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Doctoral Thesis

**A Study on Controller Design with Hysteresis Mechanism
and Sliding-Sector**

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A Study on Controller Design with Hysteresis Mechanism and Sliding-Sector*

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Abstract

The control law with high frequent switching of the input is often obtained in some systems; e.g. the system with limited input such as on/off or brake/accelerator, and the system under the sliding-mode control that constrains the state on given hypersurface against for the disturbances. Under such a control law, the chattering phenomenon occurred by the switching performance or the property of the system causes adverse effects on the target system. This study considers the chattering reduction on the input switching for the control law that requires the discontinuity of the input with respect to time. Especially, we study the obstacle avoidance control for the four-wheeled vehicle with steering limitation and the higher-order sliding-mode control.

Under the steering limitation, the movement direction of the four-wheeled vehicle is restricted to whether the forward or backward of the heading direction. Though the two-wheeled vehicle can change its heading direction without change its position, the four-wheeled vehicle cannot. In order to solve this problem, the vehicle needs to repeat going forward and backward with an appropriate steering angle. In this thesis, we design the control law that realize the n -point turn (like a three-point turn) automatically. Moreover, we introduce the hysteresis mechanism for the chattering that may occur in the n -point turn.

The higher-order sliding-mode control that is a generalized sliding mode control for the system with relative degree greater than one. The higher-order sliding-mode control has the robustness for the disturbances for and parameter uncertainty of the system. Moreover, with the negative homogeneity, the finite-time stability is guaranteed for the higher-order sliding-mode control as with the traditional sliding-mode control. Some classes of the higher-order sliding-mode control have been proposed; one of the classes have no theoretically constructive design method for the stable controller. In this thesis, we propose a novel higher-order sliding-mode control and its constructive design method with a theoretical stability. Moreover, we define the sliding sector that integrates some types of the higher-order sliding-mode controllers including the chattering rejection for binary input.

This thesis is consist of six chapters as below.

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Chapter 1 describes the background and the objective of this thesis, and Chapter 2 introduces some mathematical notations and definitions, and some preliminaries for control theory.

Chapter 3 considers the obstacle avoidance control for the four-wheeled vehicle under the steering limitation. First, we formulate the problem settings, that is, the model of the four-wheeled vehicle and the control space with a number of obstacles, and propose a design method of the navigation function. We design a control law for the vehicle based on the control Lyapunov function obtained from the navigation function. The proof of the asymptotic stability to the goal point is also provided. The proposed control law resolves the steering limitation by the n -point turn movement. Moreover, we propose the artificial hysteresis mechanism based on the control Lyapunov function, and apply the mechanism to the control law to reject the chattering between going forward and backward. End of the chapter, we show the effectiveness of the proposed obstacle avoidance controller from the experimental results using a small four-wheeled vehicle.

Chapter 4 discusses the higher-order sliding-mode control that has robustness and finite-time stability. We, first, introduce the previously-proposed higher-order sliding-mode controllers by separating into three classes. For the class that is one of the three classes, we explain the problems on the parameter design. We propose a novel higher-order sliding-mode control that solves the problems. We prove the existence of the proper parameters and its constructive method for the proposed higher-order sliding-mode control.

Chapter 5 shows that the various types of the higher-order sliding-mode control is integrated into a sliding sector control. We also discuss the actual implementation of the higher-order sliding-mode control. The sliding sector is defined as the extension of the sliding hypersurface. We show that various types of the controllers, e.g. the continuous controller except at the origin, the discontinuous controller with a signum function, and the discontinuous controller with a hysteresis mechanism are obtained by introducing the sliding sector. Moreover, the saturation function for the traditional sliding-mode control with relative degree one can be extended to the proposed higher-order sliding-mode control based on the sliding sector. We state the parameter design for first- and second-order system, and higher-order system. The results of the numerical simulation present the efficacy and the characteristic of the proposed control law based on the sliding sector.

Chapter 6 concludes this thesis by summarizing the results of the above chapters.

Keywords: Input constraint, Chattering, Hysteresis mechanism, Sliding-mode control, Finite-time stabilization, Robustness

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Chapter 1. Introduction

The control theory for the system with a polytopic input constraint is widely studied, but the disjoint input constraints is an unwelcome condition. There are, however, many systems which have a limited input such as binary input, e.g. on/off control and brake/accelerator switching system. This thesis considers the four-wheeled vehicle with steering limitation and the higher-order sliding-mode control. The control input under the steering limitation is consist of two disjoint input space, that is, the forward and backward of the heading direction on two-dimensional space. For the sliding mode, the control input must be discontinuously changed at the origin. The control law that provides the discontinuous control input for the system often cause the chattering phenomenon.

In this thesis, we study the countermeasures against the chattering phenomenon caused by the discontinuous control input. The control law for the system with input constraints may have such a discontinuity; for example, the system with discrete input, the four-wheeled vehicle with steering limitation, and so on. For the four-wheeled vehicle, we propose the CLF design method for the obstacle avoidance and the control law based on the obtained CLF. Under the steering limitation, the four-wheeled vehicle often requires the repetition of the going forward movement and the going backward movement with an appropriate steering angle. We show that an artificial hysteresis mechanism can reduce the discontinuous jump of the control input for the control law based on the CLF. By using such a hysteresis mechanism, the chattering phenomenon caused by switching between going forward and backward. Furthermore, in this thesis, we study the higher-order sliding-mode control. We provide the definition of sliding sector that integrates the some types of the chattering relaxation methods for the higher-order sliding-mode control. This representation includes three significant types of the higher-order sliding-mode controller, that is, continuous controller except at the origin, discontinuous controller with signum function and with hysteresis mechanism. The saturation function for the traditional sliding-mode control is naturally generalized for the higher-order sliding-mode control by introducing a lower limit to the positive definite function used in the control law.

1.1 Four-wheeled vehicle

The obstacle avoidance is an important function for a mobile robot. With a priori information of the obstacles, the mobile robot can reach the goal point free from straying.

For simple implementation, the global asymptotic stable controller based on the global CLF is useful. We know that there are no global continuous feedback for the obstacle avoidance problem. The control law derived from the smooth navigation function for obstacle avoidance has undesired equilibria, inevitably, such as a saddle point. Nakamura et al. have been proposed nonsmooth CLF design method called minimum projection method [1]. We use the minimum projection method to design the global navigation function for the obstacle avoidance.

This thesis studies the control law of the four-wheeled vehicle for avoiding obstacles on the known environment. The steering limitation of the four-wheeled vehicle is also considered, as with the car. Both two- and four-wheeled vehicle have a nonholonomic constraint for the state. Though the two-wheeled vehicle can move in arbitrary direction because of the capable of pivot turn, the four-wheeled vehicle under the steering limitation can move only in the direction of forward and backward movement. Such a constraint is difficult problem that cannot be solved by the navigation function. Because of the limitation of the direction of the movement, in order to move the vehicle to another point, the four-wheeled vehicle must change its heading angle. To change heading angle under the condition that the obstacles are positioned both forward and backward, the vehicle needs to alternate between going forward with left/right turn and going backward with right/left turn. Therefore, the control input may be changed discontinuously with respect to the time. We also shows that the chattering phenomenon that may occurs by the discontinuous input can be eliminated by an appropriate hysteresis mechanism.

In this thesis, we proposed the control law and the CLF design method for the four-wheeled vehicle. The global CLF is synthesized from a number of local CLFs by using multilayer minimum projection method [2]. The local CLF for the four-wheeled vehicle is obtained from the navigation function by backstepping with respect to the heading angle of the vehicle. We design the control law under the steering limitation from the obtained CLF. The proposed control law provide a n -point turn (like 3-point turn) movement for the four-wheeled vehicle, automatically. By the n -point turn, the four-wheeled vehicle changes its heading angle by repeating forward and backward movement with an appropriate steering angle. We also introduce the artificial hysteresis mechanism based on the CLF with theoretical stability. The chattering between the forward and backward movement on the n -point turn is rejected by the proposed hysteresis mechanism. We present the special controller based on the CLF around the origin that improves the convergence rate around the origin. Finally, we show that the effectiveness of the proposed method from the simulation results and the experimental results for a small four-wheeled vehicle.

1.2 Sliding mode control

Higher-order sliding-mode control has outstanding properties, such as robustness against the disturbance and uncertainty. Moreover, the finite-time stability of a sys-

tem can be realized with a homogeneous sliding-mode controller even under the input constraint. The traditional sliding-mode control also has these properties. Higher-order sliding-mode for second- and third-order have studied in many researches [3, 4, 5]. However, fourth- or higher-order sliding-mode controllers are difficult to design the control law.

Recently, arbitrary-order sliding-mode control [6, 7] has been actively studied. Kamal and Bandyopadhyay proposed higher-order sliding-mode control that uses a controllability function [8]. Polyakov et al. have proposed the higher-order sliding-mode controller using the implicit Lyapunov function (ILF) method, which is based on the homogeneous Lyapunov function with implicit form [9]. Using the controllability function or using the ILF, we can design the controller by some constructive method. However, such a controller needs to solve implicit function by some iterative calculation method. Most of the other methods require computer verification of the stability of the controller, such as quasi continuous higher-order sliding-mode control [10, 11].

In this thesis, we introduce a general strategy that includes various types of homogeneous sliding-mode control laws. We can check the stability of a system under the control law with some explicit inequalities. Therefore, we can theoretically design the controller even if the target system has input constraints, bounded disturbances, and uncertainty. We propose a novel strategy for the higher-order sliding-mode control and a parameter design method for a higher-order sliding-mode controller. We prove the existence of the parameters based on inequality conditions. Moreover, we prove the existence of the parameters based on inequality conditions, and introduce the constructive method of parameter design. The design method does not need to examine trajectories numerically for stability check because the stability is guaranteed by the properness of the parameters. We introduce the sliding sector that is an extension of the sliding hypersurface. This is a new control strategy for the higher-order sliding-mode control, that includes various types of homogeneous sliding-mode control laws, such as continuous and discontinuous input, which are demonstrated in the simulations.

Most sliding-mode controllers that use the signum function require infinitely high frequency switching of the input to constrain the state of the system on the sliding hypersurface for the higher relative degree. Such switching of the input causes the chattering phenomenon. Furuta and Pan proposed a chattering-less control that uses a linear sliding sector instead of a sliding surface; their approach traps the state in the linear sliding sector [12].

The sliding sector has some range except at the origin. We also propose a binary input controller for higher-order sliding-mode control. Under the binary input constraint, the controller for the system can only use two input such as ± 1 . The proposed method unifies the continuous and discontinuous controller for higher-order sliding-mode control into a strategy, which is applicable for a system with a binary input constraint. We provide a binary input controller for the system based on the sliding sector. Under the binary input, the state is attracted to a nonlinear sector called sliding sector, and then

the state is trapped on the sliding sector by a bang-bang input with an appropriate hysteresis. The chattering phenomenon on the sliding hypersurface is rejected except at the origin. The reduction mechanism also provides strong effect for the chattering reduction under three-valued input constraint ± 1 and zero, which is demonstrated in the simulations.

1.3 Outline

This thesis is organized as follows. Following this chapter, Chapter 2 introduces the mathematical preliminaries, including the fundamentals of system theory.

Chapter 3 shows the global obstacle avoidance controller for the four-wheeled vehicle with steering limitation. For the obstacle avoidance, we propose the design method of the navigation function, which is a CLF for a holonomic vehicle. The CLF for the four-wheeled vehicle is obtained by extending the navigation function with respect to the heading angle. We can obtain a global CLF from a number of local CLFs by using multilayer minimum projection method. From the CLF, we derive a control law for the four-wheeled vehicle and prove the asymptotic stability at the desired point of the four-wheeled vehicle. The obtained control law, however, needs switching between forward and backward movement with infinite frequency, and causes the chattering phenomenon. Therefore, we introduce the hysteresis mechanism based on the CLF, and show that the natural switching is realized by the hysteresis. Moreover, we present the special CLF around the origin, and show the experimental results for a small four-wheeled vehicle to verify the effectiveness of the proposed obstacle avoidance controller.

Chapter 4 discusses the higher-order sliding-mode control. We introduce r -sliding homogeneity ensuring the finite-time stability, and some classes of the higher-order sliding-mode controller previously proposed. The higher-order sliding-mode controller using positive definite function, called quasi-continuous sliding-mode controller, cannot design the parameters in theoretical procedures. Because of the loss of explicit stability conditions, we have to check the stability by computer simulations. We propose the novel higher-order sliding-mode controller with the constructive method for the parameter design. We prove the stability of the controller and the existence of the stable parameters.

Chapter 5 shows the higher-order sliding-mode based on the sliding sector. We, first, define the sliding sector, and show that the controller proposed in Chapter 4 satisfies the properties of the sliding sector. Next, we confirm that the sliding sector is an extension of the sliding hypersurface. We show that the controller based on sliding sector integrates the various types of the higher-order sliding-mode controllers; e.g. the quasi-continuous sliding-mode controller and the sliding-mode controller with the signum function. Moreover, the discrete-valued controller with the hysteresis mechanism for the switching of the control input can be obtained naturally. The saturation function for the traditional sliding-mode control can be generalized for the higher-order sliding-mode control in the context of the sliding sector. Finally, we discuss the parameter design for

the first-order, second-order, and higher-order system. We show the effectiveness of the proposed controllers based on the sliding sector by the numerical simulations.

Chapter 6 gives the conclusion of this thesis.

Chapter 2. Preliminaries

2.1 Notations and definitions

In this section, we introduce some general notations and definitions. Throughout this thesis, we use the following notations:

- $\mathbb{Z}$ is a set of integers.
- $\mathbb{R}$ is a set of real numbers.
- $\mathbb{R}^n$ denotes n -dimensional Euclidean space for some integer n .
- 2^X denotes the set of all nonempty subsets of some space X .
- $|z|$ denotes the absolute value of z .
- $\|z\|$ denotes the 2-norm of $z \in \mathbb{R}^n$ and the norm of $z \in S^1$ defined as

$$\|z\| = \min_{n \in \mathbb{Z}} |z + 2n\pi|.$$

- $\dot{z}$ and $\ddot{z}$ denote the time derivative of z and twice time derivative of z , respectively.
- $z^{(i)}$ denotes the i -th time derivative of z for some integer i .
- $\overline{\text{co}}S$ denotes the closure of the convex hull for given set S .

We treat $|0|^0$ as $\lim_{z \rightarrow 0} |z|^0 = 1$. We define the signum function $\text{sgn} : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\text{sgn}(z) : \begin{cases} = 1 & (z > 0) \\ \in [-1, 1] & (z = 0) \\ = -1 & (z < 0). \end{cases}$$

Throughout this thesis, we use the following notation for a real variable $z \in \mathbb{R}$ to any power $p \in \mathbb{R}$:

$$\lfloor z \rfloor^p := |z|^p \text{sgn}(z).$$

Note that $\lfloor z \rfloor^2 = |z|^2 \text{sgn}(z) = z^2 \text{sgn}(z) \neq z^2$ for $z < 0$. For any z , $\lfloor z \rfloor^0 = \text{sgn}(z)$ because $|z|^0 = 1$. Moreover, the derivative of the power functions are given by

$$\begin{aligned} \frac{d\lfloor z \rfloor^p}{dz} &= p|z|^{p-1}, \\ \frac{d|z|^p}{dz} &= p\lfloor z \rfloor^{p-1}. \end{aligned}$$

2.2 Filippov solution

Consider the following differential equation:

$$\dot{x} = f(x)$$

where $x \in \mathbb{R}^n$ is a state vector and $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is a vector field. In order to allow the discontinuity of $f(x)$ with respect to x , we consider the solution in Filippov sense. Let us consider the following differential inclusion including the differential equation with discontinuous right-hand side included

$$\dot{x} \in F(x).$$

If the set-valued function $F(x)$ satisfies below, the above differential inclusion has a continuous solution:

$$F(t, x) = \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} \overline{co}f(t, \mathcal{B}(x, \delta \setminus N))$$

where μ is a Lebesgue measure of $\mathbb{R}^n$ and $\mathcal{B}(x, \delta)$ denotes the ball of radius δ centered at x on $\mathbb{R}^n$.

2.3 Control Lyapunov function

The control Lyapunov function (CLF) is a Lyapunov function candidate for the closed-loop system. The control law for the system is designed from the candidate function to decrease the value of the function everywhere except on the origin set. The CLF approach is a common and useful method for the stabilization of nonlinear systems.

First, we introduce the definition of the CLF. Let us consider a nonlinear system

$$\dot{x} = f(x, u), \tag{2.1}$$

where $x \in X$ is a state and $u \in U \subset \mathbb{R}^m$ is a control signal. We assume that $f : X \times U \rightarrow T_x X$ is continuous with respect to x and u , and $f(x_0, 0) = 0$ for every $x_0 \in P$, where P is a set of desired equilibria.

Let $\psi(t, x; u) : T \times X \times F(T, U) \rightarrow X$ be a solution of (2.1) for an initial state x and input $u(t)$, where $t \in T = [0, T_{max}(x, u))$ indicates the time and $F(T, U)$ denotes a set of all mappings from T to U . We use the following definition of a CLF for the set of the desired equilibria P .

Definition 2.1 (Control Lyapunov function (CLF))

A global control Lyapunov function for the system (2.1) is a continuous function $V : X \rightarrow \mathbb{R}$ satisfying the following:

- (A1) V is proper; that is, the set $\{x \mid V(x) \leq \alpha\}$ is compact for every $\alpha > 0$.
- (A2) V is positive definite relative to P ; that is, $V(x) = 0$ ($\forall x \in P$) and $V(x) > 0$ $\forall x \in X \setminus P$, where P is a set of the desired equilibria.

(A3) For any $x \in X \setminus P$, there exists a time $\rho \in [0, T_{\max}(x, u))$ and an input $u(t)$ such that $V(\psi(t, x, u)) \leq V(x)$, $\forall t \in [0, \rho)$ and $V(\psi(\rho, x, u)) < V(x)$.

Let (W, η) be a local coordinate included in an atlas of X , i.e., $W \subset X$ and $\eta : W \rightarrow \Xi = \text{Im}(\eta) \subset \mathbb{R}^{d_X}$ is a homeomorphism, where d_X denotes a dimension of X . We also define a strict CLF for the set P .

Definition 2.2 (Strict control Lyapunov function (strict CLF))

A global strict control Lyapunov function for the system (2.1) is a continuous function $V : X \rightarrow \mathbb{R}$ satisfying conditions (A1) and (A2) of Definition 2.1 and the following condition:

(A3') For any $x_0 \in X \setminus P$, there exists a neighborhood S_{x_0} of x_0 , an input $u \in U$, and a local chart (W, η) such that $DV_W(\eta(x); (f(x, u))_W) \leq -Q(x)$ ($x \in S_{x_0}$), where $Q : X \rightarrow \mathbb{R}$ is a continuous positive definite function, which is independent of the choice of x_0 , the suffix $_W$ indicates a local coordinate representation, and the directional subderivative DV_W is defined as

$$DV_W(\eta(x); v) := \liminf_{\substack{t \downarrow 0 \\ v' \rightarrow v}} \frac{V_W(\eta(x) + tv') - V(x)}{t}.$$

The definition of the CLF admits a point where its time derivative for the closed-loop system is zero except the goal set, and we must use Yoshizawa-LaSalle Theorem to show the asymptotic stability via the weak (not strict) CLF. By contrast, the time derivative of a strict CLF for a closed-loop system is negative definite in the sense of directional subderivative. We introduce the multilayer minimum projection method proposed in [2] by Nakamura et al. The minimum projection method is a CLF design method on a noncontractible manifold such as a space with obstacles. It synthesizes a global CLF from a number of local CLFs.

Theorem 2.1 (Nakamura et al. [2])

Consider a finite layers $\tilde{X}_1, \dots, \tilde{X}_s$ associated with étale mappings $\phi_i : \tilde{X}_i \rightarrow X$ ($i \in S = \{1, 2, \dots, s\}$) satisfying the following conditions:

- (B1) For every $i \in S$, the continuous mapping $\phi_{i*} : T_{\tilde{x}_i} \tilde{X}_i \rightarrow T_{\phi_i(\tilde{x}_i)} X$ is bijective for all $\tilde{x}_i \in \tilde{X}_i$; here ϕ_{i*} is the differential of ϕ_i , on the local coordinate, which is represented by a linear mapping with the Jacobian matrix.
- (B2) There exists $P_i \subset X_i$ such that $\phi_i(P_i) = P$, where $P \subset X$ is the desired set of equilibria.
- (B3) The mapping $\phi : \tilde{X} \rightarrow X$ is surjective, where $\tilde{X} = \bigoplus_{i \in S} \tilde{X}_i$ and ϕ is a coproduct of $\{\phi_i\}_{i \in S}$:

$$\phi(\tilde{x}) = \begin{cases} \phi_1(\tilde{x}) & (\tilde{x} \in \tilde{X}_1) \\ \vdots \\ \phi_s(\tilde{x}) & (\tilde{x} \in \tilde{X}_s). \end{cases}$$

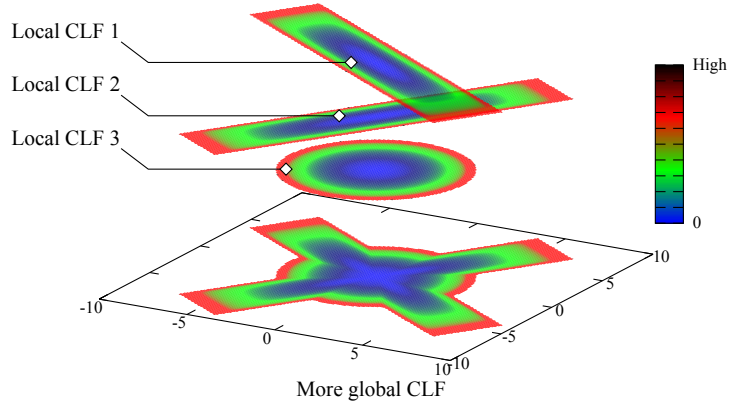


Figure 2.1: Concept image of multilayer minimum projection method

If $\tilde{V}_i$ is a CLF for global asymptotic stabilization at P_i of the induced system $\dot{\tilde{x}} = \phi_{i*}^{-1}(\phi_i(\tilde{x}), u)$ on $\tilde{X}_i$, then

$$V(x) := \min_{\tilde{x} \in \phi^{-1}(x)} \tilde{V}(\tilde{x}) \quad (2.2)$$

is a CLF on X , where $\tilde{V}$ is a coproduct of $\{\tilde{V}_i\}_{i \in S}$:

$$\tilde{V}(\tilde{x}) = \begin{cases} \tilde{V}_1(\tilde{x}) & (\tilde{x} \in \tilde{X}_1) \\ \vdots \\ \tilde{V}_s(\tilde{x}) & (\tilde{x} \in \tilde{X}_s). \end{cases}$$

If each $\tilde{V}_i$ is a strict CLF on $\tilde{X}_i$, V is also a strict CLF. Moreover, if each $\tilde{V}_i$ is locally Lipschitz (resp. semiconcave), V is a locally Lipschitz (resp. semiconcave) function on X .

The above method is modified from [2] in order to be applied for the set of desired equilibria P , as with [13]. Fig. 2.1 shows the concept of global CLF design by the multilayer minimum projection method. Note that, even for the case of $\phi_i(x_1) = \phi_j(x_2)$ ($i \neq j$), x_1 and x_2 are different points on $\bar{X}$, and $\phi^{-1}(\cdot)$ is a set-valued function.

Moreover, we can consider the singular points of V_i by using the following multilayer minimum projection method with singular point assignment.

Theorem 2.2 (H. Nakamura and N. Nakamura [14])

Consider a finite layers $\tilde{X}_1, \dots, \tilde{X}_s$ associated with étale mappings $\phi_i : \tilde{X}_i \rightarrow X$ ($i \in S = \{1, 2, \dots, s\}$) satisfying (B1), (B3) and the following conditions:

(B2'') There exists $P_1 \subset X_1$ such that $\phi_1(P_1) = P$, where $P \subset X$ is the desired set of equilibria.

(B4'') For all $i \in L \setminus \{1\}$ and any $p_i \in P_i \subset X_i$, there exists $k \in \{1, \dots, i-1\}$ such that $\phi_i(p_i) \in \text{Im}(\phi_k)$.

Let $\tilde{V}_i$ is a CLF for global asymptotic stabilization at P_i of the induced system $\dot{\tilde{x}} = \phi_{i*}^{-1}(\phi_i(\tilde{x}), u)$ on $\tilde{X}_i$. Consider

$$\begin{aligned}\hat{V}_1 &= \tilde{V}_1, \\ \hat{V}_i &= \tilde{V}_i + c_i,\end{aligned}$$

where $i \in L \setminus \{1\}$ and c_i is a positive constant satisfying

$$c_i > \min_{k=1, \dots, i-1} \sup_{p_i \in P_i} \min_{\tilde{x}_k \in \phi_k^{-1}(\phi_i(p_i))} \hat{V}_k(\tilde{x}_k).$$

Then,

$$V(x) = \min_{\tilde{x} \in \phi^{-1}(x)} \hat{V}(\tilde{x}) \quad (2.3)$$

is a CLF on X , where $\hat{V}$ is a coproduct of $\{\hat{V}_i\}_{i \in S}$.

2.4 Relative degree

Definition 2.3 (Relative degree)

We call r a relative degree for the single-input single-output (SISO) system

$$\begin{aligned}\dot{x} &= f(x) + g(x)u, \\ y &= \sigma(x),\end{aligned}$$

if r is some positive integer such that

$$\begin{aligned}\mathcal{L}_g \mathcal{L}_f^i \sigma(x) &= 0, \quad (i = 0, \dots, r-2) \\ \mathcal{L}_g \mathcal{L}_f^{r-1} \sigma(x) &\neq 0,\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}_f^0 \sigma(x) &:= \sigma(x), \\ \mathcal{L}_f^i \sigma(x) &:= \frac{\partial}{\partial x} \left(\mathcal{L}_f^{i-1} \sigma(x) \right) f(x).\end{aligned}$$

In this thesis, we call a SISO system with relative degree $r > 1$ a higher-order system because the system with relative degree one can be stabilized by the following traditional sliding-mode control with sufficiently large $A > 0$ if $L_g \sigma(x) > 0$:

$$u = -A \text{sgn}(\sigma(x)).$$

2.5 Homogeneity

We introduce some definitions on homogeneity as below.

Definition 2.4 (Dilation)

We define $D_\lambda^w : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as

$$D_\lambda^w : (x_1, x_2, \dots, x_n)^\top \mapsto (\lambda^{w_1} x_1, \lambda^{w_2} x_2, \dots, \lambda^{w_n} x_n)^\top,$$

where, the n -dimensional vector $w = (w_1, \dots, w_n)$ is called dilation coefficient and the homeomorphism D_λ^w is called dilation.

Definition 2.5 (Homogeneous function)

If the following identity is satisfied for any $\lambda > 0$, the function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a homogeneous function with homogeneity degree k for the dilation coefficient w :

$$V(D_\lambda^w(x)) \equiv \lambda^k V(x).$$

Definition 2.6 (Homogeneous map)

If the following identity is satisfied for any $\lambda > 0$, the set-valued map $F : \mathbb{R}^n \rightarrow U \subset \mathbb{R}$ is called a homogeneous map with homogeneity degree k for the dilation coefficient w :

$$F(D_\lambda^w(x)) \equiv \lambda^k F(x).$$

Definition 2.7 (Homogeneous system)

If the following identity is satisfied for any $\lambda > 0$, the system $\dot{x} = f(x)$ is called a homogeneous system of degree k with respect to the dilation coefficient w :

$$f(D_\lambda^w(x)) \equiv \lambda^k D_\lambda^w(f(x))$$

Definition 2.8 (Homogeneous differential inclusion)

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a set-valued vector field such that $F(x) = (F_1(x), \dots, F_n(x))$ where $F_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is a homogeneous map. If the following identity is satisfied for any $\lambda > 0$, the differential inclusion $\dot{x} \in F(x)$ is called a homogeneous differential inclusion of degree k with respect to the dilation coefficient w :

$$F(D_\lambda^w(x)) \equiv \lambda^k D_\lambda^w(F(x))$$

Theorem 2.3 (Nakamura et al. [15])

If the homogeneous system is strongly asymptotically stable, then

- state x of the system with negative degree of homogeneity is finite-time stable.
- state x of the system with homogeneity degree zero is exponentially stable.
- state x of the system with positive degree of homogeneity converges to an arbitrary open set which includes the origin in a constant period from each initial point.

Note that linear systems are the homogeneity system of degree 0 with respect to the dilation coefficient $(1, \dots, 1)$. The system is said to be finite-time stable if there exist $T > 0$ such that $x(t) = 0$ for any $T \geq t$ and $\forall x \in \mathbb{R}^n$.

Chapter 3. Switching control for four-wheeled vehicle

3.1 Introduction

Controlling a mobile robot in a space with obstacles is a well-known problem in robotics. The control objective is for the mobile robot to reach a goal point with obstacle avoidance. In this chapter, we specially consider the multiple obstacle avoidance problem for particularly a four-wheeled vehicle. Assuming that the shapes and locations of the obstacles are known, we propose a design method of a nondifferentiable navigation function, which guides a vehicle at the desired point. The control law for the four-wheeled vehicle is obtained from the extended function of the navigation function. As four-wheeled vehicles cannot move sideways because of their nonholonomic constraint, the proposed control law addresses this difficulty by considering a decrease in the control Lyapunov function, and can be applied to vehicles with steering limitations. Moreover, the control law makes it possible for the vehicles to achieve an n -point turn automatically without the need for any rule-based decision-making or heuristic method. The four-wheeled vehicle can reach the goal point from any initial point under the proposed control law. Our experimental results with a small four-wheeled vehicle are presented at the end of this chapter.

Generally, many obstacles are present in the control space of a mobile robot. For example, when operating indoors the robot has to avoid obstacles such as walls, desks, and devices to reach the desired point. Obstacle avoidance problem is often treated as a problem involving mobile robot localization and real-time obstacle detection. Nevertheless, the mobile robot can surely reach the desired point from any initial point without straying only when the obstacle-related information such as shapes and locations are known a priori. The analysis and design of such a global controller under the situation when full information is available is significant for the theoretical aspect of global obstacle avoidance. For achieving global reachability, it is useful to create a global strict control Lyapunov function (CLF) because it offers the one and only guidepost for the mobile robot to reach the desired point. In the obstacle avoidance problem, obstacles can be regarded as holes of the state space. Such a space with holes is known as a noncontractible manifold. Designing a global strict CLF is difficult because there are no global differentiable CLFs on a noncontractible manifold. In other words, on a noncontractible manifold, we need a discontinuous control law for global asymptotic stabilization.

Several obstacle avoidance methods have been proposed for mobile robots. When we

avoid an obstacle, we drive a vehicle by using complex rules created in the brain. Such reconstruction of rules is not easy for the mobile robots, because the rules are heuristic or arbitrarily chosen by a human. Constructing such rules requires a trial-and-error approach and consumes time. Two broad classes of this problem have been recognized: global and local. To achieve global obstacle avoidance in a known environment, many methods have been proposed [16, 17, 18, 19, 20].

On the other hand, local obstacle avoidance is implemented by different methods for various targets and works well for unknown environments. For unknown environments, sensor vision-based navigation has been proposed [21]; e.g., for the steering control [22] and four-wheeled vehicles [23]. Nonholonomic distance between a four-wheeled vehicle and an obstacle that is based on the shortest path of a four-wheeled vehicle [24, 25], plays some role in path planning and optimality [26]. Such nonholonomic distances are also used for achieving obstacle avoidance [27]. Another method for avoidance of obstacles is to design a path directly [28]. Applications for a realistic car model are also developed [29]. Optimal path planning [30] and model predictive control [31] of a realistic car are proposed to suit limited purposes and conditions.

In the case of global obstacle avoidance, virtual potential fields that enable vehicles to avoid obstacles are widely used since early research [16] on mobile robots and manipulators.. Potential fields called a navigation functions have been introduced by Rimon and Koditschek [17]. It has also been used for nonholonomic mobile robots in [20]. Laplace equation has been used for generating a navigation function in [32, 33]. The navigation function has been extended for more general obstacles in [34]. We know that there are no smooth navigation functions without undesired saddle points on a noncontractible manifold. Because of the saddle points, a smooth navigation function may cause a mobile robot to be trapped on the saddle points. Conventional methods cope with this local minima in diverse ways.

CLF-based obstacle avoidance is an alternative choice, and is an extension of virtual potential methods. This work focuses on a CLF method for the purpose of achieving global obstacle avoidance. A CLF is similar to the virtual potential fields in control theory. We have known that Lyapunov techniques can be used for the navigation of mobile robots, e.g., unicycle-like vehicles [35]. Here, we design a CLF for both navigation and obstacle avoidance instead of a virtual potential field. The traditional methods have a problem caused by the undesired local minima and saddle points. Recently, a minimum projection method has been proposed by Nakamura et al. [1]; this is a CLF construction method on a noncontractible manifold. The minimum projection method enables us to create a nonsmooth CLF; that is, the CLF has no undesired saddle points.

Fukui et al. have proposed an obstacle avoidance technique for a two-wheeled vehicle using the minimum projection method [36]. A global CLF on a noncontractible manifold can be obtained by the minimum projection method. As the CLF is nondifferentiable, the controlled object does not stop at unintended points, i.e., there are no saddle points. A two-wheeled vehicle can be stabilized at the desired point and avoid an obstacle by the

CLF proposed by Fukui et al. We call such a CLF for obstacle avoidance a navigation function.

We consider a four-wheeled vehicle, which is the most widely-distributed vehicle in the world. Because four-wheeled vehicles have the limitation of the steering angle of them, we need to suppose the existence of the steering limitation. Both two- and four-wheeled vehicles have the nonholonomic constraint. A two-wheeled vehicle is capable of performing a pivot turn, and a four-wheeled vehicle without a steering limitation is also capable of this. Nevertheless, the moving direction of a four-wheeled vehicle is restricted under the steering limitation. Such a nonholonomic constraint of a four-wheeled vehicle cannot be cleared by the navigation function. A four-wheeled vehicle can overcome this constraint by alternating between going forward/reverse with steering to the left/right.

In this chapter, we consider one or more obstacles, whose boundaries are piecewise twice differentiable, and their locations are arbitrary. Assume that the information about the obstacles is known a priori. We propose a design method for a global CLF for a four-wheeled vehicle with a steering limitation [37]. The CLF obtained by the proposed method enables the vehicle to reach the desired point via a short path while avoiding obstacles. The vehicle does not wander around because of a priori information. For the generation of the CLF, the proposed method uses a multilayer minimum projection method [2], which synthesizes a number of local CLFs to design a global CLF. Multiple obstacle avoidance needs a number of local CLFs for a number of discontinuities to avoid obstacles. These local CLFs are obtained from navigation functions by backstepping. We also propose a control law that is based on the CLF and can overcome the nonholonomic constraint and the steering limitation [38]. Moreover, we show that the nonholonomicity of the four-wheeled vehicle can be weakened by choosing the appropriate control point on the four-wheeled vehicle. Under the control law, the four-wheeled vehicle achieves an n-point turn (for instance, a three-point turn) movement automatically. Because of the discontinuous control, the signs of the obtained velocity and steering angle are changed from positive (or negative) to negative (or positive) in an infinitesimal interval, that is, the chattering phenomenon occurs. Such a chattering can be reduced by an appropriate hysteresis for the switching the sign of the input. We also propose such a hysteresis mechanism based on the CLF, and apply the hysteresis mechanism to the proposed controller. The control law is simple although multiple obstacles and the steering limitation make the problem difficult. We verify the efficacy of the proposed method by experiments, and present the results at the end of this chapter.

3.2 Problem statement

3.2.1 Four-wheeled vehicle

We focus on the obstacle avoidance problem for a four-wheeled vehicle illustrated in Fig. 3.1. The control point $(x, y)^\top = (x_0 + l \cos \theta, y_0 + l \sin \theta)^\top$ of the vehicle is a point

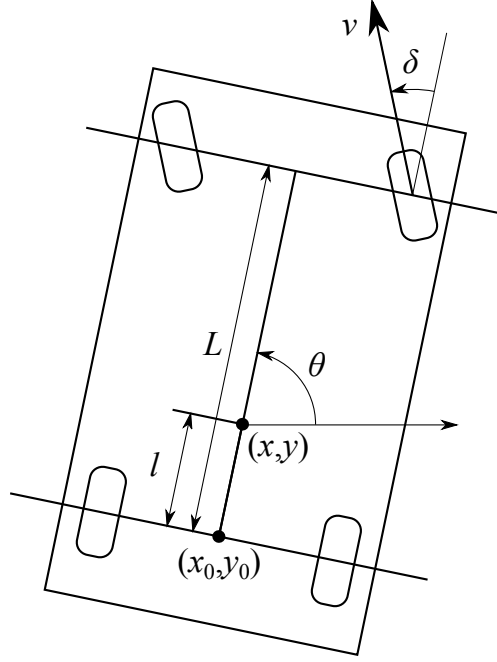


Figure 3.1: Four-wheeled vehicle model

on the perpendicular bisector of the rear shaft, where $(x_0, y_0)^\top$ is the coordinate of the center of the rear shaft, θ denotes the heading angle of the vehicle, and l is a constant denoting the distance between the control point and the center of the rear shaft. The angle θ is a point on a circle S^1 , i.e., $\theta \in [0, 2\pi]$ and the point $\theta = 2\pi$ is identified with $\theta = 0$. The usage of the offset $l \neq 0$ can reduce the distance between the control point and the circumcenter of the vehicle, and weakening the nonholonomic constraint of the vehicle at the control point. The direction of movement of the control point is determined by $\theta + \Delta(\delta)$, where $\delta \in S^1$ is the steering angle, $\Delta(\delta)$ is defined by

$$\Delta(\delta) = \begin{cases} 0 & (\cos \delta = 0 \text{ and } l = 0) \\ \arg(\cos \delta, l \sin \delta / L) & (\text{otherwise}), \end{cases}$$

and $\arg : \mathbb{R}^2 \rightarrow S^1$ denotes the argument in the 2D plane, i.e., $\arg((x, y)^\top) = \angle(x + iy)$ or $\arg(x, y) = \arctan(y/x)$ ($x > 0$). Note that $\arg(0, 0)$ is undefined. $\Delta(\delta)$ is called a mismatch angle between the heading angle and the direction of movement under the steering angle δ . We use the velocity of the vehicle as an input variable instead of the acceleration of the vehicle. Assuming that the vehicle has no sideslip and a fast responsiveness to the velocity command, its dynamics are written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x}_0 - l \sin \theta \dot{\theta} \\ \dot{y}_0 + l \cos \theta \dot{\theta} \\ v_0 \tan \delta / L \end{bmatrix} = \begin{bmatrix} v_0 \cos \theta - v_0 l \sin \theta \tan \delta / L \\ v_0 \sin \theta + v_0 l \cos \theta \tan \delta / L \\ v_0 \tan \delta / L \end{bmatrix} = \begin{bmatrix} \bar{v} \cos (\theta + \Delta(\delta)) \\ \bar{v} \sin (\theta + \Delta(\delta)) \\ v \sin \delta / L \end{bmatrix}, \quad (3.1)$$

where $L > 0$ is the distance between the front shaft and the rear shaft, the front wheel velocity v and the steering angle $\delta \in S^1$ are the control inputs, $v_0 = v \cos \delta$ is the velocity at $(x_0, y_0)^\top$ and $\bar{v} = v \cos \delta / \cos(\Delta(\delta))$ is the velocity at $(x, y)^\top$.

In this study, we consider the steering limitation

$$|\sin \delta| \leq \sin \delta_{max}, \quad (3.2)$$

where $\delta_{max} \in (0, \pi/2)$. Without loss of generality, we assume $v \geq 0$ because (3.2) admits the case of $\pi/2 < |\delta|$. General four-wheeled vehicles have steering limitations, which prohibit the vehicles from turning on a pivot. The limitation δ_{max} and L determine the minimum turning radius of the vehicle. See Remark 3.5 for the case with the velocity limitation.

We call the following system on planar surface a *simplified subsystem* of the vehicle:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \bar{v} \cos \bar{\theta} \\ \bar{v} \sin \bar{\theta} \end{bmatrix}. \quad (3.3)$$

The virtual input for the subsystem is

$$\bar{u} = \begin{bmatrix} \bar{v} \\ \bar{\theta} \end{bmatrix} = \begin{bmatrix} v \cos \delta / \cos(\Delta(\delta)) \\ \theta + \Delta(\delta) \end{bmatrix}.$$

Note that $(\bar{v}, \bar{\theta})^\top$ is different from $(v, \theta)^\top$ in (3.1). Nevertheless, a virtual input obtained as a map of $(x, y)^\top$ will be used for the desired values of the actual velocity and the direction of movement in Section 3.4, where the stability is guaranteed via the Lyapunov approach.

3.2.2 Control space and obstacle avoidance problem

We consider a number n of obstacles $\mathcal{O}_i \subset \mathbb{R}^2 \setminus \{(0, 0)^\top\}$ ($i = 1, 2, \dots, n$), whose boundaries are piecewise differentiable. The control space $\bar{X}$ of the subsystem is a plane excluding the obstacles $\bar{X} = \mathbb{R}^2 \setminus \{\mathcal{O}_i\}_{i \in \{1, \dots, n\}}$. Mathematically, a control space of the four-wheeled vehicle is expressed by $X = \bar{X} \times S^1$. These obstacles can be regarded as holes on the control space. The differential space with holes is called a noncontractible manifold, which is not equivalent to a point in the sense of topology.

From the definition of the obstacles $\mathcal{O}_i$, we can confirm that the point $x = y = 0$ is a point on $\bar{X}$. The purpose of the obstacle avoidance in this chapter is to bring the control point of the four-wheeled vehicle onto a goal set

$$P = \{(x, y, \theta)^\top \in X \mid x = y = 0\} \quad (3.4)$$

with no collision with obstacles. We do not consider the final heading angle of the vehicle on the goal point on the x - y plane. We consider a “global” obstacle avoidance problem; hence, it is supposed that the information about the obstacles, such as locations

and shapes, are known in advance, and the obstacles can be represented by piecewise twice differentiable boundaries. Moreover, the position and the angle of the vehicle are assumed to be observable.

Remark 3.1

The collision with the obstacles is detected at the control point $(x, y)^\top$ of the vehicle. For the actual implementation, we should enlarge the obstacles in consideration to the size of the vehicle.

3.3 Navigation function

In this section, we design a CLF for the simplified subsystem (3.3).

Definition 3.1 (Navigation function)

A CLF for the simplified subsystem (3.3) on the manifold $\bar{X}$ is called a navigation function.

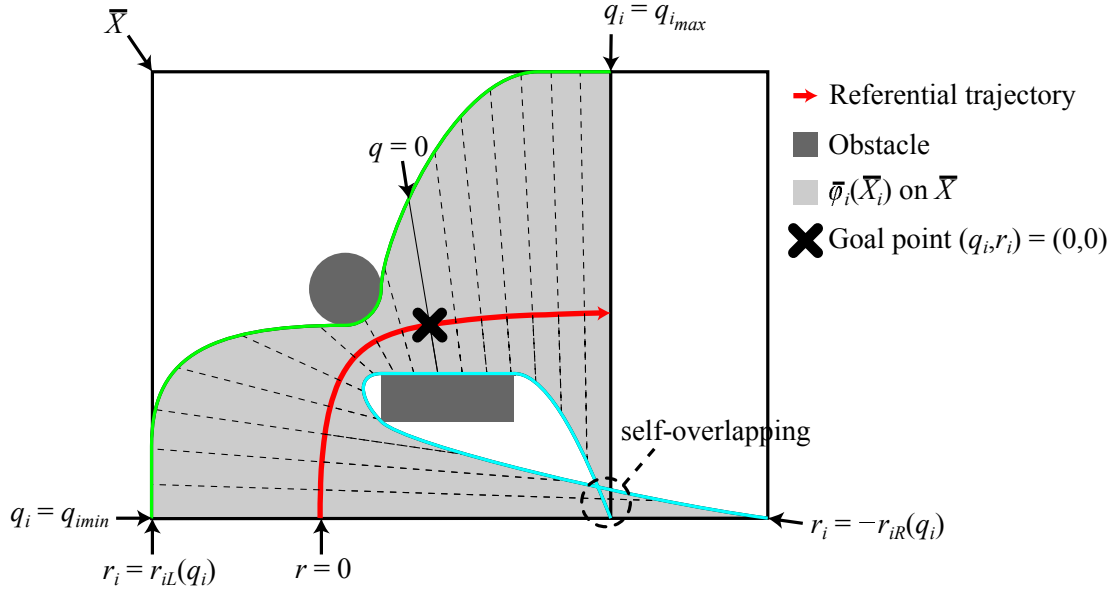
Navigation functions for obstacle avoidance are proposed in [18]. The traditional navigation function tells the mobile robot its direction to go except at the points where the gradient of the navigation function is zero. The navigation function may contain undesired saddle points, that is, a mobile robot controlled by the gradient of the navigation function may stop at such a stationary point. Without the undesired saddle points and local minima, such a problem does not occur. Allowing the navigation function to have nonsmooth points, we can remove the stationary points from navigation function.

Definition 3.2 (Strict navigation function)

A strict CLF for the simplified subsystem (3.3) on the manifold $\bar{X}$ is called a strict navigation function.

Once a strict navigation function is designed, it removes the difficulty caused by the noncontractibility of the manifolds. Nevertheless, no general design method of nonsmooth navigation function exists. Moreover, navigation functions ignore the direction of the vehicle. To design a nonsmooth global CLF of the system (3.1), we use the multilayer minimum projection method [2] described in Section 2.3.

By Theorem 2.1, we can design a global strict CLF on X from a number of strict CLFs on the other manifolds $\tilde{X}_i$ associated with étale mappings $\phi_i : \tilde{X}_i \rightarrow X$. We call X an original manifold and each manifold $\tilde{X}_i$ a simple-structured manifold. The whole noncontractible manifold X can be covered by the union of the all of the image $\phi_i(\tilde{X}_i) \subseteq X$. We prepare $\tilde{X}_i$ to be a contractible manifold because it is relatively easy to design a CLF on a contractible space. We consider that the contractible manifolds have a form $\tilde{X}_i = \bar{X}_i \times \mathbb{R}$, where $\bar{X}_i$ denotes the space of the planar position and $\mathbb{R}$ is the space of the direction of the vehicle. In the simplified manifolds, the heading angle is treated on the Euclidean space rather than the noncontractible space of circle S^1 .

Figure 3.2: Referential trajectory and $\bar{\phi}_i(\bar{X}_i)$ on $\bar{X}$

We prepare a mapping from $\bar{X}_i$ shown in Fig. 3.3 to local area around a referential trajectory shown in Fig. 3.2.

Definition 3.3 (Referential trajectory)

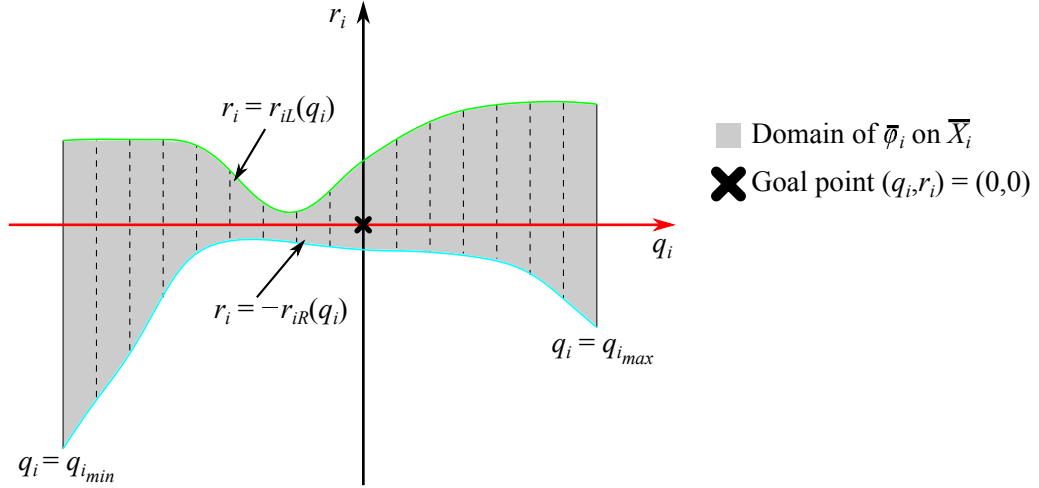
A referential trajectory is a sufficiently smooth trajectory $\mathbf{B}(q)$ on $\bar{X}$ such that $\mathbf{B}(0)$ is a point on the goal set P , where q is a parameter of the trajectory such that $q \in Q = [q_{min}, q_{max}]$ ($0 \in Q$).

We require sufficient smoothness of the referential trajectories, but do not require the feasibility of the referential trajectories on the four-wheeled vehicle with the steering limitation. The finally obtained control law does not drive the vehicle just on the trajectory. Various methods can be used to create referential trajectories. For the simulations and experiments in Section 3.7 and Section 3.8, we make several referential trajectories by using Dijkstra's method and B-spline interpolation as shown in Section 3.6.1. Let $\mathbf{B}_i(q_i)$ ($i \in S = \{1, 2, \dots, s\}$) be the referential trajectories designed in some way, where s is the number of the trajectories.

We first consider a simple-structured manifold $\bar{X}_i$, which is associated with a referential trajectory $\mathbf{B}_i$, as

$$\bar{X}_i = \{(q_i, r_i)^\top \in \mathbb{R}^2 \mid q_{i_{min}} \leq q_i \leq q_{i_{max}}, -r_{iR}(q_i) \leq r \leq r_{iL}(q_i)\}, \quad (3.5)$$

where $r_{iL}(q_i)$ is a twice differentiable left-hand-side width function that defines the domain of $\bar{X}_i$. Similarly, $r_{iR}(q_i)$ is that of the right-hand side. Note that the width functions have two requirements described in the next paragraph. We can easily design a strict CLF on the manifold $\bar{X}_i$ because $\bar{X}_i$ has a simple structure and we can choose $\dot{x}$ and $\dot{y}$ in (3.3) independently. The design method for the CLF is freely chosen by

Figure 3.3: $\bar{X}_i$ associate with the referential trajectory of Fig. 3.2

a designer. Nevertheless, we require that the CLF must be strict and be of class C^2 function for the following discussion.

Assumption 3.1

For each $i \in S$, a C^2 strict CLF $\bar{V}_i(q_i, r_i)$ is obtained.

One of the design procedures of $\bar{V}_i$ used for our experiment is described in Section 3.8.

We can consider a mapping $\bar{\phi}_i : \bar{X}_i \rightarrow \bar{X}$, which maps a point on $\bar{X}_i$ to a point on the local space around the trajectory. The mapping $\bar{\phi}_i$ from $(q_i, r_i)^\top$ to the planar position in $\bar{X}$ is defined as

$$\bar{\phi}_i(q_i, r_i) = \mathbf{B}_i(q_i) + r_i \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\mathbf{B}'_i(q_i)}{\|\mathbf{B}'_i(q_i)\|}, \quad (3.6)$$

where $\mathbf{B}'_i(q_i) = d\mathbf{B}_i(q_i)/dq_i$. Conversely, if $\bar{x} = \bar{\phi}_i(q_i, r_i)$ is given, r_i can be obtained as

$$r_i = (\bar{x} - \mathbf{B}_i(q_i))^\top \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\mathbf{B}'_i(q_i)}{\|\mathbf{B}'_i(q_i)\|}.$$

We can see that r_i represents a signed perpendicular distance from $\mathbf{B}_i(q_i)$. Therefore, $r_{iL}(q_i)$ must not exceed the distance to the nearest obstacle in the left-hand direction from $\mathbf{B}_i(q_i)$. Moreover, for the regularity condition of the Jacobian matrix of $\bar{\phi}_i$, $r_{iL}(q_i)$ must be less than or equal to the curvature radius of $\mathbf{B}_i(q_i)$ if the center of curvature lies on the left-hand side. Similar restrictions are applied for r_{iR} .

Note that we do not call $\bar{\phi}_i(\bar{X}_i)$ a simple-structured manifold because it may be a noncontractible owing to a self-overlapping, as shown in Fig. 3.2. Instead, we regard $\bar{X}_i$ as the simple-structured manifold that is contractible. Every point on $\bar{X}_i$ is mapped to a point on the local space around the trajectory on $\bar{X}$ by the mapping $\bar{\phi}_i : \bar{X}_i \rightarrow \bar{X}$ defined by (3.6). Conversely, the inverse map of $\bar{\phi}_i$ is also necessary for the implementation of

the multilayer minimum projection method. Consider a correspondence $\bar{\phi}_i^{-1} : \bar{X} \rightarrow \bar{X}_i$. As it is not required that $\bar{\phi}_i$ be a homeomorphism, $\bar{\phi}_i^{-1}$ may be a multivalued mapping. From the definition of $\bar{\phi}_i$, we know that q and r corresponding to $\bar{x} \in \bar{X}_i$ must satisfy

$$\mathbf{B}'_i(q_i) \cdot (\bar{x} - \mathbf{B}_i(q_i)) = 0, \quad (3.7)$$

$$-r_{iR}(q_i) \leq r_i \leq r_{iL}(q_i). \quad (3.8)$$

The condition (3.7) shows that $\bar{x} - \mathbf{B}_i(q_i)$ is orthogonal to $\mathbf{B}'_i(q_i)$, and (3.8) guarantees that $\bar{x}_i \in \bar{X}_i$. The set of all pairs of r_i and q_i satisfying the conditions defines the inverse map of $\bar{\phi}_i$. If no pair satisfies the conditions, $(x, y)^\top \notin \bar{\phi}_i(\bar{X}_i)$. We need to prepare a sufficient number of the referential trajectories for any $(x, y)^\top \in \bar{X}$ to be included in $\bar{\phi}_i(\bar{X}_i)$ ($\exists i \in S$). The correspondence $\bar{\phi}_i^{-1}$ is multivalued, but the uniqueness of the mapping is not required for using multilayer minimum projection method because ϕ^{-1} in (2.2) is defined as a set-valued mapping. When $\bar{\phi}_i^{-1}$ has a single value, a local navigation function on $\bar{X}$ is naturally defined as $\bar{V}_i(\bar{\phi}_i^{-1}(x, y))$.

Using the multilayer minimum projection method [2], we can design a global navigation function from the CLFs on the simple-structured manifolds. The global strict navigation function on $\bar{X}$ is obtained as

$$\bar{V}(x, y) = \min_{\tilde{x}^\top \in \bar{\phi}_S^{-1}(x, y)} \bar{V}_S(\tilde{x}),$$

where $\bar{\phi}_S$ is a coproduct of $\{\bar{\phi}_i\}_{i \in S}$ and $\bar{V}_S$ is a coproduct of $\{\bar{V}_i\}_{i \in S}$. As the local CLFs are strict, $\bar{V}$ is also a global strict navigation function.

In this study, however, we do not use this global navigation function directly in the case of $l = 0$, because we should consider the heading angle of the four-wheeled vehicle. In the following sections, we extend the CLFs on $\bar{X}_i$ with respect to the heading angle θ for the case of $l = 0$, before synthesizing the global CLF. On the other hand, in the case of $l \neq 0$, we also show that each local navigation function is a local CLF for the original system in the next section.

3.4 Control law

Let $\bar{V}_i : \bar{X}_i \rightarrow \mathbb{R}$ be a CLF on $\bar{X}_i$ and $\bar{\phi}_i : \bar{X}_i \rightarrow X$ be a mapping of $(q_i, r_i)^\top \mapsto (x, y)^\top \in X$. The actual design of $\bar{V}_i$ is discussed on the next section. For the design of the controller, there is no need to consider the actual implementation of $\bar{V}_i$ because they are independent of each other.

3.4.1 Control law for subsystem

In this subsection, we design a desired velocity v_{des} and a desired heading angle θ_{des} for the subsystem (3.3) to reach the goal point. From the subgradient of $\bar{V}_i$ at $\bar{x} \in \bar{X}$,

we define v_{des} and θ_{des} as

$$v_{des}(q_i, r_i) = k_v \left\| \frac{\partial \bar{V}_i}{\partial \bar{\mathbf{x}}} \right\|, \quad \theta_{des}(q_i, r_i) = \arg \left(-k_v \frac{\partial \bar{V}_i}{\partial \bar{\mathbf{x}}} \right), \quad (3.9)$$

where $k_v > 0$ is a velocity coefficient. Note that θ_{des} is a value on a circle S^1 . The details of the calculation of θ_{des} is discussed in Appendix A.

Remark 3.2

Let W be a differentiable function on $\bar{X}$. For fixed $\tilde{\mathbf{x}}_i = (q_i, r_i)^\top \in \bar{X}_i$, the subgradient of a W at $\bar{\mathbf{x}} = \bar{\phi}_i(q_i, r_i) \in \bar{X}$ is calculated as

$$\frac{\partial W}{\partial \bar{\mathbf{x}}} = \frac{\partial W}{\partial \tilde{\mathbf{x}}_i} \left(\frac{\partial \bar{\phi}_i}{\partial \tilde{\mathbf{x}}_i} \right)^{-1},$$

which is a map of q_i and r_i . Likewise, we regard $\partial \bar{V}_i / \partial \bar{\mathbf{x}}$ in (3.9) as map from (q_i, r_i) rather than from (x, y) .

Theorem 3.1

The input $(\bar{v}, \bar{\theta})^\top = (v_{des}, \theta_{des})^\top$ makes the vehicle of (3.3) be asymptotically stable at the origin from any initial condition on $\bar{\phi}_i(\bar{X}_i) \subseteq \bar{X}$.

Proof

Under the input $(v_{des}, \theta_{des})^\top$, the time derivative of the local navigation function is calculated as

$$\dot{\bar{V}}_i = \frac{\partial \bar{V}_i}{\partial \bar{\mathbf{x}}} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -\frac{v_{des}^2}{k_v} = -k_v \left\| \frac{\partial \bar{V}_i}{\partial \bar{\mathbf{x}}} \right\|^2.$$

Therefore, the time derivative of the CLF is negative except at the origin of $\bar{X}_i$, and the subsystem is asymptotically stable at the origin.

We can change k_v appropriately because the time derivative of k_v does not affect the negative definiteness of the time derivative of the CLF.

3.4.2 Control law for four-wheeled vehicle without steering limitation

In this subsection, we extend the CLF $\bar{V}_i$ on $\bar{X}_i$ with respect to the heading angle θ to consider the nonholonomicity of the four-wheeled vehicle with no steering limitation.

We regard the virtual input θ_{des} for the subsystem as an ideal direction of movement for the four-wheeled vehicle. The mismatch angle $\Delta(\delta)$ can change the direction of movement by the steering input δ . In the case of $l \neq 0$, we can make the actual direction of movement equal to the ideal direction θ_{des} because the range of Δ is whole S^1 under no steering limitations. Otherwise, in the case of $l = 0$, the direction of movement is determined by θ where $\delta \in (-\pi/2, \pi/2)$ or $\theta + \pi$ where $\delta \in (\pi/2, 3\pi/2)$. Note that for $l = 0$,

$$\Delta(\delta) = \begin{cases} 0 & (\cos \delta \geq 0) \\ \pi & (\cos \delta < 0). \end{cases}$$

Let $\mathcal{Z} \subseteq \mathbb{R}$ be a candidate set of error angles between the heading angle and the actual direction of movement at $(q_i, r_i, \theta)^\top$:

$$\begin{aligned} \mathcal{Z}(q_i, r_i, \theta) &= \{z = \theta - \theta_{des}(q_i, r_i) + \Delta(\delta_{ref}) + 2n'\pi \mid \delta_{ref} \in S^1, n' \in \mathbb{Z}\} \\ &= \begin{cases} \{z = \theta - \theta_{des}(q_i, r_i) + n\pi \mid n \in \mathbb{Z}\} & (l = 0) \\ \mathbb{R} & (l \neq 0), \end{cases} \end{aligned} \quad (3.10)$$

where $\mathcal{Z}$ is undefined at $(q_i, r_i)^\top = (0, 0)^\top$ because $\theta_{des}(0, 0) = \arg(0, 0)$ is not defined. Now, we can obtain the following CLF $\tilde{V}_i$ extended with respect to θ by using the multilayer minimum projection method with infinite layers:

$$\begin{aligned} \bar{V}_i(q_i, r_i, z) &:= \bar{V}_i(q_i, r_i) + \frac{k_z}{2} z^2, \\ \tilde{V}_i(q_i, r_i, \theta) &:= \min_{z \in \mathcal{Z}} \bar{V}_i(q_i, r_i, z) = \begin{cases} 0 & (l = 0, (q_i, r_i) = (0, 0)) \\ \bar{V}_i(q_i, r_i) + \min_{z \in \mathcal{Z}} \frac{k_z}{2} z^2 & (l = 0, (q_i, r_i) \neq (0, 0)) \\ \bar{V}_i(q_i, r_i) & (l \neq 0), \end{cases} \end{aligned}$$

where $k_z (> 0)$ is a coefficient with respect to the error angle. Note that, in the case of $l \neq 0$, the strict local navigation function becomes a strict local CLF for the four-wheeled vehicle. The nonzero offset dissolves the nonholonomicity problem as long as there are no steering limitations.

Theorem 3.2

There exists a pair of steering angle δ and the vehicle velocity v that can make $\dot{\tilde{V}}_i = -v_{des}^2/k_v$.

Proof

The time derivative of $\tilde{V}_i$ is calculated as

$$\dot{\tilde{V}}_i = \begin{cases} \frac{\partial \bar{V}_i}{\partial \mathbf{x}} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + k_z z (\dot{\theta} - \dot{\theta}_{des}) & (l = 0) \\ \frac{\partial \bar{V}_i}{\partial \mathbf{x}} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} & (l \neq 0), \end{cases} \quad (3.11)$$

where $z = \underset{z \in \mathcal{Z}(q_i, r_i, \theta)}{\operatorname{argmin}} z^2$. Substituting (3.1) into (3.11), we get

$$\begin{aligned} \dot{\tilde{V}}_i &= \begin{cases} v \cos \delta \left\{ \left(\frac{\partial \tilde{V}_i}{\partial \bar{\mathbf{x}}} - k_z z \frac{\partial \theta_{des}}{\partial \bar{\mathbf{x}}} \right) \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} + k_z z \frac{\tan \delta}{L} \right\} & (l = 0) \\ v \frac{\cos \delta}{\cos(\Delta(\delta))} \frac{\partial \tilde{V}_i}{\partial \bar{\mathbf{x}}} \begin{bmatrix} \cos(\theta + \Delta(\delta)) \\ \sin(\theta + \Delta(\delta)) \end{bmatrix} & (l \neq 0) \end{cases} \\ &= \begin{cases} v z W^\top \begin{bmatrix} \cos \delta \\ \sin \delta \end{bmatrix} - v \cos \delta \left\| \frac{\partial \tilde{V}_i}{\partial \bar{\mathbf{x}}} \right\| & (l = 0) \\ v \frac{\cos \delta}{\cos(\Delta(\delta))} \frac{\partial \tilde{V}_i}{\partial \bar{\mathbf{x}}} \begin{bmatrix} \cos(\theta + \Delta(\delta)) \\ \sin(\theta + \Delta(\delta)) \end{bmatrix} & (l \neq 0), \end{cases} \end{aligned}$$

where $\theta_{cur} = \arg(-\partial \theta_{des} / \partial \bar{\mathbf{x}})$, $\|\partial \theta_{des} / \partial \bar{\mathbf{x}}\|$ are functions of q_i and r_i as in Remark 3.2 and W is given by

$$W = \begin{bmatrix} \frac{1 - \cos(\theta - \theta_{des})}{z} \left\| \frac{\partial \tilde{V}_i}{\partial \bar{\mathbf{x}}} \right\| + k_z \left\| \frac{\partial \theta_{des}}{\partial \bar{\mathbf{x}}} \right\| \cos(\theta - \theta_{cur}) \\ \frac{k_z}{L} \end{bmatrix}.$$

We transform the above equation into

$$\dot{\tilde{V}}_i = -v \frac{\cos \delta}{\cos(\Delta(\delta))} \left\| \frac{\partial \tilde{V}_i}{\partial \bar{\mathbf{x}}} \right\|$$

under the steering input $\delta = \delta_{des}$ that is obtained by

$$\delta_{des} = \begin{cases} \arg((-1)^n W) & (l = 0) \\ \underset{\delta \in S^1}{\operatorname{argmin}} \min_{n \in \mathbb{Z}} (\theta + \Delta(\delta) - \theta_{des}(q_i, r_i) + 2n\pi)^2 = \Delta^{-1}(\theta_{des}(q_i, r_i) - \theta) & (l \neq 0), \end{cases}$$

where n in the case of $l = 0$ is determined by (3.10) and the selection of z , and $((-1)^n \cos(\theta - \theta_{des}) - 1) / (\theta - \theta_{des} + n\pi)$ is an analytical function. Therefore, $\dot{\tilde{V}}_i = -v_{des}^2 / k_v$ is accomplished by the control input

$$\delta = \delta_{des}, \quad v = v_{des} \frac{\cos(\Delta(\delta_{des}))}{\cos \delta_{des}}.$$

For the case of $l = 0$, $\cos \delta_{des}$ is nonzero. Moreover, for a positive l , $\cos(\Delta(\delta)) / \cos \delta$ can be considered as an analytic function. Hence, the above control law is well defined. Note that $\cos(\Delta(\delta_{des})) / \cos \delta_{des}$ is always positive. Without the steering limitation, the time derivative of $\tilde{V}_i$ can be negative except at the goal point under the input

$$(v, \delta)^\top = (v_{des}, \delta_{des})^\top, \quad (3.12)$$

because the desired velocity v_{des} is nonzero except at the goal point, and the velocity coefficient k_v can be chosen freely. We call δ_{des} the desired steering angle. The desired steering angle is also used in the next subsection.

3.4.3 Control law under steering limitation

We can control the four-wheeled vehicle without steering limitation by the control law (3.12). Nonetheless, the actual vehicles have steering limitation $|\sin \delta| \leq \sin \delta_{max}$. If $|\sin \delta_{des}| \leq \sin \delta_{max}$, we can adopt (3.12) as the control input. In this subsection, we consider the control law for the case where $|\sin \delta_{des}| > \sin \delta_{max}$.

Under the steering limitation $|\sin \delta| \leq \sin \delta_{max}$, the steering range consists of two disjoint ranges:

$$\delta \in U_{\geq 0} \cup U_{\leq 0},$$

where $U_{\geq 0}$ and $U_{\leq 0}$ defined below correspond to forward and backward movement, respectively:

$$\begin{aligned} U_{\geq 0} &= \{\delta \in S^1 \mid \cos \delta \geq \cos \delta_{max}\}, \\ U_{\leq 0} &= \{\delta \in S^1 \mid \cos \delta \leq -\cos \delta_{max}\}. \end{aligned}$$

Any controller cannot be continuous with respect to time, because the vehicle must go both forward and backward in order to change the heading angle under the steering limitation. In other words, the steering angle for the vehicle switches between $U_{\geq 0}$ and $U_{\leq 0}$.

Theorem 3.3

For the four-wheeled vehicle with steering limitation (3.2), there exists a steering angle that can make $\dot{\tilde{V}}_i < 0$ except on the goal set under the velocity v_{des} .

Proof

We can rewrite (3.11) as

$$\begin{aligned} \dot{\tilde{V}}_i &= \begin{cases} -v \cos \delta \left(\left\| \frac{\partial \bar{V}_i}{\partial \bar{\mathbf{x}}} \right\| \cos(\theta - \theta_{des}) - k_z z \left\| \frac{\partial \theta_{des}}{\partial \bar{\mathbf{x}}} \right\| \cos(\theta - \theta_{cur}) \right) + v \sin \delta k_z \frac{z}{L} & (l = 0) \\ -v \frac{\cos \delta}{\cos(\Delta(\delta))} \left\| \frac{\partial \bar{V}_i}{\partial \bar{\mathbf{x}}} \right\| \cos(\theta + \Delta(\delta) - \theta_{des}) & (l \neq 0) \end{cases} \\ &= \begin{cases} -v \mathbf{p}^\top \begin{bmatrix} \cos \delta \\ \sin \delta \end{bmatrix} & (l = 0) \\ -v \frac{\cos \delta}{\cos(\Delta(\delta))} \left\| \frac{\partial \bar{V}_i}{\partial \bar{\mathbf{x}}} \right\| \mathbf{p}^\top \begin{bmatrix} \cos(\Delta(\delta)) \\ \sin(\Delta(\delta)) \end{bmatrix} & (l \neq 0) \end{cases} \\ &= \begin{cases} -v \cos \delta \mathbf{p}^\top \begin{bmatrix} 1 \\ \tan \delta \end{bmatrix} & (l = 0) \\ -v \cos \delta \left\| \frac{\partial \bar{V}_i}{\partial \bar{\mathbf{x}}} \right\| \mathbf{p}^\top \begin{bmatrix} 1 \\ \tan(\Delta(\delta)) \end{bmatrix} & (l \neq 0), \end{cases} \end{aligned}$$

where $\mathbf{p}$ is define as

$$\mathbf{p} = \begin{cases} \begin{bmatrix} \left\| \frac{\partial \bar{V}_i}{\partial \bar{\mathbf{x}}} \right\| \cos(\theta - \theta_{des}) - k_z z \left\| \frac{\partial \theta_{des}}{\partial \bar{\mathbf{x}}} \right\| \cos(\theta - \theta_{cur}) \\ -k_z \frac{z}{L} \end{bmatrix} & (l = 0) \\ \begin{bmatrix} \cos(\Delta(\delta_{des})) \\ -\sin(\Delta(\delta_{des})) \end{bmatrix} & (l \neq 0). \end{cases}$$

Note that $\dot{\bar{V}}$ can be expressed by an inner product of a vector $\mathbf{p}$ and $(\cos \delta, \sin \delta)^\top$ ($l = 0$) or $(\cos(\Delta(\delta)), \sin(\Delta(\delta)))^\top$ ($l \neq 0$). The vector $\mathbf{p}$ is nonzero for nonzero v except on the goal set (Fig. 3.4). As shown in Fig. 3.4, the inner product can be both positive and negative depending on the steering input δ . The following control law α_i makes $\dot{V}_i(q, r, z)$ be negative except at the goal point:

$$\begin{bmatrix} v \\ \delta \end{bmatrix} = \alpha_i(q_i, r_i, \theta) = \begin{cases} (v_{des}, \delta_{des})^\top & (\delta_{des} \in U) \\ (v_{des}, \underset{\delta \in U}{\operatorname{argmin}} \dot{V}_i|_{v=v_{des}/|\cos \delta|})^\top & (\text{otherwise}), \end{cases} \quad (3.13)$$

where $U = U_{\geq 0} \cup U_{\leq 0} = \{\delta \in S^1 \mid |\sin \delta| \leq \sin \delta_{max}\}$.

Note that we should use δ_{des} wherever $|\sin \delta_{des}| \leq \sin \delta_{max}$. Moreover, the above control law has some robustness for the configuration error of L ; the negative definiteness of the time derivative of V is reserved as long as the actual distance between the rear shaft and the front shaft is smaller than given L .

Remark 3.3

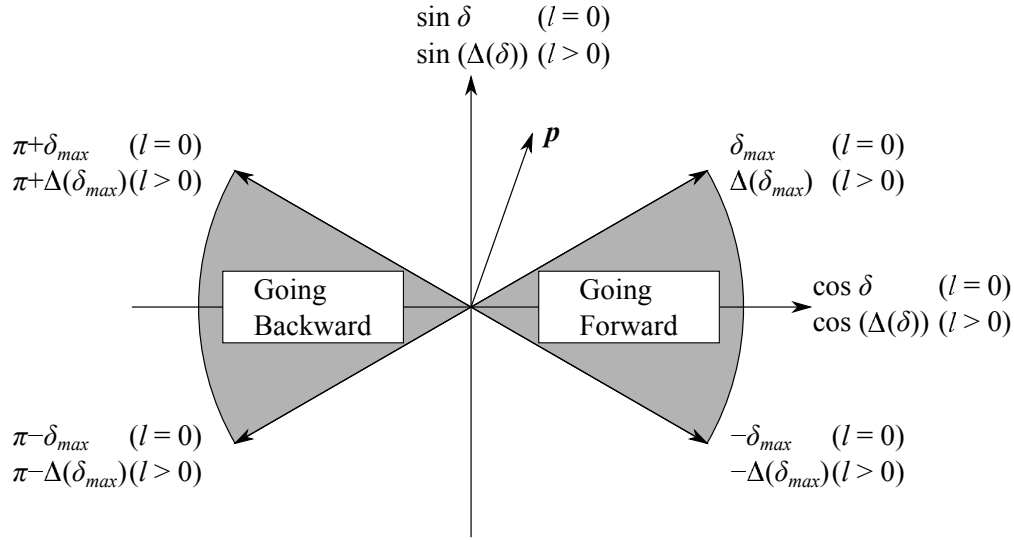
Under the proposed controller, the four-wheeled vehicle faces forward for $l > 0$ and faces backward for $l < 0$. Moreover, if $l = 0$ then the vehicle faces both forward and backward. Therefore, $l > 0$ is natural for the car. The backward of the forklift is correspond to the forward of the four-wheeled vehicle (3.1). By setting the baggage as the control point, $l < 0$ is natural because for the forklift.

Remark 3.4

When $|\sin \delta_{des}| > \sin \delta_{max}$, the input $\delta \in U$ that minimizes $\dot{V}_i|\cos \delta|/v$ is always one of δ_{max} , $-\delta_{max}$, $\pi - \delta_{max}$ or $-\pi + \delta_{max}$.

Remark 3.5

Because the time-dependent velocity coefficient $k_v > 0$ is allowed, we can replace the velocity input $v = v_{des}$ by $v = \min\{v_{des}, v_{max}\}$ to satisfy the velocity limitation $|v| \leq v_{max}$ by choosing some appropriate $k_v(t)$, for some $v_{max} > 0$.

Figure 3.4: Inner product in the calculation of $\dot{\tilde{V}}_i$

3.5 Global controller

3.5.1 CLF synthesis

In Section 3.3, we design the strict CLFs for the simplified subsystem (3.3) on the other spaces $\bar{X}_i$ ($i \in S$) with the referential trajectories. We also design the CLF V_i extended for the four-wheeled vehicle (3.1) and its control law in Section 3.4. A global CLF synthesized from a number of CLFs can globally asymptotically stabilize the four-wheeled vehicle. To synthesize the CLFs, we employ the multilayer minimum projection method described in Section 2.3.

We can obtain a set of strict CLFs $\{\tilde{V}_i\}_{i \in S}$ generated from referential trajectories and the corresponding mappings $\phi_i : \tilde{X}_i \rightarrow X$, which define a coproduct ϕ of mappings $\{\tilde{\phi}_i : (q_i, r_i, \theta)^\top \mapsto (x, y, \theta)^\top\}$, where $\tilde{X}_i = \bar{X}_i \times S^1$ and each $\tilde{\phi}_i$ is given by

$$(x, y, \theta)^\top = \tilde{\phi}_i(q_i, r_i, \theta) := (\bar{\phi}_i(q_i, r_i)^\top, \theta)^\top.$$

We get a global strict CLF V on X from the CLFs $\tilde{V}_i$ on $\tilde{X}_i$ by using the multilayer minimum projection method [2].

Proposition 3.1

The function $V(x, y, \theta)$ is a CLF for the four-wheeled vehicle:

$$V(x, y, \theta) := \min_{(q_j, r_j, \theta)^\top \in \phi^{-1}(x, y, \theta)} \tilde{V}(q_j, r_j, \theta), \quad (3.14)$$

where ϕ is a coproduct of $\{\tilde{\phi}_i\}_{i \in S}$ and $\tilde{V}$ is a coproduct of $\{\tilde{V}_i\}_{i \in S}$.

The control law based on the global CLF V is one for the minimum CLF, i.e.,

$$\begin{bmatrix} v \\ \delta \end{bmatrix} = \alpha(x, y, \theta) = \begin{cases} \alpha_1 \left(\underset{(q_j, r_j, \theta)^\top \in \phi^{-1}(x, y, \theta)}{\operatorname{argmin}} \tilde{V} \right) & ((q_j, r_j, \theta)^\top \in \tilde{X}_1 \Leftrightarrow j = 1) \\ \vdots \\ \alpha_s \left(\underset{(q_j, r_j, \theta)^\top \in \phi^{-1}(x, y, \theta)}{\operatorname{argmin}} \tilde{V} \right) & ((q_j, r_j, \theta)^\top \in \tilde{X}_s \Leftrightarrow j = s), \end{cases}$$

where each α_i ($i = 1, \dots, s$) is given by (3.13). $\dot{V}$ in the sense of directional subderivative is negative except on the goal set P because the control law is selected to suit the minimum CLF $\tilde{V}_i$ at the moment and $\dot{V}_i < 0$ except on P under the control law. The CLF V always decreases under the control law, and the vehicle reaches the goal point. The selection of the control law α_i is arbitrary where $\phi^{-1}(x, y, \theta)$ has multiple values [39].

3.5.2 Hysteresis mechanism

Note that the longitudinal velocity of the vehicle, that is $v \operatorname{sgn}(\cos \delta)$, becomes both positive and negative, and causes the chattering phenomenon under the control law α_i . To reduce the switching of the actual sign of the velocity, we introduce an appropriate hysteresis mechanism using the following proposition.

Proposition 3.2

Consider

$$\dot{x} = f(x, u)$$

where $x \in \mathbb{R}^n$ is a state and $u \in U \subseteq \mathbb{R}^m$ is a control input.

$$u \in U = U_1 \cup U_2 \cup \dots \cup U_k$$

where each U_i ($i = 1, \dots, k$) are disjoint subsets of U . Let $V(x)$ be a CLF for the system, $\alpha : \mathbb{R}^n \rightarrow 2^U$ be a set of the control input that makes $\dot{V} < 0$. Then, the system under the following control law is asymptotically stable:

$$u = \begin{cases} \underset{u \in \alpha(x) \cap U_1}{\operatorname{argmin}} \dot{V} & u_t \in U_1 \text{ and } \alpha(x) \cap U_1 \neq \emptyset \\ \vdots \\ \underset{u \in \alpha(x) \cap U_k}{\operatorname{argmin}} \dot{V} & u_t \in U_k \text{ and } \alpha(x) \cap U_k \neq \emptyset \\ \underset{u \in \alpha(x)}{\operatorname{argmin}} \dot{V} & \text{otherwise,} \end{cases}$$

where $u_t \in U_t$ is the last input.

It is obvious that the above control law satisfies $\dot{V} < 0$ along the trajectory.

The local control law with the hysteresis can be expressed as

$$\begin{bmatrix} v \\ \delta \end{bmatrix} = \alpha'_i(q_i, r_i, \theta) = \begin{cases} (v_{des}, \delta_{des})^\top & (|\sin \delta_{des}| \leq \sin \delta_{max}) \\ \left(v_{des}, \underset{\delta \in U_{\geq 0}}{\operatorname{argmin}} \dot{V}_i \Big|_{v=v_{des}/|\cos \delta|} \right)^\top & (\min_{\delta \in U_{\geq 0}} \dot{V}_i < 0 \text{ and } \delta_t \in U_{\geq 0}) \\ \left(v_{des}, \underset{\delta \in U_{\leq 0}}{\operatorname{argmin}} \dot{V}_i \Big|_{v=v_{des}/|\cos \delta|} \right)^\top & (\min_{\delta \in U_{\leq 0}} \dot{V}_i < 0 \text{ and } \delta_t \in U_{\leq 0}) \\ \left(v_{des}, \underset{\delta \in U}{\operatorname{argmin}} \dot{V}_i \Big|_{v=v_{des}/|\cos \delta|} \right)^\top & (\text{otherwise}), \end{cases} \quad (3.15)$$

where δ_t is the last steering angle of the vehicle.

Remark 3.6

A special CLF around the goal point may be useful for the actual implementation of the control law, because the high rate of change in θ_{des} around the goal point reduces a convergence performance. Using the offset technique for CLF synthesis summarized in Theorem 2.2, we can synthesize such a special CLF V_0 with the proposed global CLF V easily:

$$\hat{V}(x, y, \theta) = \begin{cases} V(x, y, \theta) + c & (V(x, y, \theta) + c < V_0(x, y, \theta)) \\ V_0(x, y, \theta) & (\text{otherwise}), \end{cases}$$

where $c > 0$ is an offset. The control law for $\hat{V}$ can be obtained in the same manner as for V .

We give an example of V_0 in the next section.

3.6 CLF design

3.6.1 Local CLF design

In this subsection, we introduce an implementation of the referential trajectory creation, and a design of the CLF $\bar{V}_i$ on the manifold $\bar{X}_i$.

We employ Dijkstra's method and B-spline interpolation for the design of the referential trajectories. First, we placed a number of nodes on the target space, ensuring that one of the nodes is on the goal point, and connected each node the others by an edge with a positive cost. When the edge contacts an obstacle, its cost becomes infinity; otherwise, the cost of an edge is basically determined by its length. By Dijkstra's method, the shortest path and the lowest cost to the goal node is calculated for each node on the graph. Moreover, by taking a weighted cost of an edge around the obstacles, the shortest path is at a sufficient distance from obstacles. Similarly, we can consider the roughness of the ground and slopes by imposing penalties for bad conditions. Connecting two shortest paths smoothly at the goal point, we obtained a path composed of a sequence

of nodes, which passed through the goal point. We obtained a reference trajectory $\mathbf{B}_i(q)$ by B-spline interpolation, where $\mathbf{B}_i(0)$ is the goal point. We also interpolated the costs for the nodes along the path, and called it a cost function $C_i(q)$, where we changed the sign of the cost of the node to be negative when $q < 0$. Therefore, $C_i(q)$ monotonically increases, and $C_i(0) = 0$. A number of referential trajectories and cost functions are prepared to cover $\bar{X}$ with $\bar{\phi}_i(\bar{X}_i)$ ($i \in S$).

We design strict CLF $\bar{V}_i$ on the manifold $\bar{X}_i$ as

$$\begin{aligned}\bar{V}_i(q, r) &= k_q V_q(q) + k_r V_r(q, r), \\ V_q(q) &= \frac{4}{\pi^2} C_i(q_m)^2 \tan^2 \left(\frac{\pi}{2} \frac{C_i(q)}{C_i(q_m)} \right) + k_{q2} \left(\sqrt{C_i(q)^2 + a^2} - a \right), \\ V_r(q, r) &= \frac{4}{\pi^2} r_m(q)^2 \tan^2 \left(\frac{\pi}{2} \frac{r}{r_m(q)} \right) + k_{r2} \left(\sqrt{r^2 + b^2} - b \right),\end{aligned}$$

where $V_q(q)$ is a cost function for the distance from the goal point, $V_r(q, r)$ is a penalty cost function with respect to the distance from the referential trajectory, k_q and k_r are weight coefficients, and k_{q2} and k_{r2} are nonnegative coefficients, $a > 0$ and $b > 0$ are design parameters, and $C_i(q)$ is the cost function associated with $\mathbf{B}_i(q)$. Note that $V_q(q)$ and $V_r(q, r)$ are positive definite with respect to q and r , respectively. Moreover, these functions are differentiable functions satisfying $\partial V_q / \partial q \neq 0$ ($q \neq 0$) and $\partial V_r / \partial r \neq 0$ ($r \neq 0$), and reach infinity on the edge of the domain of $\bar{X}_i$. The above CLF is used in the simulations and the experiments in Section 3.7 and Section 3.8.

3.6.2 Special CLF around the origin

From nature of the proposed controller, the high change rate of θ_{des} results in adverse effect. Especially, $\dot{\theta}_{des}$ becomes a big value around the origin because $\partial V / \partial x = 0$ iff $x = 0$. In this subsection, we proposed a helper CLF on the following space:

$$X_0 = \{(x, y, \theta)^\top \mid \sqrt{x^2 + y^2} \leq r_{m0}\}, \quad (3.16)$$

where r_{m0} represents the radius of the domain of X_0 centered at $x = y = 0$. We assume that r_{m0} is smaller than the distance to any obstacles from $x = y = 0$. Therefore, there are no obstacles on the space X_0 .

First, we define the desired heading angle on X_0 as

$$\theta_{des}(x, y) = \text{atan2}(-y, -x).$$

We also define the signed distance from the goal point as

$$r(x, y) = \sqrt{x^2 + y^2}.$$

Note that it can be written as $(x, y)^\top = (-r \cos \theta_{des}, -r \sin \theta_{des})^\top$.

We consider two cases: the case of the vehicle can reach the goal point with going forward or backward only, and the case of the switching of the sign of velocity needed for change the heading angle. For classifying the two cases, we define the lower limit steering angle $\delta_{min}(r, z)$ as

$$\begin{aligned}\delta_{min}(r, z) &= -\text{atan}\left(\frac{L}{R_{max}}\right) \\ &= -\tan^{-1}\left(\frac{2L \sin z}{r + 2l \cos z}\right),\end{aligned}$$

where $\text{atan} : S^1 \rightarrow [-\pi/2, \pi/2]$ is the inverse tangent function, z is an error angle, $R_{max}(r, z)$ is a turning radius under $\delta_{min}(r, z)$, and z and R_{max} are determined as

$$\begin{aligned}z(x, y, \theta) &= \theta - \theta_{des}(x, y) + 2\pi \underset{n \in \mathbb{Z}}{\text{argmin}} |\theta - \theta_{des}(x, y) + 2n\pi|, \\ R_{max}(r, z) &= \frac{\frac{l}{2} + l \cos z}{\sin z}.\end{aligned}$$

As shown in Fig. 3.5, lower limit steering angle $\delta_{min}(r, z)$ is the steering angle such that the vehicle reaches the goal point from a given point under $\tan \delta = \tan \delta_{min}(r, z)$, that is, $\delta = \delta_{min}(r, z)$ or $\delta = \delta_{min}(r, z) \pm \pi$. The vehicle can reach the goal point without k-turn if $|\delta_{min}(r, z)| \leq \delta_{max}$.

We recall that the vehicle can reach the goal point with no change of sign of the velocity if $|\delta_{min}(r, z)| \leq \delta_{max}$. Consider the CLF V_0 as

$$V_0(x, y, \theta) = \frac{k_{r0}}{2} \left(\frac{2r_{m0}}{\pi}\right)^2 \tan^2\left(\frac{\pi r(x, y)}{2 r_{m0}}\right) + \frac{k_{z0}}{2} \tan^2\left(\frac{\pi \delta_{min}(r(x, y), z(x, y, \theta))}{2 \delta_{max}}\right), \quad (3.17)$$

where $k_{r0} > 0$ is a coefficient of the cost function with respect to the distance from the goal point and $k_{z0} > 0$ is a coefficient of the penalty for the steering angle limitation. The domain of V_0 is given by

$$\text{Dom}(V_0) = \{(x, y, \theta)^\top \in X_0 \mid |\delta_{min}(r(x, y), z(x, y, \theta))| \leq \delta_{max}\} \subset X_0.$$

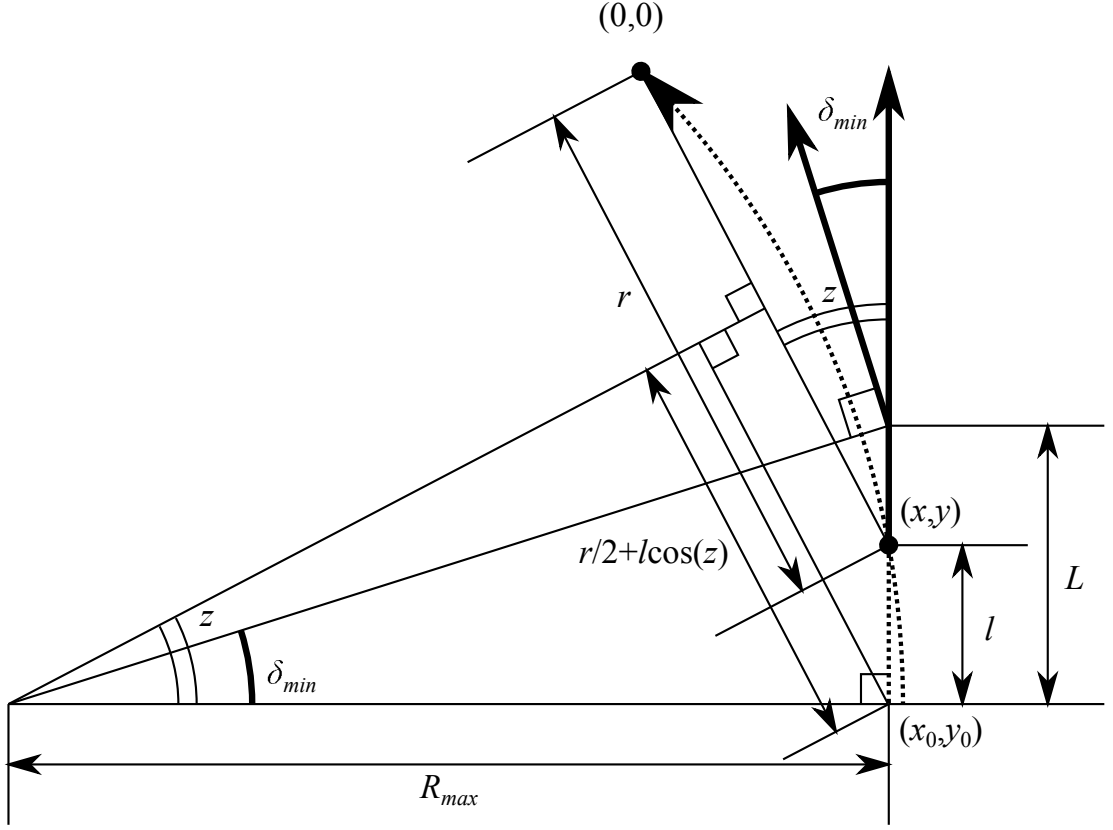
Lemma 3.1

The time derivative of V_0 becomes negative definite under the following control input:

$$\begin{aligned}(v, \delta)^\top &= \alpha_0(x, y, \theta) = \alpha'_0(r(x, y), z(x, y, \theta)) \\ \alpha'_0(r, z) &= \begin{cases} (k_{v0}r, -\delta_{max} \text{sgn}((l + r \cos z)\delta_{min}(r, z)))^\top & (\cos z \geq 0) \\ (k_{v0}r, \pi + \delta_{max} \text{sgn}((l + r \cos z)\delta_{min}(r, z)))^\top, & (\cos z < 0) \end{cases}\end{aligned} \quad (3.18)$$

where $k_{v0} > 0$ is a velocity coefficient.

Note that the steering angle in the case of $\cos z < 0$ belongs to $U_{\leq 0}$.

Figure 3.5: Concept image of lower limit steering angle $\delta_{min}(r, z)$ **Proof**

The time derivative of δ_{min} is obtained as

$$\dot{\delta}_{min} = \frac{\frac{4lL \sin^2 z}{(2l \cos z + r)^2} + \frac{2L \cos z}{2l \cos z + r}}{\frac{4L^2 \sin^2 z}{(2l \cos z + r)^2} + 1} \dot{z} - \frac{2L \sin z}{(2l \cos z + r)^2 \left(\frac{4L^2 \sin^2 z}{(2l \cos z + r)^2} + 1 \right)} \dot{r}.$$

Substituting $\dot{r} = v_0(\tan \delta l \sin z / L - \cos z)$, $\dot{z} = v_0(\sin z + \cos z \tan \delta l / L + r \tan \delta / L) / r$ and $v = k_{v0} r \cos \delta$

$$\dot{\delta}_{min} = -2k_{v0} \frac{2L \sin z + (2l \cos z + r) \tan \delta}{(2L \sin z)^2 + (2l \cos z + r)^2} (r \cos z + l) \cos \delta.$$

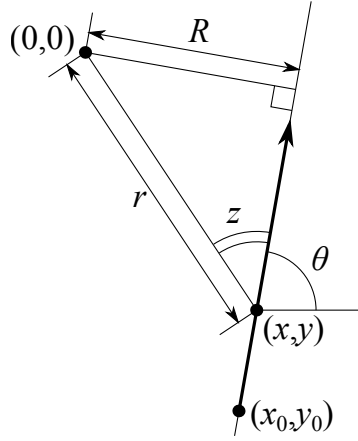
Substituting $\tan \delta = \tan \delta_{min}$ into the above equation, we have

$$\dot{\delta}_{min} = 0.$$

From a simple calculation, we can check that both $\tan^2 \left(\frac{\pi}{2} \frac{r}{r_{m0}} \right)$ and $\tan^2 \left(\frac{\pi}{2} \frac{\delta_{min}(r, z)}{\delta_{max}} \right)$ are decreased under the control law.

For the case of changing the heading angle, we define the horizontal distance R as

$$R(x, y, \theta) = r(x, y) \sin z(x, y, \theta).$$

Figure 3.6: Concept image of horizontal distance R

R is a distance between the vehicle and the goal point in the transverse direction of the vehicle as shown in Fig. 3.6. From the definition of R , the error angle $z(x, y, \theta) = 0$ iff $R(x, y, \theta) = 0$. We design CLF V_R as

$$V_R(x, y, \theta) = \begin{cases} \frac{k_R}{2} R(x, y, \theta)^2 & |r(x, y)| \leq r_0 \\ \frac{k_R}{2} R(x, y, \theta)^2 + \frac{k_{r0}}{2} \left(\frac{2(r_{m0}-r_0)}{\pi} \right)^2 \tan^2 \left(\frac{\pi}{2} \frac{|r(x, y)-r_0|}{r_{m0}-r_0} \right) & |r(x, y)| > r_0, \end{cases} \quad (3.19)$$

where k_R is a coefficient of the cost function with respect to the horizontal distance and r_0 is a nonnegative design parameter satisfying $r_0 < r_{m0}$.

Assumption 3.2

If $l = 0$, k_r and K_R satisfy

$$\frac{k_{r0}L}{4k_R \tan \delta_{max}} < r_0.$$

Lemma 3.2

Under Assumption 3.2, the time derivative of V_R becomes negative except on the area such that $V_R(x, y, \theta) = 0$ under the following control input:

$$(v, \delta)^\top = \alpha_R(x, y, \theta) \quad (3.20)$$

$$\begin{cases} = u_+ & \left(\dot{V}_R \Big|_{(v, \delta)^\top = u_-} \geq 0 \right) \\ = u_- & \left(\dot{V}_R \Big|_{(v, \delta)^\top = u_+} \geq 0 \right) \\ \in \{u_+, u_-\}, & \text{(otherwise)} \end{cases} \quad (3.21)$$

where $k_{v0} > 0$ is a velocity coefficient, u_+ and u_- are inputs value given by

$$\begin{aligned} u_+ &= \begin{bmatrix} k_{v0}r \\ -\delta_{max}\text{sgn}((l + r \cos z) \sin z) \end{bmatrix}, \\ u_- &= \begin{bmatrix} k_{v0}r \\ \pi + \delta_{max}\text{sgn}((l + r \cos z) \sin z) \end{bmatrix}. \end{aligned}$$

Proof

From $\dot{r} = v_0(\tan \delta l \sin z/L - \cos z)$, $\dot{z} = v_0(\sin z + \cos z \tan \delta l/L + r \tan \delta/L)/r$ and $v_0 = k_{v0}r \cos \delta$, both time derivatives of R^2 under u_+ and u_- become

$$\begin{aligned} \frac{d}{dt}(R^2) &= 2R\dot{R} \\ &= 2R(\dot{r} \sin z + r \cos z \dot{z}) \\ &= 2k_{v0}r^2 \sin z \sin \delta (l + r \cos z)/L \\ &= -2k_{v0}r^2 |\sin z \sin \delta_{max} (l + r \cos z)|/L. \end{aligned}$$

Therefore, the control law decreases the horizontal distance R . Moreover, if $l \neq 0$ then r is decreased by u_+ or u_- or both, otherwise Assumption 3.2 guarantees that $\dot{V}_R$ can be negative definite u_+ or u_- or both.

The velocity v should be selected by taking into consideration of the previous velocity as with the hysteresis mechanism on (3.15). Then, the sign of the velocity is just flipped when $\dot{V}_R = 0$.

Note that the set P_R such that $V_R = 0$ is given by

$$P_R(x, y, \theta) = \{(x, y, \theta)^\top \mid |r(x, y)| = \sqrt{x^2 + y^2} \leq r_0, \theta = \theta_{des}(x, y)\} \subset X_0.$$

Therefore, the control law (3.21) does not guarantee that the vehicle reaches on the goal set P . We can eliminate the singular area of V_R by combining with V_0 because $P_R \subset \text{Dom}(V_0)$, that is, decreasing $|R(x, y, \theta)|$ means that the error angle $|z(x, y, \theta)|$ and $|\delta_{min}(r, z)|$ becomes smaller.

Proposition 3.3

Define the CLFs with offset as

$$\begin{aligned} \hat{V}_0(x, y, \theta) &= V_0(x, y, \theta), \\ \hat{V}_R(x, y, \theta) &= V_R(x, y, \theta) + c_R, \\ \hat{V}_S(x, y, \theta) &= V(x, y, \theta) + c_S, \end{aligned}$$

where $c_R > 0$ is an offset that makes $V_0 < V_R + c_R$ on the set of singular points $P_R \setminus P$, c_S is an offset and $V(x, y, \theta)$ is given in Proposition 3.1. Using multilayer minimum projection method in Theorem 2.2, we obtain new global CLF $\hat{V}$ for four-wheeled vehicle:

$$\hat{V} = \min_{i \in \{0, R, S\}} \hat{V}_i \quad (3.22)$$

The following control law makes $\dot{\hat{V}}$ be negative except at the origin of V_0 :

$$\begin{bmatrix} v \\ \delta \end{bmatrix} = \alpha(x, y, \theta) = \begin{cases} \alpha_0(x, y, \theta) & (\hat{V}(x, y, \theta) = \hat{V}_0(x, y, \theta)) \\ \alpha_R(x, y, \theta) & (\hat{V}(x, y, \theta) = \hat{V}_R(x, y, \theta)) \\ \alpha(x, y, \theta) & (\hat{V}(x, y, \theta) = \hat{V}_S(x, y, \theta)). \end{cases}$$

If $\hat{V}_0 < \hat{V}_S$ or $\hat{V}_R < \hat{V}_S$ around the goal point, then we can obtain the smooth control law. Therefore, we should make c_S be large enough to satisfy $\hat{V}_0 < \hat{V}_S$ or $\hat{V}_R < \hat{V}_S$ around the goal point.

Remark 3.7

Generally, the mobile robot may have the special purpose is required around the goal point. For example, the case of the goal set P is $x = y = \theta = 0$. We can use a CLF for such a purpose instead of V_0 and V_R .

Under the proposed control law, the vehicle reaches the goal point without straying and stopping at unintended points, because the global strict CLF does not contain any local minima and saddle points. As the value of the global CLF is strictly decreasing with respect to time except on P , the reachability of the vehicle to the goal point is completely proven mathematically. With the proposed method, the vehicle attains perfect ability to avoid all obstacles even with the steering limitation. While rule-based approaches often need extensive tuning of parameters and are based on a non-versatile approach, for the proposed method we need only have to construct a global strict CLF. The concept of obstacle avoidance by the proposed method is not specific to the four-wheeled vehicle, and is applicable to the other vehicles as well as to the cases in three-dimensional spaces by a natural extension. The obstacle avoidance problem includes decision making to choose turning left or right and going backward or forward. Our method gives a mathematical realization of the decision making mechanism, which is versatile.

3.7 Simulation results

This section presents the simulation results of the proposed controller. The simulation results show the effectiveness of the proposed hysteresis mechanism for the chattering rejection.

The steering limitations of the vehicle is set to ± 30 [degree]. The distance L between the front and rear shafts is set to 10 [cm]. We set the offset l at $L/2$. The coefficients of the CLF used in this experiment are $k_q = 1.0$, $k_r = 5.0$, $k_{q2} = 0.5$, $k_{r2} = 1.0$, $a = 3.0$, $b = 3.0$, $k_z = 0.5$, $r_0 = 4.5$, $r_{m0} = 5.0$, $k_{r0} = 0.5$, $k_{z0} = 0.5$, $k_R = 0.5$, $c_S = 800$, $c_R = 200$. The obstacles in the simulation space can be seen in the simulation result.

The simulation results are shown in Fig. 3.7 and Fig. 3.8; Fig. 3.7(a) and Fig. 3.8(a) show the trajectories of the four-wheeled vehicle on the x - y plane, and Fig. 3.7(b) and

Fig. 3.8(b) indicate the time responses of the control input for the four-wheeled vehicle. Note that, for easy to see and understand, we show $v_0 = v \cos \delta$ and δ_0 instead of v and δ , where v_0 is the velocity on the center of the rear shaft and δ_0 defined as

$$\delta_0 = \begin{cases} \delta + \pi & (\delta < -\pi/2) \\ \delta - \pi & (\delta > \pi/2) \\ \delta & (\text{otherwise}). \end{cases} \quad (3.23)$$

Both simulations has the same initial condition and the same parameters as described above. In Fig. 3.7, the vehicle is controlled under the control law (3.13). On the other hand, the vehicle is controlled under the control law (3.15) in Fig. 3.8. The control law (3.15) for Fig. 3.8 has hysteresis mechanism. As compared to Fig. 3.7(b), the switching of the velocity and the steering angle is significantly reduced in Fig. 3.8(b). We can see that the hysteresis mechanism proposed in Section 3.5.2 is effective for the chattering rejection on the four-wheeled vehicle with the steering limitation.

3.8 Experiment results

In this section, we present an implementation of the method and experimental results. The experimental results validate the reliability of the proposed method, not only in a mathematical way.

We performed experiments for the proposed method with a miniature four-wheeled vehicle, RoboCar (ZMP Inc.), shown in Fig. 3.9(a). The steering and velocity limitations of the car are ± 30 [degree] and ± 2800 [mm/s], respectively. The distance L between the front and rear shafts is 256 [mm]. The vehicle is 429.0 [mm] long, 195.0 [mm] wide, and 212.2 [mm] tall. A positioning sensor, StarGazer (Fig. 3.9(b)), manufactured by HAGISONIC Co. Ltd., calculates its two-dimensional position and angle from markers on the ceiling, and the nominal measurement accuracy of the position and angle is ± 20 [mm] and ± 1 [degree], respectively. The nominal measurement sampling rate is 20 [times/s]. For the controller, we used a state observer to remove the measurement noises and to interpolate during the sampling intervals. We set the offset l at $L/2$. An experiment terminates when the vehicle reaches within a range of 10 [mm] from the goal point.

The coefficients of the CLF used in this experiment are $k_q = 10.0$, $k_r = 0.1$, $k_{q2} = 1.0$, $k_{r2} = 1.0$, $a = 10.0$, $b = 10.0$ and $k_z = 1.0$, $r_0 = 40.0$, $r_{m0} = 5.0$, $k_{r0} = 0.5$, $k_{z0} = 0.5$, $k_R = 0.5$, $c_S = 52000$, $c_R = 12000$.

The global CLF obtained from the referential trajectories is shown in Fig. 3.10. Note that the global CLF shown in Fig. 3.10 is the case of $\theta = \theta_{des}$, and it indicates a strict navigation function. We also use the special CLF $\tilde{V}_0$, which improves the convergence of the vehicle around the goal point (Remark 3.6).

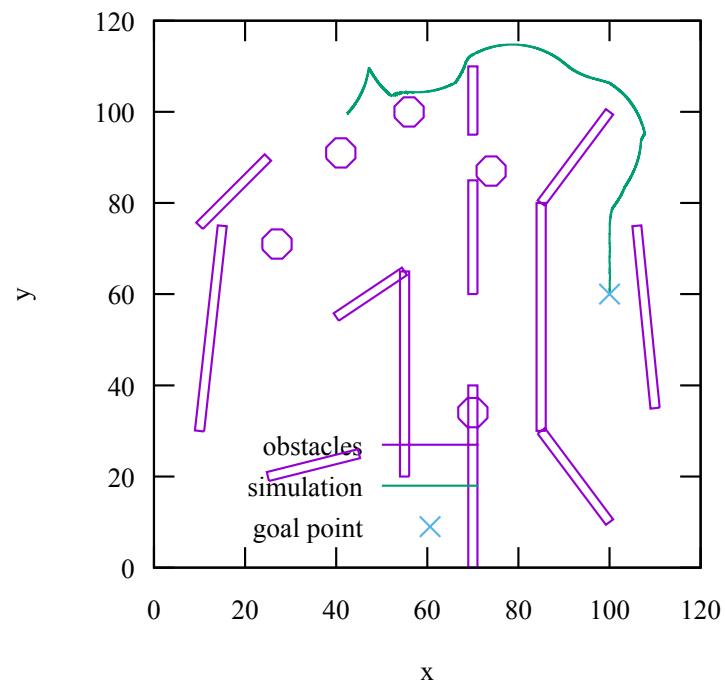
Figures 3.11, 3.12, and 3.13 show the experimental results from various initial states. Figures 3.11(b), 3.12(b) and 3.13(b) show the time responses of the input for the vehicle.

Note that, for easy to see and understand, we show $v_0 = v \cos \delta$ and δ_0 defined by (3.23). In the right side figures, the solid lines and the dotted lines indicate the command values and the sensor values on the vehicle, respectively. By the frequent repetition of k-turn like movement, the vehicle changes its heading angle automatically; see the sign of the velocity and the steering in 2-4 [s] and 6-9 [s] of Fig. 3.11(b), 1-6 [s] of Fig. 3.12(b), or 1-4 [s] of Fig. 3.13(b). For example, in Fig. 3.12(b), the vehicle repeats going backward with right steering and going forward with left steering. Figures 3.11(a), 3.12(a) and 3.13(a) show the trajectory of the four-wheeled vehicle. Although the seven obstacles are arranged densely, the four-wheeled vehicle reaches the goal point. As shown in the figures, the vehicle goes forward as long as the navigation function decreases. Otherwise, the vehicle changes the heading angle by k-turn like movement as mentioned above.

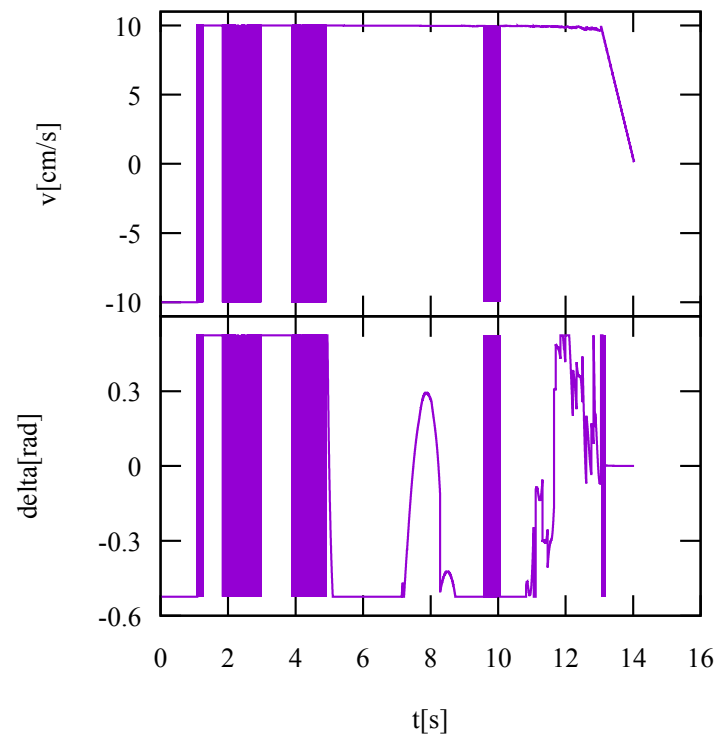
Fig. 3.14 presents eight results other than that shown above. The vehicle reaches the 10 [mm]-circle centered at the goal point for all cases. The experiments were successful as long as the location information was correct.

3.9 Summary

In this chapter, we present the control law that leads the four-wheeled vehicle to the goal point avoiding all obstacles on the control space under the steering limitation. We first propose a method for designing navigation functions for a mobile robot without nonholonomicity via a referential trajectory. Next, we formulated a control law that enabled a four-wheeled vehicle to reach a goal point using a CLF synthesized from the local navigation functions. Because the control law may contain the sliding-mode between going forward and going backward, we also propose the hysteresis mechanism that reduces the chattering phenomenon based on the CLF. The four-wheeled vehicle achieves an n-point turn automatically by the mathematical approach and does not involve any (meta-)heuristic methods or rule-based controls. The proposed control law is based on the global strict CLF for the four-wheeled vehicle obtained by synthesizing a number of local strict CLFs; each local strict CLF is an extension of a local strict navigation function. In order to obtain a local strict CLF for a four-wheeled vehicle, we extend the strict navigation function because the nonholonomicity and the steering limitation are not considered in the navigation function. By the proposed control law, the vehicle reached the goal point from various initial points, as can be seen in the experimental results.

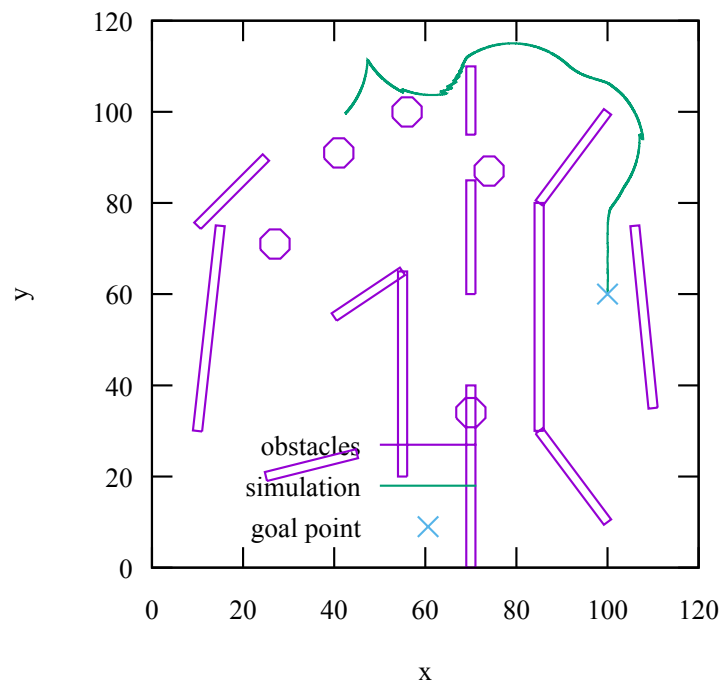


(a) Tracks of four-wheeled vehicle

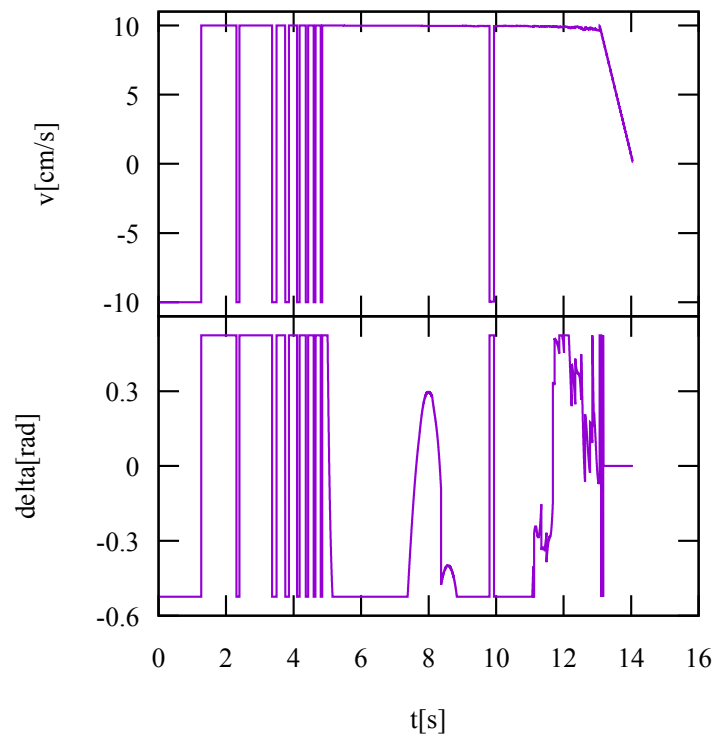


(b) Input for four-wheeled vehicle

Figure 3.7: Simulation result (without hysteresis)

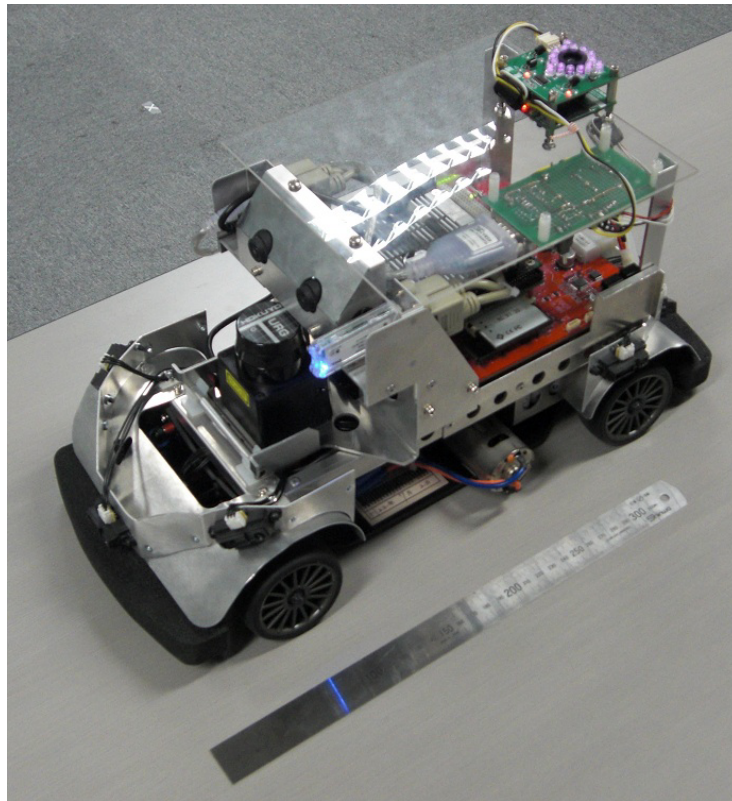


(a) Tracks of four-wheeled vehicle

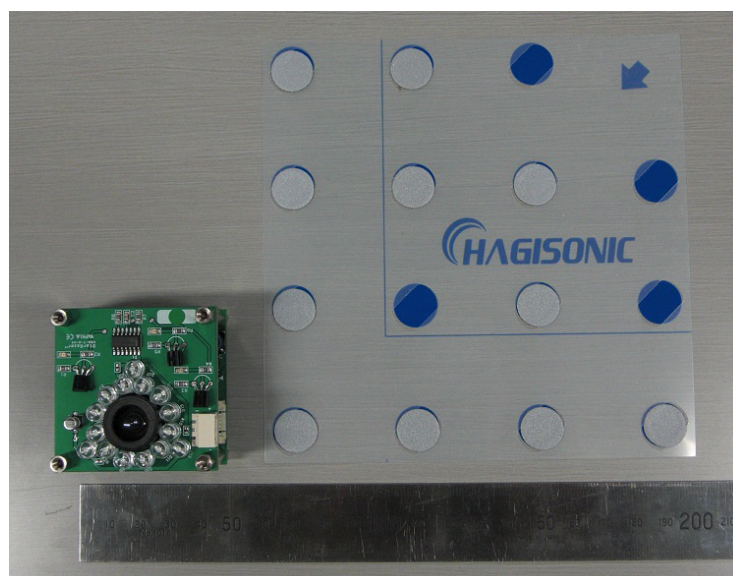


(b) Input for four-wheeled vehicle

Figure 3.8: Simulation result (with hysteresis)



(a) RoboCar



(b) StarGazer

Figure 3.9: Experimental environment

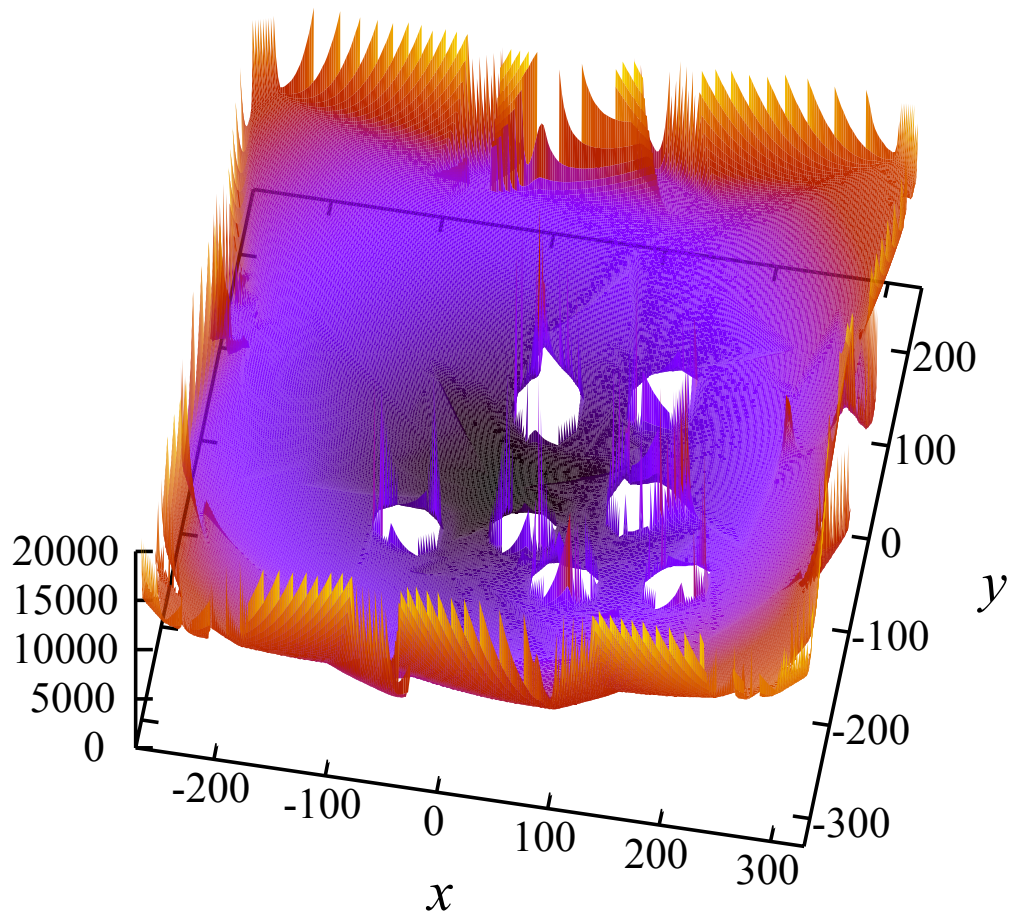
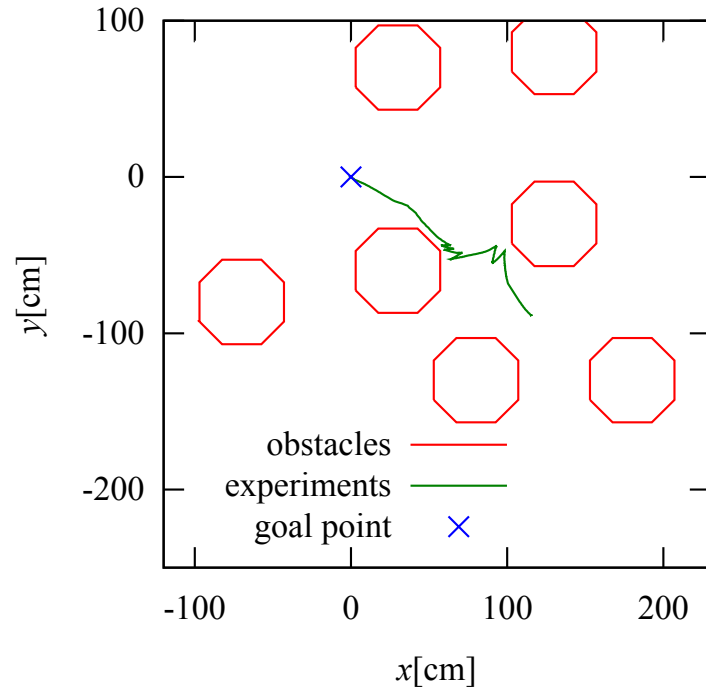
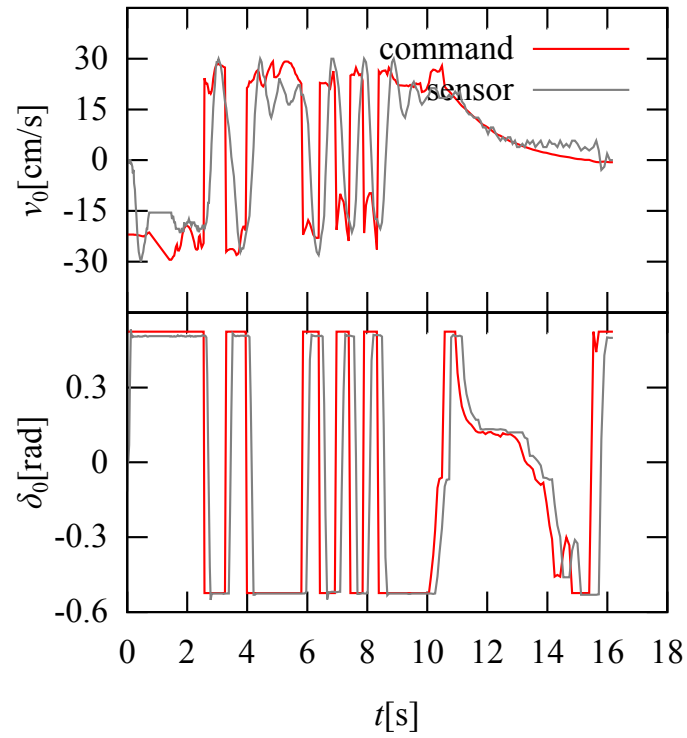


Figure 3.10: CLF used for experiments (in case of $\theta = \theta_{des}$)

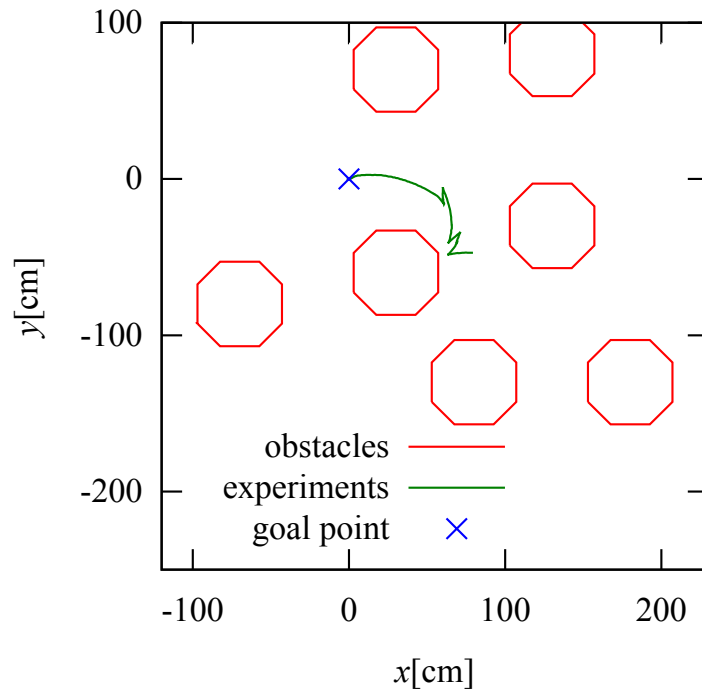


(a) Tracks of four-wheeled vehicle

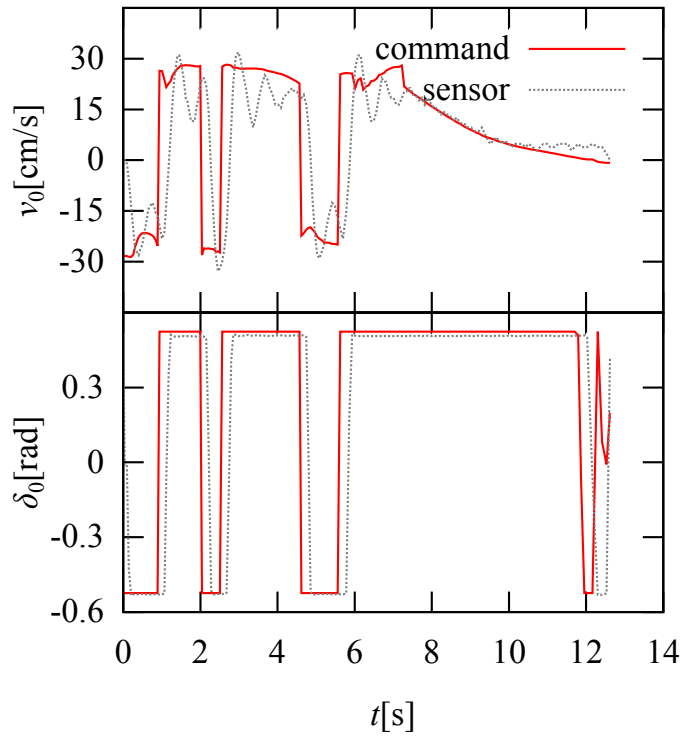


(b) Input for four-wheeled vehicle

Figure 3.11: Experimental result (1)

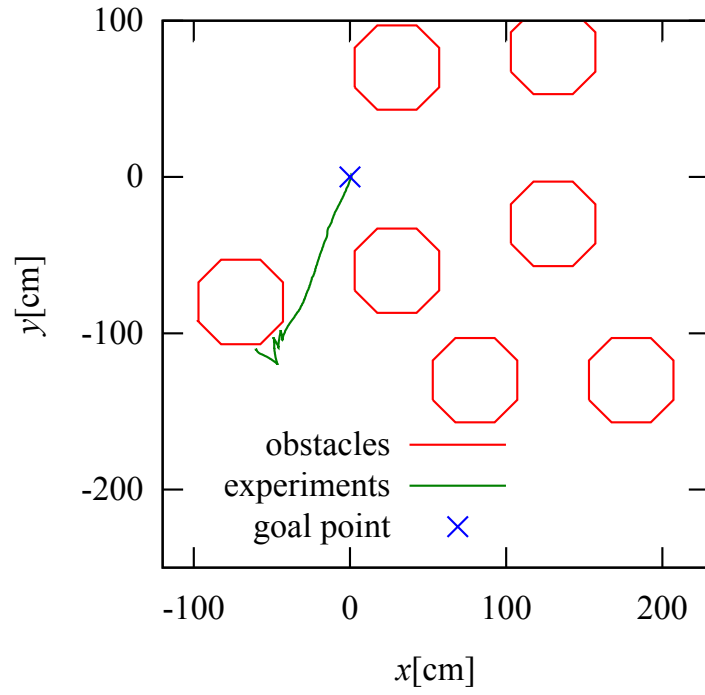


(a) Tracks of four-wheeled vehicle

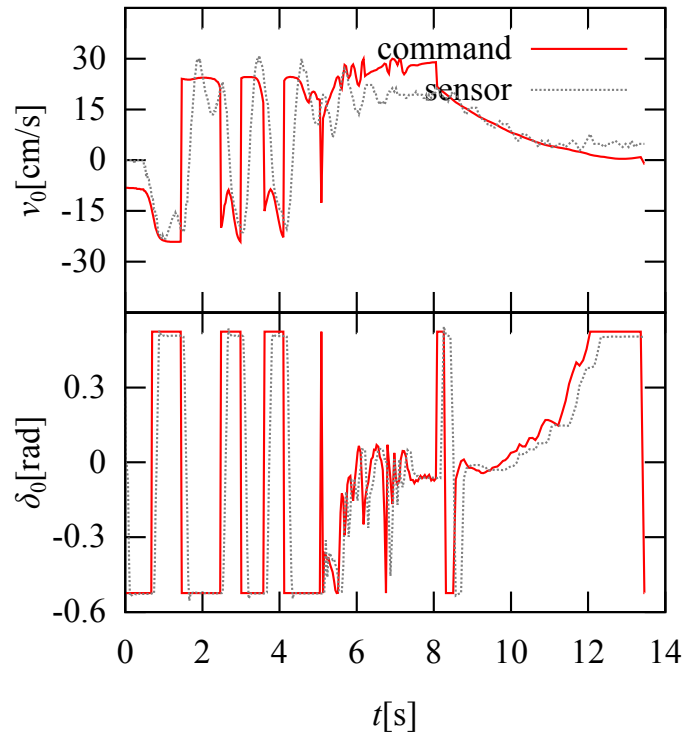


(b) Input for four-wheeled vehicle

Figure 3.12: Experimental result (2)



(a) Tracks of four-wheeled vehicle



(b) Input for four-wheeled vehicle

Figure 3.13: Experimental result (3)

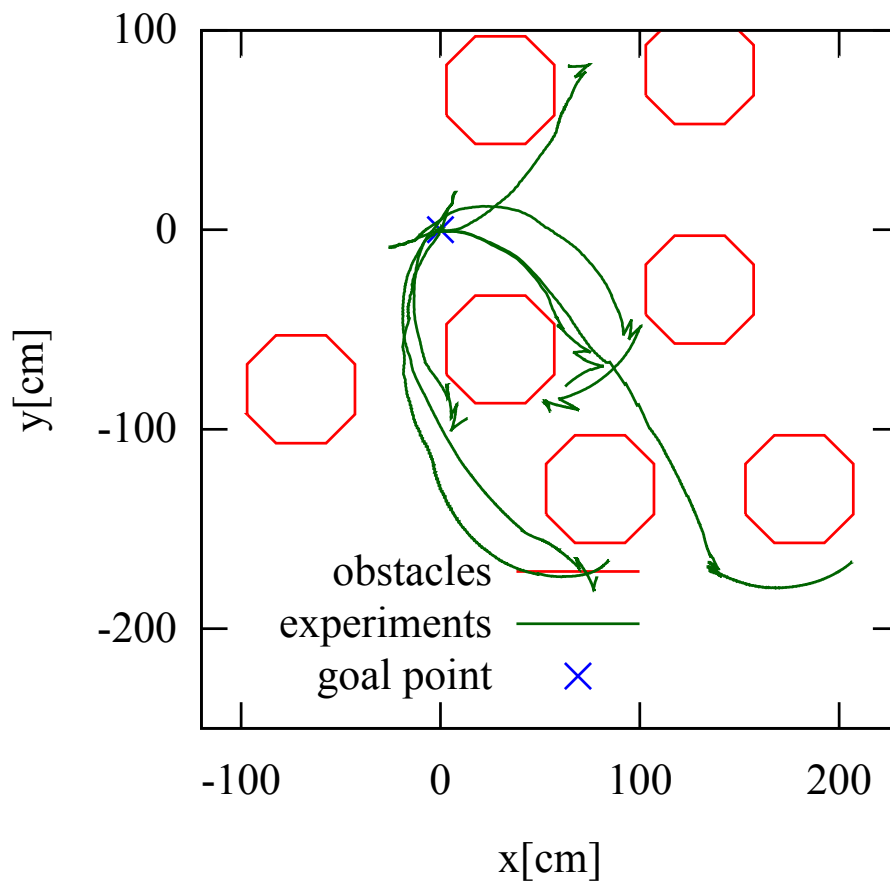


Figure 3.14: Other experimental results

Chapter 4. Higher-order sliding-mode controller

4.1 Introduction

Sliding-mode control is robust against disturbances and uncertainty of a target system. The traditional sliding-mode control has the form of $u = -k\text{sgn}(\sigma(x))$ for the output function $\sigma(x)$ with a relative degree 1 [40]. The control law consists of two modes: one where the state is approaching a boundary called a sliding hypersurface and a sliding mode where the state is forced to glide along the sliding hypersurface. The state reaches the sliding hypersurface in finite time, and then it is restricted on the sliding hypersurface by infinitely high-frequency switching of the input.

Higher-order sliding-mode control achieves finite-time stabilization having a robustness for disturbances and uncertainties. The higher-order sliding-mode for second- and third-order have studied in many researches [41, 42, 4, 43, 5]. Especially, because the second-order systems occur widely throughout mechanical systems, many researches have been done [44, 3, 45, 46, 47, 48, 49, 49, 50]. However, for fourth- or higher-order, it is difficult to construct the higher-order sliding-mode controller. Recently, for its finite-time stability and robustness for disturbances and an uncertainties, arbitrary-order sliding-mode control has been studied and developed [51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63]. Bhat et al. have proposed the finite time stabilization for the higher-order chained integrator based on Hurwitz polynomial [64, 64, 7]. Hong et al. have studied the finite time stabilization for the chained integrator by backstepping based on its homogeneity [65, 66, 67, 68, 69, 70]. These results, however, cannot be applicable to the matched disturbance [71, 72]. Levant et al. proposed a quasi-continuous sliding-mode control law that is continuous except at the origin [10, 73, 11]. Quasi-continuous sliding-mode control makes the output zero in finite time under the bounded matched disturbances. Unfortunately, the stability of quasi-continuous sliding-mode control has been only guaranteed in simulations; it is difficult to design the controller parameters for high-order cases. Polyakov et al. have proposed the higher-order sliding-mode controller using the Implicit Lyapunov Function (ILF) method, which is based on the homogeneous Lyapunov function with implicit form [74, 75, 9]. Moreover, the higher-order super-twisting algorithm that is effective for unmatched disturbance also have been studied [71, 72, 76, 77, 78]. Though most of the higher-order sliding-mode controllers are a homogeneous controller [79, 80], some non-homogeneous higher-order sliding-mode controllers have been proposed to be close to a desired behavior [81, 82]. In the context of

the homogeneity in the bi-limit, global fixed time stable controllers have been proposed [83, 84, 85].

In this chapter, we propose a novel sliding-mode control framework that can explicitly check the stability based on some inequalities, and that needs no extra computational resource for the calculation of the implicit function for the controller implementation.

4.2 Problem statement

Consider the following nonlinear single-input single-output system:

$$\begin{cases} \dot{\xi} = f(\xi, t) + g(\xi, t)u \\ \sigma = y(\xi, t), \end{cases} \quad (4.1)$$

where $\xi \in \mathbb{R}^n$ is the state of the system, n is an unknown positive integer, f and g are unknown vector fields, $y : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is the output function of the system, and $u \in \mathbb{R}$ is the input for the system.

Assumption 4.1

We assume that the relative degree r of the system is known a priori and that $\sigma, \dot{\sigma}, \ddot{\sigma}, \sigma^{(3)}, \dots, \sigma^{(r-1)}$ are observable and continuous functions with respect to the state, where $\sigma^{(i)}$ denotes the i -th time derivative of the output σ .

From these assumptions, the control input appears explicitly in the r -th derivative of σ . Therefore, we can obtain the following control system with respect to the output function:

$$\sigma^{(r)} = a(\xi, t) + b(\xi, t)u, \quad (4.2)$$

where a and b are both unknown functions. The solution of the system (4.1) is understood in the Filippov sense [86].

The state equation of the output system (4.2) is the form of

$$\dot{x} = Ax + B(x, t)u + d(x, t), \quad (4.3)$$

where $x = [\sigma, \dots, \sigma^{(r-1)}]^\top \in \mathbb{R}^r$ is the state, $u \in \mathbb{R}$ is the input, $d : \mathbb{R} \times \mathbb{R}^r \rightarrow \mathbb{R}^r$ is the disturbance term, and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B(x, t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(x, t) \end{bmatrix}, \quad d(x, t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a(x, t) \end{bmatrix}.$$

Assumption 4.2

For the functions a and b in (4.2), there exist some positive constants C and K_m , and some constant $K_M \in [K_m, \infty)$ such that, for any $t > 0$ and $x \in \mathbb{R}^n$, the following inequalities hold:

$$|a(x, t)| \leq C, \quad K_m \leq b(x, t) \leq K_M.$$

We suppose that (4.1) is forward complete and that its zero dynamics are asymptotically stable.

Under Assumption 4.2, the system (4.2) can be transformed into the following differential inclusion form:

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u. \quad (4.4)$$

Note that the solution of the differential inclusion (4.4) includes the solution of the output system (4.2). In this chapter, we consider the stabilization problem of (4.4) instead of (4.2).

It is known that most higher-order sliding-mode controller has the following particular homogeneity:

Definition 4.1 (r -sliding homogeneity [79])

The system (4.4) becomes a homogeneous system under the control law α that satisfying the following identity:

$$\alpha(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}) \equiv \alpha(\lambda\sigma, \lambda^{\frac{r-1}{r}}\dot{\sigma}, \dots, \lambda^{\frac{1}{r}}\sigma^{(r-1)}),$$

where the dilation coefficient is $(1, \frac{r-1}{r}, \dots, \frac{2}{r}, \frac{1}{r})$ and the homogeneity degree of the system is $-\frac{1}{r}$. Both the system (4.4) and the control law α are called r -sliding homogeneous.

Remark 4.1

For any $-\frac{1}{r} \leq \rho \leq 0$, the system (4.4) becomes a homogeneous system under the control law α that satisfying the following identity:

$$\alpha(\sigma, \dots, \sigma^{(r-1)}) \equiv \lambda^{1+r\rho}\alpha(\lambda\sigma, \lambda^{1+\rho}\dot{\sigma}, \dots, \lambda^{1+(r-1)\rho}\sigma^{(r-1)}),$$

where the dilation coefficient is $(1, 1+\rho, \dots, 1+(r-1)\rho)$ and the homogeneity degree of the system is ρ . This is a nature of the integrator.

Obviously, the case of $\rho = -\frac{1}{r}$ is r -sliding homogeneity. Note that if $\rho < -\frac{1}{r}$ then the control input becomes infinity on the origin.

If the system is r -sliding homogeneous then the control input on the origin is discontinuous, as with the traditional sliding-mode controller $u = -\text{sgn}(\sigma)$. This discontinuity guarantees the robustness for the disturbances. The remainder this section introduces three classes of the higher-order sliding-mode controllers:

- using power function,
- using implicit Lyapunov function, and
- using positive definite function.

Using power function

In this subsection, we consider the chained integrator system with no disturbances

$$\begin{aligned}
 \dot{x}_1 &= x_2, \\
 \dot{x}_2 &= x_3, \\
 &\vdots \\
 \dot{x}_{r-1} &= x_r, \\
 \dot{x}_r &= u.
 \end{aligned} \tag{4.5}$$

Bhat and Bernstein et al. proposed the finite-time controller as below in Proposition 8.1 in [7]:

Proposition 4.1 (Bhat and Bernstein [7])

Let $k_1, \dots, k_r$ be a such that the polynomial $s^n + k_n s^{n-1} + \dots + k_2 s + k_1$ is Hurwitz, and consider the chained integrator system (4.5). There exist such that $\epsilon \in (0, 1)$, for every $\alpha \in (1 - \epsilon, 1)$, the origin is globally finite-time-stable equilibrium for the system under the feedback

$$u = -k_1[\sigma]^{\alpha_1} - k_2[\dot{\sigma}]^{\alpha_2} - \dots - k_{r-1}[\sigma^{(r-2)}]^{\alpha_{r-1}} - k_r[\sigma^{(r-1)}]^{\alpha_r}$$

where $\alpha_1, \dots, \alpha_r$ satisfy

$$\begin{aligned}
 \alpha_r &= \alpha, \\
 \alpha_{i-1} &= \frac{\alpha_i \alpha_{i+1}}{2\alpha_{i+1} - \alpha_i}.
 \end{aligned}$$

with $\alpha_{r+1} = 1$.

The existence of ϵ is guaranteed by the continuous transformation from the linear feedback with $\alpha = 1$, that is, from the stability of the feedback law $u = -k_1 x_1 - k_2 x_2 - \dots - k_{r-1} x_{r-1} - k_r x_r$. Therefore, it is difficult to obtain the stability for the small $\alpha \in (0, 1)$. Moreover, we choose $\alpha_1 = \dots = \alpha_r = 0$ then the above feedback becomes r -sliding homogeneous, though the stability for $\alpha = 0$ is out of the application range of the above feedback.

Hong and Jiang proposed the finite-time controller based on the backstepping as below:

Proposition 4.2 (Hong and Jiang [70])

Let us consider the chained integrator system (4.5), and the Lyapunov functions defined

by

$$\begin{aligned}
V_0 &:= 0 \\
V_j &:= W_j + V_{j-1}^{\frac{(\beta_{j-1}+1)r_j}{(\beta_{j-2}+1)r_{j-1}}}, \\
W_j &:= \int_{u_{j-1}}^{x_j} ([x_j]^{\beta_j} - [u_{j-1}]^{\beta_j}) ds \\
&= \frac{1}{\beta_{j-1}} (|x_j|^{\beta_{j-1}+1} + \beta_{j-1}|u_{j-1}|^{\beta_{j-1}+1}) - x_j [u_{j-1}]^{\beta_{j-1}}.
\end{aligned}$$

The following controller with sufficiently large $l_i > 0$ decreases the Lyapunov function V_i :

$$\begin{aligned}
u_0 &= 0, \\
u_i &= -l_i \left[[x_i]^{\beta_{i-1}} - [u_{i-1}]^{\beta_{i-1}} \right]^{\frac{r_{i+1}+k}{r_{i+1}\beta_i}}
\end{aligned}$$

where $k \in (-\frac{1}{r}, 0)$, r_i and β_i are determined by

$$\begin{aligned}
r_1 &= 1, \\
r_i &= r_{i-1} + k, \\
\beta_0 &= r_2, \\
(\beta_i + 1)r_{i+1} &\geq (\beta_{i-1} + 1)r_i > 0.
\end{aligned}$$

Then, V_r be a Lyapunov function of system under the control law

$$u = u_r.$$

The stability is given by the time derivative of the Lyapunov function:

$$\dot{V}_j := \dot{W}_j + \frac{(\beta_{j-1} + 1)r_j}{(\beta_{j-2} + 1)r_{j-1}} V_{j-1}^{\frac{(\beta_{j-1}+1)r_j}{(\beta_{j-2}+1)r_{j-1}}-1} \dot{V}_{j-1}.$$

Let $\dot{V}_{j-1}$ be a negative definite under $\sigma^{(j-1)} = u_{j-1}$. From the homogeneous continuity of $\dot{V}_{j-1}$, there exist some area that $\dot{V}_{j-1} < 0$ around $\sigma^{(j-1)} = u_{j-1}$. Therefore, sufficiently large l_i make $\dot{W}_j$ be small enough such that cancels out $\dot{V}_{j-1} > 0$, and $\dot{V}_j$ becomes negative definite. Both of the above controllers do not include the r -sliding homogeneous controller. However, we can obtain the following r -sliding homogeneous controller similar to Hong's controller with $k = -1/r$ for some $l_i > 0$:

$$\begin{aligned}
u &= -l_r \left[[x_r]^{\beta_{r-1}} - [u_{r-1}]^{\beta_{r-1}} \right]^{r_r+k} \\
&= -l_r \text{sgn}([x_r]^{\beta_{r-1}} - [u_{r-1}]^{\beta_{r-1}}),
\end{aligned}$$

where $r_r = 1 + (r - 1)k = -k$.

Using implicit Lyapunov function

Here, we briefly introduce the Implicit Lyapunov Function (ILF) method proposed by Polyakov et al. [9].

Theorem 4.1 (Polyakov et al. [9])

Let us consider the following single input system with the relative degree n :

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r-1} \\ x_r \end{bmatrix} &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r-1} \\ x_r \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} d_1(x, t) \\ d_2(x, t) \\ \vdots \\ d_{r-1}(x, t) \\ d_r(x, t) \end{bmatrix} \\ &= Ax + Bu + d(x, t). \end{aligned} \quad (4.6)$$

For given $\mu \in (0, 1]$ and $\beta > 0$, let the following matrix inequalities be feasible for some $X = X^\top \in \mathbb{R}^{r \times r}$, $Y \in \mathbb{R}^{1 \times r}$ and $\alpha \in \mathbb{R}$:

$$\begin{aligned} AX + XA^\top + BY + Y^\top B^\top + \alpha X + \beta I &\leq 0, \\ X(I + \mu R) + (I + \mu R)X &> 0, \\ X &> 0, \\ \alpha - \beta &> 0, \end{aligned}$$

where

$$R = \begin{bmatrix} r-1 & 0 & \cdots & 0 & 0 \\ 0 & r-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

We define a Lyapunov function as

$$V(x) := \sqrt{x^\top D_r(V(x)^{-\mu}) P D_r(V(x)^{-\mu}) x},$$

where $P = X^{-1}$ and

$$D_r(\lambda) := \begin{bmatrix} \lambda^{r-1} & 0 & \cdots & 0 & 0 \\ 0 & \lambda^{r-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

If the disturbance $d(x, t)$ satisfies the inequality

$$\beta V^{1-\mu} \geq \sup_{t \geq 0, x \in \mathbb{R}^r} \|D_r(V^{-\mu}) d(x, t)\|,$$

then the following control law stabilize the system (4.6) in finite time:

$$\begin{aligned} u &= V^{-\mu} K D_r (V^{-\mu}) x, \\ K &= Y X^{-1}. \end{aligned}$$

Note that $\mu = 0$ means that the linear control law.

The proofs of the existence and uniqueness of the ILF V and its negative definiteness of $\dot{V}$ are given by [9]. For brief confirmation, let us consider the time derivative of V in the case of $d(x, t) = 0$. Denoting $z = D_r (V^{-\mu}) x$, the time derivative of V is calculated as

$$\begin{aligned} \dot{V} &= \frac{d}{dt} \sqrt{z^\top P z} = \frac{\frac{d}{dt} (z^\top P z)}{2\sqrt{z^\top P z}} = \frac{\dot{z}^\top P z + z^\top P \dot{z}}{2V}, \\ \dot{z} &= D_r (V^{-\mu}) (A + V^{-\mu} B K) x - \mu \frac{\dot{V}}{V} R D_r (V^{-\mu}) x, \\ &= V^{-\mu} (A + B K) z + \mu \frac{\dot{V}}{V} R z \\ 2V\dot{V} &= z^\top \left(V^{-\mu} (A + B K) + \mu \frac{\dot{V}}{V} R \right)^\top P z + z^\top P \left(V^{-\mu} (A + B K) + \mu \frac{\dot{V}}{V} R \right) z, \\ \dot{V} &= V^{1-\mu} \frac{z^\top ((A + B K)^\top P + P(A + B K)) z}{z^\top ((I + \mu R) P + P(I + \mu R)) z}. \end{aligned}$$

From Theorem 4.1, the time derivative of V is negative definite because $(A + B K)^\top P + P(A + B K)$ is negative definite and $(I + \mu R) P + P(I + \mu R)$ is positive definite from the given LMIs in Theorem 4.1.

Note that the ILF V must be calculated numerically each time because V is implicit function. Moreover, the existence of the solution for the given LMI is not guaranteed. The controller of Theorem 4.1 is r -sliding homogeneous if we use $\mu = 1$.

Using positive definite function

We introduced the higher-order sliding-mode controllers using power function and implicit Lyapunov function. The higher-order sliding-mode controller using power function cannot be applicable to r -sliding controller. The implicit Lyapunov function need to be solved numerically in each cycle of the controller. The alternative method is to use the explicit positive definite function.

Levant and Pavlov proposed the r -sliding homogeneous control law summarized below [73].

Levant's r -sliding homogeneous control law: Consider r -sliding homogeneous positive definite functions with the homogeneity degree $(r - i)/r$ defined by

$$M_0 = |\sigma|, \quad M_i = |\sigma^{(i)}| + \beta_i M_{i-1}^{\frac{r-i}{r-i+1}} \quad (i = 1, \dots, r-1).$$

Using the positive definite function M_i , we can design the following recursive function:

$$\begin{aligned}\varphi_0 &= 0, \\ \varphi_i &= -\beta_i \frac{y_{i-1}}{M_{i-1}^{\frac{1}{r-i+1}}}, \\ y_i &= \sigma^{(i)} - \varphi_i,\end{aligned}$$

where each β_i is of a sufficiently large number compared to β_{i-1} ; β_r is a coefficient for the input magnitude. The following r -sliding control law with well-chosen β_i establishes the global finite-time stability of the system (4.4):

$$u = \varphi_r(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}). \quad (4.7)$$

The above control law moves the state $\sigma^{(r-1)}$ close to φ_{r-1} and confines the state to some neighborhood of $\sigma^{(r-1)} = \varphi_{r-1}$. Note that $\sigma^{(r-1)} = \varphi_{r-1}$ cannot be a sliding hypersurface because the time derivative of φ_{r-1} is not bounded along the trajectory. When the control law keeps $\sigma^{(r-1)}$ close to φ_{r-1} , the state $\sigma^{(r-2)}$ approaches and stays around φ_{r-2} . Then, $\sigma^{(r-3)}$ approaches and stays around φ_{r-3} , and each $\sigma^{(i)}$ ($i = r-4, \dots, 1$) behaves under the same mechanism. After the state of the system is trapped in the area close to φ_1 , the system finally establishes $\sigma \equiv 0$ in finite time.

The stability condition of the control law (4.7) is not written in explicit form. In order to analyze the stability of the above control law, we have to find some invariant set around $\sigma^{(r-1)} = \varphi_{r-1}$. Unfortunately, the time derivative of the function φ_{r-1} is not bounded at $\sigma = 0$; that is, $\dot{M}_i$ ($i = 1, \dots, r-1$) are unbounded because

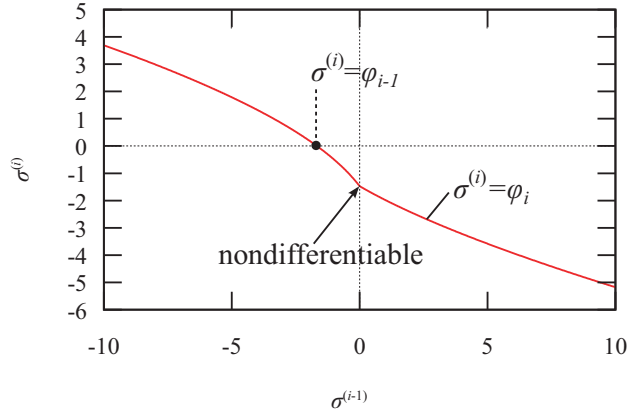
$$\frac{d}{dt} M_0^{\frac{r-1}{r}} = \frac{r-1}{r} \frac{\dot{\sigma}}{|\sigma|^{\frac{1}{r}}}$$

is unbounded where $\sigma = 0$, and φ_{r-1} has nondifferentiable points, as shown in Fig. 4.1. Moreover, we have to find such an invariant manifold for each φ_i ($i = 1, \dots, r-1$). For these reasons, finding or estimating the invariant set is difficult. Therefore, we cannot design the parameters of the control law explicitly, and we should check whether or not the parameters are stable through a heuristic approach, such as numerical simulations.

In the next section, we propose a novel strategy for higher-order sliding-mode control that allows us to check the stability using some simple inequality conditions, and its design procedure is also given explicitly. The r -sliding homogeneous controllers introduced in this section are based on the form like

$$u = -k_1 \frac{x_1}{\mathcal{V}_1(x)} - k_2 \frac{x_2}{\mathcal{V}_2(x)} - \dots - k_r \frac{x_r}{\mathcal{V}_r(x)},$$

where each $\mathcal{V}_i(x)$ is some positive homogeneous function that makes u be a homogeneous function. The higher-order sliding-mode controller proposed in this chapter also has the above form.

Figure 4.1: Function φ_i for Levant's controller

4.3 Control law

In this section, we show a novel high-order sliding-mode controller using homogeneous positive definite function. The stability of the high-order sliding-mode controller is guaranteed by the theoretically with no simulations.

4.3.1 Main result

We design some r -sliding homogeneous functions. First, we obtain the following homogeneous positive definite functions whose time derivative is locally bounded.

Proposition 4.3

Let $p \geq r$ and consider

$$N_0(\sigma) = |\sigma|_r^{\frac{p}{r}},$$

$$N_i(\sigma, \dot{\sigma}, \dots, \sigma^{(i)}) = |\sigma^{(i)}|^{\frac{p}{r-i}} + b_i N_{i-1}(\sigma, \dot{\sigma}, \dots, \sigma^{(i-1)}),$$

where b_i ($i = 1, \dots, r-1$) are positive constants. Then, the functions $N_i(\sigma, \dots, \sigma^{(i)})$ ($i = 0, \dots, r-1$) are homogeneous positive definite functions with the homogeneity degree p with dilation coefficient $(r, r-1, \dots, r-i)$ for $(\sigma, \dots, \sigma^{(i)})^\top$.

The homogeneity of N_i is directly confirmed by substituting $(\lambda^r \sigma, \lambda^{r-1} \dot{\sigma}, \dots, \lambda^{r-i} \sigma^{(i)})^\top$ into $N_i(\sigma, \dots, \sigma^{(i)})$. For simplicity, $N_i(\sigma, \dots, \sigma^{(i)})$ is denoted by N_i . The positive definiteness of N_i is obvious from its definition. The time derivative of N_i is locally bounded as long as $p \geq r$ because all of the exponents are equal to or greater than 1.

$$\begin{aligned} \dot{N}_i &= \frac{p}{r-i} [\sigma^{(i)}]^{\frac{p-r+i}{r-i}} \sigma^{(i+1)} + b_i \dot{N}_{i-1} \\ &= \sum_{j=0}^i b_i \dots b_{j+1} \frac{p}{r-j} [\sigma^{(j)}]^{\frac{p-r+j}{r-j}} \sigma^{(j+1)} \end{aligned}$$

The sliding hypersurface $\sigma^{(r-1)} = \phi_{r-1}(\sigma, \dots, \sigma^{(r-1)})$ is designed recursively as follows:

$$\begin{aligned}\phi_0 &= 0, \\ \phi_i(\sigma, \dots, \sigma^{(i)}) &= -a_i \frac{z_{i-1}(\sigma, \dots, \sigma^{(i-1)})}{N_i(\sigma, \dots, \sigma^{(i)})^{\frac{1}{p}}}, \\ z_i(\sigma, \dots, \sigma^{(i)}) &= \sigma^{(i)} - \phi_i(\sigma, \dots, \sigma^{(i)}),\end{aligned}$$

where a_i ($i = 1, \dots, r-1$) are positive constants that determine the slope of the curve $\sigma^{(i)} = \phi_i(\sigma, \dots, \sigma^{(i)})$. For simplicity, $\phi_i(\sigma, \dots, \sigma^{(i)})$ and $z_i(\sigma, \dots, \sigma^{(i)})$ is denoted by ϕ_i and z_i , respectively. Note that each ϕ_i is homogeneous function with the homogeneity degree $r-i$ for $(\sigma, \dots, \sigma^{(i)})^\top$ with dilation coefficient $(r, r-1, \dots, r-i)$. The middle curve in Fig. 4.2 indicates the curve $\sigma^{(i)} = \phi_i$.

In this thesis, we consider the following nonlinear sliding sector instead of the sliding hypersurface $\sigma^{(r-1)} = \phi_{r-1}$:

$$\Omega_{r-1}(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}) = \{\sigma^{(r-1)} \in \mathbb{R} \mid |\sigma^{(r-1)} - \phi_{r-1}| \leq c_{r-1} N_{r-1}^{\frac{1}{p}}\},$$

where $c_{r-1} < 1$ is a positive constant. We design the Ω_{r-1} to guarantee the stability of the system as long as the system satisfies $\sigma^{(r-1)} \in \Omega_{r-1}$. It means that the nonlinear sliding sector is the extension of the sliding hypersurface. The details on the property of the sliding sector and the actual controller implementation are discussed in the next chapter.

Note that $\sigma^{(r-1)} \in \Omega_i$ is described by implicit form, however, there are no need to solve this equation for the implementation of the controller. Moreover, $\sigma^{(i)} = \phi_i$ is well-defined for fixed $\sigma, \dots, \sigma^{(i-1)}$.

Lemma 4.1

There exist a unique solution to $\sigma^{(i)} = \phi_i$.

Proof

Multiplying $N_i^{\frac{1}{p}}$ on both sides of $\sigma^{(i)} = \phi_i$, we obtain

$$\sigma^{(i)} N_i^{\frac{1}{p}} = -a_i z_{i-1}.$$

The right-hand side of the above equation is a constant that is independent of $\sigma^{(i)}$. Because the left-hand side of the equation is monotonically increasing function, there exists a unique solution.

Lemma 4.2

For any $c_i \in [0, 1)$, there exist a unique solution to $\sigma^{(i)} = \phi_i + c_i N_i^{\frac{r-i}{p}}$.

Proof

Multiplying $N_i^{\frac{1}{p}}$ on both sides of $\sigma^{(i)} - c_i N_i^{\frac{r-i}{p}} = \phi_i$, we obtain

$$\sigma^{(i)} N_i^{\frac{1}{p}} - c_i N_i^{\frac{r-i+1}{p}} = -a_i z_{i-1}.$$

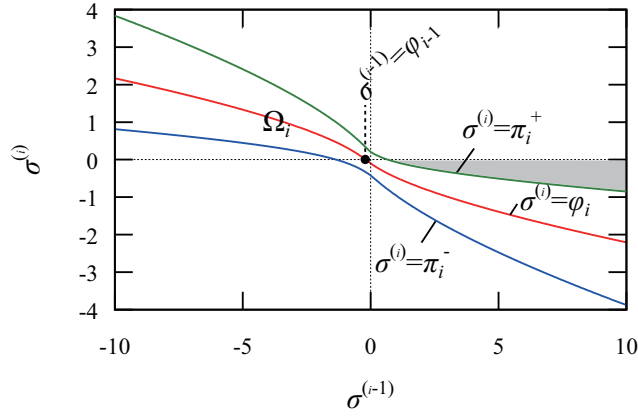


Figure 4.2: The curve $\sigma^{(i)} = \phi_i$ and the area $\sigma^{(i)} \in \Omega_i$

The right-hand side of the above equation is a constant that is independent of $\sigma^{(i)}$.

$$\begin{aligned}
 \frac{\partial}{\partial \sigma^{(i)}} (\sigma^{(i)} N_i^{\frac{1}{p}} - c_i N_i^{\frac{r-i+1}{p}}) &= N_i^{\frac{1}{p}} + \frac{1}{p} \frac{\sigma^{(i)}}{N_i^{\frac{p-1}{p}}} \frac{\partial N_i}{\partial \sigma^{(i)}} - c_i \frac{r-i+1}{p} \frac{1}{N_i^{\frac{p-r+i-1}{p}}} \frac{\partial N_i}{\partial \sigma^{(i)}} \\
 &= N_i^{\frac{1}{p}} + \left(\frac{1}{p} \frac{\sigma^{(i)}}{N_i^{\frac{p-1}{p}}} - c_i \frac{r-i+1}{p} \frac{1}{N_i^{\frac{p-r+i-1}{p}}} \right) \frac{p}{r-i} [\sigma^{(i)}]^{\frac{p-r+i}{r-i}} \\
 &= N_i^{\frac{1}{p}} + \frac{1}{r-i} \frac{|\sigma^{(i)}|^{\frac{p}{r-i}}}{N_i^{\frac{p-1}{p}}} - c_i \left(1 + \frac{1}{r-i} \right) \frac{[\sigma^{(i)}]^{\frac{p-r+i}{r-i}}}{N_i^{\frac{p-r+i-1}{p}}} \\
 &\geq (1 - c_i) N_i^{\frac{1}{p}}.
 \end{aligned}$$

Because the left-hand side of the equation is monotonically increasing function, there exists a unique solution.

Corollary 4.1

For any $c_i \in [0, 1)$, there exist a unique solution to $\sigma^{(i)} = \phi_i - c_i N_i^{\frac{r-i}{p}}$.

We also define the following set Ω_i ($i = 1, \dots, r-2$), which is similar definition of Ω_{r-1} :

$$\Omega_i := \{\sigma^{(i)} \in \mathbb{R} \mid |\sigma^{(i)} - \phi_i| \leq c_i N_i^{\frac{r-i}{p}}\},$$

where $c_i < 1$ is a positive constant. The area Ω_i is the area between $\sigma^{(i)} = \pi_i^-(\sigma, \dot{\sigma}, \dots, \sigma^{(i)})$ and $\sigma^{(i)} = \pi_i^+(\sigma, \dot{\sigma}, \dots, \sigma^{(i)})$ as shown in Fig. 4.2, where π_i^+ and π_i^- are defined as

$$\begin{aligned}
 \pi_i^+(\sigma, \dot{\sigma}, \dots, \sigma^{(i)}) &:= \phi_i + c_i N_i^{\frac{r-i}{p}}, \\
 \pi_i^-(\sigma, \dot{\sigma}, \dots, \sigma^{(i)}) &:= \phi_i - c_i N_i^{\frac{r-i}{p}}.
 \end{aligned}$$

Definition 4.2

For $i = 1, \dots, r-1$, consider the following subsystem:

$$\frac{d}{dt}\sigma^{(i)} \in \Omega_{i+1}.$$

where the parameters $a_i, b_i \in [0, \infty)$ and $c_i \in [0, 1)$ of Ω_{i+1} are given. If the set Ω_i is an invariant set of the above subsystem, that is, $\sigma^{(i)}$ converges to Ω_i under the above differential inclusion, the parameter $(a_{i+1}, b_{i+1}, c_{i+1})$ of Ω_{i+1} is said to be proper for Ω_i . Note that we define $\Omega_0 := \{0\}$.

The following theorem is the improved version of Theorem 7 in [87] with respect to the existence of the parameters for the controller.

Theorem 4.2

Consider the control law $u = \alpha(\sigma, \dots, \sigma^{(r-1)})$ satisfying the following inequality:

$$\begin{aligned} \sigma^{(r)} &= a(x, t) + b(x, t)\alpha(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}) \\ &\begin{cases} < -\mu_r & (\sigma^{(r-1)} > \pi_{r-1}^+) \\ > \mu_r & (\sigma^{(r-1)} < \pi_{r-1}^-) \\ \in \mathbb{R} & (\text{otherwise}). \end{cases} \end{aligned} \quad (4.8)$$

Then, there exist some proper parameters $(a_1, b_1, c_1), \dots, (a_{r-1}, b_{r-1}, c_{r-1})$ and μ_r for the control law such that the system (4.2) establishes $\sigma \equiv 0$ in finite time under the control law $u = \alpha(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)})$.

The positive constant μ_r guarantees that the state of the system converges on $\sigma^{(r-1)} \in \Omega_{r-1}$, and that there exists a sufficiently large magnitude A of the input u for the system (4.4) satisfying $A > (\mu_r + C)/K_m$ along the trajectory: where C and K_m are both positive constants given by Assumption 4.2. For example, the following simple controllers satisfy (4.8):

- $u = -A \operatorname{sgn}(\sigma^{(r-1)} - \phi_{r-1}),$
- $u = -A(\sigma^{(r-1)} - \phi_{r-1})/(c_{r-1}N_{r-1}^{\frac{1}{p}}).$

More details on the controller design is discussed in the next chapter.

4.3.2 Proof of Theorem 4.2**Lemma 4.3**

The subsystem $\frac{d}{dt}\sigma \in \Omega_1$ establishes $\sigma = 0$ in finite time for any positive parameters a_1, b_1 and $c_1 < 1$.

Lemma 4.4

If the system (4.1) start from the area where $\sigma^{(i)} > \pi_i^+$, then the system reaches on the area $\sigma^{(i)} \leq 0$ or $\sigma^{(i)} \in \Omega_i$ in finite time under the condition $\sigma^{(i+1)} \leq -A_{i+1}N_i^{\frac{r-i-1}{p}}$ with some positive constant A_{i+1} .

The time derivative of the distance between $\sigma^{(i)}$ and the upper boundary π_i^+ of the set Ω_i is calculated as

$$\frac{d}{dt} (\sigma^{(i)} - \pi_i^+) = f_i(\sigma, \dots, \sigma^{(i)}) + g_i(\sigma, \dots, \sigma^{(i)}) \sigma^{(i+1)}, \quad (4.9)$$

$$f_i(\sigma, \dots, \sigma^{(i)}) = - \left(\frac{a_i}{p} \frac{\sigma^{(i-1)} - \phi_{i-1}}{N_i^{\frac{r-i+1}{p}}} + \frac{r-i}{p} c_i \right) \frac{b_i \dot{N}_{i-1}}{N_i^{\frac{p-r+i}{p}}} + a_i \frac{\sigma^{(i)} - \dot{\phi}_{i-1}}{N_i^{\frac{1}{p}}}, \quad (4.10)$$

$$g_i(\sigma, \dots, \sigma^{(i)}) = \left(1 - \left(\frac{a_i}{r-i} \frac{\sigma^{(i-1)} - \phi_{i-1}}{N_i^{\frac{r-i+1}{p}}} + c_i \right) \frac{\lfloor \sigma^{(i)} \rfloor^{\frac{p-r+i}{p}}}{N_i^{\frac{p-r+i}{p}}} \right). \quad (4.11)$$

Lemma 4.4 guarantees that the subsystem $\frac{d}{dt} \sigma^{(i)} \leq -A_i N_i^{\frac{r-i-1}{p}}$ can reach on the following area from any initial point:

$$\begin{aligned} O_i^+ : &= \{(\sigma, \dots, \sigma^{(i)})^\top \in \mathbb{R}^i \mid \sigma^{(i)} = \pi_i^+(\sigma, \dots, \sigma^{(i)})\} \\ &\cup \{(\sigma, \dots, \sigma^{(i)})^\top \in \mathbb{R}^i \mid \pi_i^+(\sigma, \dots, \sigma^{(i)}) \leq \sigma^{(i)} \leq 0\}. \end{aligned}$$

The gray area in Fig. 4.2 indicates the area $\pi_i^+ \leq \sigma^{(i)} \leq 0$.

In order to prove that $\sigma^{(i)} \in \Omega_i$ is an invariant set of the subsystem $\frac{d}{dt} \sigma^{(i)} \in \Omega_{i+1}$, we have to check the $\max_{O_i^+} f_i/g_i$.

Lemma 4.5

There exists some positive constant α_i such that the following inequality is satisfied for given parameters $(a_1, b_1, c_1), \dots, (a_i, b_i, c_i)$:

$$\max_{(\sigma, \dots, \sigma^{(i)})^\top \in O_i^+} f_i(\sigma, \dots, \sigma^{(i)}) \leq \alpha_i N_i^{\frac{r-i-1}{p}}.$$

Note that, in the case of $i = r - 1$, the right hand side of the inequality becomes a constant, that is, $\alpha_{r-1} N_{r-1}^0 = \alpha_{r-1}$.

Lemma 4.6

For any given parameters $(a_1, b_1, c_1), \dots, (a_i, b_i, c_i)$, the following inequality is always satisfied:

$$\min_{(\sigma, \dots, \sigma^{(i)})^\top \in O_i^+} g_i(\sigma, \dots, \sigma^{(i)}) \geq 1 - c_i.$$

Lemma 4.7

For given $A_{i+1} > 0$, there exists $a_{i+1}, b_{i+1} > 0$ and $0 < c_{i+1} < 1$ that satisfying the following equality if the system remains on the area $\sigma^{(i)} \geq \pi_i^+$:

$$\max_{\sigma^{(i+1)} \in \Omega_{i+1}} \sigma^{(i+1)} = -A_{i+1} N_i^{\frac{r-i-1}{p}}. \quad (4.12)$$

Note that the proof of all of Lemmas 4.3–4.7 are summarized in Appendix B. Using the above lemmas, the proof of the Theorem 4.2 is given as below.

Proof of Theorem 4.2: From Lemma 4.3, the subsystem $\frac{d}{dt}\sigma \in \Omega_1$ establishes $\sigma = 0$ in finite time for any $a_1 > 0$, $b_1 > 0$ and $0 < c_1 < 1$. From Lemma 4.4, the state of the system reaches on $\dot{\sigma} \leq 0$ or $\dot{\sigma} = \pi_1^+$ as long as $\ddot{\sigma} \leq -A_2 N_1^{\frac{r-2}{p}}$ for any $A_2 > 0$. From Lemma 4.5 and Lemma 4.6, there exists some A_2 such that $\dot{\sigma} \leq \pi_1^+$ becomes an invariant set as long as $\ddot{\sigma} \leq -A_2 N_1^{\frac{r-2}{p}}$. Similarly, replacing π_1^+ with π_1^- in the lemmas, we confirm that $\dot{\sigma} \geq \pi_1^-$ can be also an invariant set as long as $\ddot{\sigma} \geq A_2 N_2^{\frac{r-2}{p}}$. Therefore, from such A_2 and Lemma 4.7, for a given (a_1, b_1, c_1) , there exists proper parameter (a_2, b_2, c_2) , that is, $\dot{\sigma} \in \Omega_1$ becomes an invariant set of the subsystem $\frac{d}{dt}\dot{\sigma} \in \Omega_2$. Similarly, for each $i = 2, \dots, r-2$, there exists some set of parameters $(a_{i+1}, b_{i+1}, c_{i+1})$ such that $\sigma^{(i)} \in \Omega_i$ becomes a invariant set of the subsystem $\frac{d}{dt}\sigma^{(i)} \in \Omega_{i+1}$. Moreover, there exists some positive constant μ_r such that $\sigma^{(r-1)} \in \Omega_{r-1}$ becomes an invariant set of the system (4.1) under the control law α satisfying (4.8). Therefore, any control law α with the proper parameters determined in the above procedures establishes $\sigma^{(r-1)} \in \Omega_{r-1}$ in finite time and keeps $\sigma^{(r-1)} \in \Omega_{r-1}$ permanently. If the system remains on $\sigma^{(r-1)} \in \Omega_{r-1}$, the system reaches on $\sigma^{(r-2)} \in \Omega_{r-2}$ in finite time. For each i (from $r-3$ to 1), $\sigma^{(i)} \in \Omega_i$ is fulfilled in finite time in the same manner as $i = r-2$. As long as $\dot{\sigma} \in \Omega_1$, the system converge to $\sigma = 0$. If the system establishes $\sigma = 0$ and $\sigma^{(i)} \in \Omega_i$ for every $i = 1, \dots, r-1$, then, $\Omega_1 = \Omega_2 = \dots = \Omega_{r-1} = \{0\}$ is guaranteed by their definitions. Consequently, $\sigma = \dot{\sigma} = \dots = \sigma^{(r-1)} = 0$ can be established in finite time by the control law with some parameters. The existence of the proper parameters for the control law is proven. ■

The parameter design procedures for the proposed controller are given in the next chapter.

Remark 4.2

We can design the r -sliding controller by using some r -sliding homogeneous positive definite function $\mathcal{N}(\sigma, \dots, \sigma^{(r-1)})$ instead of $N_i(\sigma, \dots, \sigma^{(i)})$. For any $i = 1, \dots, r-1$, as long as the state of the system satisfies $\sigma^{(j)} \in \{s \mid |s - \phi_j| < c_j V_j^{\frac{r-j}{p}}\}$ for each $j = i+1, \dots, r-1$, if the following inequality is satisfied along the trajectory of the system then the system is finite-time stable:

$$\frac{d}{dt}(\phi_i + c_i V^{\frac{r-i}{p}}) < 0.$$

Note that, in this case, we have no constructive method for designing stable parameters of the controller with $\mathcal{N}$ of explicit form.

4.4 Summary

In this chapter, we propose a general strategy for designing of higher-order sliding-mode control laws. The stability under the proposed controller is guaranteed by simple conditions: the control input for the system must have an absolute value larger than

some positive constant and have the same sign as $-z_{r-1}$ as long as the state of the system stays outside the area $|z_{r-1}| > c_{r-1}N_{r-1}^{\frac{1}{r}}$. From the stability condition, there are no requirements for the input as long as the state of the system stays in the area. Moreover, we prove the existence of such a stable parameters for the proposed higher-order sliding-mode controller. Compared to Levant's controller summarized in Section 4.2, the stability condition can be easily obtained because the inequality conditions are given explicitly and the constructive method for the parameters design is proposed. For future works, the robustness for the disturbance into $\sigma^{(i)}$ ($i \neq r$), and extension of the stability condition with non-backstepping approach are exists.

Chapter 5. Controller design and sliding sector

In order to reduce the chattering phenomenon, we can use integral sliding mode that treats the actual input as an artificial state and the derivative of the input as a new input [51, 88]. There are, however, many systems which have a limited input such as on/off control or brake/accelerator switching system. In such cases, the chattering cannot be relaxed by using continuous input [89]. The higher-order sliding-mode for binary input system have been proposed [6], but the parameter design method of the stable controller is not provided explicitly. The higher-order sliding-mode controller for binary input based on the controllability function has been proposed [8]. The controllability function is constructive but has few design parameters [90, 91, 92]. Moreover, the proof in [8] is incomplete in the boundedness of the time derivative of a switching function.

The main objective of this chapter is to design a chattering-less control law for the system with a discrete-valued input because the switching of the control input is not necessary for the achievement of finite-time stability and robustness for the disturbance and uncertainty except at the origin $\sigma = \dots = \sigma^{(r-1)} = 0$. The other objective is to define the nonlinear sliding sector. We can design a control law to make a input hold the previous input in a nonlinear area like a linear sliding sector, which is an extension of the traditional sliding mode control aimed at chattering free [93, 12]. By defining the arbitrary-order sliding sector, we can integrate the various types of the higher-order sliding-mode control into one strategy based on the sliding sector. The proposed controller design strategy based on the sliding sector includes, (i) the traditional controller using a signum function, (ii) a continuous controller similar to the quasi-continuous sliding-mode controller, and (iii) a chattering-less controller with a hysteresis mechanism. In this chapter, we also discuss the parameter design for the proposed sliding-mode controllers. The quasi-continuous sliding-mode control proposed by Levant et al. cannot design the sliding sector because the stability condition is not given by explicitly but depends on the target system and disturbances. Similarly, though the homogeneous controller using power function is constructive, we cannot check the area that the time derivative of the Lyapunov function becomes negative can be an invariant set. We introduce the parameter design procedure for the proposed controller. We know that the homogeneous finite time controller can be redesigned by using its homogeneity [11, 94, 95]. The similar redesign method adjusting to given input constraint is also introduced in this chapter. Furthermore, we can treat the input saturation without complex mechanism as in [96] because the globally boundedness of the control signal of

the r -sliding controller is naturally obtained. The general representation of the saturation function for the higher-order sliding-mode control is also derived from the context of the sliding sector. The simulation results using the proposed controllers are presented to verify the effectiveness of the control laws based on the sliding sector.

5.1 Sliding sector

In this chapter, we consider the following system

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u, \quad (5.1)$$

where σ is an output function, $u \in \mathbb{R}$ is an input and C , K_m and K_M are given positive constant. The above system is same with the system (4.4) in the previous chapter under the same assumptions. Theorem 4.2 shows that there exist many degrees of freedom for the controller design for the above system. By making use of the flexibility of the controller, in this chapter, we design controllers for various target systems.

Definition 5.1 (Sliding sector)

Let the following subsystem be finite-time stable:

$$\sigma^{(r-1)} \in \Omega(\sigma, \dots, \sigma^{(r-2)}).$$

If there exists some positive constant μ such that the area $\sigma^{(r-1)} \in \Omega(\sigma, \dots, \sigma^{(r-2)})$ becomes an invariant set for the system (5.1) under

$$\sigma^{(r)} \begin{cases} \leq -\mu, & (\sigma^{(r-1)} > \max\{\Omega\}) \\ \geq \mu, & (\sigma^{(r-1)} < \min\{\Omega\}) \end{cases}$$

then we say that the area Ω is a sliding sector for the system (5.1).

Proposition 5.1

The system (5.1) is finite time stable under the following control law:

$$u \begin{cases} \leq -A & (\sigma^{(r-1)} > \max\{\Omega\}) \\ \geq A & (\sigma^{(r-1)} < \min\{\Omega\}) \\ \in [-A, A] & (\sigma^{(r-1)} \in \Omega), \end{cases} \quad (5.2)$$

where Ω is a sliding sector and $A > 0$ is sufficient large constant such that the sliding sector becomes an invariant. The following A makes Ω be an invariant under the control input for system (5.1):

$$A \geq \frac{\mu + C}{K_m}.$$

Theorem 5.1

The following area Ω obtained from Theorem 4.2 is a sliding sector:

$$\Omega(\sigma, \dots, \sigma^{(r-2)}) := \{\sigma^{(r-1)} \in \mathbb{R} \mid \sigma^{(r-1)} \in \Omega_{r-1}(\sigma, \dots, \sigma^{(r-1)})\}.$$

Proof

Theorem 4.2 says that there exist some positive constant μ_r that makes the set $\sigma^{(r-1)} \in \Omega_{r-1}(\sigma, \dots, \sigma^{(r-1)})$ be an invariant set for the system (5.1), and that the subsystem $\sigma^{(r-1)} \in \Omega_{r-1}(\sigma, \dots, \sigma^{(r-1)})$ is finite-time stable. Moreover, Ω defined in the statement is well-defined because each boundary of Ω_{r-1} , that is, $\sigma^{(r-1)} = \pi_{r-1}^+$ and $\sigma^{(r-1)} = \pi_{r-1}^-$, has a unique solution for fixed $(\sigma, \dots, \sigma^{(r-2)})$. Therefore, Ω is a sliding sector for the system (5.1).

The input can be freely chosen by the controller designer as long as the system remains on the area $\sigma^{(r-1)} \in \Omega$, because there is no requirement for input in the case of $\sigma^{(r-1)} \in \Omega$ from definition of the sliding sector Ω . In this chapter, we propose the some controller based on the nonlinear sliding sector Ω_{r-1} :

$$\Omega_{r-1} = \{\sigma^{(r-1)} \mid \phi_{r-1} - c_{r-1}N_{r-1}^{\frac{1}{r}} \leq \sigma^{(r-1)} \leq \phi_{r-1} + c_{r-1}N_{r-1}^{\frac{1}{r}}\}.$$

The above nonlinear sliding sector Ω is a sliding hypersurface if $c_{r-1} = 0$:

$$\sigma^{(r-1)} = \phi_{r-1}.$$

If $c_{r-1} > 0$, the sliding sector has width $2c_{r-1}N_{r-1}^{\frac{1}{r}} > 0$ except at the origin of the system (5.1). Therefore, the control law (5.2) with $c_{r-1} > 0$ can reduce the chattering phenomenon except at $N_{r-1} = 0$, that is, the origin. How to choose a $u \in [-A, A]$ where $\sigma^{(r-1)} \in \Omega$ depends on the target system. Note that the discontinuity of the controller cannot avoid because the origin of the system (5.1) is not stable under $u = 0$ due to the disturbance.

5.2 Controller design

5.2.1 Binary input controller

In this subsection, we consider the system (5.1) with binary input $u \in \{-A, A\}$ for some $A > 0$.

Corollary 5.1

The system (5.1) is finite time stable under the following control law:

$$u \begin{cases} = -A & (\sigma^{(r-1)} > v, \quad \forall v \in \Omega) \\ = A & (\sigma^{(r-1)} < v, \quad \forall v \in \Omega) \\ \in [-A, A] & (\sigma^{(r-1)} \in \Omega), \end{cases} \quad (5.3)$$

where Ω is a sliding sector and $A > 0$ is sufficient large constant such that the sliding sector becomes an invariant. By defining the output

$$y = x_r - \underset{v \in \Omega(x')}{\operatorname{argmin}} |x_r - v|,$$

we can rewrite the above control law as

$$u = \text{sgn}(y).$$

Note that $\text{sgn}(0) \in [-1, 1]$. How to choose a $u \in \{-A, A\}$ where $x_r \in \Omega$ depends on the target system.

The sliding sector $\Omega_{r-1} = \{\sigma^{(r-1)} \mid \sigma^{(r-1)} = \phi_{r-1}\}$ obtained from Theorem 4.2 with $c_{r-1} = 0$ becomes a sliding hypersurface $\sigma^{(r-1)} = \phi_{r-1}$. The sliding hypersurface $\sigma^{(r-1)} = \phi_{r-1}$ is an invariant of the system.

Corollary 5.2

The following controller with proper parameters establishes $\sigma \equiv 0$ in finite time:

$$u = \alpha(\sigma, \dots, \sigma^{(r-1)}) = -A \text{sign} z_{r-1}, \quad (5.4)$$

where $\text{sign} z$ is defined as

$$\text{sign} z := \begin{cases} 1 & (z \geq 0) \\ -1 & (z < 0). \end{cases}$$

The above controller is the simplest one and is the same as the traditional sliding-mode controller with the output function z_{r-1} . In this study, we regard the control law (5.4) as the special case of Corollary 5.1, because the control law (5.3) is equivalent to (5.4) given by Corollary 5.2 if we choose $c_{r-1} = 0$. Then, $\sigma^{(r-1)} = \phi_{r-1}$ becomes a proper sliding hypersurface, and the system (5.1) reaches and slides on the hypersurface. Note that Corollary 5.2 requires infinite frequency switching that causes chattering problem on the switching surface $x_r = \phi_{r-1}$ as with the traditional sliding-mode control. Although the sliding hypersurface $\sigma^{(r-1)} = \phi_{r-1}$ is not expressed in an explicit form, there are no obstacles to the implementation of the control law because $z_{r-1} = \sigma^{(r-1)} - \phi_{r-1}$ is well-defined for given $(\sigma, \dots, \sigma^{(r-1)})$.

5.2.2 Binary and ternary input controller with hysteresis

Corollary 5.3

The following controller with proper parameters establishes $\sigma \equiv 0$ in finite time:

$$u = \alpha(\sigma, \dots, \sigma^{(r-1)}) = \begin{cases} -A \text{sgn}(z_{r-1}) & (|z_{r-1}| \geq c_{r-1} N_{r-1}^{\frac{1}{r}}) \\ u_t & (\text{otherwise}), \end{cases}$$

where u_t is the previous input at the time $\lim_{\delta \rightarrow +0} t - \delta$.

If the actuator of the control system can provide only $\pm A$, then the above controller is helpful for reducing the chattering problem. The controller in Corollary 5.3 is an extension of Corollary 5.2 with respect to the hysteresis for the input switching. Theorem 4.2 requires no condition for the input as long as the system remains on the sliding sector

Ω . Therefore, there is no need to switch the input on the hypersurface, and the Theorem 4.2 admits the time dependency of the controller such as a hysteresis mechanism in Corollary 5.3. Such a hysteresis can eliminate the chattering phenomenon except at the origin.

If we can use three-valued control input $u \in \{-A, 0, A\}$, the next corollary is useful.

Corollary 5.4

The following controller with proper parameters establishes $\sigma \equiv 0$ in finite time:

$$u = \alpha(\sigma, \dots, \sigma^{(r-1)}) = \begin{cases} -A \operatorname{sgn}(z_{r-1}) & (|z_{r-1}| \geq c_{r-1} N_{r-1}^{\frac{1}{r}}) \\ 0 & (z_{r-1} = 0) \\ u_t & (\text{otherwise}), \end{cases}$$

where u_t is the previous input at the time $\lim_{\delta \rightarrow +0} t - \delta$.

The above says that the following sequence is effective to reduce the switching of the control input:

1. employ current control input as long as the system does not satisfy the below conditions,
2. employ $u = 0$ if the system reaches on ϕ_{r-1} ,
3. employ $u = -A \operatorname{sgn}(x_r - \phi_{r-1})$ if the system reaches on the boundaries of Ω .

As long as the system remains on the sliding sector, the finite time stability of the system does not depend on the control law because the finite stability is guaranteed by the definition of the sliding sector, which is says that the subsystem $\sigma^{(r-1)} \in \Omega$ is finite time stable. The controller can keep $u = 0$ for nonzero time from $\sigma^{(r-1)} = \phi_{r-1}$ to the boundary of Ω except at the origin.

The input discontinuity at the origin cannot be removed because the origin may not be an equilibrium point for (5.1) with $u = 0$, where the discontinuity cancels the disturbance and uncertainty for the system. This discontinuity is common for the above three controllers.

5.2.3 Quasi-continuous controller

Corollary 5.5

If $x_{r-1} > 0$, the following controller with proper parameters establishes $\sigma \equiv 0$ in finite time:

$$\begin{aligned} u = \alpha(\sigma, \dots, \sigma^{(r-1)}) &= -A \frac{z_{r-1}}{c_{r-1} N_{r-1}^{\frac{1}{r}}} \\ &= -\frac{A}{c_{r-1} N_{r-1}^{\frac{1}{r}}} \left(\sigma^{(r-1)} + \frac{a_{r-1}}{N_{r-1}^{\frac{1}{r}}} \sigma^{(r-2)} + \dots + \frac{a_{r-1} \dots a_1}{N_{r-1}^{\frac{1}{r}} \dots N_1^{\frac{1}{r}}} \sigma \right), \end{aligned}$$

where A satisfies the inequality $A \geq (\mu_r + C)/K_m$.

When $z_{r-1} = c_{r-1}N_{r-1}^{\frac{1}{r}}$, u becomes A , because z_{r-1} is the distance between $\sigma^{(r-1)}$ and ϕ_{r-1} . The control input given by the above control law is smaller than A when $|z_{r-1}| < c_{r-1}N_{r-1}^{\frac{1}{r}}$, and greater than A when $|z_{r-1}| > c_{r-1}N_{r-1}^{\frac{1}{r}}$. On the hypersurface $\sigma^{(r-1)} = \phi_{r-1}$, the input for the system is zero. These properties guarantee that the condition (5.2) holds. This controller is similar to the quasi-continuous sliding-mode controller summarized in Section 4.2 in that the input for the system changes smoothly almost everywhere except at the origin.

Because the quasi-continuous sliding-mode controller proposed by Levant et al. does not provide the explicit conditions for the stability, very large control input for the system is required at a distance from the hypersurface. The proposed higher-order sliding-mode controller guarantees the finite-time stability of the system as long as the system remains on the sliding sector. Moreover, we can obtain the maximum control input required for making the system reach and remain on the sliding sector. For example, we can replace the controller the following controller for the input constraint system:

$$u = \alpha(\sigma, \dots, \sigma^{(r-1)}) = \begin{cases} -A \operatorname{sgn}(z_{r-1}) & (|z_{r-1}| \geq c_{r-1}N_{r-1}^{\frac{1}{r}}) \\ -A \frac{z_{r-1}}{c_{r-1}N_{r-1}^{\frac{1}{r}}} & (\text{otherwise}). \end{cases}$$

5.2.4 Chattering rejection at the origin

Introducing the lower limit for N_{r-1} as below, we can reduce the chattering on the origin in exchange for the complete performance in the vicinity of the origin given by $\{(\sigma, \dots, \sigma^{(r-1)})^\top \in \mathbb{R}^r \mid N_{r-1}(\sigma, \dots, \sigma^{(r-1)}) < N_{\min}\}$:

$$u = \begin{cases} -A \frac{z_{r-1}}{c_{r-1}N_{r-1}^{\frac{1}{p}}} & (N_{r-1} \geq N_{\min}) \\ -A \frac{z_{r-1}}{c_{r-1}N_{\min}^{\frac{1}{p}}}, & (N_{r-1} < N_{\min}) \end{cases}$$

or

$$u \begin{cases} = -A \operatorname{sgn}(z_{r-1}) & (|z_{r-1}| \geq c_{r-1}N_{\min}^{\frac{1}{r}}) \\ \in \{-A, 0, A\}, & (|z_{r-1}| < c_{r-1}N_{\min}^{\frac{1}{r}}) \end{cases}$$

where $N_{\min} > 0$ is a constant, which is designed from the control interval and the magnitude of $\sigma^{(r-1)}$.

5.3 Parameter design

5.3.1 For the first-order systems

The sliding sector for the first-order system is determined by

$$\Omega_0(\sigma) = \{\dot{\sigma} \in \mathbb{R} \mid |\dot{\sigma} - \phi_0| \leq c_0 N_0^{\frac{1}{p}}\} = \{0\},$$

where $0 < c_0 < 1$ and $N_0 = |\sigma|^p$ with $p \geq 1$. Then, the control law for the first-order system is same as the traditional sliding mode controller:

$$u = -\text{Asgn}(\sigma - \phi_0) = -\text{Asgn}(\sigma),$$

where $A > C/K_m$. If we use the lower limit for N_0 then the controller is same as the traditional sliding mode controller with saturation function:

$$u = \begin{cases} -A \frac{\sigma}{N_0^{\frac{1}{p}}} & (N_0 \geq N_{min}) \\ -A \frac{\sigma}{N_{min}^{\frac{1}{p}}} & (N_0 < N_{min}), \end{cases}$$

$$= \begin{cases} -\text{Asgn}(\sigma) & (|\sigma| \geq N_{min}^{\frac{1}{p}}) \\ -A \frac{\sigma}{N_{min}^{\frac{1}{p}}} & (\text{otherwise}), \end{cases}$$

5.3.2 For second-order systems

The sliding sector for the system with relative degree of 2 is given by:

$$\Omega_1 = \{\dot{\sigma} \mid \phi_1 - c_1 N_1^{\frac{1}{p}} \leq \dot{\sigma} \leq \phi_1 + c_1 N_1^{\frac{1}{p}}\}.$$

We can redefine Ω_1 by the following explicit form:

$$\Omega(\sigma) = \{\dot{\sigma} \in \mathbb{R} \mid (\dot{\sigma} + a^- \lfloor \sigma \rfloor^{\frac{1}{2}})(\dot{\sigma} + a^+ \lfloor \sigma \rfloor^{\frac{1}{2}}) \leq 0\},$$

where $a^- < a^+$ are some positive constants determined from a_1 , b_1 and c_1 of the definition of Ω_1 . Both a^- and a^+ can be solved numerically from the following relation:

$$\dot{\sigma} = -a_1 \frac{\sigma}{N_1(\sigma, \dot{\sigma})^{\frac{1}{p}}} + c_1 N_1(\sigma, \dot{\sigma})^{\frac{1}{p}} = \begin{cases} -a^- \lfloor \sigma \rfloor^{\frac{1}{2}} & (\sigma \geq 0) \\ -a^+ \lfloor \sigma \rfloor^{\frac{1}{2}} & (\sigma < 0), \end{cases}$$

$$\dot{\sigma} = -a_1 \frac{\sigma}{N_1(\sigma, \dot{\sigma})^{\frac{1}{p}}} - c_1 N_1(\sigma, \dot{\sigma})^{\frac{1}{p}} = \begin{cases} -a^+ \lfloor \sigma \rfloor^{\frac{1}{2}} & (\sigma \geq 0) \\ -a^- \lfloor \sigma \rfloor^{\frac{1}{2}} & (\sigma < 0). \end{cases}$$

We have proposed such a sliding sector and obtained the homogeneous Lyapunov function for unperturbed system [97].

5.3.3 For higher-order systems

A set of parameters stabilizing the system can be numerically calculated. We can construct the stable parameters by the procedures below:

1. from Lemma 4.3, (a_1, b_1, c_1) is freely chosen;
2. repeating 3) and 4), for every $i = 2, \dots, r-1$, proper parameter (a_i, b_i, c_i) is determined from $(a_i, b_1, c_1), \dots, (a_{i-1}, b_{i-1}, c_{i-1})$;

3. from Lemma 4.5, for a given $(a_1, b_1, c_1), \dots, (a_{i-1}, b_{i-1}, c_{i-1})$, some positive constant α_{i-1} can be calculated by a computer;
4. from $A_i = \alpha_{i-1}/(1 - c_{i-1})$ and Lemma 4.7, proper parameter (a_i, b_i, c_i) can be determined;
5. and finally, μ_r can be replaced by $A_r = \alpha_{r-1}/(1 - c_{r-1})$.

5.3.4 Controller adjustment

Once a set of proper parameters is obtained, we can get another set of parameters from its homogeneity [11]. Consider the time transformation $t = \lambda\tau$. The system (5.1) on the new time-axis τ is given by

$$\frac{\sigma^{(r)}}{\lambda^r} \in [-C, C] + [K_m, K_M]\alpha(T_\lambda \Sigma_r),$$

where $\Sigma_r := (\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)})^\top$ and

$$T_\lambda : (\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)})^\top \mapsto (\sigma, \frac{\dot{\sigma}}{\lambda}, \dots, \frac{\sigma^{(r-1)}}{\lambda^{r-1}})^\top$$

is the time-axis transformation. The time-axis transformation does not change the stability of the system. Moreover, there exists some set of parameters such that the identity

$$\alpha(\Sigma_r) \equiv \bar{\alpha}(T_\lambda \Sigma_r)$$

holds for any Σ_r and any $\lambda > 0$, where $\bar{\alpha}$ is a new controller with the new parameters $(\bar{a}_1, \bar{b}_1, \bar{c}_1), \dots, (\bar{a}_{r-1}, \bar{b}_{r-1}, \bar{c}_{r-1})$ and $\bar{\mu}_r$ that derived from proposition below.

Proposition 5.2 ([87])

Let a_i , b_i , and c_i be proper parameters, and μ_r be a constant in (4.8). For any $\lambda > 0$, the following set of parameters derived from a proper set of parameters is also proper:

$$\bar{a}_i = \frac{a_i}{\lambda^{\frac{r}{r-i}}}, \quad \bar{b}_i = \frac{b_i}{\lambda^{\frac{r}{r-i} \frac{p}{r-i+1}}}, \quad \bar{c}_i = c_i, \quad \bar{\mu}_r = \frac{\mu_r}{\lambda^r},$$

where $i = 1, \dots, r-1$.

By substituting both the new parameters $(\bar{a}_i, \bar{b}_i, \bar{c}_i)$ and the state $T_\lambda \Sigma_r$ into the inequalities in the lemmas in previous chapter, we obtain inequalities equivalent to the original inequalities. Because the inequalities are satisfied for any $\Sigma'_r = T_\lambda \Sigma_r$, they are also satisfied for any Σ_r . The redesigned parameters in the manner of Proposition 5.2 are proper if the original parameters are proper. Consequently, the controller in Theorem 4.2 with the new parameters of Proposition 5.2 establishes $\sigma \equiv 0$ in finite time.

We can use the new controller with

- $\lambda > 1$ if K_m of the system is not large enough to make the original sliding sector be an invariant set, and

- $\lambda < 1$ if the fast convergence time or high robustness is necessary.

The obtained controller with $\lambda = 1$ is equivalent to the original controller. For actual implementation, we should search appropriate $\lambda > 0$ suitable for the specification and the purpose of the target system such as K_m , C and the given input constraints.

Example 5.1

For the case with the input constraint $u \in [-1, 1]$, λ satisfying $\mu/\lambda^r < K_m - C$ is needed to cancel out the disturbance.

5.3.5 Sliding sector based on ILF method

The higher-order sliding-mode controller using the implicit Lyapunov function have been proposed by Polyakov et al. [9]. Original ILF method is only applicable to the system with continuous input. Kamal and Bandyopadhyay have proposed the higher-order sliding-mode controller for binary input [8], which is based on one of the implicit Lyapunov function called controllability function [90]. The controllability function is constructive but has few design parameters. Moreover, the proof is incomplete in the boundedness of the time derivative of a switching function. In this subsection, we design the sliding sector based on the ILF method with taking into account of a robustness and an input magnitude [98].

The control law obtained by using ILF expect that the control input is continuously changeable, and is not a set-valued. Let us consider the following $(r - 1)$ -dimensional subsystem of the r -dimensional system (4.6):

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r-2} \\ x_{r-1} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r-2} \\ x_{r-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} x_r + \begin{bmatrix} d_1(t, x) \\ d_2(t, x) \\ \vdots \\ d_{r-2}(t, x) \\ d_{r-1}(t, x) \end{bmatrix} \\ &= A'x' + B'x_r + d'(t, x). \end{aligned} \quad (5.5)$$

Then, we can design the binary input controller for the original system by regarding the obtained control input for the subsystem as a desired state of x_r .

The following propositions are both proven by ILF method (Theorem 4.1).

Proposition 5.3

Let x_r be a virtual input for the subsystem. By applying Theorem 4.1 for the subsystem (5.5) with $n = r - 1$, $\mu = 1/2$ and β such that

$$\beta V^{\frac{1}{2}} \geq \sup_{t \geq 0, x' \in \mathbb{R}^{r-1}} \|D_{r-1}(V^{-\frac{1}{2}})d'_v(t, x')\|,$$

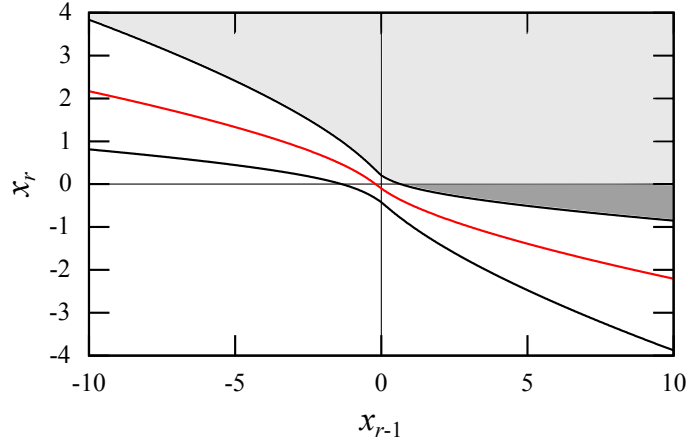


Figure 5.1: The curve $x_r = v_{r-1}(x')$ and the area $x_r \in \Omega(x')$ (in the case of $x' \neq 0$)

where $d'_v(t, x') := d'(t, x_1, \dots, x_{r-1}, v_{r-1}(x'))$. Then, the virtual control law with the finite time stability for the subsystem is obtained as

$$x_r = v_{r-1}(x') := V^{-\frac{1}{2}} K D_{r-1}(V^{-\frac{1}{2}}) x',$$

where K and V are given in Theorem 4.1.

Similar to the manner of Proposition 5.3, by considering the distance between x_r and $v_{r-1}(x')$ as disturbance for the subsystem (5.5), the following proposition can be obtained.

Proposition 5.4

Choosing β that satisfies

$$\beta V^{\frac{1}{2}} \geq \lambda V^{\frac{1}{2}} + \sup_{t \geq 0, x' \in \mathbb{R}^{r-1}} \|D_{r-1}(V^{-\frac{1}{2}}) d'_v(t, x')\|,$$

where $\lambda \geq 0$ is a design parameter and $d'_v(x')$ is defined in Proposition 5.3, then the following set-valued virtual control law establishes $x' = 0$ for the subsystem (5.5) in finite time:

$$x_r \in \Omega(x') := \left\{ v \mid |v - V^{-\frac{1}{2}} K D_{r-1}(V^{-\frac{1}{2}}) x'| \leq \lambda V^{\frac{1}{2}} \right\}.$$

The central solid line and the white area of Fig. 5.1 indicate the curve $x_r = v_{r-1}(x')$ and the area $x_r \in \Omega(x')$ for some V , respectively.

For simplicity, we define the set Ω_0 and a function $Q : \mathbb{R} \times \mathbb{R}^r \rightarrow \mathbb{R}$:

$$\Omega_0(x') := \{x_r \in \mathbb{R} \mid v_{r-1}(x') + \lambda V^{\frac{1}{2}} \leq x_r \leq 0\} \cup \{v_{r-1}(x') + \lambda V^{\frac{1}{2}}\},$$

$$Q(t, x) := \frac{d}{dt}(v_{r-1} + \lambda V^{\frac{1}{2}}),$$

where $Q(t, x)$ is the time derivative of the upper boundary of the area $\Omega(x')$ along the trajectory of the system (4.1). Note that the area $\Omega_0(x') \setminus \{v_{r-1}(x') + \lambda V^{\frac{1}{2}}\}$ is indicated by the dark gray area in Fig. 5.1.

Proposition 5.5

The area Ω obtained from Proposition 5.4 is a sliding sector, where μ is given by

$$\mu = \inf_{t \geq 0, x' \in \mathbb{R}^{r-1}, x_r \in \Omega_0(x')} |Q(t, x)|.$$

Note that the proof of the above proposition is given in Appendix C. The concept of the above proposition have been obtained in [87].

We consider the redesign of the virtual control law $v_{r-1}(x')$ for given K_m . The scale of the constant q cannot be considered in the virtual controller design based on ILF method. If we suppose that $d'(t, x) = 0$, we can redesign the parameters P and K of the controller because of its homogeneity as shown by [11]. Consider the time transformation $t = \lambda\tau$ and the time-axis transformation

$$T_\lambda : (x_1, x_2, \dots, x_{r-1})^\top \mapsto (x_1, \frac{x_2}{\lambda}, \dots, \frac{x_{r-1}}{\lambda^{r-2}})^\top.$$

The change of the time-axis does not change the stability of the system. Moreover, there exists some virtual controller $\bar{v}_{r-1}$ such that the identity $\lambda^{r-1}\bar{v}_{r-1}(T_\lambda(x')) = v_{r-1}(x')$ holds for any x' and any $\lambda > 0$, where

$$\begin{aligned} \bar{v}_{r-1}(x') &= \bar{V}(x')^{-\frac{1}{2}} \bar{K} D_{r-1}(\bar{V}(x')^{-\frac{1}{2}})x', \\ \bar{V}(x') &= \sqrt{x'^\top D_{r-1}(\bar{V}(x')^{-\frac{1}{2}}) \bar{P} D_{r-1}(\bar{V}(x')^{-\frac{1}{2}})x'}, \\ \bar{P}_{ij} &= \lambda^{i+j-2} P_{ij}, \\ \bar{K}_i &= \lambda^{-(r-i+1)} K_i, \end{aligned}$$

where $\bar{P}_{ij}$, P_{ij} , $\bar{K}_i$ and K_i are the (i, j) -th component of $\bar{P}$ and P and the i -th component of $\bar{K}$ and K , respectively. Note that $\bar{V}(T_\lambda(x')) \equiv V(x')$ for any x' .

5.4 Numerical simulations

5.4.1 Fourth-order system

In this subsection, we show some simulation results. We consider the following nonlinear system with relative degree $r = 4$:

$$\begin{aligned} \sigma^{(4)} &= a(t) + b(t)u, \\ a(t) &= 5(\sin(t) + \sin(10t)), \\ b(t) &= \left(1 + \frac{\sin(0.71t) + \sin(1.71t)}{4}\right). \end{aligned} \tag{5.6}$$

where $|a(t)| \leq C = 10$ and $b(t) \in [K_m, K_M] = [0.5, 1.5]$. Moreover, we consider the input constraint $|u| \leq A = 50$.

First, we design the parameters for the nonlinear sliding sector. The parameters are designed by the following procedures:

- $p = 2r = 8$,
- $(a_1, b_1, c_1) = (0.1, 0.1, 0.2)$,
- $\alpha_1 = 0.0601966$, which is calculated by using Mathematica,
- $(a_2, b_2, c_2) = (0.705984, 0.421697, 0.2)$, where $b_2 = a_1^{\frac{3}{8}}$ and a_2 is calculated by Lemma 4.7,
- $\alpha_2 = 1.15303$, which is calculated by using Mathematica,
- $(a_3, b_3, c_3) = (4.66333, 0.91664, 0.2)$, where $b_3 = a_2^{\frac{2}{8}}$ and a_3 is calculated by Lemma 4.7,
- $\alpha_3 = 24.0706$, which is calculated by using Mathematica,
- $\mu_4 = \alpha_3/(1 - c_3) = 30.0882$.

Next, we redesign the parameter based on Proposition 5.2 with

$$\lambda = \left(\frac{\mu_4}{K_m A - C} \right)^{\frac{1}{4}} = \left(\frac{\mu_4}{15} \right)^{\frac{1}{4}}.$$

In the below subsections, we show two different type of control laws that satisfy the requirement of the Theorem 4.2. Note that

$$\begin{aligned} N_3 &= |\sigma^{(3)}|^8 + b_3 N_2 \\ &= |\sigma^{(3)}|^8 + b_3 |\ddot{\sigma}|^4 + b_3 b_2 |\dot{\sigma}|^{\frac{8}{3}} + b_3 b_2 b_1 |\sigma|^2, \\ \phi_3 &= -a_3 \frac{\ddot{\sigma} - \phi_2}{N_3^{\frac{1}{8}}} \\ &= -a_3 \frac{\ddot{\sigma}}{N_3^{\frac{1}{8}}} - a_3 a_2 \frac{\dot{\sigma}}{(N_3 N_2)^{\frac{1}{8}}} - a_3 a_2 a_1 \frac{\sigma}{(N_3 N_2 N_1)^{\frac{1}{8}}}, \\ \pi_3^+ &= \phi_3 + c_3 N_3^{\frac{1}{8}}, \\ \pi_3^- &= \phi_3 - c_3 N_3^{\frac{1}{8}}, \\ \Omega_3 &= \{\sigma^{(3)} \in \mathbb{R} \mid \pi_3^- \leq \sigma^{(3)} \leq \pi_3^+\}. \end{aligned}$$

Discontinuous controller

Consider the following controller:

$$\alpha \begin{cases} = -A & (\sigma^{(3)} > \pi_3^+) \\ \in \{-A, 0, A\} & (\sigma^{(3)} \in \Omega_3) \\ = A & (\sigma^{(3)} < \pi_3^-), \end{cases}$$

where $A = 50$ and we suppose that the input constraint $u \in \{-A, 0, A\}$. As long as the system remains on $\sigma^{(3)} \in \Omega_3$, Theorem 4.2 guarantees the finite time stability of the

system. Therefore, in this simulation, we introduce the hysteresis mechanism proposed in Corollary 5.4.

Fig. 5.2 shows the simulation result of the system (5.6) under the above controller. Fig. 5.2(c), Fig. 5.2(d) and Fig. 5.2(e) are time responses of the state $(\sigma, \dot{\sigma}, \ddot{\sigma}, \sigma^{(3)})^\top$, the input u and the positive definite function N_3 , respectively. Especially, Fig. 5.2(e) indicates the value of the positive definite function N_3 in semi-logarithmic scale, and shows the results for several control interval $T = 0.001$, $T = 0.0001$ and $T = 0.00001$. As shown in Fig. 5.2(e), N_3 converges to zero against for the disturbances, and the convergence limit is dependent on the control interval.

Continuous controller (except at the origin)

Consider the following controller:

$$\alpha = \begin{cases} -A & (\sigma^{(3)} > \pi_3^+) \\ -A \frac{\sigma^{(3)} - \phi_3}{c_3 N_3^{\frac{1}{p}}} & (\sigma^{(3)} \in \Omega_3) \\ A, & (\sigma^{(3)} < \pi_3^-) \end{cases}$$

where $A = 50$.

Fig. 5.3 shows the simulation result of the system (5.6) under the above controller. Fig. 5.3(c), Fig. 5.3(d) and Fig. 5.3(e) are time responses of the state $(\sigma, \dot{\sigma}, \ddot{\sigma}, \sigma^{(3)})^\top$, the input u and positive definite function N_3 , respectively. Fig. 5.3(e) indicates the value of the positive definite function N_3 in semi-logarithmic scale, and shows the results for several control interval $T = 0.001$, $T = 0.0001$ and $T = 0.00001$. Fig. 5.3(e) shows similar results to Fig. 5.2(e). As shown in Fig. 5.3(d), the control input is chattering at the origin (after 55 [s]). In such a case, the following mechanism reduces the chattering phenomenon around the origin:

$$\alpha' = \begin{cases} -A & (\sigma^{(3)} > \phi_3 + c_3 \mathcal{N}_3^{\frac{1}{8}}) \\ -A \frac{\sigma^{(3)} - \phi_3}{c_3 \mathcal{N}_3^{\frac{1}{p}}} & (\phi_3 - c_3 \mathcal{N}_3^{\frac{1}{8}} \leq \sigma^{(3)} \leq \phi_3 + c_3 \mathcal{N}_3^{\frac{1}{8}}) \\ A, & (\sigma^{(3)} < \phi_3 - c_3 \mathcal{N}_3^{\frac{1}{8}}) \end{cases}$$

where $\mathcal{N}_3 := \max\{N_3, N_{min}\}$ and the small positive constant N_{min} should be appropriately chosen from the input magnitude and the control interval. Strictly speaking, the above control law does not finite time stable around the origin. Fig. 5.4 shows the results, and we can confirm that the chattering phenomenon around the origin is completely avoided as shown in Fig. 5.4(c).

5.4.2 Cart-beam system

In this subsection, we consider the cart-beam system have studied by Sankaranarayanan et al. in [99], as shown in Fig. 5.5. In Fig. 5.5, q_1 is the beam angle, q_2

is the cart position of from the center of the beam, u is the force applied to the cart, l_c is the position of the center-of-gravity of the beam, m is mass of the beam, and M is mass of the cart. Let J be the moment of inertia of the beam about the pivot. The state equation (see [99]) is given by

$$\dot{x} = f(x) + g(x)u,$$

where $x = [x_1 \ x_2 \ x_3 \ x_4]^\top = [q_1 \ q_2 + ml_c/M \ \dot{q}_1 \ \dot{q}_2]^\top$,

$$f(x) = \begin{bmatrix} x_3 \\ x_4 \\ -\frac{2Mx_3x_4\alpha(x_2)+Mgx_2\cos x_1}{\beta(x_2)} \\ x_3^2\alpha(x_2) - g\sin x_1 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\alpha(x_2) = x_2 - \frac{ml_c}{M},$$

$$\beta(x_2) = J + M\alpha^2(x_2).$$

The output function $S(x)$ with relative degree 3 have obtained by Muralidharan et al. in [59] as

$$\sigma(x) := x_1 + x_3M\beta(x_2).$$

Then, the output system is obtained by

$$\sigma^{(3)} = a(x) + b(x)u,$$

where

$$\begin{aligned} a(x) = & x_4 \left(\frac{-2Mx_3x_4 - Mgx_2\cos x_1}{\beta(x_2)} - \frac{2M\alpha(x_2)x_3\dot{x}_2}{\beta(x_2)} + M^2gx_3\sin x_1 \right) \\ & + x_3 \left(M^2g(x_4 + x_2x_3\sin x_1\cos x_1) + \frac{Mgx_2\sin x_1}{\beta(x_2)} \right) \\ & + \dot{x}_3 \left(gM^2x_2\sin x_1 - \frac{2M\alpha(x_2)x_4}{\beta(x_2)} \right) \\ & + \left(-\frac{2M\alpha(x_2)x_3}{\beta(x_2)} - M^2g\cos x_1 \right) (\alpha(x_2)x_3^2 - g\sin x_1) \\ b(x) = & -\frac{2M\alpha(x_2)x_3}{\beta(x_2)} - M^2g\cos x_1 < 0. \end{aligned}$$

We use the following continuous controller with saturation:

$$u = -\frac{a(x) + v(x)}{b(x)},$$

$$v = \begin{cases} A, & (z_2 > c_2\mathcal{N}_2^{\frac{1}{p}}) \\ -A, & (z_2 < -c_2\mathcal{N}_2^{\frac{1}{p}}) \\ -A \frac{\ddot{\sigma} - \phi_3}{c_2\mathcal{N}_2^{\frac{1}{p}}}, & (\text{otherwise}) \end{cases},$$

where $\mathcal{N}_2 = \max\{N_2, 2.44 \times 10^{-10}\}$.

Fig. 5.6 shows that the simulation result of the cart-beam system under the disturbance such that

$$\dot{x} = \begin{cases} f(x) + g(x)u + [0 & 0 & 0 & +0.5]^\top & (t \in [0, 1) \cup [2, 3) \cup [4, 5) \cup \dots) \\ f(x) + g(x)u + [0 & 0 & 0 & -0.5]^\top & (\text{otherwise}) \end{cases}$$

with the following parameters; $M = 1$, $m = 2$, $J = 2$, $l_c = 0.1$, $g = 9.81$ and the initial state is $x(0) = [\pi/8 \quad -0.3 \quad -0.1 \quad 0.1]^\top$. In the simulation, we choose the parameters of the controller as $p = 6$, $a_1 = 0.25$, $b_1 = 0.0625$, $c_1 = 0.1$, $a_2 = 0.5625$, $b_2 = 0.000244$, $c_2 = 0.1$, $A = 5.0$. Fig. 5.6(a) is the time responses of each state x_1 , x_2 , x_3 and x_4 . Fig. 5.6(b) indicates the time responses of the control input and the disturbance term. As the figure illustrates, the system is stabilized at the origin against for the disturbance.

5.5 Summary

We define the sliding sector based on the higher-order sliding-mode control. Such flexibility allows us to design controllers with various properties. Using the sliding sector, we introduce various types of the higher-order sliding-mode controllers. We proposed three sliding-mode controllers;

- a controller that uses the signum function same as the traditional sliding-mode control (Corollary 5.2),
- a controller with a hysteresis mechanism for switching of the control input (Corollary 5.3 and Corollary 5.4), and
- a controller that can provide continuous input except at the origin (Corollary 5.5).

The stability of the system under the controller can be confirmed theoretically by some given inequalities. Using the homogeneous property of the controller, we can redesign proper parameters to adjust for the specifications of a target system, such as the magnitude of a disturbance and the input constraints of the system. Moreover, setting the lower limit for the positive definite function, we can introduce the saturation function for the higher-order sliding-mode control as with the traditional sliding-mode control. We verified the efficiency of the proposed control strategy for higher-order sliding-mode control through simulations. The parameters for the controllers used in the simulations are determined by the proposed design method.

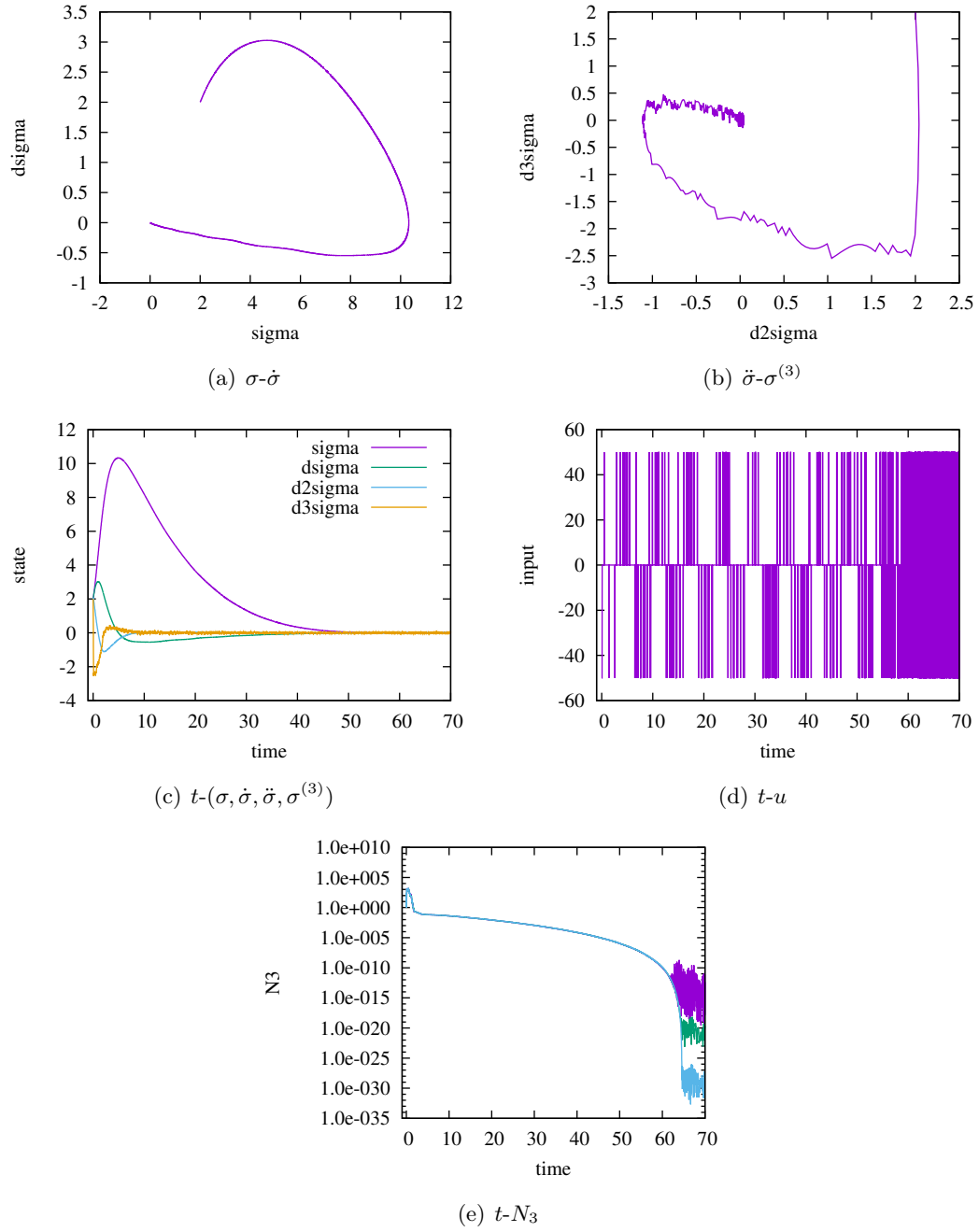


Figure 5.2: Simulation result (discontinuous controller)

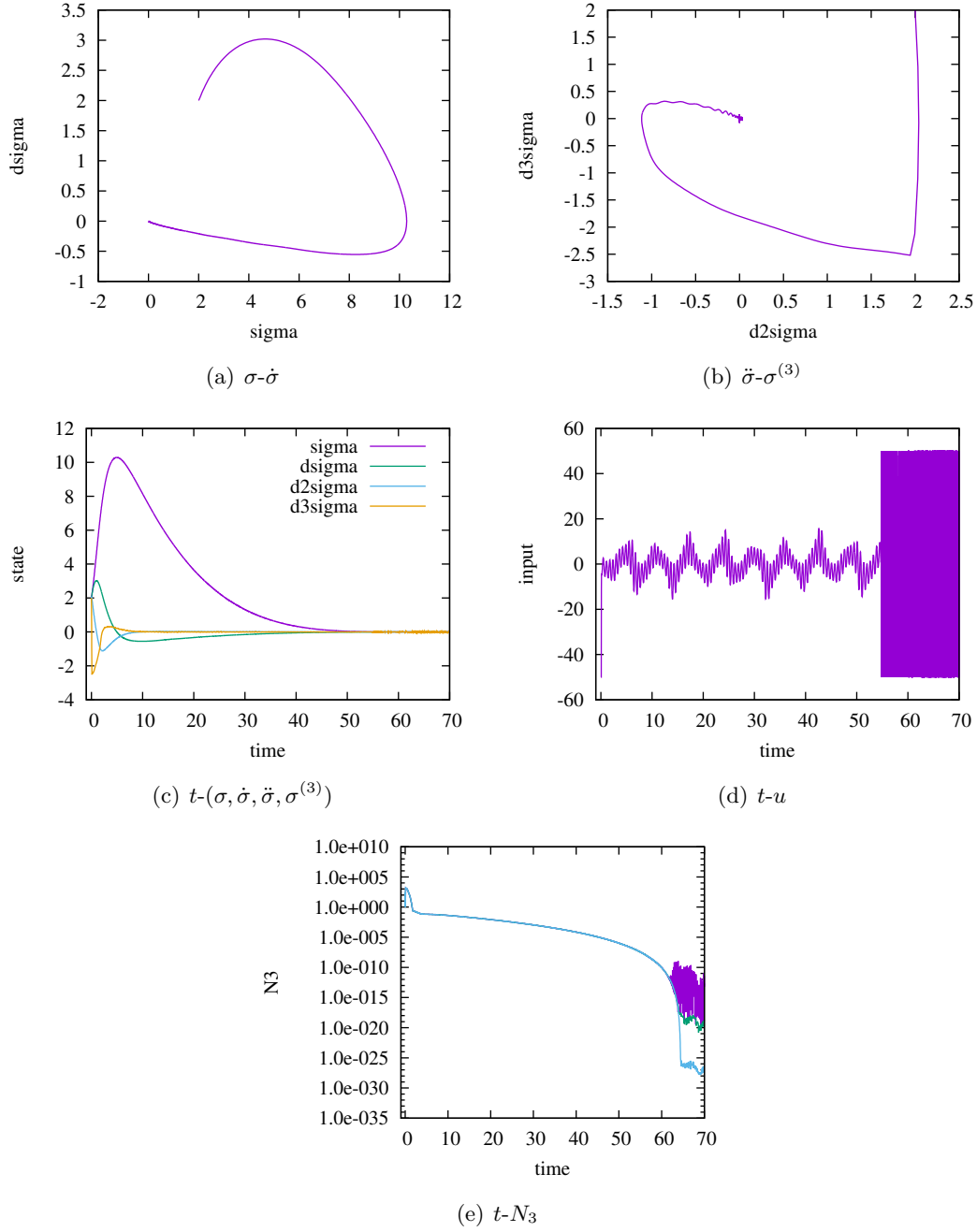
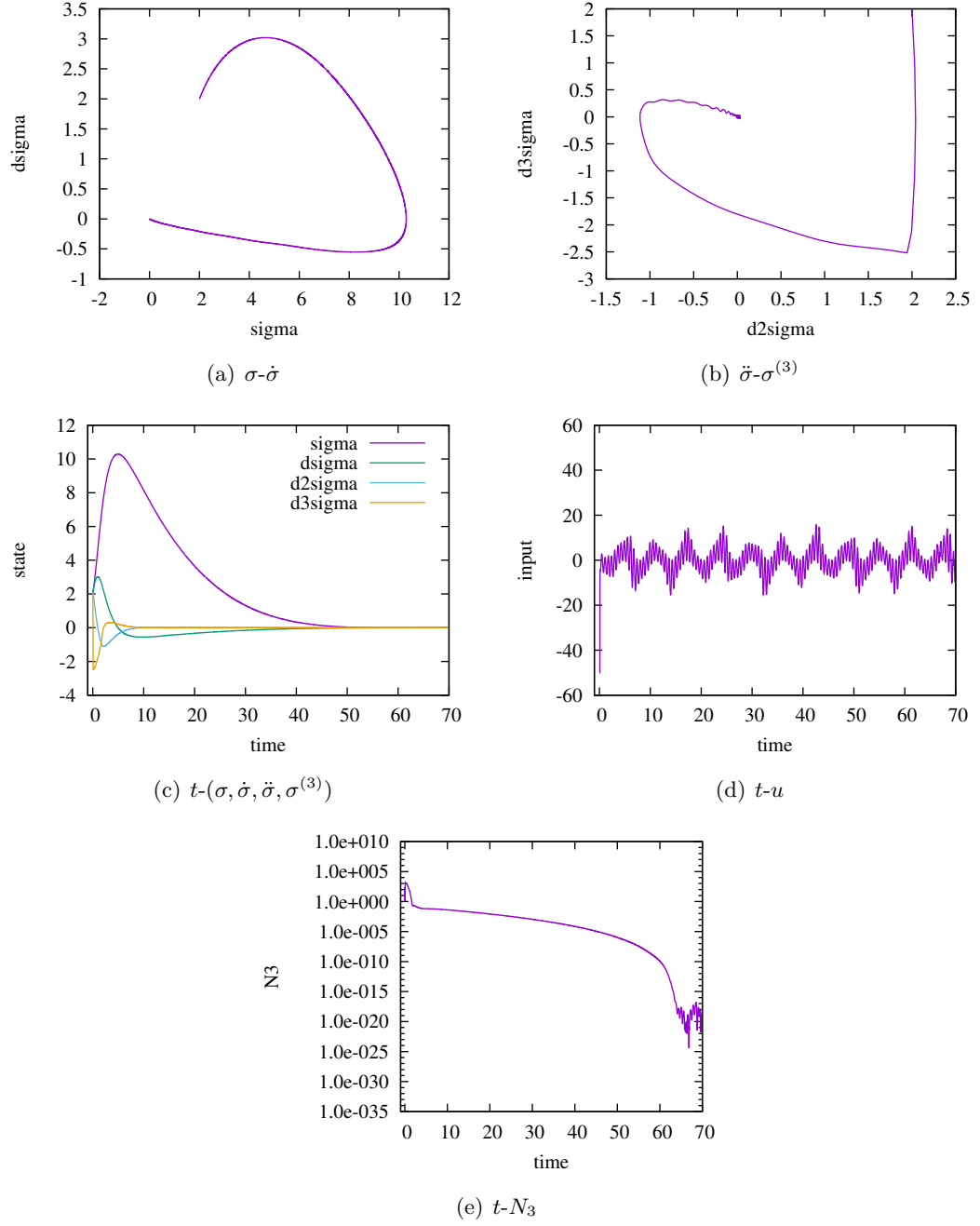


Figure 5.3: Simulation result (continuous controller)

Figure 5.4: Simulation result (continuous controller with $N_{min} = 10^{-3}$)

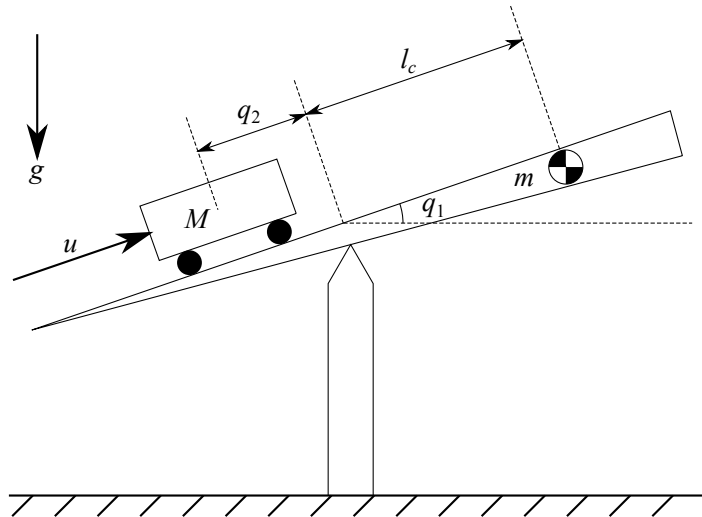


Figure 5.5: Cart-beam system model

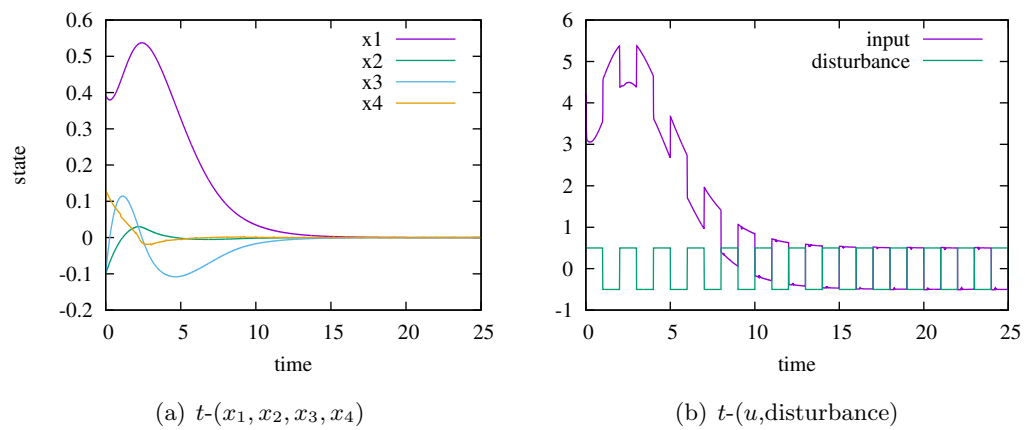


Figure 5.6: Simulation result (Cart-beam system)

Chapter 6. Conclusion

This thesis proposed the hysteresis mechanism based on the CLF, and the higher-order sliding-mode controllers based sliding sector.

The control of the four-wheeled vehicle on noncontractible space has a number of difficulties. This thesis solved the difficulties by designing the control law based on the control Lyapunov function. The control law obtained from the CLF leads the vehicle to the goal point avoiding all obstacles on the control space. We proposed the design method of a local strict navigation function via a referential trajectory, and extended the strict navigation function to obtain a local strict CLF for a four-wheeled vehicle, because the nonholonomicity and the steering limitation are not considered in the navigation function. A global strict CLF for the four-wheeled vehicle was obtained by synthesizing a number of local strict CLFs; each local strict CLF is an extension of a local strict navigation function. Moreover, in this thesis, the proposed hysteresis mechanism based on a CLF solved the chattering phenomenon occurred by the discontinuity on the switching from forward movement to backward movement. With the hysteresis mechanism, the four-wheeled vehicle achieved a natural n -point turn automatically by the mathematical approach and does not involve any (meta-)heuristic methods or rule-based controls. By the proposed control law, the vehicle reached the goal point from various initial points, as can be seen in the experimental results.

This thesis proposed a novel higher-order sliding-mode controller and proved the existence of the stable parameters for the proposed controller. The stability under the proposed controller is guaranteed by simple conditions: the control input for the system must have an absolute value larger than some positive constant with an appropriate sign as long as the state of the system stays outside the sliding sector. On the other hand, there are no requirements for the input as long as the state of the system stays in the sliding sector. Such flexibility allows us to design controllers with various properties. We designed three sliding-mode controllers based on the sliding sector; a controller that can provide continuous input except at the origin, a controller that uses the signum function same as the traditional sliding-mode control, and a controller with a hysteresis mechanism for switching of the control input. Moreover, we provided the constructive parameter design method based on the proven way. Using the homogeneous property of the controller, we can redesign proper parameters to adjust for the specifications of a target system, such as the magnitude of a disturbance and the input constraints of the system. We verified the effectiveness of the proposed control strategy for higher-order sliding-mode control through simulations. The parameters for the controllers used in the

simulations were determined by the proposed design method. We proposed a general strategy for designing higher-order sliding-mode controllers that has finite-time stability and the robustness against for the disturbances and the uncertainties for the system.

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Appendix A On calculation of desired heading angle

A.1 Desired heading angle θ_{des}

The desired heading angle θ_{des} is defined by the direction of the gradient of the navigation function. We denote $\mathbf{B}_i(q) = (B_x(q), B_y(q))^T$. First, differentiating (3.7) and the square of (3.8), we obtain

$$2rdr = 2 \{ (x - B_x)(dx - B_x'dq) + (y - B_y)(dy - B_y'dq) \},$$

$$B_x''(x - B_x)dq + B_y''(y - B_y)dq + B_x'(dx - B_x'dq) + B_y'(dy - B_y'dq) = 0$$

where $\mathbf{B}_i(q) = (B_x(q), B_y(q))^T$, $B'_x = dB_x/dq$, $B'_y = dB_y/dq$, $B''_x = d^2B_x/dq^2$ and $B''_y = d^2B_y/dq^2$. From the above equations, we get dr and dq as

$$dr = \frac{x - B_x}{r} (dx - B_x'dq) + \frac{y - B_y}{r} (dy - B_y'dq), \quad (\text{A.1})$$

$$dq = -\frac{B_x'dx + B_y'dy}{B_x''(x - B_x(q)) - B_x'^2 + B_y''(y - B_y(q)) - B_y'^2}. \quad (\text{A.2})$$

From the definition of r , with no division by zero, $(x - B_x)/r$ and $(y - B_y)/r$ in (A.1) are calculated as

$$\frac{x - B_x}{r} = \frac{B'_y}{\sqrt{B_x'^2 + B_y'^2}},$$

$$\frac{y - B_y}{r} = -\frac{B'_x}{\sqrt{B_x'^2 + B_y'^2}}.$$

From the above equations and (A.1), we have

$$dr = \frac{B'_y}{\sqrt{B_x'^2 + B_y'^2}} (dx - B_x'dq) - \frac{B'_x}{\sqrt{B_x'^2 + B_y'^2}} (dy - B_y'dq).$$

Substituting (A.2) and the above into

$$d\bar{V}_i = \frac{\partial \bar{V}_i}{\partial q} dq + \frac{\partial \bar{V}_i}{\partial r} dr,$$

we have

$$\begin{aligned}
\frac{\partial \bar{V}_i}{\partial x} &= \frac{\partial \beta}{\partial r} \left(\frac{B'_y(-B_x'^2/(B''_y y + B''_x x - B_y B''_y - B_y'^2 - B_x B''_x - B_x'^2) - 1)}{\sqrt{B_y'^2 + B_x'^2}} \right. \\
&\quad \left. + \frac{B_x'^2 B'_y}{\sqrt{B_y'^2 + B_x'^2}(B''_y y + B''_x x - B_y B''_y - B_y'^2 - B_x B''_x - B_x'^2)} \right) \\
&\quad - \frac{B'_x \frac{\partial \beta}{\partial s}}{B''_y y + B''_x x - B_y B''_y - B_y'^2 - B_x B''_x - B_x'^2} \\
&\quad - \frac{B'_x \frac{\partial \alpha}{\partial s}}{B''_y y + B''_x x - B_y B''_y - B_y'^2 - B_x B''_x - B_x'^2} \\
&= -\frac{\partial \beta}{\partial r} \frac{B'_y}{\sqrt{B_y'^2 + B_x'^2}} - \frac{B'_x \frac{\partial \alpha}{\partial s} + B'_x \frac{\partial \beta}{\partial s}}{B''_y y + B''_x x - B_y B''_y - B_y'^2 - B_x B''_x - B_x'^2} \\
\frac{\partial \bar{V}_i}{\partial y} &= -\frac{\partial \beta}{\partial r} \left(\frac{B'_x(-B_y'^2/(B''_y y + B''_x x - B_y B''_y - B_y'^2 - B_x B''_x - B_x'^2) - 1)}{\sqrt{B_y'^2 + B_x'^2}} \right. \\
&\quad \left. + \frac{B'_x B_y'^2}{\sqrt{B_y'^2 + B_x'^2}(B''_y y + B''_x x - B_y B''_y - B_y'^2 - B_x B''_x - B_x'^2)} \right) \\
&\quad - \frac{B'_y \frac{\partial \beta}{\partial s}}{B''_y y + B''_x x - B_y B''_y - B_y'^2 - B_x B''_x - B_x'^2} \\
&\quad - \frac{B'_y \frac{\partial \alpha}{\partial s}}{B''_y y + B''_x x - B_y B''_y - B_y'^2 - B_x B''_x - B_x'^2} \\
&= \frac{\partial \beta}{\partial r} \frac{B'_x}{\sqrt{B_y'^2 + B_x'^2}} - \frac{B'_y \frac{\partial \alpha}{\partial s} + B'_y \frac{\partial \beta}{\partial s}}{B''_y y + B''_x x - B_y B''_y - B_y'^2 - B_x B''_x - B_x'^2}
\end{aligned}$$

A.2 Time derivative of θ_{des}

The desired heading angle θ_{des} is determined from the subgradient of the navigation function, the calculation of $\dot{\theta}_{des}$ requires the time derivatives of $\frac{\partial V_i}{\partial x}$ and $\frac{\partial V_i}{\partial y}$. The time

derivatives of $\frac{\partial V_i}{\partial x}$ and $\frac{\partial V_i}{\partial y}$ are obtained by

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\partial \bar{V}_i}{\partial x} \right) &= \frac{B'_x \left(\frac{\partial \beta}{\partial s} + \frac{\partial \alpha}{\partial s} \right) (B''_y \dot{y} + B''_x \dot{x})}{(B''_y y + B''_x x - B_y B''_y - B'_y{}^2 - B_x B''_x - B'_x{}^2)^2} \\
&\quad - \left(\frac{B'_x \left(\frac{\partial \beta}{\partial s} + \frac{\partial \alpha}{\partial s} \right) (B'''_y y + B'''_x x - B_y B'''_y - 3B'_y B''_y - B_x B'''_x - 3B'_x B''_x)}{(B''_y y + B''_x x - B_y B''_y - B'_y{}^2 - B_x B''_x - B'_x{}^2)^2} \right. \\
&\quad - \frac{B'_x \frac{\partial^2 \beta}{\partial s^2} + B''_x \frac{\partial \beta}{\partial s} + B'_x \frac{\partial^2 \alpha}{\partial s^2} + B''_x \frac{\partial \alpha}{\partial s}}{B''_y y + B''_x x - B_y B''_y - B'_y{}^2 - B_x B''_x - B'_x{}^2} + \frac{B'_y (B'_y B''_y + B'_x B''_x) \frac{\partial \beta}{\partial r}}{(B'_y{}^2 + B'_x{}^2)^{3/2}} \\
&\quad \left. - \frac{B'_x \frac{\partial}{\partial s} \frac{\partial \beta}{\partial r} + B''_x \frac{\partial \beta}{\partial r}}{\sqrt{B'_y{}^2 + B'_x{}^2}} \right) \frac{B'_y \dot{y} + B'_x \dot{x}}{B''_y y + B''_x x - B_y B''_y - B'_y{}^2 - B_x B''_x - B'_x{}^2} \\
&\quad - \left(\frac{B'_x \frac{\partial}{\partial s} \frac{\partial \beta}{\partial r}}{B''_y y + B''_x x - B_y B''_y - B'_y{}^2 - B_x B''_x - B'_x{}^2} + \frac{B'_y \frac{\partial^2 \beta}{\partial r^2}}{\sqrt{B'_y{}^2 + B'_x{}^2}} \right) \frac{B'_x \dot{y} - B'_y \dot{x}}{\sqrt{B'_y{}^2 + B'_x{}^2}}, \\
\frac{d}{dt} \left(\frac{\partial \bar{V}_i}{\partial y} \right) &= \frac{B'_y \left(\frac{\partial \beta}{\partial s} + \frac{\partial \alpha}{\partial s} \right) (B''_y \dot{y} + B''_x \dot{x})}{(B''_y y + B''_x x - B_y B''_y - B'_y{}^2 - B_x B''_x - B'_x{}^2)^2} \\
&\quad - \left(\frac{B'_y \left(\frac{\partial \beta}{\partial s} + \frac{\partial \alpha}{\partial s} \right) (B'''_y y + B'''_x x - B_y B'''_y - 3B'_y B''_y - B_x B'''_x - 3B'_x B''_x)}{(B''_y y + B''_x x - B_y B''_y - B'_y{}^2 - B_x B''_x - B'_x{}^2)^2} \right. \\
&\quad - \frac{B'_y \frac{\partial^2 \beta}{\partial s^2} + B''_y \frac{\partial \beta}{\partial s} + B'_y \frac{\partial^2 \alpha}{\partial s^2} + B''_y \frac{\partial \alpha}{\partial s}}{B''_y y + B''_x x - B_y B''_y - B'_y{}^2 - B_x B''_x - B'_x{}^2} - \frac{B'_x (B'_y B''_y + B'_x B''_x) \frac{\partial \beta}{\partial r}}{(B'_y{}^2 + B'_x{}^2)^{3/2}} \\
&\quad \left. + \frac{B'_x \frac{\partial}{\partial s} \frac{\partial \beta}{\partial r} + B''_x \frac{\partial \beta}{\partial r}}{\sqrt{B'_y{}^2 + B'_x{}^2}} \right) \frac{B'_y \dot{y} + B'_x \dot{x}}{B''_y y + B''_x x - B_y B''_y - B'_y{}^2 - B_x B''_x - B'_x{}^2} \\
&\quad - \left(\frac{B'_y \frac{\partial}{\partial s} \frac{\partial \beta}{\partial r}}{B''_y y + B''_x x - B_y B''_y - B'_y{}^2 - B_x B''_x - B'_x{}^2} - \frac{B'_x \frac{\partial^2 \beta}{\partial r^2}}{\sqrt{B'_y{}^2 + B'_x{}^2}} \right) \frac{B'_x \dot{y} - B'_y \dot{x}}{\sqrt{B'_y{}^2 + B'_x{}^2}},
\end{aligned}$$

where $B'''_x = d^3 B_x / dq^3$, $B'''_y = d^3 B_y / dq^3$.

Appendix B Proofs of lemmas

In this appendix, we prove Lemmas 4.3– 4.7.

B.1 Proof of Lemma 4.3:

For any pair of $a_1 > 0$, $b_1 > 0$ and $0 < c_1 < 1$, the subsystem $\frac{d}{dt}\sigma \in \Omega_1$ is asymptotically stable because $\dot{\sigma} \in \Omega_1$ is always negative for $\sigma > 0$ and positive for $\sigma < 0$. Therefore, the subsystem is finite time stable from the negative homogeneity of the subsystem. ■

B.2 Proof of Lemma 4.4:

For any $A_{i+1} > 0$, the system satisfies the following inequality:

$$\frac{d}{dt}\sigma^{(i)} = \sigma^{(i+1)} < -A_{i+1}N_i^{\frac{r-i-1}{p}} < -A_{i+1}|\sigma^{(i)}|^{\frac{r-i-1}{r-i}}.$$

From the negative homogeneity degree, the system establishes $\sigma^{(i)} = 0$ in finite time as long as the above inequality is satisfied. Moreover, in the case of $\pi_i^+ > 0$, it is obvious that the system reaches on Ω_i before $\sigma^{(i)} = 0$. ■

B.3 Proof of Lemma 4.5:

Using the inequality

$$|\sigma^{(j)}|^{\frac{1}{r-j}} \leq (b_k \dots b_{j+1} N_k)^{\frac{1}{p}},$$

we can estimate $\mathcal{A}_i$ such that $|f_i(\sigma, \dots, \sigma^{(i)})| \leq \mathcal{A}_i N_i^{\frac{r-i-1}{p}}$ as

$$\begin{aligned} \frac{|f_i(\sigma, \dots, \sigma^{(i)})|}{N_i^{\frac{r-i-1}{p}}} &= - \left(\frac{a_i}{p} \frac{\sigma^{(i-1)} - \phi_{i-1}}{N_i^{\frac{r-i+1}{p}}} + \frac{r-i}{p} c_i \right) \frac{b_i \dot{N}_{i-1}}{N_i^{\frac{p-1}{p}}} + a_i \frac{\sigma^{(i)} - \dot{\phi}_{i-1}}{N_i^{\frac{r-i}{p}}} \\ &< \left(\frac{1}{p} \left(\frac{a_i}{b_i^{\frac{r-i+1}{p}}} + \frac{a_i a_{i-1}}{b_i^{\frac{r-i+1}{p}} b_{i-1}^{\frac{r-i+2}{p}}} + \dots + \frac{a_i a_{i-1} \dots a_1}{b_i^{\frac{r-i+1}{p}} b_{i-1}^{\frac{r-i+2}{p}} \dots b_1^{\frac{r}{p}}} \right) + \frac{r-i}{p} c_i \right) \\ &\quad \times \left(\frac{p b_i^{\frac{r-i+1}{p}}}{r-i+1} + \frac{p b_{i-1}^{\frac{r-i+2}{p}}}{r-i+2} + \dots + \frac{p b_1^{\frac{r}{p}}}{r} \right) \end{aligned}$$

$$\begin{aligned}
& + a_i + \frac{a_i a_2}{(b_i^{r-i+1} b_{i-1}^{r-i+2})^{\frac{1}{p}}} + \dots + \frac{a_i a_2 \dots a_1}{(b_i^{r-i+1} b_{i-1}^{r-i+2} \dots b_2^{r-1})^{\frac{1}{p}}} \\
& + \frac{a_i}{p N_i^{\frac{r-i}{p}}} \left(\frac{a_{i-1}}{N_{i-1}^{\frac{1}{p}}} |\sigma^{(i-2)}| \left(\frac{|\dot{N}_{i-1}|}{N_{i-1}} \right) + \frac{a_{i-1} a_{i-2}}{(N_{i-1} N_{i-2})^{\frac{1}{p}}} |\sigma^{(i-3)}| \left(\frac{|\dot{N}_{i-1}|}{N_{i-1}} + \frac{|\dot{N}_{i-2}|}{N_{i-2}} \right) + \dots \right. \\
& \left. + \frac{a_{i-1} a_{i-2} \dots a_1}{(N_{i-1} N_{i-2} \dots N_1)^{\frac{1}{p}}} |\sigma| \left(\frac{|\dot{N}_{i-1}|}{N_{i-1}} + \dots + \frac{|\dot{N}_1|}{N_1} \right) \right)
\end{aligned}$$

Moreover, we substitute the following inequality into $\dot{N}_k$ ($k = 1, \dots, i-1$) in the above inequality:

$$|\dot{N}_k| \leq \sum_{j=0}^k \frac{p}{r-j} b_{j+1} \dots b_k N_{j+1}^{\frac{r-i-1}{p}} N_j^{\frac{p-r+i}{p}}.$$

Then, we have some constant $\mathcal{A}_i > 0$ such that $|f_i|/N_i^{\frac{r-i-1}{p}} \leq \mathcal{A}_i$ for any $(\sigma, \dots, \sigma^{(i)}) \in \mathbb{R}^{i+1}$. Therefore, there exists some $\alpha_i \leq \mathcal{A}_i$ such that

$$\max_{(\sigma, \dots, \sigma^{(i)})^\top \in O_i^+} f_i \leq \alpha_i N_i^{\frac{r-i-1}{p}}.$$

■

B.4 Proof of Lemma 4.6:

We separate O_i^+ into the following areas:

A1) $\sigma^{(i)} = \pi_i^+ > 0$ and $\phi_i > 0$,

A2) $\sigma^{(i)} = \pi_i^+ > 0$ and $\phi_i \leq 0$,

A3) $\pi_i^+ \leq \sigma^{(i)} \leq 0$.

For the case of A1), g_i is transformed into

$$g_i = 1 + \frac{1}{r-i} \frac{|\phi_i|}{N_i^{\frac{r-i}{p}}} \frac{|\sigma^{(i)}|^{\frac{p-r+i}{r-i}}}{N_i^{\frac{p-r+i}{p}}} - c_i \frac{|\sigma^{(i)}|^{\frac{p-r+i}{r-i}}}{N_i^{\frac{p-r+i}{p}}} \geq 1 - c_i.$$

For the case of A2), from $\sigma^{(i)} - \phi_i = c_i N_i^{\frac{r-i}{p}}$, we obtain

$$\begin{aligned}
\sigma^{(i)} &= q c_i N_i^{\frac{r-i}{p}}, \\
\phi_i &= -(1-q) c_i N_i^{\frac{r-i}{p}},
\end{aligned}$$

for some $q \in [0, 1]$. Substituting the above equality, from $i > 0$ and $c_i < 1$, the inequality can be transformed into

$$g_i = 1 + \left(\frac{1-q}{r-i} - 1 \right) q^{\frac{p-r+i}{r-i}} c_i^{\frac{p}{r-i}} \geq 1 - c_i^{\frac{p}{r-i}} \geq 1 - c_i.$$

Moreover, for the case of A3), from $\phi_i < 0$ and $\sigma^{(i)} \leq 0$, we have

$$g_i = 1 + \left(\frac{1}{r-i} \frac{|\phi_i|}{N_i^{\frac{r-i}{p}}} + c_i \right) \frac{|\sigma^{(i)}|^{\frac{p-r+i}{r-i}}}{N_i^{\frac{p-r+i}{p}}} \geq 1.$$

Therefore, $g_i > 1 - c_i$ is always satisfied for all A1), A2) and A3), that is, for all point in O_i^+ . ■

B.5 Proof of Lemma 4.7:

From $\sigma^{(i)} \geq \pi_i^+$, we obtain

$$\sigma^{(i)} - \phi_i \geq c_i N_i^{\frac{r-i}{p}}.$$

Substituting the above inequality into $\pi_{i+1}^- \leq \sigma^{(i+1)} \leq \pi_{i+1}^+$, we obtain

$$\begin{aligned} \sigma^{(i+1)} &\leq \phi_{(i+1)} + c_{i+1} N_i^{\frac{r-i-1}{p}} \\ &\leq -a_{i+1} \frac{c_i N_i^{\frac{r-i}{p}}}{N_{i+1}^{\frac{1}{p}}} + c_{i+1} N_{i+1}^{\frac{r-i-1}{p}}. \end{aligned}$$

Denoting

$$\max_{s \in \Omega_{i+1}} s = -A_{i+1} N_i^{\frac{r-i-1}{p}},$$

we have

$$\begin{aligned} -A_{i+1} N_i^{\frac{r-i-1}{p}} &= -a_{i+1} \frac{c_i N_i^{\frac{r-i}{p}}}{(A_{i+1}^{\frac{p}{r-i-1}} + b_{i+1})^{\frac{1}{p}} N_i^{\frac{1}{p}}} + c_{i+1} (A_{i+1}^{\frac{p}{r-i-1}} + b_{i+1})^{\frac{r-i-1}{p}} N_i^{\frac{r-i-1}{p}} \\ a_{i+1} c_i N_i^{\frac{r-i}{p}} &= A_{i+1} (A_{i+1}^{\frac{p}{r-i-1}} + b_{i+1})^{\frac{1}{p}} N_i^{\frac{r-i}{p}} + c_{i+1} (A_{i+1}^{\frac{p}{r-i-1}} + b_{i+1})^{\frac{r-i}{p}} N_i^{\frac{r-i}{p}} \\ a_{i+1} &= \frac{A_{i+1}}{c_i} (A_{i+1}^{\frac{p}{r-i-1}} + b_{i+1})^{\frac{1}{p}} + \frac{c_{i+1}}{c_i} (A_{i+1}^{\frac{p}{r-i-1}} + b_{i+1})^{\frac{r-i}{p}}. \end{aligned}$$

Therefore, there exists a_{i+1} that satisfying the equality(4.12) if the system remains on the area $\sigma^{(i)} \geq \pi_i^+$. ■

Appendix C Proof of sliding sector based on ILF

Because of the unboundedness of the time derivative of the virtual input where $x' = 0$ and $x_r \neq 0$, the proof of the earlier study by [8] is incomplete. To avoid the unboundedness, we distinguish between several situations, and prove the reachability of the system in each situation. Theorem 4.2 is proved after the following lemmas.

Lemma C.1

As long as the system (4.1) remains on $x_r \in \Omega_0(x')$, there exist a upper bound of the time derivatives of the boundaries of the area $\Omega(x')$.

Proof

$Q(t, x)$ is the time derivative of the upper boundary of the area $\Omega(x')$, and is calculated as

$$\begin{aligned} Q(t, x) &= \frac{d}{dt} \left(V^{-\frac{1}{2}} K z + \kappa V^{\frac{1}{2}} \right) \\ &= \frac{\dot{V}}{2V^{\frac{1}{2}}} \left(\kappa - \frac{K}{V} \frac{R_{r-1}}{2} z \right) + \frac{K}{V^{\frac{1}{2}}} D_{r-1}(V^{-\frac{1}{2}}) \dot{x}', \end{aligned}$$

where $z = D_{r-1}(V^{-\frac{1}{2}})x'$. We have

$$\begin{aligned} Q(t, x) &= \frac{V^{\frac{1}{2}} \dot{x}'^\top D_{r-1}(V^{-\frac{1}{2}}) P z + z^\top P D_{r-1}(V^{-\frac{1}{2}}) \dot{x}'}{2 z^\top (I + R_{r-1}/2) P + P(I + R_{r-1}/2) z} \left(\kappa - \frac{K}{V} \frac{R_{r-1}}{2} z \right) \\ &\quad + \frac{K}{V^{\frac{1}{2}}} D_{r-1}(V^{-\frac{1}{2}}) \dot{x}' \\ &= \mathcal{Q}(x) D_{r-1}(V^{-\frac{1}{2}}) \dot{x}' \\ &= \mathcal{Q}(x) (V^{-\frac{1}{2}} A z + B x_r + D_{r-1}(V^{-\frac{1}{2}}) d'(t, x)). \end{aligned}$$

Therefore, the maximum absolute value is obtained by

$$\begin{aligned} |Q(t, x)| &\leq \left| \mathcal{Q}(x) (V^{-\frac{1}{2}} A z + B x_r) \right| + \|\mathcal{Q}(x)\| \|D_{r-1}(V^{-\frac{1}{2}}) d'(t, x)\| \\ &\leq \left| \mathcal{Q}(x) (V^{-\frac{1}{2}} A z + B x_r) \right| + \|\mathcal{Q}(x)\| \beta V^{\frac{1}{2}} \\ &=: Q_{max}(x). \end{aligned}$$

Because $Q_{max}(x)$ is homogeneous of degree 0, that is, $Q_{max}(x_1, x_2, \dots, x_r) \equiv Q_{max}(\lambda^r x_1, \lambda^{r-1} x_2, \dots, \lambda x_r)$ for any $\lambda > 0$. From the homogeneity, in order to check the boundedness

of $Q(t, x)$, we only have to check the boundedness of the $Q_{\max}(x)$ where $V = 1$. From $V = 1$ and $x_r \in \Omega_0(x')$, we obtain $|x_r| \leq |Kx'| + \kappa$ and

$$Q_{\max}(x) = \left| \left(\frac{(\kappa - KR_{r-1}x'/2)x'^{\top}P}{x'^{\top}P_{\mathcal{R}}x'} + K \right) (Ax' + Bx_r) \right| \\ + \beta \left\| \frac{(\kappa - KR_{r-1}x'/2)x'^{\top}P}{x'^{\top}P_{\mathcal{R}}x'} + K \right\|,$$

where $P_{\mathcal{R}} = (I + R_{r-1}/2)P + P(I + R_{r-1}/2) > 0$. Since $x'^{\top}Px' = 1$ and $P > 0$, we have $0 < 1/\sqrt{\lambda_{\max}(P)} \leq \|x'\| \leq 1/\sqrt{\lambda_{\min}(P)}$. Therefore, the time derivative of the upper boundary of the area Ω is bounded for any V . The same result is obtained for the lower boundary.

Lemma C.2

If the system (4.1) remains on the area where $x_r > \Omega_0(x')$ or $x_r < \Omega_0(x')$, then the system establishes $x_r = 0$ or $x_r \in \Omega$ in finite time under the control law such that

$$\begin{cases} \frac{d}{dt}x_r \leq -c & (x_r > \max\{\Omega(x')\}) \\ \frac{d}{dt}x_r \geq c & (x_r < \min\{\Omega(x')\}), \end{cases}$$

for some $c > 0$.

Proof

Under the control law, the system reaches $x_r = 0$ or $x_r \in \Omega$ from any $x_r(0)$ in $|x_r(0)|/c$ seconds or less.

Lemma C.3

If the system (4.1) remains on $0 \geq x_r > \min\{\Omega(x')\}$ or $0 \leq x_r < \max\{\Omega(x')\}$, then the system reaches $x_r \in \Omega$ in finite time under the control law such that

$$\begin{cases} \frac{d}{dt}x_r \leq -\mu - c & (x_r > \max\{\Omega(x')\}) \\ \frac{d}{dt}x_r \geq \mu + c & (x_r < \min\{\Omega(x')\}), \end{cases}$$

where $c > 0$ and μ is some upper bound of the time derivatives of the boundaries of the area Ω which is determined by Lemma C.1.

Proof

The following inequalities are satisfied along the trajectory of the system (4.1) as long as the system remains on $0 \geq x_r > \max\{\Omega(x')\}$ or $0 \leq x_r < \min\{\Omega(x')\}$, which is indicated by the dark gray area in Fig. 5.1:

$$\begin{cases} \frac{d}{dt}(x_r - v_{r-1}(x') - \kappa V(x')^{\frac{1}{2}}) \leq -c & (0 \geq x_r > \max\{\Omega(x')\}) \\ \frac{d}{dt}(x_r - v_{r-1}(x') + \kappa V(x')^{\frac{1}{2}}) \geq c & (0 \leq x_r < \min\{\Omega(x')\}). \end{cases}$$

Therefore the system reaches on $x_r \in \Omega$ in finite time under the control law.

Proof of Theorem 5.5. Lemma C.2 and C.3 guarantee that the system (4.1) establishes $x_r \in \Omega(x')$ in finite time under the control law:

$$\begin{cases} \frac{d}{dt}x_r < -\mu & (x_r > \max\{\Omega(x')\}) \\ \frac{d}{dt}x_r > \mu & (x_r < \min\{\Omega(x')\}). \end{cases}$$

The existence of $\mu \in \mathbb{R}$ is guaranteed by Lemma C.1. Under the above control law, once the system (4.1) reaches on $x_r \in \Omega$, the system does not leave from the area where $x_r \in \Omega$. Therefore, from Proposition 5.4, the subsystem (5.5) achieves $V = 0$ in finite time. From these, we prove that the area $x_r \in \Omega$ is a sliding sector. ■