_1^{(i)} e_1 + x_2^{(i)} e_2 + \dots + x_M^{(i)} e_M \\ &= (x_1^{(i)}, x_2^{(i)}, \dots, x_M^{(i)}). \end{aligned} \quad (3.25)$$

Then, we assume that each company $\mu = 1, \dots, K$ possesses their own valuation basis

(weight) as a M -dimensional vector,

$$\begin{aligned}\mathbf{a}_\mu &= a_{\mu 1}e_1 + a_{\mu 2}e_2 + \cdots + a_{\mu M}e_M \\ &= (a_{\mu 1}, a_{\mu 2}, \cdots, a_{\mu M}).\end{aligned}\tag{3.26}$$

Here, $a_{\mu m}$ are the weight coefficient of company k corresponding to the student evaluative features $x_m^{(i)}$. Then, in order to reflect the ‘diversity’ both in students and companies, we define that $x_m^{(i)}, (m = 1, 2, \cdots, M)$ are identically independent distributed variables in terms of a lognormal distribution, and the probability density function is defined by

$$f(x_m^{(i)}) = \frac{1}{x_m^{(i)} \tau \sqrt{2\pi}} e^{-\log^2 x_m^{(i)} / 2\tau^2}.\tag{3.27}$$

And $a_{\mu m}$ are in terms of a normal distribution, and the probability density function is defined by

$$f(a_{\mu m}) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(a_{\mu m} - 1)^2 / 2\sigma^2}.\tag{3.28}$$

Therefore, the score of student i evaluated by the company $\mu = 1, \cdots, K$ is given by

$$y_\mu^{(i)} = a_{\mu 1}x_1^{(i)} + a_{\mu 2}x_2^{(i)} + \cdots + a_{\mu M}x_M^{(i)}.\tag{3.29}$$

It is naturally accepted that the company μ selects the students who are the v_μ^* -top score candidates. Therefore, For a given threshold θ_μ , the decision by companies is given by

$$\hat{y}_\mu^{(i)} = \Theta(y_\mu^{(i)} - \theta_\mu) = \begin{cases} 1 & (\text{accept}) \\ 0 & (\text{reject}) \end{cases}\tag{3.30}$$

where $\Theta(\cdots)$ is a unit step function.

Thus, for N students and K companies, the situation is determined by the fol-

lowing linear equation:

$$\begin{pmatrix} y_1^{(i)} \\ \vdots \\ y_K^{(i)} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{K1} & \cdots & \cdots & \cdots & a_{KN} \end{pmatrix} \begin{pmatrix} x_1^{(i)} \\ \vdots \\ \vdots \\ \vdots \\ x_N^{(i)} \end{pmatrix}, \quad i = 1, \dots, N \quad (3.31)$$

namely,

$$\mathbf{y}^{(i)} = \mathbf{A}\mathbf{x}^{(i)}, \quad i = 1, \dots, N. \quad (3.32)$$

When we have enough number of data sets $(\mathbf{y}^{(i)}, \mathbf{x}^{(i)})$, $i = 1, \dots, N$, one might estimate the valuation base $\mathbf{A}$ using suitable learning algorithm. When we notice that the above problem is described by ‘learning of a linear perception’, one might introduce the following cost function:

$$E = \frac{1}{2NK} \sum_{i=1}^N \sum_{\mu=1}^K \delta_{s_\mu^{(i)}, 1} \left\{ y_\mu^{(i)} - \sum_{l=1}^N a_{\mu l} x_l^{(i)} \right\}^2 \quad (3.33)$$

where we defined $\delta_{a,b}$ as Kronecker’s delta and

$$s_\mu^{(i)} = \begin{cases} 1 & \text{(student } i \text{ sends an application letter to company } \mu) \\ 0 & \text{(otherwise)} \end{cases} \quad (3.34)$$

Then, we construct the learning equation as

$$\frac{da_{\mu j}}{dt} = -\eta \frac{\partial E}{\partial a_{\mu j}} = \frac{\eta}{NK} \sum_{i=1}^N \delta_{s_\mu^{(i)}, 1} \left\{ y_\mu^{(i)} - \sum_{l=1}^N a_{\mu l} x_l^{(i)} \right\} x_j^{(i)} \quad (3.35)$$

for $\mu = 1, \dots, K$, $j = 1, \dots, N$.

We show an example of the learning dynamics through the error:

$$\epsilon(t) = \frac{1}{NK} \sum_{\mu=1}^K \sum_{j=1}^N (a_{\mu j}^* - a_{\mu j}(t))^2, \quad (3.36)$$

where $a_{\mu j}^*$ denotes a ‘true weight’, for artificial data sets in Fig. 3-2.

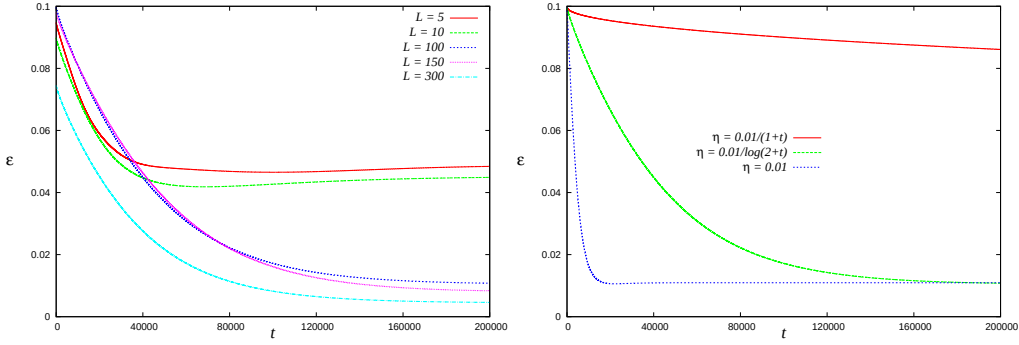


Figure 3-2: Time-dependence of error $\epsilon(t) = (1/NK) \sum_{\mu=1}^K \sum_{j=1}^N (a_{\mu j}^* - a_{\mu j}(t))^2$ for the learning (3.35) using artificial data sets. We choose the learning rate as $\eta = 0.01/\log(2+t)$. $N = K = 10$.

Here we showed just an example of learning from artificial data sets for demonstration. however, it should be addressed as our future work to apply the learning algorithm to realistic situation using empirical data set collected from [30, 31] or large-scale survey. Finally, it would be important for us to mention that it could be treated as ‘dictionary learning’ [35] when the vector $\mathbf{x}^{(i)}, i = 1, \dots, N$ is ‘sparse’ in the context of *compressive sensing* [36, 37, 38].

Then, to assess the impact on $\mathbf{y}^{(i)}$ change in this selection process, in Fig. 3-3, we plot the relationship between unemployment U and the standard deviations τ (left), σ (right). As seen in figure, with the other parameter settings remaining unchanged, U remained about the same with the increasing of τ , but decreased with the increasing of σ . That means for unemployment U , there is a much more impact on the change of ‘valuation basis’ $\mathbf{a}_{\mu}$ than the change of $\mathbf{x}^{(i)}$.

In addition, in the left panel, we can find when σ has a larger value, for example, when $\sigma = 10$, there is a lower value of U . That is because of $\mathbf{a}_{\mu}$ obey the normal distribution, when σ is larger, the values of $a_{\mu m}$ are highly dispersed. In other words,

for the given student i , the values of $y_\mu^{(i)}$ are highly dispersed. And this kind of diversity can increase the possibility of students to get an acceptance successfully. The current selection process is relative equity, and close to the method of ‘random selection’. Especially, when the set of $(\tau = 0.01, \sigma = 10)$, because of $\mathbf{x}^{(i)}$ obey the lognormal distribution of $(\tau = 0.01)$ as shown in (3.27), the values of $x_m^{(i)}$ are concentrated in the value of 1. Therefore, almost all the $\mathbf{y}^{(i)}$ approximately equal in value, there is similar to the method of ‘random selection’.

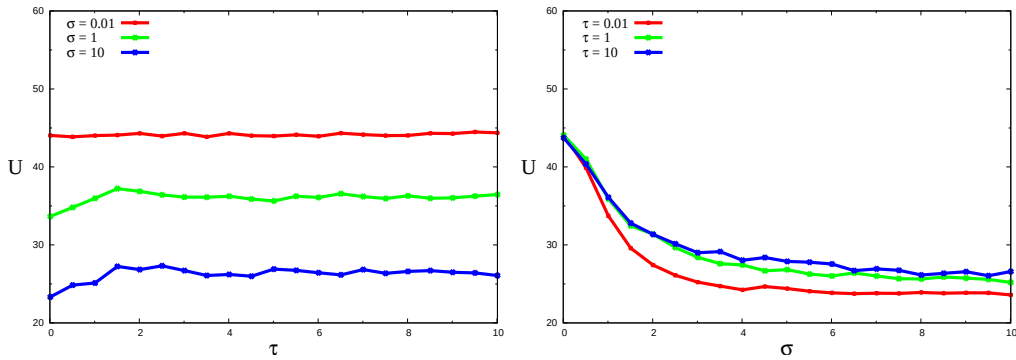


Figure 3-3: The relationship between unemployment U and standard deviations τ (left), σ (right). The system parameters are set to $a = 5$, $\alpha = 1.61$, $\gamma = 5$, $\beta = 1$, $N = 42,000$, $K = 1,350$, and $M = 10$. These results are plot for $\tau = 0.01, 1, 10$ and $\sigma = 0.01, 1, 10$, and calculated by 10-times independent trials.

On the contrary, when σ takes a smaller value, for example, $\sigma = 0.01$, there is a higher value of U . That is because of when σ is smaller, the values of $a_{\mu m}$ are concentrated in the value of 1. That is, for the given student i , the values of $y_\mu^{(i)}$ are approximately equal, and only depend on $\mathbf{x}^{(i)}$. This makes each student nearly has the unique ranking distribution, the selection process is close to the ‘ranking selection’.

Finally, except for these exceptional cases, when there are other sets of τ and σ , the selection process is a complex selection and more conform to the realistic society.

In conclusion, we introduced the components of our model system. Especially, from simple to complex, we proposed three different methods for companies to select students, random selection, ranking selection, and complex selection based on M -

dimensional vector. In following, we will give some numerical simulation results and analysis which are connected with these job matching processes.

3.6 Acceptance rate

From microcosmic point of view, the ideal situation of labor markets is that each student could get at least one acceptance from companies. However, this equal right does not adapt to the objective reality. In order to evaluate this kind of ‘unequal employment opportunity’, we define a microscopic variable of acceptance rate $r_i (i = 1, \dots, N)$, which represents the ability of students obtain the acceptances from companies, and it is given by

$$r_i = \frac{s_i(t)}{a_i(t)} \quad (3.37)$$

$$a_i(t) = \left[\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{t=1}^M \sum_{k=1}^K a_{ik}(t) \right] \quad (3.38)$$

$$s_i(t) = \left[\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{t=1}^M \sum_{k=1}^K s_{ik}(t) \right]. \quad (3.39)$$

Here, $a_i(t)$ means the total number of applications for student i post to companies at time t , and is given by $a_i(t) = \sum_{k=1}^K a_{ik}(t)$. And $s_i(t)$ means the total number of acceptances for the student i (as shown in Sec. 3.5). Then M is the time step and set as $M \rightarrow \infty$. To put it another way, the success or failure of the job hunting is not simply deciding to the number of acceptances that the student i could obtained, but based on the ratio of the acceptances to the applications for the student i .

In Fig. 3-4, we plot the relationship between the acceptance r_i and the student i . Here, $i = 0$ (or 1) is the lowest(or highest) ranking student. We find the scale of the students whose acceptance rate r_i equal 100%($r_i = 1$) decreases with the increase of the average applications a (left) and the companies’ ranking preference γ (right). It is because with the increase of these two parameters, the higher ranking students become to monopolize the acceptances, meanwhile, the lower ranking students become harder to obtain a position from companies. In addition, when $a = 1$, because of the

system parameters ($\alpha = 1.61$) > 1 , in other words, the total job vacancy in society V is far exceeded the number of students N , r_i takes a higher value as a whole, and on account of taking an average of 10-times, there is ladder-like distribution.

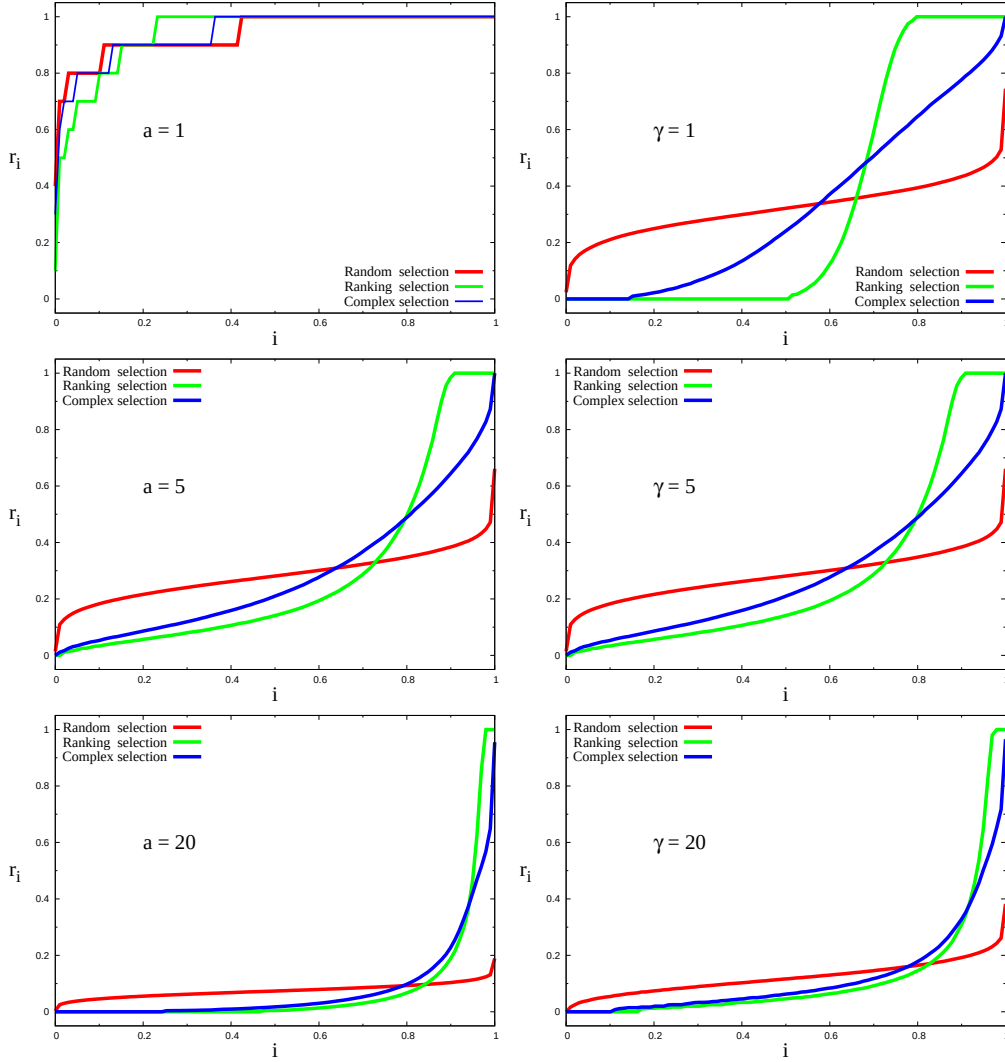


Figure 3-4: The relationship between acceptance rate r_i and students i . Here, $i = 0$ (or 1) is the lowest (or highest) ranking student. The system parameters are set to $\alpha = 1.61$, $\beta = 1$, $\tau = 1$, $\sigma = 1$, $N = 42,000$, $K = 1,350$, and $\{a = 1, 5, 20, \gamma = 5\}$ (left), $\{a = 5, \gamma = 1, 5, 20\}$ (right). The results are calculated by 10-times independent trials.

Then, we define the critical value of students ranking by

$$\epsilon_c \equiv \min\{\epsilon_i | r_i = 1\}. \quad (3.40)$$

It also can be found near the critical value ϵ_c , the acceptance rate r_i sharply changes. There is existing a sort of ‘phase transition’. This trend generates a gap in job hunting, and further aggravates the social inequality.

3.7 Distribution of microscopic quantities

For the model introduced in the previous section, we evaluate the distribution of several microscopic physical quantities. To calculate the distribution numerically, we define the distribution of such quantity $A(t)$ by

$$P(A) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \delta_{A(t), A}. \quad (3.41)$$

Using the above definition, the distribution of the number of applications $v_k(t)$ which the company $k(= 1, \dots, K)$ obtains is evaluated by substituting $A(t) = v_k(t)$ into the definition (3.41) for finite system size $K, N(\gg 1)$. After recursively updating (3.5), (3.6) for $T(\gg 1)$ -times, we obtain the distribution of any micro or macro quantities through (3.41).

In Fig. 3-5(left), we show the distribution of the microscopic quantity a_{ik} . The system size is chosen as $N = 10000, K = 1000$ and fix the quota for each company so as to be a company-independent variable as $v_k^* = v \equiv 30$ (homogeneous quota) for simplicity. This choice leads to $\alpha = 3$. The time step to evaluate (3.41) is set to $T = 10^4$ and the ranking factor γ is fixed as $\gamma = 1$. From this figure, we find that a_{ik} is distributed around the average a . In our computer simulations to be given in following, we use the $P(a_{ik})$ to determine the number of applications posted by each student.

We next consider the distribution of the number of applications obtained by company k . We plot the result in Fig. 3-5 (right). From this figure, we clearly find that in the regime of market history actually works (the market history is switched on) $\beta = \gamma = 1$, the distribution has a single peak around the relatively large value of v_k . However, as one increases the ranking effect γ as $\gamma \gg \beta$, the peak eventually

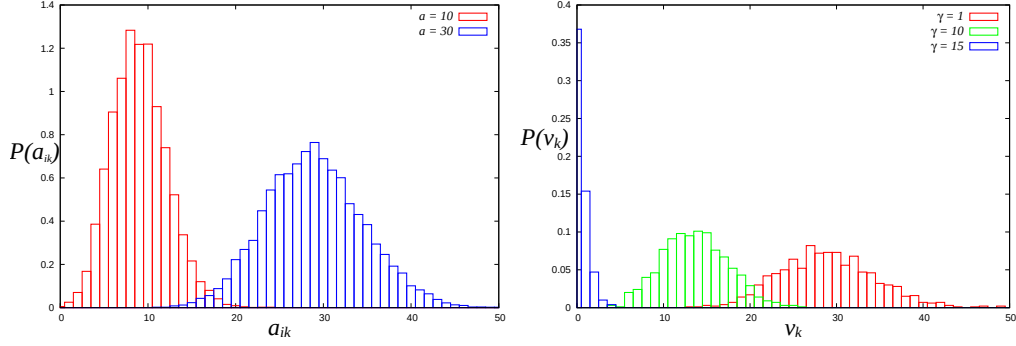


Figure 3-5: The distribution of a_{ik} (left) and v_k (right). In our simulations, we set $N = 10000$, $K = 1000$ and for simplicity $v_k^* = v = 30$, which reads $\alpha = 3$. The time step to evaluate (3.41) is set to $T = 10^4$. In the left panel, we set $\beta = \gamma = 1$, $a = 10$ and 30 . In the right panel, we set $\beta = 1$ and change γ as $\gamma = 1, 10, 15$.

moves to zero. The fraction (probability) of companies getting no application is easily evaluated as follows.

Obviously, the probability $P(v_k = \bar{v})$ follows a binomial distribution $P(v_k = \bar{v}) = {}^{aN}C_{\bar{v}} P_k^{\bar{v}} (1 - P_k)^{aN - \bar{v}}$, where we neglected the time-dependent part in $P_k(t)$, that is $\sim -\beta|v_k^* - v_k(t)|$ for $\gamma \gg \beta$. Then, $P(v_k = 0)$ is written as

$$P(v_k = 0) = (1 - P_k)^{aN} \simeq \exp(-aN P_k), \quad (3.42)$$

where P_k is roughly estimated in the thermodynamic limit as

$$P_k = \frac{(1 + k/K)^\gamma}{\sum_{k=1}^K (1 + k/K)^\gamma} \simeq \frac{(1 + k/K)^\gamma}{\int_1^K (1 + k/K)^\gamma dk} = \frac{(\gamma + 1)(1 + k/K)^\gamma}{K(2^{\gamma+1} - 1)}. \quad (3.43)$$

Hence, in the limit of $K \propto N \rightarrow \infty$ (in other words, $K, N \rightarrow \infty$ keeping $\rho \equiv K/N = \mathcal{O}(1)$), the highest ranking company has $P_{k=K} \simeq (\gamma + 1)/2K$, whereas the lowest ranking company gets $P_{k=1} \simeq (\gamma + 1)/2^{\gamma+1}K$. Substituting these results into $P(v_k = 0)$, we immediately have

$$P(v_K = 0) = \exp[-aN(\gamma + 1)/2K], \quad (3.44)$$

$$P(v_1 = 0) = \exp[-aN(\gamma + 1)/2^{\gamma+1}K]. \quad (3.45)$$

These results imply that in the limit of $\gamma \rightarrow \infty$, the lowest ranking company gets no application with probability $P(v_1 = 0) = \exp(0) = 1$ from (3.44), whereas the highest ranking company always can get macroscopic order of application as $P(v_k = 0) = 0$ from (3.45). We easily notice that the argument for (3.45) should be valid for any companies which satisfy $k \ll K$. Therefore, for large $\gamma \gg 1$, macroscopic number of companies completely lose their applicants by this probabilistic nature. We can actually confirm this result indirectly from Fig. 3-5 as $P(v_k) = 0$ for $\gamma = 15$ which is relatively a large value in our simulations.

In following, we investigate the macroscopic behavior of this model system through several physical quantities.

Chapter 4

Numerical simulation results

4.1 Unemployment rate

In the previous section, we modeled the microscopic job matching process between companies and students by means of the probability distribution $P_k(t)$ of Boltzmann-Gibbs-type. As our main purpose is to reconstruct the macroscopic behavior of labor markets from the microscopic description, we should calculate the macroscopic quantities by means of the microscopic variables. To reconsider the results in the previous section from the macroscopic viewpoint, here we calculate the unemployment rate U as a function of t as

$$U_t = \frac{1}{N} \sum_{i=1}^N \delta_{s_i(t),0}. \quad (4.1)$$

Namely, the unemployment rate at business year t is defined as a ratio of persons who could not get any acceptance ($s_i(t) = 0$) to the total number of students N .

4.1.1 The order parameter

To discuss the macroscopic quantity, we consider the long time average of U_t as an ‘order parameter’ U as usually used in statistical physics. Namely, we define the order parameter as

$$U = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} U_t. \quad (4.2)$$

Here it should be noted that the above time average should be identical to the ensemble average $\langle U \rangle$, where the bracket $\langle \cdots \rangle$ stands for the average over the joint probability for the microscopic quantities $P(a_1, \cdots, a_K; s_1, \cdots, s_N)$ with $a_k \equiv (a_{1k}, \cdots, a_{Nk})$, $a_{ik} \in \{0, 1\}$, $s_i \in \{0, 1, 2, \cdots, a\}$, $i = 1, \cdots, N$, $k = 1, \cdots, K$ when the system can reach the equilibrium state.

It is not trivial whether the equilibrium state actually exists especially when we consider a coupled dynamics with the macroscopic Neugart model [8]. Nevertheless, in our modeling, we shall calculate the order parameter (4.2) for large T ($\gg 1$) and regard it as one of the relevant macroscopic quantities describing the ‘thermodynamics’ of our labor market.

4.1.2 The Beveridge curve

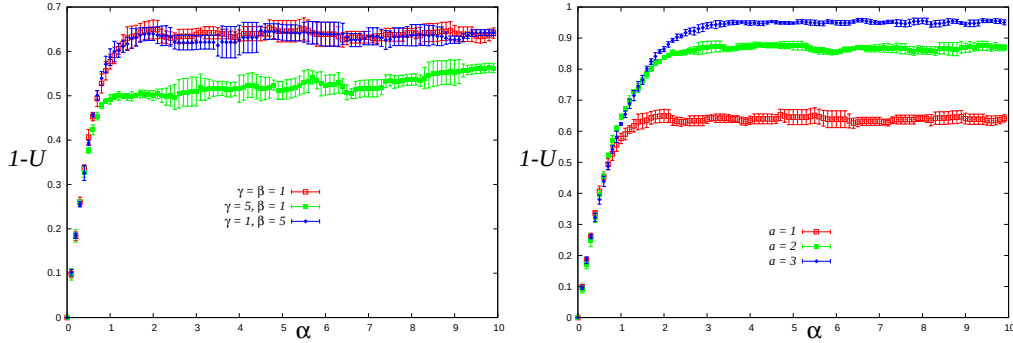


Figure 4-1: The employment rate $1 - U$ as a function of α . We evaluated the rate for the cases $(\gamma, \beta) = (1, 1), (1, 5), (5, 1)$ (left), and $a = 1, 2, 3$, $\gamma = \beta = 1$ (right). In our simulations, we set the system sizes as $N = 500, K = 50$. The error-bars are calculated by 5 independent trials.

In Fig. 4-1 (left), we plot the employment rate $1 - U$ as a function of α for several choices of $(\gamma, \beta) = (1, 1), (1, 5), (5, 1)$. From this figure, we certainly find that the employment rate is lower than 0.7. In these numerical simulations, we assumed that each student posts their applications just only once on average. However, if the number of posting increases the situation might be changed. Thus, we next consider the case in which each student posts their applications a times on average. Therefore, we increase the number of applications a for each student to post to the market. We

check the case $a = 1, 2$ and $a = 3$. The result is shown in Fig. 4-1 (right). From this figure, as we expected, we find that the employment rate increases up to near 1 (perfect employment) when we increase the number a .

We should notice that the plots shown in Fig. 4-1 correspond to the so-called *Beveridge curve* in economics (labor economics) [39, 40]. Usually, the Beveridge curve is defined as the behavior of the number of vacancy V against the unemployment rate U (hence, it is sometimes referred to as *UV curve*). However, as we already mentioned, in the thermodynamic limit $V, N \rightarrow \infty$, the relevant system parameter is job-offer ratio α rather than the number of vacancies V . Thus, we might regard the α as the effective number of vacancies. Hence, the U - α curves shown in Fig. 4-1 correspond to the conventional Beveridge curves.

4.1.3 Phase transitions in labor markets

We next consider the γ -dependence of the employment rate for several values of a . In Fig. 4-2, we show the employment rate $1 - U$ as a function of γ for a fixed β value ($\beta = 1$). The results are plotted for $a = 1, 2, 3$ and $a = 10$. The left panel is given for $\alpha = 1$, whereas the right panel is given for $\alpha = 10$. From this figure, as we expected, we find that the employment rate for a relatively high job-offer ratio $\alpha = 10$ increases up to near 1 when we increase the number a .

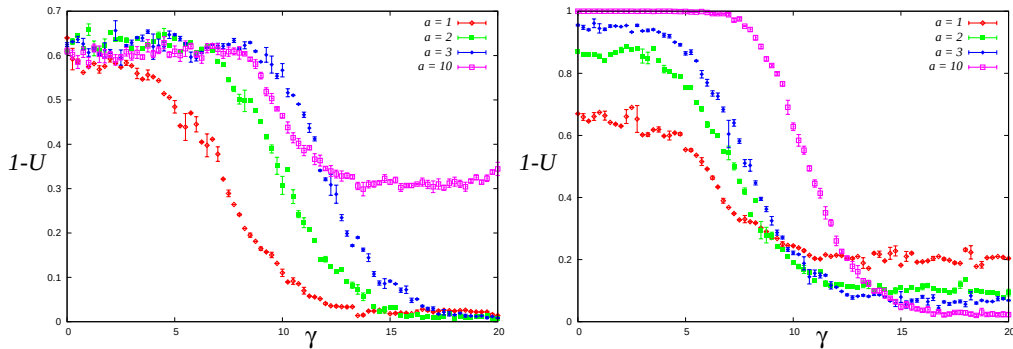


Figure 4-2: The employment rate $1 - U$ as a function of γ for a fixed β value ($\beta = 1$). The results are plotted for $a = 1, 2, 3, 10$, and $\alpha = 1$ (left), $\alpha = 10$ (right).

One of the remarkable features of the results is existing a sort of ‘phase transitions’ in our probabilistic labor market. In the right panel of Fig. 4-2, we clearly find that there exist two distinct phases. Namely, ‘perfect employment phase’ ($1 - U \simeq 1$) and ‘perfect unemployment phase’ ($1 - U \simeq 0$, that is, $U = 1$), and the system changes gradually from the perfect employment phase to the poor employment phase around $\gamma \simeq 10 (\equiv \gamma_c)$. From statistical mechanical point of view, the system shows ‘spontaneous symmetry breaking’ when the phase transition takes place. To evaluate the residual employment rate, we consider the extreme limit $\gamma \rightarrow \infty$ for the simplest case $\alpha = a = 1$. In this limit, all students want to post their own single application to the highest ranking company $k = K$. As the result, $v_K^* = v = 10$ students get their jobs and the residual employment rate is approximately evaluated as $1 - U \simeq vN = 10/500 = 0.02$.

Then, the microscopic variables take $\Theta(s_i) = 1$ for only the selected v -students ($v \ll N$) and $\Theta(s_i) = 0$ for the other $(N - v)$ -students. For this microscopic state, the employment rate as ‘spontaneous magnetization’: $1 - U = (1/N) \sum_{i=1}^N \langle \Theta(s_i) \rangle$ leads to $1 - U \simeq (1/N) \times v = v/N = 0$ in the thermodynamic limit $N \rightarrow \infty$. This means that in the limit of $\gamma \rightarrow \infty$, the labor market exists in the ‘symmetry breaking phase’. On the other hand, in the limit of $\gamma \rightarrow 0$, the students post their applications to randomly selected companies and a macroscopic fraction of students $\sim N \langle \Theta(s_i) \rangle$ can get the jobs. This reads $1 - U \simeq (1/N) \times N \langle \Theta(s_i) \rangle = \langle \Theta(s_i) \rangle > 0$ and the labor market exists in the ‘symmetric phase’. Therefore, the phase transition is taken place at the critical point $\gamma = \gamma_c$ with this type of spontaneous symmetry breaking.

To see these phenomena and find the critical point of γ_c , we consider the following ‘susceptibility’ χ defined by

$$\chi \equiv -\frac{\partial(1 - U)}{\partial \gamma} = \frac{\partial U}{\partial \gamma}. \quad (4.3)$$

On the analogy of magnetic systems, we might regard $1 - U$ as a sort of ‘magnetization’ and γ as an ‘external field’. Then, (4.3) denotes the response of the magnetization to the external field. As the $1 - U$ is dependent on the γ implicitly, we use the discretized

definition of (4.3) as $\chi = \Delta U / \Delta \gamma = \{U(\gamma + \Delta \gamma) - U(\gamma)\} / \Delta \gamma$. We show the result in Fig. 4-3.

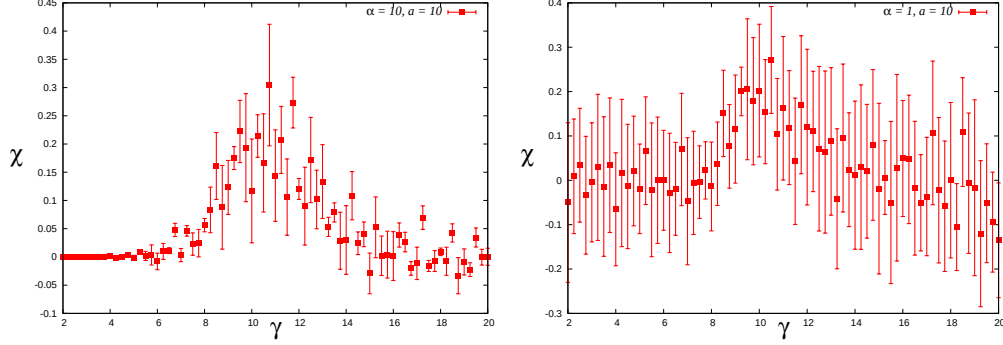


Figure 4-3: The susceptibility χ as a function of γ . We set $\alpha = 10, a = 10$ (left), and $\alpha = 1, a = 10$ (right). In our computer simulations, the infinitesimal interval $\Delta \gamma$ is set to $\Delta \gamma = 0.25$. The error-bars are calculated by 20-independent trials.

We set $\alpha = 10, a = 10$ (left), and $\alpha = 1, a = 10$ (right). In our computer simulations, the infinitesimal interval $\Delta \gamma$ is set to $\Delta \gamma = 0.25$. The error-bars are calculated by 20-independent trials. From this figure, we clearly find that the susceptibility χ has a single peak around the critical point $\gamma_c \simeq 10$, which is an indicator of phase transition.

From the viewpoint of labor markets, the phase transition might be understood as follows. For the case of a society with quite high job-offer ratio, such as $\alpha = 10$, the employment rate can be almost 1 when each student can post $a = 10$ applications on almost randomly to companies. However, as students start to take into account the ranking of company, namely, for $\gamma \simeq \gamma_c$, the employment rate is gradually decreased to the zero-level. Apparently, this result is induced due to the job-worker mismatch job-worker mismatch between the students and companies as observed in recent Japanese labor market.

4.2 The Philips curve

We next consider the Philips curve for our labor market. To obtain the Philips curve, we should evaluate the inflation rate separately and it needs some information about the production process of the companies and consumption procedure by consumers. Moreover, the bargaining process of wages of labors for each company also should be taken into account to determine the inflation rate. In our study, we shall use the macroscopic formula for the inflation rate given by Neugart [8]. Then, by making coupled equations with our result of unemployment rate, we attempt to draw the Philips curve.

4.2.1 Macroscopic Neugart model

We shall briefly explain the derivation of non-linear maps with respect to the unemployment and inflation rates according to Neugart [8].

We first defined the update rule for the unemployment rate U as

$$U_{t+1} = U_t + \xi(1 - U_t) - o_t U_t \quad (4.4)$$

where the second term in the right hand side denotes the contribution of labors who are fired (lost) their jobs at time t and the ratio is controlled by a single parameter ξ . The third term means the contribution of labors who get their job at time t and o_t is a time-dependent rate. The o_t is explicitly given by

$$o_t = \frac{J_s + \Gamma(m - \pi_t)}{U_t + d(1 - U_t)} \quad (4.5)$$

where the denominator denotes the total amount of labors who seek for the jobs at time t and $d(1 - U_t)$ corresponds to the labors who are ‘on the job searching’. On the other hand, in the numerator, J_s denotes the time-independent number of job openings and the time-dependent part of job openings comes from the second term $\Gamma(m - \pi_t)$. In this term, π_t denotes the inflation rate at time t and m stands for a constant growth rate for the value of money. Hence, (4.5) means that the probability

getting jobs o_t decreases when the number of labors who seek for jobs increases, and increases when the inflation rate is smaller than the growth rate for the value of money.

Moreover, the inflation rate at time t is written in terms of the ‘expected inflation rate’ $\pi_{e,t}$ as

$$\pi_t = \frac{1}{\delta} \left(\pi_{e,t} + \frac{w_{b,t} - w_p}{w_p} \right) = \frac{1}{\delta} \left(\pi_{e,t} + \frac{\mu - (1 - c_2)U_t}{1 - \mu} \right) \quad (4.6)$$

where δ stands for a scaling factor and $w_{b,t}$ is bargaining wages and we naturally set $w_{b,t} = 1 - (1 - c_2)U_t$ with a constant $0 \leq c_2 \leq 1$. The justification of this choice depends on the validity of our assumption that the bargaining should go well when the unemployment rate is low. The union having enough number of labors can negotiate with management for wage increases well. The w_p is a base wage and it is controlled a single parameter μ as $w_p = 1 - \mu$.

Then, the expected inflation rate is updated by means of a linear combination of inflation rate π_t and the expected inflation rate $\pi_{e,t}$ with $0 \leq c_1 \leq 1$ as

$$\pi_{e,t+1} = c_1 \pi_t + (1 - c_1) \pi_{e,t} . \quad (4.7)$$

From (4.4), (4.5), (4.6) and (4.7), we obtain the following non-linear maps with respect to U and π :

$$U_{t+1} = U_t + \xi(1 - U_t) - U_t \frac{J_s + \Gamma(m - \pi_t)}{U_t + d(1 - U_t)}, \quad (4.8)$$

$$\begin{aligned} \pi_{t+1} = & \frac{1}{\delta} \left(\frac{\mu}{1 - \mu} + c_1 \pi_t + (1 - c_1) \left(\delta \pi_t - \frac{\mu - (1 - c_2)U_t}{1 - \mu} \right) \right) \\ & - \frac{1}{\delta} \left(\frac{1 - c_2}{1 - \mu} \left(U_t + \xi(1 - U_t) - U_t \frac{J_s + \Gamma(m - \pi_t)}{U_t + d(1 - U_t)} \right) \right). \end{aligned} \quad (4.9)$$

The fixed point of the above non-linear maps is easily obtained as $(U^*, \pi^*) = (\{\mu - m(\delta - 1)(1 - \mu)\}/(1 - b), m)$. Then, according to Neugart [8], we set the value of J_s in terms of the above fixed point, namely, by inserting the fixed point $U_{t+1} = U_t = U^*, \pi_{t+1} = \pi_t = \pi^*$ into (4.8), we obtain $J_s = J_s^* \equiv \xi(1 - U^*)(U^* + d(1 - U^*))/U^*$.

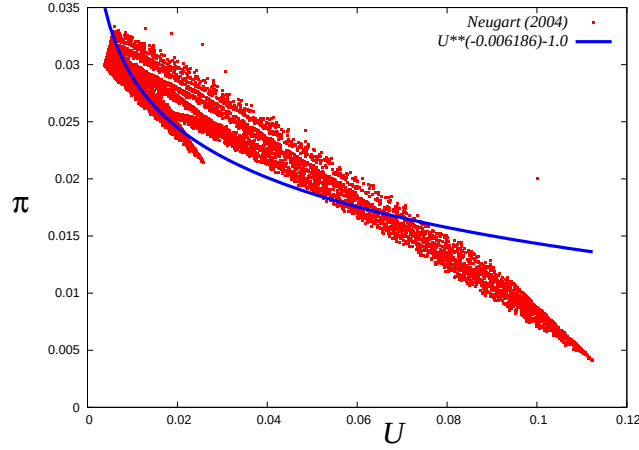


Figure 4-4: Philips curve obtained by the Neugart model [8]. We set $\xi = 0.18, d = 0.01, c_1 = c_2 = 0.5, \mu = 0.04, \Gamma = 0.5, \delta = 2$ and $m = 0.03$. The chaotic attractor follows the scaling form: $\pi + 1.0 \propto U^{-0.006186}$.

The chaotic attractor (U_t, π_y) gives the Philips curve. In Fig. 4-4, we plot the Philips curve obtained by the set of parameters: $\xi = 0.18, d = 0.01, c_1 = c_2 = 0.5, \mu = 0.04, \Gamma = 0.5, \delta = 2$ and $m = 0.03$. From this figure, we observe that the chaotic attractor follows the scaling form: $\pi + 1.0 \propto U^{-0.006186}$.

4.2.2 Coupling with our probabilistic model

As is shown in the previous sections, we can make a model for the probabilistic labor market which described by $a_{ik}(t)$ (in (3.18)) and $s_i(t)$ (in (3.20)) with $s_i(t) = \sum_k s_{ik}(t)$ microscopically. The behavior of the system is macroscopically written in terms of the time-dependence of unemployment rate U_t or the order parameter U which is defined as long-time average of the U_t as $U = \lim_{T \rightarrow \infty} (1/T) \sum_{t=0}^{T-1} U_t$. To draw the Philips curve, here we consider the coupled equations for U_t obtained in our model and π_t which is one of the no-linear map in the Neugart model. Namely, here we use (4.9) with $J_s = J_s^*$ for the update rule for the inflation rate. The other parameters in the Neugart model are set to $\xi = 0.18, d = 0.01, c_1 = c_2 = 0.5, \mu = 0.04, \Gamma = 0.5, \delta = 2$ and $m = 0.03$.

We first show the typical dynamics of the unemployment rate U_t and inflation rate

π_t in Fig. 4-5. From this figure we find that both U_t and π_t are ‘clustered’ during the interval $100 \sim 200$. Within each interval, these quantities behave periodically.

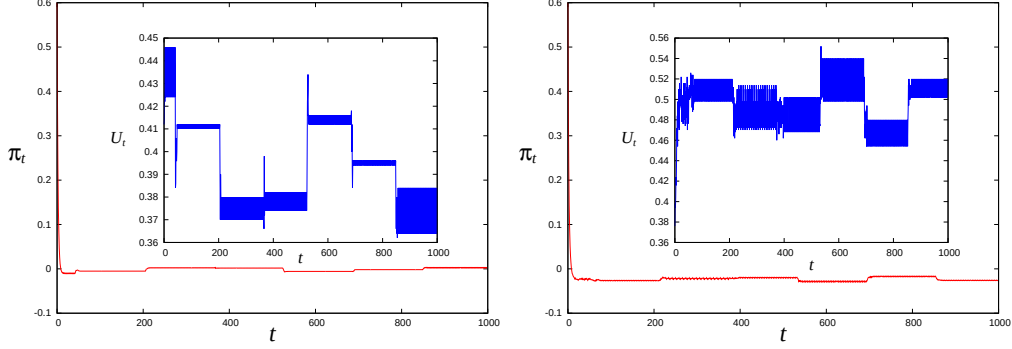


Figure 4-5: Typical dynamics of the unemployment rate U_t and inflation rate π_t . The left panel is given for $\beta = 1, \gamma = 1$, whereas the right panel is plotted for $\beta = 10, \gamma = 1$. We set $N = 500, K = 50, v = 10 (\alpha = 1)$ and $a = 10$. We find that both U_t and π_t are ‘clustered’ during the interval $100 \sim 200$. Within each interval of ‘clustering’, these quantities behave periodically.

We next plot the Philips curve as a trajectory (U_t, π_t) of the dynamics. The results are shown in Fig. 4-6. We find that the negative correlation between U_t and π_t is actually observed for both cases $\beta = 1, \gamma = 1$ and $\beta = 10, \gamma = 1$. And the curve is ‘well-fitted’ by the form: $\pi + 1.49 \propto U^{-0.54}$.

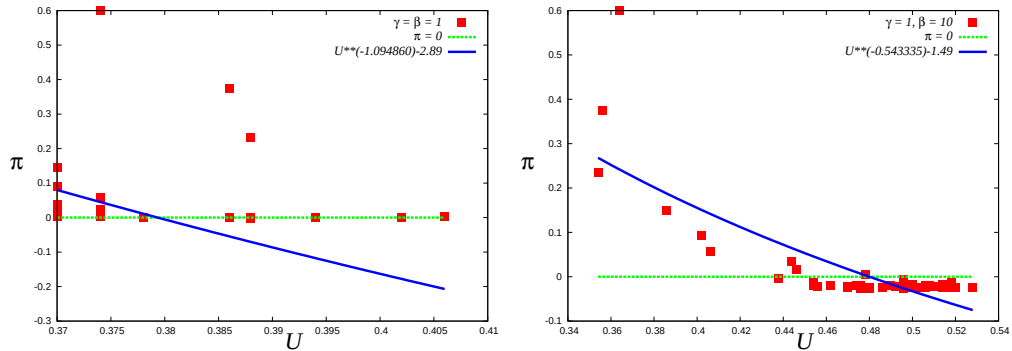


Figure 4-6: The Philips curve for our probabilistic model. The left panel is given for $\beta = 1, \gamma = 1$, whereas the right panel is plotted for $\beta = 10, \gamma = 1$. We set $N = 500, K = 50, v = 10 (\alpha = 1)$ and $a = 10$. The curve is ‘well-fitted’ by the form: $\pi + 1.49 \propto U^{-0.54}$.

We should notice from the right panel of Fig. 4-6 that the inflation rate becomes negative $\pi < 0$ in some parts of the Philips curve, especially, in the regime of relatively

high unemployment rate. The negative inflation rate might be an indicator to detect a sort of social crisis and it might depend on the parameters appearing in our model system. To check it for several parameters, we plot the $\pi \equiv \lim_{T \rightarrow \infty} (1/T) \sum_{t=0}^{T-1} \pi_t$ as a function of c_1 and c_2 . The results are shown in Fig. 4-7. From this figure, we find that for some specific set of the parameters $(c_1, c_2), 0 \leq c_1, c_2 \leq 1$, the inflation rate becomes negative. Of course, in our model system, there exist other parameters and we should take into account the effects on the negative π . The extensive analysis might be needed to figure out the mechanism of the negative inflation rate.

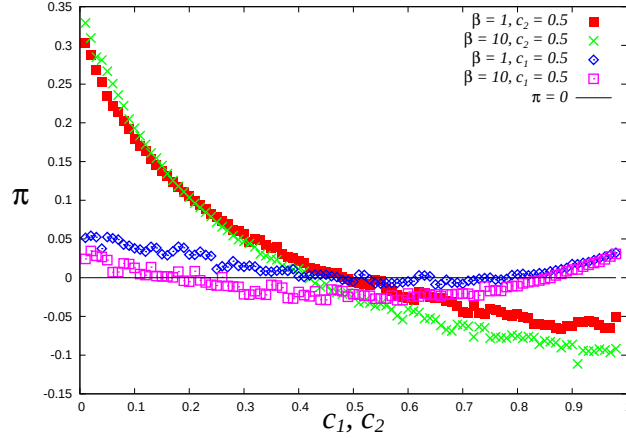


Figure 4-7: The parameters c_1, c_2 -dependence of the long time average of the inflation rate $\pi \equiv \lim_{T \rightarrow \infty} (1/T) \sum_{t=0}^{T-1} \pi_t$. The other parameters and system sizes are the same as in Fig. 4-6.

4.3 Theoretical $U - \Omega$ curve

In this section, we describe the U (unemployment rate) $-\Omega$ (labor shortage ratio) curve based on our probabilistic model. Before we simulate the artificial labor market, we should determine (estimate) several parameters such as the degree of ranking preference γ , market history preference β and average number of applications a from the available empirical data.

4.3.1 Model calibration

To determine the parameters appearing in the system, we focus on the following unemployment rate U as an order parameters shown in (4.1) and (4.2).

$$U \equiv U_0 = U^{(0)} = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} U_t, \quad U_t = \frac{1}{N} \sum_{i=1}^N \delta_{s_i(t), 0}. \quad (4.10)$$

The labor shortage ratio $\Omega \equiv \Omega_0 = \Omega^{(0)}$ is evaluated by inserting the above result (4.10) into (2.11). As we mentioned, in order to draw the $U - \Omega$ curve, we need information about the system parameters such as strength of ranking preference γ and market history β . Here we fix $\beta = 1$ and look for the best possible γ as a solution of $U_{\text{empirical}} = U(\gamma)$, namely,

$$\gamma = U^{-1}(U_{\text{empirical}}), \quad (4.11)$$

where $U_{\text{empirical}}$ denotes the empirical unemployment rate for a given job-offer ratio $\alpha \equiv \alpha^{(0)}$, which is opened for the public by *MEXT* & *MHLW* [15, 16] in every October (in Fig. 2-2 (left)). In the reference [17, 18], we showed that the cumulative employment rate $1 - U = (1 - U)_0$ monotonically decreases as γ increases. It means that there exists a unique solution of (4.11) for a given empirical data $U_{\text{empirical}}$. In our simulations, we set $N = 2000$, $K = 100$ and $n_k^* = \eta, \forall k$, and it turns out to be

$$\eta = \frac{N}{K} \alpha = 20\alpha \quad (4.12)$$

for a given α . Hence, we can control the η to realize any positive value of α .

We plot the resulting γ and $U - \Omega$ curve in Fig. 4-8. From this figure, we find that the large γ apparently pushes the $U - \Omega$ curve toward the upper right direction where the job-worker mismatch between the students and companies is extremely large. The gap between the theoretical and empirical $U - \Omega$ curve comes from the uncertainties in the calibration of average number of applications a . Obviously, it is observable quantity and a large scale survey tells us the precise statistics. However, up to know,

there is no authorized reliable statistics although several private researches provided the data. Hence, we simply chose the value as $a = 10$ in our simulation.

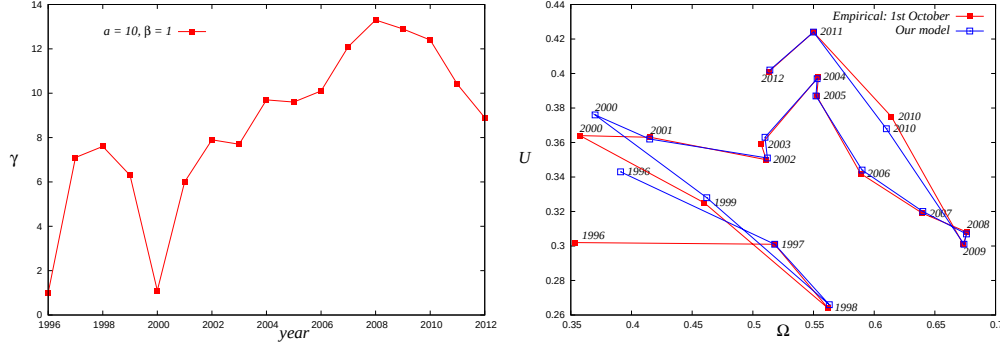


Figure 4-8: Time dependence of ranking preference γ (left). By using this panel accompanying with the right panel in Fig. 2-2 (right), one can account for the appearance of job-worker mismatch. The right panel shows the empirical and theoretical $U - \Omega$ curves. We clearly find that the large γ apparently pushes the $U - \Omega$ curve toward the upper right direction where the job-worker mismatch between the students and companies is large.

We should bear in mind that the parameter γ is not observable in principle. However, our approach enables us to mention the annual strength of students' ranking preferences γ against that of the market history β . In 2010 when the job-worker mismatch was the largest, the point (Ω, U) is located at the upper right position in the $U - \Omega$ curve. However, the corresponding γ does not take the peak in the annual γ trends and the peak appeared in 2008. At a glance, it seems to be inconsistent, however from Fig. 2-2 (right), we find that the job-offer ratio α in 2010 is larger than that in 2008. This implies that the market in 2010 was more 'seller's market' than in 2008, and the market supplied a lot of positions to job-seeking students. Actually, unemployment rate U in 2008 was relatively low in Fig. 2-2 (left). On the other hand, in 2010, the unemployment rate increases by 0.6 from 2009 due to the decreases of α although the strength of students' ranking preferences was getting smaller during 2009-2010. As the result, the point (Ω, U) moved toward the upper right position in the $U - \Omega$ curve in Fig. 4-8 (right), which resulted in the worst job-worker mismatch between the students and companies in 2010.

4.3.2 Analysis for the general case

As we already mentioned before, both U and Ω are not scale-invariant in general. Therefore, we should investigate the flow of relevant quantities $(\alpha^{(n)}, U^{(n)}, \Omega^{(n)})$ by means of our probabilistic labor market [17, 18]. After several complicated algebra, we obtain the following coupled ‘dynamical’ equations.

$$P_k(t) = \frac{\exp[\gamma \log(1 + \frac{k}{K}) - \beta |v_k^* - v_k(t-1)|]}{\sum_{k=1}^K \exp[\gamma \log(1 + \frac{k}{K}) - \beta |v_k^* - v_k(t-1)|]} \quad (4.13)$$

$$\alpha^{(t)} = \frac{\sum_{k=1}^K (v_k^*(t) - m_k(t)) \delta_{v_k^*(t), m_k(t)}}{\sum_{i=1}^N \prod_{l \leq t} \delta_{s_i(l), 0}} \quad (4.14)$$

$$U^{(t)} = 1 - \frac{\sum_{i=1}^N \prod_{l \leq t} \delta_{s_i(l), 0}}{\sum_{i=1}^N \prod_{l \leq t-1} \delta_{s_i(l-1), 0}} \quad (4.15)$$

$$\Omega^{(t)} = \frac{U^{(t)} + \alpha^{(t-1)} - 1}{\alpha^{(t-1)}} \quad (4.16)$$

$$(1 - U)_n = \frac{\sum_{i=1}^N \prod_{l \leq t} \delta_{s_i(l), 0}}{N} \quad (4.17)$$

We carry out computer simulations based on the above equations (4.13)-(4.17).

In our simulation, we calibrate the parameters γ, β by making use of the empirical data which is opened to the public in every October. After the calibration, we do not change the variables. In this sense, the students do not ‘learn’ at all from the past markets (stages). In real labor market for university graduates, students who failed to get the position at $n = 0$ might change their strategies. For instance, they might decrease the strength of ranking preference γ to look for much broadly including small enterprises. However, we do not take into account such ‘intelligence’ of the students and we investigate to what extent the collective behavior of such ‘non-intelligent’ students can reduce the job-worker mismatch.

In Fig. 4-9, we plot $\alpha^{(n)}$ and $1 - U^{(n)}, \Omega^{(n)}$ in the last three years, and the cumulative employment rate $(1 - U)_n$ as a function of n in 2012. From these panels, we clearly find that the relevant quantities $(\alpha^{(n)}, U^{(n)}, \Omega^{(n)})$ are not scale-invariant at all and each of them possesses a rather complicated n -dependence. It should be stressed that the empirical data set for $(\alpha^{(n)}, U^{(n)}, \Omega^{(n)})$ is not available (has never

been released to the public) and we just have the data for the cumulative employment rate $(1 - U)_n$ instead of $U^{(n)}$ as shown in Fig. 2-2. In this sense, our result might be addressed as the ‘first attempt’ to show the stage n -dependence of the quantities $(\alpha^{(n)}, U^{(n)}, \Omega^{(n)})$ explicitly in Japanese labor market.

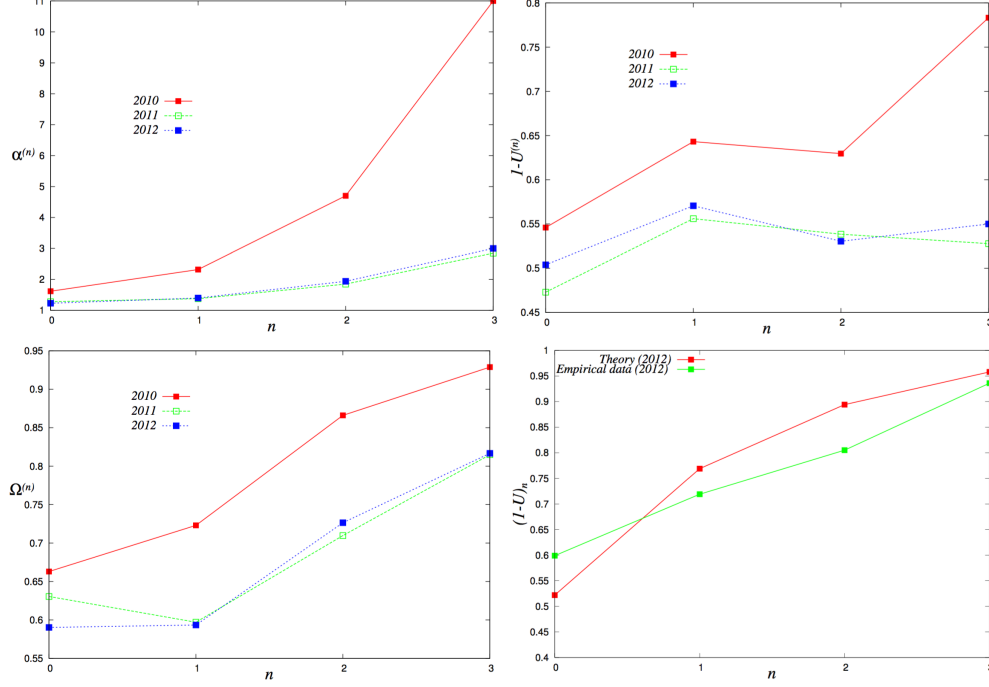


Figure 4-9: From the upper left to the lower right, $\alpha^{(n)}$ and $1 - U^{(n)}$, $\Omega^{(n)}$ in the last three years, and the cumulative employment rate $(1 - U)_n$ as a function of n in 2012 are shown.

In the lower right panel of Fig. 4-9, we show the cumulative employment rate $(1 - U)_n$ with the counter empirical plot. We are confirmed that the theoretical result is remarkably improved from the simple observation $(1 - U)_n = 1 - U^n$ as shown in Fig. 2-2 (right). However, there still exists a gap between empirical and theoretical results. The gap might come from wrong calibration of system parameters such as the average number of application letters a for a student *etc.* (In the above simulations, we set $a = 10$).

Finally, in Fig. 4-10, we show the learning curves as the cumulative employment rate $(1 - U)_n$ for the case of $a = 5$ (left) and $a = 10$ (right). In these panels, we examine for $\alpha = 0.5$ and 2. From this figure, we clearly observe that the $(1 - U)_n$ converges to 1 for $\alpha = 2(> 1)$ and α for $\alpha = 0.5(< 1)$ as we have discussed in the

previous sections. Here we should focus on the speed of convergence. From both panels, we find that the speed of the convergence of $(1 - U)_n$ to the theoretical upper bound unity for the case of $\alpha > 1$ is much faster than that of the case for $\alpha < 1$ (the theoretical upper bound is α). This fact might imply that the job-matching between the students and companies in ‘buyer’s market’ is much harder than that in ‘seller’s market’ especially in the large n regime. It is quite reasonable because $\alpha^{(n)}$ decays to zero for $\alpha < 1$. Another remarkable finding is about average number of applications, and actually we find that the increase of a causes tremendous speed down in the convergence.

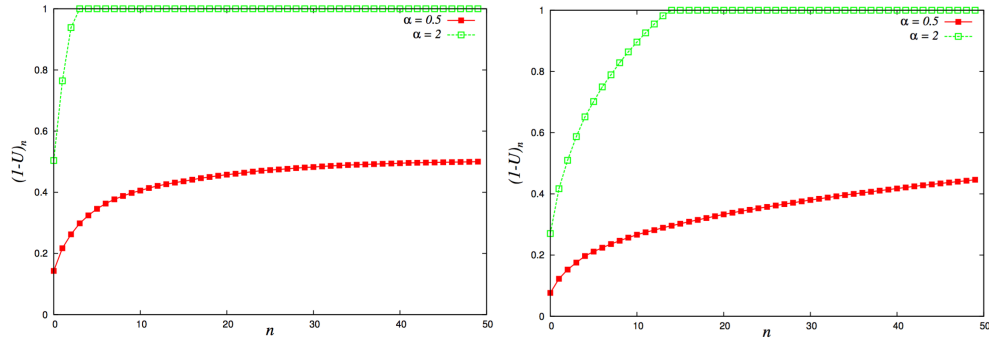


Figure 4-10: The learning curves as the cumulative employment rate $(1 - U)_n$ for the case of $a = 5$ (left) and $a = 10$ (right). We set the system parameters simply as $\gamma = \beta = 1$.

Chapter 5

Statistical mechanics of human resource allocation

5.1 Urn models and Bose condensation: A short review

As a candidate of describing the resource allocation problem, we might use the urn models. In this model, one can show that a sort of Bose condensation takes place. Hence, here we introduce the urn model with a disorder and explain several macroscopic properties according to [41].

We first introduce the Boltzmann weight for the system as

$$p(\varepsilon_i, n_i) = \begin{cases} \frac{\exp[-\beta E(\varepsilon_i, n_i)]}{n_i!} & \text{(Each ball is distinguishable)} \\ \exp[-\beta E(\varepsilon_i, n_i)] & \text{(Each ball is NOT distinguishable)} \end{cases} \quad (5.1)$$

where β stands for the inverse temperature. The former is called *Ehrenfest class*, whereas the latter is referred to as *Monkey class*.

$E(\varepsilon_i, n_i)$ denotes the energy function for the urn i possessing a disorder ε_i and n_i balls. Obviously, in the system with $E(\varepsilon_i, n_i) \propto n_i(>0)$, each urn (agent) is affected by attractive forces and they attempt to gather the balls (resources), whereas in the system of $E(\varepsilon_i, n_i) \propto -n_i$, each urn is affected by repulsive force and they refuse to

collect the balls. The job-matching process in labor market is well-described by the former case. On the other hand, the problem of spent-nuclear-fuel reprocessing plant in Japan is a good example to consider using the latter case, namely, balls are ‘wastes’ and urns are ‘prefectures’.

In the thermodynamic limit: $N, M \rightarrow \infty, M/N = \rho = \mathcal{O}(1)$, the averaged occupation probability $P(k)$, which is a probability that an arbitrary urn possesses k balls is given by

$$\rho = \left\langle \frac{\sum_{n=0}^{\infty} n \phi_{E,\mu,\beta}(\varepsilon, n)}{\sum_{n=0}^{\infty} \phi_{E,\mu,\beta}(\varepsilon, n)} \right\rangle, \quad P(k) = \left\langle \frac{\phi_{E,\mu,\beta}(\varepsilon, k)}{\sum_{n=0}^{\infty} \phi_{E,\mu,\beta}(\varepsilon, n)} \right\rangle, \quad z_s = \exp(\beta\mu) \quad (5.2)$$

where z_s is a solution of the saddle point equation (S.P.E.) and we defined

$$\phi_{E,\mu,\beta}(\varepsilon, n) = \begin{cases} \frac{\exp[-\beta(E(\varepsilon, n) - n\mu)]}{n!} & \text{(Ehrenfest class)} \\ \exp[-\beta(E(\varepsilon, n) - n\mu)] & \text{(Monkey class)} \end{cases} \quad (5.3)$$

In following, we consider the case of Monkey class with the cost function:

$$E(\varepsilon, n) = \varepsilon n, \quad (5.4)$$

which leads to the Boltzmann weight:

$$\phi_{E,\mu,\beta}(\varepsilon, n) = \exp[-\beta n(\varepsilon - \mu)]. \quad (5.5)$$

We choose the distribution of disorder: $D(\varepsilon) = \varepsilon_0 \varepsilon^\alpha$. Then, the saddle point equation is given by

$$\rho = \int_0^\infty \frac{\varepsilon_0 \varepsilon^\alpha d\varepsilon}{z_s^{-1} \exp(\beta\varepsilon) - 1} + \rho_{\varepsilon=0} \quad (5.6)$$

where we should notice that $\rho_{\varepsilon=0}$ is negligibly small before condensation. We increase the density ρ keeping the temperature β^{-1} constant. Then, the possible scenario is shown in Table 5.1. It should be noted that we defined the critical density as

$$\rho_c = \int_0^\infty \frac{\varepsilon_0 \varepsilon^\alpha d\varepsilon}{\exp(\beta\varepsilon) - 1}. \quad (5.7)$$

Density of balls	Solution of S.P.E.	# of condensation/# of non-condensation
$\rho < \rho_c$	$z_s < 1$	$0/N\rho$
$\rho = \rho_c$	$z_s = 1$	$0/N\rho_c$
$\rho > \rho_c$	$z_s = 1$	$N(\rho - \rho_c)/N\rho_c$

Table 5.1: The possible scenario of Bose condensation controlled by the density ρ

After simple algebra, we have

$$P(k) = \frac{z_s^k \varepsilon_0 \Gamma(3/2)}{\beta^{3/2}} k^{-3/2} - \frac{z_s^{k+1} \varepsilon_0 \Gamma(3/2)}{\beta^{3/2}} (k+1)^{-3/2} \quad (5.8)$$

for $\alpha = 1/2$. We show the $P(k)$ for several values of z_s in Fig. 5-1. From this figure, we find that before condensation, namely, for $z_s < 1, \rho < \rho_c$, the occupation probability is given by

$$P(k) = \frac{(1 - z_s) \varepsilon_0}{\beta^{3/2}} k^{-3/2} e^{-k \log(1/z_s)} \quad (5.9)$$

On the other hand, after condensation, that is, for $z_s = 1, \rho \geq \rho_c$, we have

$$P(k) = \frac{3\varepsilon_0 \Gamma(3/2)}{2\beta^{3/2}} k^{-5/2} + \frac{1}{N} \delta(k - k_*) \quad (5.10)$$

The important remarks here are the fact that the condensation is specified by the power-law behavior of the occupation probability and for the case of without disorder, namely, for $D(\varepsilon) = \delta(\varepsilon - \varepsilon_0)$, the power-law behavior disappears.

As we saw, the urn model with disorder exhibits a rich physical phenomena such as condensation; however, there is no explicit interaction between agents (balls and urns). Actually, when we consider the job-matching process, it is impossible to accept the assumption that there is no correlation between urns (companies), balls (students), or between urns and balls. Hence, we should use a different description of the system. In the next section, we use the so-called *Potts model* to describe the problem of human resource allocation.

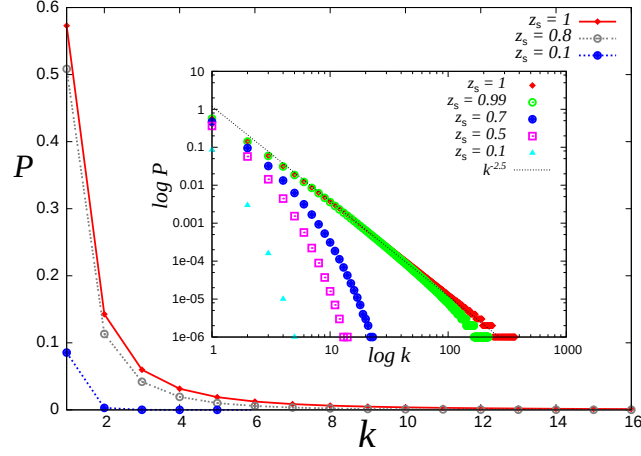


Figure 5-1: The occupation probability of the Monkey class urn model with disorder. This figure was quoted from [41].

5.2 The Potts model descriptions

The basic model proposed here for this purpose is described by the so-called Potts spin glass model which is well-known in the research field of statistical physics. In the model, each Potts spin represents the action of each student, and it takes a discrete value (integer) corresponding to the company he/she applies for. The pairwise interaction term in the energy function describes cross-correlations between students, and it makes our previous model [17, 18, 19] more realistic. Obviously, labor science deals with empirical evidence in labor markets and it is important for us to look for the so-called ‘stylized facts’ which have been discussed mainly in financial markets [42, 43]. We also should reproduce the findings from data-driven models to forecast the market’s behavior.

In following, we show the limited results. Here we consider the system of labor market having N students and K companies. To make the problem mathematically tractable, we construct the energy (Hamiltonian) to include three distinct effects on

the students' behavior:

$$H(\boldsymbol{\sigma}_t) = -\frac{J}{N} \sum_{ij} c_{ij} \delta_{\sigma_i^{(t)}, \sigma_j^{(t)}} - \gamma \sum_{i=1}^N \sum_{k=0}^{K-1} \epsilon_k \delta_{k, \sigma_i^{(t)}} + \sum_{i=1}^N \sum_{k=0}^{K-1} \beta_k |v_k^* - v_k(t-1)| \delta_{k, \sigma_i^{(t)}}, \quad (5.11)$$

where $\delta_{a,b}$ denotes a Kronecker's delta and a Potts spin $\sigma_i^{(t)}$ stands for the company which student i posts their applications to at stage (or time) t , namely,

$$\sigma_i^{(t)} \in \{0, \dots, K-1\}, \quad i = 1, \dots, N. \quad (5.12)$$

Therefore, the first term in (5.11) denotes a collective effect, the second corresponds to the ranking of companies and the third term is a market history. In order to include the cross-correlations between students, we describe the system by using the Potts spin glass (see the 'quenched' random variables c_{ij} in (5.11)) as a generalization of the Sherrington-Kirkpatrick model, which is well-known as an exactly solvable model for spin glass so far. The overall energy function of probabilistic labor market is written explicitly by (5.11). c_{ij} is an adjacency matrix standing for the 'interpersonal relationship' of students, and one can choose an arbitrary form, say

$$c_{ij} = \begin{cases} c & \text{(students } i, j \text{ are 'friendly')} \\ 0 & \text{(students } i, j \text{ are 'independent')} \\ -c & \text{(students } i, j \text{ are 'anti-friendly')} \end{cases} \quad (5.13)$$

for $c > 0$ and the ranking of the company k is defined by ϵ_k (as shown in Sec. 3.3 for the detail).

Before investigating some specific cases below, we shall first provide a general setup. Let us introduce a microscopic variable, which represents the decision making of companies for a student as

$$\xi_i^{(t)} = \begin{cases} 1 & \text{(student } i \text{ receives an acceptance at stage } t) \\ 0 & \text{(student } i \text{ is rejected at stage } t) \end{cases} \quad (5.14)$$

Then, the conditional probability is given by

$$P\left(\xi_i^{(t)}|\sigma_i^{(t)}\right) = 1 - A\left(\sigma_i^{(t)}\right) - \left(1 - 2A\left(\sigma_i^{(t)}\right)\right)\xi_i^{(t)} \quad (5.15)$$

with the acceptance ratio

$$A\left(\sigma_i^{(t)}\right) \equiv \sum_{k=0}^{K-1} \delta_{k,\sigma_i^{(t)}} \Theta\left(v_k^* - v_k(t)\right) + \sum_{k=0}^{K-1} \delta_{k,\sigma_i^{(t)}} \frac{v_k^*}{v_k(t)} \Theta\left(v_k(t) - v_k^*\right) \quad (5.16)$$

where $v_k^*(=1/K, \text{ for simplicity})$ and $v_k(t)$ denote the quota and actual number of applicants to the company k per student at stage t , respectively. $\Theta(\dots)$ is a conventional step function. Hence, when we assume that selecting procedure by companies is independent of students, we immediately have

$$\begin{aligned} P(\boldsymbol{\xi}_t|\boldsymbol{\sigma}_t) &= \prod_{i=1}^N P\left(\xi_i^{(t)}|\sigma_i^{(t)}\right) \cdots P\left(\xi_N^{(t)}|\sigma_N^{(t)}\right) \\ &= \exp\left[\sum_{i=1}^N \log\left\{1 - A\left(\sigma_i^{(t)}\right) - \left(1 - 2A\left(\sigma_i^{(t)}\right)\right)\xi_i^{(t)}\right\}\right]. \end{aligned} \quad (5.17)$$

Thus, we calculate the joint probability $P(\boldsymbol{\xi}_t, \boldsymbol{\sigma}_t)$ by means of $P(\boldsymbol{\xi}_t|\boldsymbol{\sigma}_t) P(\boldsymbol{\sigma}_t)$ as

$$\begin{aligned} P(\boldsymbol{\xi}_t, \boldsymbol{\sigma}_t) &= P(\boldsymbol{\xi}_t|\boldsymbol{\sigma}_t) P(\boldsymbol{\sigma}_t) \\ &= \frac{\exp\left[\sum_{i=1}^N \log\left\{1 - A\left(\sigma_i^{(t)}\right) - \left(1 - 2A\left(\sigma_i^{(t)}\right)\right)\xi_i^{(t)}\right\} - H(\boldsymbol{\sigma}_t)\right]}{\sum_{\boldsymbol{\xi}_t, \boldsymbol{\sigma}_t} \exp\left[\sum_{i=1}^N \log\left\{1 - A\left(s_i^{(t)}\right) - \left(1 - 2A\left(s_i^{(t)}\right)\right)\xi_i^{(t)}\right\} - H(\boldsymbol{\sigma}_t)\right]} \end{aligned} \quad (5.18)$$

where we assumed that the $P(\boldsymbol{\sigma}_t)$ obeys a Gibbs-Boltzmann distribution for the energy function (5.11) as $\sim e^{-H(\boldsymbol{\sigma}_t)}$.

Therefore, the employment rate as a macroscopic quantity:

$$1 - U(t) = \frac{1}{N} \sum_{i=1}^N \xi_i^{(t)} \quad (5.19)$$

is evaluated as an average over the joint probability $P(\boldsymbol{\xi}_t, \boldsymbol{\sigma}_t)$, and in the thermodynamic limit $N \rightarrow \infty$, it leads to

$$\begin{aligned} 1 - U(t) &= \frac{\sum_{\boldsymbol{\xi}_t, \boldsymbol{\sigma}_t} \xi_i \exp \left[\sum_{i=1}^N \log \left\{ 1 - A \left(\sigma_i^{(t)} \right) - \left(1 - 2A \left(\sigma_i^{(t)} \right) \right) \xi_i^{(t)} \right\} - H(\boldsymbol{\sigma}_t) \right]}{\sum_{\boldsymbol{\xi}_t, \boldsymbol{\sigma}_t} \exp \left[\sum_{i=1}^N \log \left\{ 1 - A \left(\sigma_i^{(t)} \right) - \left(1 - 2A \left(\sigma_i^{(t)} \right) \right) \xi_i^{(t)} \right\} - H(\boldsymbol{\sigma}_t) \right]} \\ &= \frac{\sum_{\boldsymbol{\sigma}_t} A \left(\sigma_i^{(t)} \right) \exp [-H(\boldsymbol{\sigma}_t)]}{\sum_{\boldsymbol{\sigma}_t} \exp [-H(\boldsymbol{\sigma}_t)]} \equiv \left\langle A \left(\sigma_i^{(t)} \right) \right\rangle, \end{aligned} \quad (5.20)$$

where we defined the bracket:

$$\langle \cdots \rangle \equiv \frac{\sum_{\boldsymbol{\sigma}_t} (\cdots) \exp [-H(\boldsymbol{\sigma}_t)]}{\sum_{\boldsymbol{\sigma}_t} \exp [-H(\boldsymbol{\sigma}_t)]}. \quad (5.21)$$

From the resulting expression (5.21), we are confirmed that the employment rate $1 - U(t)$ is given by an average of the acceptance ratio (5.16) over the Gibbs–Boltzmann distribution for the energy function (5.11). Using the above general formula, we shall calculate the employment rate exactly for several limited cases.

5.3 Preliminary results

In following, we show our several limited contributions. Before we show our main result, we shall give a relationship between the Potts model and our labor market model which are obtained by simply setting J in (5.11).

5.3.1 For the case of $J = 0$

We first consider the case of $J = 0$. For this case, the energy function (5.11) is completely ‘decoupled’ as follows.

$$H(\boldsymbol{\sigma}_t) = \sum_i H_i, \quad (5.22)$$

$$H_i = - \sum_{k=0}^{K-1} \{ \gamma \epsilon_k - \beta |v_k^* - v_k(t-1)| \} \delta_{\sigma_i^{(t)}, k} \quad (5.23)$$

where we set $\beta_k = \beta$ ($\forall_k$) for simplicity. Hence, the $v_k(t)$ is evaluated in terms of the definition (5.21) as

$$v_k(t) \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \delta_{\sigma_i^{(t)}, k} = \langle \delta_{\sigma_i^{(t)}, k} \rangle = \frac{\exp[-\gamma \epsilon_k + \beta |v_k^* - v_k(t-1)|]}{\sum_{k=0}^{K-1} \exp[-\gamma \epsilon_k + \beta |v_k^* - v_k(t-1)|]} \quad (5.24)$$

and from the expression of employment rate (5.20), we have

$$1 - U(t) = \frac{\sum_{k=0}^{K-1} \left\{ \frac{v_k^*}{v_k(t)} + \left(1 - \frac{v_k^*}{v_k(t)} \right) \Theta(v_k^* - v_k(t)) \right\} \exp[-\gamma \epsilon_k + \beta |v_k^* - v_k(t-1)|]}{\sum_{k=0}^{K-1} \exp[-\gamma \epsilon_k + \beta |v_k^* - v_k(t-1)|]}. \quad (5.25)$$

By solving the nonlinear equation (5.24) recursively and substituting the solution $v_k(t)$ into (5.25), we obtain the time-dependence of the employment rate $1 - U(t)$. In Fig. 5-2, we plot the time-dependence of the employment rate for the case of $K = 3$ (left) and the γ -dependence of the employment rate at the steady state at $t = 10$ for $K = 3$ and $K = 50$ (right). We set the job-offer ratio $\alpha = 1$, the ranking factor is also selected by $\epsilon_k = 1 + k/K$, and assume that each agent posts only a single application $a = 1$ to the market.

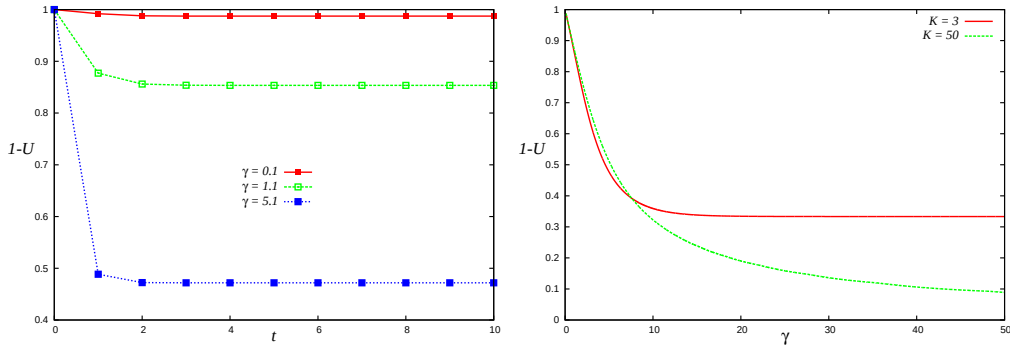


Figure 5-2: The time-dependence of the employment rate for the case of $K = 3$ (left) and the γ -dependence of the employment rate at the steady state at $t = 10$ for $K = 3$ and $K = 50$ (right). We set the job offer ratio $\alpha = 1$ and assume that each agent posts only a single application $a = 1$ to the market.

It should be important for us to remind (5.24) is exactly the same as the update rule for the aggregation probability $P_k(t)$. However, when we restrict ourselves to the case of $\alpha = a = 1$, one can obtain the time-dependence of the employment rate

exactly by (5.25). This is an advantage of this approach. It also should be noted that from the relationship:

$$U = \alpha\Omega + 1 - \alpha \quad (5.26)$$

(as shown in Sec. 2.3), we have $U = \Omega$, namely, the unemployment rate is exactly the same as the labor shortage ratio for $\alpha = 1$.

It is important for us to notice that the aggregation probability of the system $P(\boldsymbol{\sigma}_t)$ is rewritten in terms of $P_k(t)$ as

$$P(\boldsymbol{\sigma}_t) = \left\{ \prod_{k=1}^K P_k(t) \right\}^N \quad (5.27)$$

with $P_k(t) = v_k(t)$ (in (5.24)) even for $\alpha \neq 1$. For this case, the system parameters are only γ and β , and these unknown parameters are easily calibrated from the empirical data, as the result, we obtained U - Ω curve for the past years in Japanese labor market for university graduates(as shown in Sec. 4.3).

5.3.2 For the case of $J \neq 0$

We next consider the case of $J \neq 0$. Then, we should note that some ‘chiral representation’ of the energy function (5.11) by means of the chiral Potts spin [44, 45] (Note: ‘ i ’ appearing in ‘ $2\pi i$ ’ below is an imaginary unit):

$$\lambda_i = \exp\left(\frac{2\pi i}{K}\sigma_i^{(t)}\right), \sigma_i^{(t)} = 0, \dots, K-1 \quad (5.28)$$

enables us to obtain some analytical insights into our labor markets.

The case of $\gamma = \beta = 0$: without ranking and market history:

As a preliminary, we show the employment rate $1 - U$ as a function of $J(>0)$ for the simplest case $\gamma = \beta = 0$ and $c_{ij} = 1 (\forall_{ij})$ in Fig. 5-3 (right), and the actual number of applicants the company k obtains in Fig. 5-3 (left). We should keep in mind that

for this simplest case with local energy

$$H_{ij} \equiv -J\delta_{\sigma_i, \sigma_j} = -\frac{J}{K} \sum_{r=0}^{K-1} \lambda_i^r \lambda_j^{K-r} = -\frac{J}{K} \left\{ 1 + \sum_{r=1}^{K-1} \lambda_i^r \lambda_j^{K-r} \right\} \quad (5.29)$$

under the transformation (5.28) leading to the total energy $H(\boldsymbol{\sigma}) \equiv \sum_{ij} H_{ij}$, by evaluating the partition function:

$$Z = \sum_{\boldsymbol{\sigma}} \exp \left[\frac{J}{NK} \sum_{r=1}^{K-1} \sum_{ij} \cos \frac{2\pi r(\sigma_i - \sigma_j)}{K} \right] \quad (5.30)$$

in the limit of $N \rightarrow \infty$, one can obtain the employment rate $1 - U = \langle A(\boldsymbol{\sigma}) \rangle$ (in (5.20)) exactly as

$$\begin{aligned} 1 - U &= \frac{\sum_{\boldsymbol{\sigma}} A(\boldsymbol{\sigma}) \exp[-H(\boldsymbol{\sigma})]}{\sum_{\boldsymbol{\sigma}} \exp[-H(\boldsymbol{\sigma})]} \\ &= \frac{\left\{ \frac{v_0^*}{v_0} + \left(1 - \frac{v_0^*}{v_0} \right) \Theta(v_0^* - v_0) \right\}}{1 + (K-1)e^{-\frac{Jx}{K-1}}} + \frac{(K-1) \left\{ \frac{v_k^*}{v_k} + \left(1 - \frac{v_k^*}{v_k} \right) \Theta(v_k^* - v_k) \right\} e^{-\frac{Jx}{K-1}}}{1 + (K-1)e^{-\frac{Jx}{K-1}}} \end{aligned} \quad (5.31)$$

with

$$v_k \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \delta_{\sigma_i, k} = \langle \delta_{\sigma, k} \rangle = \frac{\delta_{0, k} + \sum_{\sigma=1}^{K-1} \delta_{\sigma, k} e^{-\frac{Jx}{K-1}}}{1 + (K-1)e^{-\frac{Jx}{K-1}}}, \quad k = 0, \dots, K-1 \quad (5.32)$$

where an order parameter x is determined as a solution of the following nonlinear equation:

$$x = (K-1) \left(\frac{1 - e^{-\frac{J}{K-1}x}}{1 + (K-1)e^{-\frac{J}{K-1}x}} \right). \quad (5.33)$$

It should be noted that the above x is given by the extremum of the free energy density:

$$f = -\frac{Jx^2}{K(K-1)} + \log \sum_{\sigma=0}^{K-1} \exp \left[\frac{Jx}{K(K-1)} \sum_{r=1}^{K-1} \cos \left(\frac{2\pi r}{K} \sigma \right) \right]. \quad (5.34)$$

The acceptance ratio $A(\boldsymbol{\sigma})$ is now given by

$$A(\boldsymbol{\sigma}) \equiv \sum_{i=1}^N A(\sigma_i) = \sum_{i=1}^N \sum_{k=0}^{K-1} \delta_{\sigma_i, k} \left\{ \frac{v_k^*}{v_k} + \left(1 - \frac{v_k^*}{v_k} \right) \Theta(v_k^* - v_k) \right\}, \quad (5.35)$$

and we omit the time t -dependence in the above expressions because the system is no longer dependent on the market history, namely $v_k(t-1)$, for the choice of $\beta = \gamma = 0$ in the energy function (5.11).

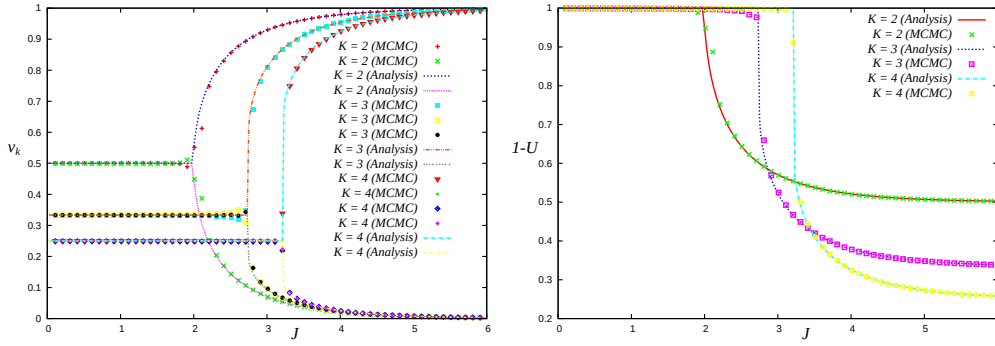


Figure 5-3: The actual number of applicants v_k (left) and employment rate $1 - U$ (right) as a function of the strength of cooperation J . We find that the system undergoes a phase transition at the critical point. The transition is the second order for $K = 2$, whereas it is the first order for $K \geq 3$. These critical points are given by $J_c = 2$ for $K = 2$, $J_c = 2.73$ for $K = 3$ and $J_c = 3.21$ for $K = 4$. We should mention that the analytic results (*lines*) and the corresponding Monte Carlo simulations (MCMC) with the finite number of students $N = 1000$ (*dots*) are in an excellent agreement. We should notice that perfect employment phase is a ‘disordered phase’, whereas the poor employment phase corresponds to an ‘ordered phase’ in the literature of order-disorder phase transition. For large strength of cooperation J , as a company occupies all applications up to the quota, $\lim_{J \rightarrow \infty} (1 - U) = v_k^* = 1/K$ (the quota per student) is satisfied.

In Fig. 5-3, we easily find that phase transitions take place when the strength of ‘cooperation’ J increases beyond the critical point J_c . Namely, for weak J regime, ‘random search’ by students is a good strategy to realize the perfect employment state ($1 - U = 1$), however, once J increases beyond the critical point, the perfect state is no longer stable and system suddenly goes into the extremely worse employment phase for $K \geq 3$ (first order phase transition). The critical point of the second-order

phase transition for $K = 2$ is easily obtained by expanding (5.33) around $x = 0$ as

$$x = \frac{1 - e^{-Jx}}{1 + e^{-Jx}} \simeq Jx/2 \quad (5.36)$$

and this reads $J_c = 2$. For the first order phase transition, we numerically obtain the critical values, for instance, we have $J_c = 2.73$ for $K = 3$ and $J_c = 3.21$ for $K = 4$. As the number K is quite large far beyond $K = 3$ in real labor markets, hence the above finding for the discontinuous transition might be useful for discussing a mismatch between students and companies, which is a serious issue in recent Japanese labor markets.

We also carried out computer simulations to examine the efficiency of the model. We should mention that the analytic results (lines) and the corresponding Monte Carlo simulations (dots) with finite system size $N = 1000$ are in an excellent agreement in the figures. This preliminary result is a justification for us to conform that one can make a mathematically rigorous platform to investigate the labor market along this direction.

We next consider the case of $\beta, \gamma \neq 0$.

Ranking effects:

For the case of $\gamma \neq 0, \beta_k = 0 (\forall_k)$, the saddle point equation is given by the following two-dimensional vector form:

$$(x_r, y_r) = \langle \mathbf{u}_r(s) \rangle_* = \left(\left\langle \cos \frac{2\pi r}{K} s \right\rangle_*, \left\langle \sin \frac{2\pi r}{K} s \right\rangle_* \right), \quad r = 0, \dots, K-1 \quad (5.37)$$

where we defined the bracket $\langle \dots \rangle_*$ as

$$\langle \dots \rangle_* \equiv \frac{\sum_{s=0}^{K-1} (\dots) \exp [\psi_r(s : \{x_r\}, \{y_r\})]}{\sum_{s=0}^{K-1} \exp [\psi_r(s : \{x_r\}, \{y_r\})]}, \quad (5.38)$$

$$\psi_r(s : \{x_r\}, \{y_r\}) \equiv \sum_{r=0}^{K-1} X_r \cdot u_r(s) \quad (5.39)$$

with the following two vectors:

$$X_r = \left(\frac{J}{K}x_r + \frac{r}{K} \sum_{k=0}^{K-1} \varepsilon_k \cos \frac{2\pi r}{K}k, \frac{J}{K}y_r + \frac{r}{K} \sum_{k=0}^{K-1} \varepsilon_k \sin \frac{2\pi r}{K}k \right), \quad (5.40)$$

$$u_r(s) = \left(\cos \frac{2\pi r}{K}s, \cos \frac{2\pi r}{K}s \right). \quad (5.41)$$

From the energy function (5.11) and the above formula, we should notice that the ranking factor ϵ_k is regarded as a ‘state-dependent field’ affecting each spin and the symmetry in the ‘perfect employment phase’ for small J (in Fig. 5-3 (right)) might be broken by these unbiased effects. We also should keep in mind that for the case of $\gamma = 0$ or $\epsilon_k = \epsilon (\forall_k)$, we find that (5.37) possesses the solution of the type: $x_0, \dots, x_{K-1} \neq 0, y_0 = \dots = y_{K-1} = 0$. It should be also bear in mind that $K = 2$ is rather a special case and the solution of the above type is obtained simply as

$$x_0 = 1, x \equiv x_1 = \frac{1 - e^{-Jx+\gamma(\epsilon_1-\epsilon_0)}}{1 + e^{-Jx+\gamma(\epsilon_1-\epsilon_0)}}, y_0 = y_1 = 0. \quad (5.42)$$

However, for general case, we must deal with two-dimensional vectors (x_r, y_r) , $r = 0, \dots, K-1$ with each non-zero component $x_r, y_r \neq 0$ to specify the equilibrium properties of the system.

For the solution $(x_r, y_r), r = 0, \dots, K-1$, we obtain the order parameters and employment rate as

$$v_r = \langle \delta_{r,s} \rangle_* \quad (5.43)$$

$$1 - U = \langle A(s) \rangle_*, r = 0, \dots, K-1. \quad (5.44)$$

In Fig. 5-4, we plot the J -dependence of the employment rate for $K = 2$ (left) and $K = 3$ (right). From this figure, we find that the employment rate decreases monotonically, however, within intermediate range of J , the $1 - U$ behaves discontinuously. We should notice that in this regime, the ‘ergodicity’ of the system might be broken because the realized value of $1 - U$ by Monte Carlo simulation is strongly dependent on the choice of initial configuration (pattern) of Potts spins.

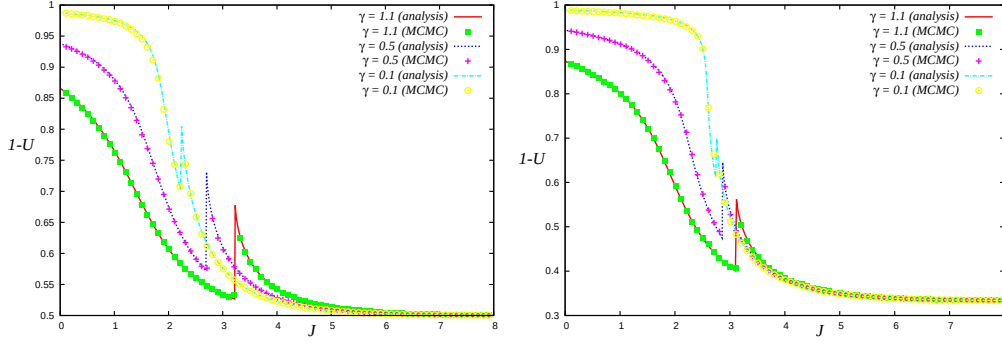


Figure 5-4: The strength of cooperation J -dependence of the employment rate for the case of $\gamma \neq 0, \beta_k = 0 (\forall_k)$. We plot the case of $K = 2$ (left) and $K = 3$ (right). We find that the phase transition as shown in Fig. 5-3 disappears. however, the ergodicity breaking phase appears within intermediate range of J . We are confirmed that $\lim_{J \rightarrow \infty} (1 - U) = 1/K$ is satisfied even for this case. The simulations (MCMC) are carried out for the system of size $N = 1000$.

To see the result more explicitly, we should draw our attention to the initial condition dependence of the J -($1 - U$) curve. Actually, here we carry out Monte Carlo simulation to examine the initial configuration dependence of the $1 - U$ numerically and show the results in Fig. 5-5. From this figure, we confirm that the value of the $1 - U$ depends on the initial configuration of the Potts spins although the $1 - U$ is independent of the initial condition for $J < 3$ and $J \gg 1$. In this figure, we chose two distinct initial conditions so as to make the gap of order parameters $\mathcal{O}(1)$ object, that is,

$$\Delta x_r (\equiv x_r^{(a)} - x_r^{(b)}), \Delta y_r (\equiv y_r^{(a)} - y_r^{(b)}) \sim \mathcal{O}(1) \quad (5.45)$$

for $r = 0, \dots, K - 1$.

It might be important for us to investigate the basin of attraction for the matching dynamics analytically [46], however, it is far beyond the scope of the current study and it should be addressed our future study.

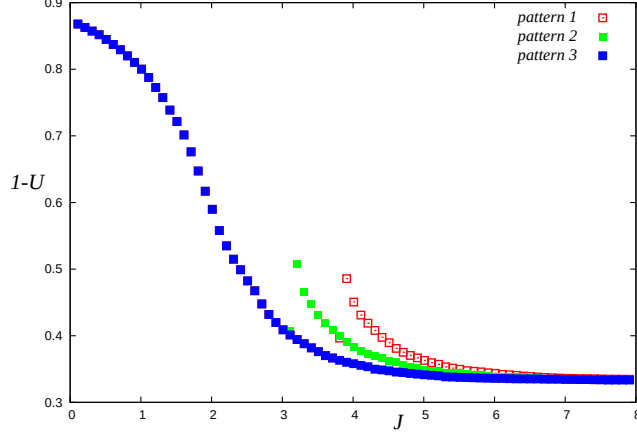


Figure 5-5: The initial configuration dependence of the $1 - U$. We set $K = 3, \gamma = 1.1$ and choose three distinct initial configurations for Monte Carlo simulations. We find that $1 - U$ is strongly dependent on the initial condition ('pattern 1 ~ 3') within intermediate range of J . In this plot, we chose the two distinct initial conditions so as to make the gap of order parameters $\mathcal{O}(1)$ object, that is, $\Delta x_r (\equiv x_r^{(a)} - x_r^{(b)})$, $\Delta y_r (\equiv y_r^{(a)} - y_r^{(b)}) \sim \mathcal{O}(1)$ for $r = 0, \dots, K - 1$, ($a, b = \{pattern1, pattern2, pattern3\}$)

Market history effects:

We next consider the case of $\beta_k \neq 0 (\forall_k)$. For this case, we should replace the $\mathbf{X}_r$ in the saddle point equation (5.37) by

$$\mathbf{X}_r = \left(\frac{J}{K} x_r + \frac{1}{K} \sum_{k=0}^{K-1} (\gamma \varepsilon_k - \beta_k |v_k^* - v_k(t-1)|) \cos \frac{2\pi r}{K} k, \right. \\ \left. \frac{J}{K} y_r + \frac{1}{K} \sum_{k=0}^{K-1} (\gamma \varepsilon_k - \beta_k |v_k^* - v_k(t-1)|) \sin \frac{2\pi r}{K} k \right). \quad (5.46)$$

It should be noticed that the v_k at the previous stage $t - 1$ is regarded as an 'external field' which affects the spin system at the current stage t . Hence, by substituting $v_k(0)$ as an initial state into (5.37) with (5.46), we can solve the equation with respect to $v_k(1)$. By repeating the procedures, we obtain the 'time series' as $v_k(0) \rightarrow v_k(1) \rightarrow \dots v_k(t) \rightarrow$ for all k and $1 - U(t)$ as a function of t .

In Fig. 5-6, we plot the time (stage) dependence of the employment rate $1 - U$ for the case of $K = 2, J = 1, \gamma = 0.1$ and $(\beta_1, \beta_0) = (1, 4)$ (left) and $(\beta_1, \beta_0) = (4, 1)$ (right). From this figure, we find that the larger weight of the market history effect

for the highest ranking company β_1 in comparison with β_0 induces the periodical change of the order for v_1, v_0 due to the negative feedback (a sort of ‘minority game’ [47] for the students). Namely, from the ranking gap $\varepsilon_1 - \varepsilon_0 = 1/2$ for $K = 2$, the company ‘1’ attracts a lot of applications at time t even for a relatively small strength of the preference $\gamma = 0.1$. However, at the next stage, the ability of the aggregation for the company ‘1’ remarkably decreases due to the large β_1 . As the result, the inequality $v_1 > v_0$ is reversed as $v_0 > v_1$, and the company ‘0’ obtains much more applications than the company ‘1’ at this stage. After several time steps, the amount of $\beta_1|v_1^* - v_1(t-1)|$ becomes small enough to turn on the switch of the preference for the high ranking company ‘1’, and eventually the inequality $v_1 > v_0$ should be recovered again. The ‘zigzag behavior’ due to the above feedback mechanism in $v_k(t)$ is actually observed in Fig. 5-6 (right). On the other hand, when the strength of the history effect β_0 for the lower ranking company is larger than that of the higher ranking company β_1 , the zigzag behavior disappears and v_0, v_1 converge monotonically to the steady states reflecting the ranking $\varepsilon_0 < \varepsilon_1$.

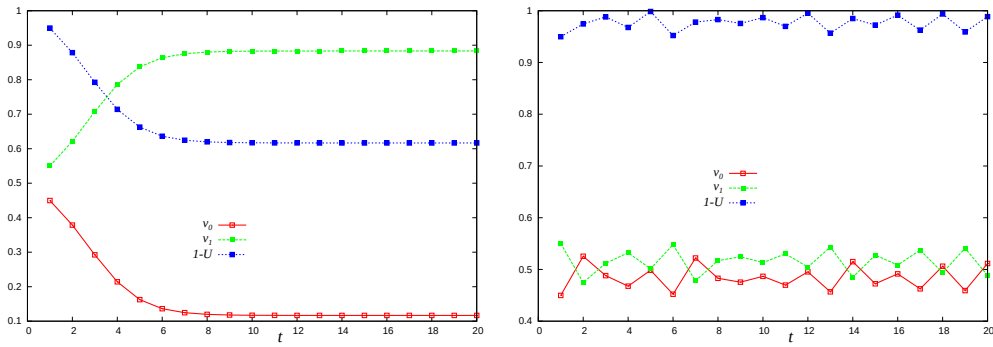


Figure 5-6: The time (stage) dependence of the employment rate $1 - U$ for the case of $K = 2, J = 1, \gamma = 0.1$ and $(\beta_1, \beta_0) = (1, 4)$ (left) and $(\beta_1, \beta_0) = (4, 1)$ (right). The ‘zigzag behavior’ in $v_k(t)$ is observed for $(\beta_1, \beta_0) = (4, 1)$.

5.4 Inverse problem of the Potts model

However, from the view point of empirical science, in this model system, the cross-correlations (the adjacent matrix) between students and companies are unknown and not yet specified. Hence, we should estimate these elements by using appropriate empirical data sets. For instance, if we obtain the ‘empirical correlation’ $\langle \delta_{\sigma_i, \sigma_j} \rangle_{\text{empirical}}$ from the data, we can determine c_{ij} so as to satisfy the following relationship:

$$\begin{aligned} \langle \delta_{\sigma_i, \sigma_j} \rangle &= \frac{\partial}{\partial c_{ij}} \log \sum_{\boldsymbol{\sigma}} \exp[-H(\boldsymbol{\sigma} : \{c_{ij}\})] \\ &= \frac{\sum_{\boldsymbol{\sigma}} \delta_{\sigma_i, \sigma_j} \exp[-H(\boldsymbol{\sigma} : \{c_{ij}\})]}{\sum_{\boldsymbol{\sigma}} \exp[-H(\boldsymbol{\sigma} : \{c_{ij}\})]} = \langle \delta_{\sigma_i, \sigma_j} \rangle_{\text{empirical}} \end{aligned} \quad (5.47)$$

where $\langle \delta_{\sigma_i, \sigma_j} \rangle_{\text{empirical}}$ might be evaluated empirically as a time-average by

$$\langle \delta_{\sigma_i, \sigma_j} \rangle_{\text{empirical}} = (1/\tau) \sum_{t=t_0}^{\tau+t_0} \delta_{\sigma_i^{(t)}, \sigma_j^{(t)}}. \quad (5.48)$$

We might also use the EM (Expectation and Maximization)-type algorithm [48] to infer the interactions. Those extensive studies in this directions (the ‘inverse Potts problem’) including collecting the empirical data are now working in progress.

Chapter 6

Summary and discussion

In this thesis, we proposed a probabilistic model based on the concept of statistical mechanics for probabilistic labor markets, in particular, Japanese labor markets for university (college) graduates. In this agent-based modeling, we introduced the energy function for each company. The energy function describes the power of each company to gather the applications of the students. The energy consists of two distinct factors, namely, ranking preference γ and market history effects β . The later acts as ‘negative feedback’ on the ranking preference. Then, we evaluated the relevant macroscopic quantity, namely, unemployment rate by means of the microscopic variables. We also drew the so-called Beveridge curve (UV curve: ‘Unemployment-Vacancy’) which shows the relationship between the unemployment rate and the number of vacancies (job-offer ratio). We found that the system undergoes a phase transition with spontaneous symmetry breaking from the good employment rate phase to the poor employment rate phase when one increases the degree of ranking preference. We also utilized the macroscopic Neugart model which describes the Philips curve as an attractor of chaotic maps. We used the update equation for the inflation rate accompanying with our probabilistic model to derive the Philips curve for our model system. We found that the Philips curve is actually obtained by the ‘mixture model’.

In Japanese labor markets for university graduates, the students have several successive chances to obtain their positions within an academic (business) year. After

each opportunity (stage) is over, the numbers of both students and positions (namely, the so-called ‘effective system size’) are reduced remarkably because several excellent students get their job immediately after they apply for. Accordingly, the macroscopic quantities such as unemployment rate U , labor shortage ratio Ω or job offer ratio α changes as $U^{(0)} \rightarrow \dots U^{(n)}$, $\Omega^{(0)} \rightarrow \dots \Omega^{(n)}$ or $\alpha^{(0)} \rightarrow \dots \alpha^{(n)}$. However, it is highly non-trivial whether some of them could be scale invariant or not, that is to say, it is not easy for us to conclude that there exists the condition on which a macroscopic quantity, say, $U^{(n)}$ satisfies $U^{(n)} = \text{constant}$ for any job-hunting stage n . Here we focused on this issue and clarify the n -dependence (or independence) of Japanese labor markets for university graduates by making use of both extensive computer simulations and empirical data analysis. In order to investigate these aspects of labor markets, Inspired by the unsupervised learning or self-organization in the machine learning context, we drew ‘learning curve’ for the collective behavior of job-seeking ‘non-intelligent’ labors in successive job-hunting process.

Our labor market is supposed to be opened especially for university graduates in Japan, where the students have several successive chances $n = 0, 1, 2, \dots$ to obtain their positions within an academic (business) year. In this sense, the ‘cumulative unemployment rate’ in our model system is regarded as an error-measurement in the collective intelligence of students and the job-hunting stage n -dependence of the error constructs a learning curve. In our toy-model of probabilistic labor market, the diversity of students behavior is built-in by means of the Jaynes-Shannon’s MaxEnt (Maximum Entropy) principle. We discussed the speed of convergence for the error measurement, where we consider a scenario in which the students do not use any information about the result of job-hunting processes in the previous stages.

We also discussed the scale invariance of the macroscopic quantities in probabilistic labor markets. We obtained the flow $(\alpha^{(n)}, U^{(n)}, \Omega^{(n)})$ with the assistance of computer simulations. Obviously, students in real labor markets are much more ‘intelligent’ than the agents in our model system. In this sense, we should modify our model by taking into account the students’ tendency of job-hunting process. Especially, ‘intelligent’ learning algorithm of system parameters such as γ or β might

play important role in the next direction of our studies. These parameters might be dependent on the business cycle in the society and in this sense, the labor market should be coupled with the other kinds of markets such as financial market or housing market. These issues should be addressed as our future problems.

However, our studies depend on numerical (computer) simulations for relatively small system size to calculate these quantities, and we definitely need some mathematically rigorous approaches to find the universal fact underlying in the job-matching process of labor markets. It is also important issue to be considered that we should take into account correlation between agents (students) when we consider the job-matching process in realistic labor markets.

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