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Radial Bargmann representation for the Fock space of type B

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Abstract

Let $\nu_{\alpha,q}$ be the probability and orthogonality measure for the q -Meixner-Pollaczek orthogonal polynomials, which has appeared in [BEH15] as the distribution of the (α, q) -Gaussian process (the Gaussian process of type B) over the (α, q) -Fock space (the Fock space of type B). The main purpose of this paper is to find the radial Bargmann representation of $\nu_{\alpha,q}$. Our main results cover not only the representation of q -Gaussian distribution by [LM95], but also of q^2 -Gaussian and symmetric free Meixner distributions on $\mathbb{R}$. In addition, non-trivial commutation relations satisfied by (α, q) -operators are presented.

Keywords: Radial Bargmann representation, deformation, Fock spaces, q -orthogonal polynomials.

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1 Introduction

Bożejko-Ejsmont-Hasebe [BEH15] considered a deformation of the (algebraic) full Fock space with two parameters α and q , namely, the (α, q) -Fock space (or the Fock space of type B) $\mathcal{F}_{\alpha,q}(H)$ over a complex Hilbert space H . The deformation with $\alpha = 0$ is equivalent to the q -deformation by Bożejko-Speicher [BS91] and Bożejko-Kümmerer-Speicher [BKS97], and the corresponding q -Bargmann-Fock space has been constructed by van Leeuwen-Maassen [LM95].

For the construction of $\mathcal{F}_{\alpha,q}(H)$, their starting point is to replace the Coxeter group of type A, that is, symmetric group S_n for the q -Fock space by the Coxeter group of type B, $\Sigma_n := \mathbb{Z}_2^n \rtimes S_n$ in (A.1) of the Appendix A. This replacement provides us a more general symmetrization operator on $H^{\otimes n}$ than that of [BS91] to define the (α, q) -inner product $\langle \cdot, \cdot \rangle_{\alpha,q}$ in (A.3). One can define annihilation $B_{\alpha,q}^-(f)$ and creation $B_{\alpha,q}^+(f)$ operators acting on $\mathcal{F}_{\alpha,q}(H)$, and the (α, q) -Gaussian process (the Gaussian process of type B) $G_{\alpha,q}(f)$ for $f \in H$ as the sum of them, $G_{\alpha,q}(f) := B_{\alpha,q}^-(f) + B_{\alpha,q}^+(f)$. It is one of their main interests to find a probability distribution $\mu_{\alpha,q,f}$ on $\mathbb{R}$ of $G_{\alpha,q}(f)$, $\|f\|_H = 1$, with respect to the vacuum state $\langle \Omega, \cdot \rangle_{\alpha,q}$. $\mathcal{F}_{\alpha,q}(H)$ equipped with $\langle \cdot, \cdot \rangle_{\alpha,q}$, $B_{\alpha,q}^-(f)$, and $B_{\alpha,q}^+(f)$ is a typical example of interacting Fock spaces in the sense of Accardi-Bożejko [AB98]. It suggests that the theory of orthogonal polynomials plays intrinsic roles in all previous works mentioned above. In fact, the measure $\mu_{\alpha,q,f}$ given in [BEH15, Theorem 3.3] is derived essentially from the orthogonality measure $\nu_{\alpha,q}$ associated with the q -Meixner-Pollaczek orthogonal polynomials $\{P_n^{(\alpha,q)}(x)\}$ for $\alpha, q \in (-1, 1)$ given by the recurrence relation,

$$\begin{cases} P_0^{(\alpha,q)}(x) = 1, & P_1^{(\alpha,q)}(x) = x, \\ xP_n^{(\alpha,q)}(x) = P_{n+1}^{(\alpha,q)}(x) + (1 + \alpha q^{n-1})[n]_q P_{n-1}^{(\alpha,q)}(x), & n \geq 1 \end{cases}$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ is the q number. However, the Bargmann representation (measure on $\mathbb{C}$) of $\nu_{\alpha,q}$ has not been obtained yet except the case of $\alpha = 0$ for $0 \leq q < 1$ [LM95], for $q = 1$ [Barg61][AKK03], for $q = 0$ [Bi97], and t -deformed cases of these [AKW16][KW14], and for $q > 1$ [Kr98].

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Therefore, the main purpose of this paper is to find the radial Bargmann representation of the probability measure $\nu_{\alpha,q}$ on $\mathbb{R}$. Our results cover the radial Bargmann representations of q -Gaussian, symmetric free Meixner (Kesten) and q^2 -Gaussian distributions on $\mathbb{R}$.

The organization of this paper will be as follows. In Section 2, we shall explain how the (α, q) -Fock space is related to the notion of one-mode interacting Fock spaces and Bargmann representation. In Section 3, the radial Bargmann representation of $\nu_{\alpha,q}$ is constructed explicitly in Theorem 3.11. In Section 4, commutation relations satisfied by one-mode (α, q) -annihilation and creation operators will be treated. In the Appendix, we shall give a minimum reference on the Coxeter group of type B extracted from [BEH15].

2 Key ideas and our purpose

Let us point out some of the keys to calculate the distribution of $G_{\alpha,q}(f)$ in [BEH15]. It is shown that a linear map, $\Phi : \text{Span}\{f^{\otimes n} \mid f \in H, n \geq 0\} \rightarrow L^2(\mathbb{R}, \mu_{\alpha,q,f})$ given by $\Phi(f^{\otimes n}) = P_n^{(\alpha\langle f, \bar{f} \rangle_H, q)}(x)$, is an isometry and a relation under $\|f\|_H = 1$,

$$\begin{aligned} G_{\alpha,q}(f)f^{\otimes n} &= (B_{\alpha,q}^+(f) + B_{\alpha,q}^-(f))(f^{\otimes n}) \\ &= f^{\otimes(n+1)} + (1 + \alpha\langle f, \bar{f} \rangle_H)q^{n-1}[n]_q f^{\otimes(n-1)}, \end{aligned}$$

is satisfied where $\bar{f}$ denotes a self-adjoint involution of $f \in H$ in (A.2). This corresponds to the three terms recursion relation satisfied by $P_n^{(\alpha\langle f, \bar{f} \rangle_H, q)}(x)$ through Φ . Then, it is proved that $\mu_{\alpha,q,f} = \nu_{\alpha\langle f, \bar{f} \rangle_H, q}$ (see $\nu_{\alpha,q}$ in (3.3)) in the sense of

$$\langle \Omega, G_{\alpha,q}(f)^n \Omega \rangle_{\alpha,q} = \int x^n \mu_{\alpha,q,f}(dx) \quad (2.1)$$

where Ω denotes the vacuum vector. Therefore, in order to get the Bargmann representation of $\nu_{\alpha\langle f, \bar{f} \rangle_H, q}$, it is enough to consider that of $\nu_{\alpha,q}$ in the sense of Definition 2.2 given later.

Since the structure mentioned above can be reduced to the one-mode analogue of (α, q) -Fock spaces, let us recall fundamental relationships between one-mode interacting Bargmann-Fock spaces and the theory of orthogonal polynomials of one variable.

Definition 2.1. Let $\{\omega_n\}_{n=0}^\infty$ with $\omega_0 := 1$ be an infinite sequence of positive real numbers and $\{\alpha_n\}_{n=0}^\infty$ be of real numbers. A one-mode interacting Bargmann-Fock space $\mathcal{B}$ is defined as $\bigoplus_{n=0}^\infty \mathbb{C}\Phi_n$ equipped with $\Phi_n := z^n/[\omega_n]!$, $[\omega_n]! := \prod_{k=0}^n \omega_k$, the inner product $\langle \Phi_m, \Phi_n \rangle_{\mathcal{B}} = \delta_{m,n}$ for all $m, n \in \mathbb{N} \cup \{0\}$, operators of creation a^+ , annihilation a^- , and conservation a° defined by

$$\begin{cases} a^+ \Phi_n := \sqrt{\omega_{n+1}} \Phi_{n+1}, & n \geq 0, \\ a^- \Phi_0 = 0, \quad a^- \Phi_n := \sqrt{\omega_n} \Phi_{n-1}, & n \geq 1, \\ a^\circ \Phi_n := \alpha_n \Phi_n, & n \geq 0. \end{cases} \quad (2.2)$$

Let $(\{\omega_n\}_{n=0}^\infty, \{\alpha_n\}_{n=0}^\infty)$ be a pair of sequences in Definition 2.1 and define a sequence of monic polynomials $\{P_n(x)\}$ recurrently by

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \alpha_0, \\ xP_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x) + \alpha_n P_n(x), \quad n \geq 1. \end{cases} \quad (2.3)$$

Then there exists a probability measure μ on $\mathbb{R}$ with finite moments of all orders such that $\{P_n(x)\}$ is the orthogonal polynomials with $\langle P_m(x), P_n(x) \rangle_{L^2(\mathbb{R}, \mu)} = \delta_{m,n} [\omega_n]!$ for all $m, n \in \mathbb{N} \cup \{0\}$. (See [Chi78][HO07], for example.)

It is easy to see that a linear map

$$U : \mathcal{B} = \bigoplus_{n=0}^\infty \mathbb{C}\Phi_n \rightarrow L^2(\mathbb{R}, \mu)$$

defined by $U\left(\sqrt{[\omega_n]!}\Phi_n\right) = P_n(x)$ is an isometry and in addition $a^+ + a^- + a^\circ = U^*XU$ is satisfied due to (2.2) and (2.3), where X represents the multiplication operator by x in $L^2(\mathbb{R}, \mu)$. This intertwining relation provides a notion of the quantum decomposition of a classical random variable X and

$$\langle \Phi_0, (a^+ + a^- + a^\circ)^n \Phi_0 \rangle_{\mathcal{B}} = \int x^n \mu(dx). \quad (2.4)$$

Therefore, if $\omega_n = (1 + \alpha q^{n-1})[n]_q$, $\alpha_n = 0$, the equality in (2.4) is a one-mode analogue of (2.1).

Now it is interesting to consider the following moment problem to realize the inner product by the integral:

Problem 1. For a given $\{\omega_n\}$ of μ , find a probability measure γ_μ satisfying the equality,

$$\int_{\mathbb{C}} \bar{z}^m z^n \gamma_\mu(d^2 z) = \delta_{m,n} [\omega_n]! \quad (2.5)$$

for all $m, n \in \mathbb{N} \cup \{0\}$.

Definition 2.2. A measure γ_μ satisfying the equality (2.5) is called a Bargmann representation (measure on $\mathbb{C}$) of a measure μ on $\mathbb{R}$.

It was proved in [Sz07] (see also [AKW16][KW14]) that if a measure μ admits any Bargmann representation, then it also admits a radial (rotation invariant) Bargmann representation

$$\gamma_\mu(d^2 z) = \frac{1}{2\pi} \lambda_{[0,2\pi)}(d\theta) \rho_\mu(dr), \quad z = re^{i\theta}, \quad r \geq 0, \quad \theta \in [0, 2\pi),$$

where $\lambda_{[0,2\pi)}$ is the Lebesgue measure on $[0, 2\pi)$. It says that the angular part takes care of orthogonality of (2.5). Therefore, Problem 1 can be transformed into the following Problem 2:

Problem 2. Find a positive radial measure ρ_μ satisfying

$$\int_0^\infty r^{2n} \rho_\mu(dr) = [\omega_n]!$$

for all $m, n \in \mathbb{N} \cup \{0\}$.

Main Purpose: We shall consider Problem 2 associated with $\omega_n = (1 + \alpha q^{n-1})[n]_q$, $\alpha_n = 0$ of $\nu_{\alpha,q}$ in Section 3. Furthermore, commutation relations satisfied by a^+, a^- acting on $\mathcal{B}$ associated with $\omega_n = (1 + \alpha q^{n-1})[n]_q$ will be presented in Section 4.

Remark 2.3. (1) One can notice that γ_μ and ρ_μ are determined only by $[\omega_n]!$. Therefore, it is enough in general for the Bargmann representation in the above sense to consider the symmetric measure μ with $\alpha_n = 0$ for all n , which implies that a° is a zero operator.

(2) If μ is symmetric, then $\alpha_n = 0$ for all n is implied. The converse statement is true if μ is determined by its moments.

3 (α, q) -Bargmann representation

3.1 q -Meixner-Pollaczek polynomials

Let us recall standard notations from q -calculus, which can be found in [GR04][KLS10] for example. The q -shifted factorials are defined by

$$(a; q)_0 := 1, \quad (a; q)_k := \prod_{\ell=1}^k (1 - aq^{\ell-1}), \quad k = 1, 2, \dots, \infty,$$

and the product of q -shifted factorials is defined by

$$(a_1, a_2; q)_k := (a_1; q)_k (a_2; q)_k, \quad k = 1, 2, \dots, \infty.$$

Remark 3.1. The q -shifted factorials are a natural extension of the Pochhammer symbol $(\cdot)_n$ because one can see that $\lim_{q \rightarrow 1} [k]_q = k$ implies

$$\lim_{q \rightarrow 1} \frac{(q^k; q)_n}{(1 - q)^n} = (k)_n, \quad (3.1)$$

where $(k)_0 := 1$, $(k)_n := k(k+1) \cdots (k+n-1)$, $n \geq 1$.

As we have mentioned, $\{P_n^{(\alpha, q)}(x)\}$ for $\alpha, q \in (-1, 1)$ is the q -Meixner-Pollaczek polynomials satisfying the recurrence relation,

$$\begin{cases} P_0^{(\alpha, q)}(x) = 1, & P_1^{(\alpha, q)}(x) = x, \\ xP_n^{(\alpha, q)}(x) = P_{n+1}^{(\alpha, q)}(x) + (1 + \alpha q^{n-1})[n]_q P_{n-1}^{(\alpha, q)}(x), & n \geq 1. \end{cases} \quad (3.2)$$

It is known in [KLS10, 14.9.2] and [BEH15, page 1781] that the orthogonality measure $\nu_{\alpha, q}$ for such polynomials has the density of the form,

$$\frac{(q, \gamma^2; q)_\infty}{2\pi} \sqrt{\frac{1-q}{4-(1-q)x^2}} \left(\frac{g(x, 1; q)g(x, -1; q)g(x, \sqrt{q}; q)g(x, -\sqrt{q}; q)}{g(x, i\gamma; q)g(x, -i\gamma; q)} \right), \quad (3.3)$$

supported on the interval $(-2/\sqrt{1-q}, 2/\sqrt{1-q})$ where

$$g(x, b; q) = \prod_{k=0}^{\infty} (1 - 4bx(1-q)^{-1/2}q^k + b^2q^{2k}),$$

and

$$\gamma = \begin{cases} \sqrt{-\alpha}, & \alpha < 0, \\ i\sqrt{\alpha}, & \alpha \geq 0. \end{cases}$$

Example 3.2. (1) If $\alpha = 0$, then q -Meixner-Pollaczek polynomials get back to the q -Hermite polynomials $H_n^{(q)}(x)$ whose orthogonality measure is the standard q -Gaussian measure on $(-2/\sqrt{1-q}, 2/\sqrt{1-q})$ given by

$$\nu_q(dx) := \frac{\sqrt{1-q}}{\pi} \sin \theta \prod_{n=1}^{\infty} (1 - q^n) |1 - q^n e^{2i\theta}|^2 dx,$$

where $x\sqrt{1-q} = 2 \cos \theta$, $\theta \in [0, \pi]$. Furthermore, one can get the standard Gaussian law as $q \rightarrow 1$, the Bernoulli law as $q \rightarrow -1$, and the standard Wigner's semi-circle law if $q = 0$. See [BKS97][BS91].

(2) The measure $\nu_{\alpha, 0}$ is the symmetric free Meixner law [An03][BB06][SY01].

(3) The measure $\nu_{q, q}$ is the q^2 -Gaussian law scaled by $\sqrt{1+q}$.

(4) If $\alpha = -q^{2\beta}$ as suggested in Remark 3.1, then the measure $\nu_{-q^{2\beta}, q}$ under a certain scaling converges to the classical symmetric Meixner law as $q \uparrow 1$,

$$\frac{2^{2\beta}}{2\pi\Gamma(2\beta)} |\Gamma(\beta + ix)|^2 dx, \quad x \in \mathbb{R}. \quad (3.4)$$

See also [KLS10, 14.9.15].

3.2 Problem

For $\alpha, q \in (-1, 1)$, we would like to know when there exists a radial measure $\rho_{\nu_{\alpha, q}}$ satisfying

$$\int_0^\infty r^{2k} \rho_{\nu_{\alpha, q}}(dr) = (-\alpha; q)_k [k]_q!, \quad k \in \mathbb{N} \cup \{0\}. \quad (3.5)$$

Here $[k]_q!$ denotes the q -factorials defined by

$$[0]_q! := 1, \quad [k]_q! := \prod_{\ell=1}^k [\ell]_q = \frac{(q; q)_k}{(1-q)^k}, \quad k \geq 1.$$

It is easy to get the inequality for $\alpha, q \in (-1, 1)$,

$$|(-\alpha; q)_k [k]_q!| \leq \left(\frac{4}{1-|q|} \right)^k, \quad k \in \mathbb{N} \cup \{0\}. \quad (3.6)$$

Due to Carleman criterion for the moment problem, this inequality implies that a radial measure $\rho_{\nu_{\alpha,q}}$ is determined uniquely by the sequence $\{(-\alpha; q)_k [k]_q!\}$.

We shall follow the procedure below to construct $\rho_{\nu_{\alpha,q}}$ in (3.5).

- (1) Recall the radial part of the q -Gaussian measure on $\mathbb{C}$ (q -Bargmann measure), $\rho_{\nu_q} = \rho_{\nu_{0,q}}$, obtained in [LM95],

$$\int_0^\infty r^{2k} \rho_{\nu_q}(dr) = [k]_q!. \quad (3.7)$$

- (2) Find a radial (possibly signed) measure $\rho_{\alpha,q}$ having the moment $(-\alpha; q)_k$.

- (3) Compute the multiplicative (Mellin) convolution $\rho_{\nu_q} \circledast \rho_{\alpha,q}$ to get $\rho_{\nu_{\alpha,q}}$.

Remark 3.3. It is known [LM95] that a radial measure ρ_{ν_q} in (3.7) does not exist for $q < 0$. However, one can see that the positivity assumption on q can be relaxed for $\rho_{\nu_{\alpha,q}}$ if $\alpha = q$. It will be discussed right after the proof of Proposition 3.6 and in Proposition 3.7.

3.3 Construction of (α, q) -radial measures

Lemma 3.4. Suppose that $\alpha \in (-1, 1)$ and $q \in [0, 1)$. Let

$$\rho_{\alpha,q} := (-\alpha; q)_\infty \sum_{n=0}^\infty \frac{(-\alpha)^n}{(q; q)_n} \delta_{q^{n/2}},$$

which is a signed measure. Then we have

$$\int_0^\infty r^{2k} \rho_{\alpha,q}(dr) = (-\alpha; q)_k, \quad k \in \mathbb{N} \cup \{0\}.$$

In particular, if taking $\alpha = -q$, then one can see $\rho_{\nu_q} = D_{(1-q)^{-1/2}}(\rho_{-q,q})$, namely,

$$\int_0^\infty r^{2k} D_{(1-q)^{-1/2}}(\rho_{-q,q})(dr) = \frac{(q; q)_k}{(1-q)^k} = [k]_q!,$$

where $D_t(\lambda)$ is the push-forward of λ by the map $x \mapsto tx$ for a measure λ on $\mathbb{R}$.

Proof. Firstly, we have

$$\int_0^\infty r^{2k} \rho_{\alpha,q}(dr) = (-\alpha; q)_\infty \sum_{n=0}^\infty \frac{(-\alpha q^k)^n}{(q; q)_n}.$$

Since Euler's formula (see [GR04, 1.3.15]),

$$\frac{1}{(a; q)_\infty} = \sum_{n=0}^\infty \frac{a^n}{(q; q)_n}, \quad (3.8)$$

is known, we replace a by $-\alpha q^k$ in (3.8) to obtain

$$\begin{aligned} \int_0^\infty r^{2k} \rho_{\alpha,q}(dr) &= \frac{(-\alpha; q)_\infty}{(-\alpha q^k; q)_\infty} \\ &= (-\alpha; q)_k. \end{aligned}$$

The proof is complete. □

Remark 3.5. (1) The last equality in proof is due to the formula

$$(a; q)_k = \frac{(a; q)_\infty}{(aq^k; q)_\infty}.$$

See [GR04, 1.2.30], for example.

(2) Euler's formula is considered as the q -analogue of exponential function e^a due to

$$\lim_{q \rightarrow 1} \frac{1}{((1-q)a; q)_n} = e^a.$$

Let

$$\begin{bmatrix} n \\ \ell \end{bmatrix}_q := \frac{[n]_q!}{[\ell]_q! [n-\ell]_q!} = \frac{(q; q)_n}{(q; q)_\ell (q; q)_{n-\ell}}$$

be the q -binomial coefficients and $h_n(z | q)$ be the Rogers-Szegö polynomials [GR04][S05] defined by

$$h_n(z | q) = \sum_{\ell=0}^n \begin{bmatrix} n \\ \ell \end{bmatrix}_q z^\ell.$$

Proposition 3.6. *Suppose that $\alpha \in (-1, 1)$ and $q \in [0, 1)$. Let*

$$\rho_{\nu_{\alpha, q}} := \begin{cases} (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q)^{-1/2} q^{n/2}}, & q > 0, \\ -\alpha \delta_0 + (1+\alpha) \delta_1, & q = 0, \end{cases} \quad (3.9)$$

which is a signed measure in general. Then we have

$$\int_0^\infty r^{2k} \rho_{\nu_{\alpha, q}}(dr) = \frac{(-\alpha, q; q)_k}{(1-q)^k} = (-\alpha; q)_k [k]_q!, \quad k \in \mathbb{N} \cup \{0\}. \quad (3.10)$$

Proof. First of all, it is easy to show (3.10) for the case $q = 0$. Therefore, let us assume $q > 0$.

One can compute the multiplicative (Mellin) convolution $\circledast$ to get $\rho_{\nu_{\alpha, q}}$ as follows:

$$\begin{aligned} \rho_{\nu_{\alpha, q}} &= \rho_{\alpha, q} \circledast D_{(1-q)^{-1/2}}(\rho_{-q, q}) \\ &= (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^n \frac{(-\alpha)^\ell q^{n-\ell}}{(q; q)_\ell (q; q)_{n-\ell}} \right) \delta_{(1-q)^{-1/2} q^{n/2}} \\ &= (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q)^{-1/2} q^{n/2}}. \end{aligned}$$

On the other hand, by Lemma 3.4, we have

$$\int_0^\infty r^{2k} D_{(1-q)^{-1/2}}(\rho_{-q, q})(dr) = \frac{(q; q)_k}{(1-q)^k} = [k]_q!.$$

Therefore, one can get

$$\begin{aligned} \int_0^\infty r^{2k} \rho_{\nu_{\alpha, q}}(dr) &= \int_0^\infty r^{2k} \rho_{\alpha, q}(dr) \int_0^\infty r^{2k} D_{(1-q)^{-1/2}}(\rho_{-q, q})(dr) \\ &= (-\alpha; q)_k [k]_q!, \quad k \in \mathbb{N} \cup \{0\}. \end{aligned}$$

□

In Proposition 3.6, we have obtained $\rho_{\nu_{\alpha, q}}$ for $\alpha \in (-1, 1)$ and $q \in (0, 1)$. Due to the term

$$\delta_{(1-q)^{-1/2} q^{n/2}} \text{ in } \rho_{\nu_{\alpha, q}},$$

it seems impossible for $q \in (-1, 0)$ to define $\rho_{\nu_{\alpha, q}}$. However, if $-1 < \alpha = q < 0$ then $\nu_{q, q}$ coincides with a scaled q^2 -Gaussian measure, and hence the Bargmann measure exists.

Proposition 3.7. Suppose $-1 < \alpha = q < 0$. We define

$$\begin{aligned}\rho_{\nu_{q,q}} &:= D_{(1+q)^{1/2}}(\rho_{\nu_{q^2}}) \\ &= (q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2; q^2)_n} \delta_{(1-q)^{-1/2}(-q)^n}.\end{aligned}\tag{3.11}$$

Then one can see

$$\int_0^\infty r^{2k} \rho_{\nu_{q,q}}(dr) = (1+q)^k [k]_{q^2}! = (-q; q)_k [k]_q!.$$

Proof. One can see by direct computations

$$\begin{aligned}(-q; q)_k [k]_q! &= \left\{ \prod_{\ell=1}^k (1 - (-q)q^{\ell-1}) \right\} \left\{ \prod_{\ell=1}^k \frac{1 - q^\ell}{1 - q} \right\} \\ &= (1+q)^k \prod_{\ell=1}^k \frac{1 - q^{2\ell}}{1 - q^2} \\ &= (1+q)^k [k]_{q^2}!.\end{aligned}$$

Thus $\rho_{\nu_{q,q}}$ can be defined as the radial measure for q^2 -Gaussian measure on $\mathbb{C}$ scaled by $(1+q)^{1/2}$, namely, $\rho_{\nu_{q,q}} = D_{(1+q)^{1/2}}(\rho_{\nu_{q^2}})$. $\square$

Remark 3.8. If we use the fact that $h_n(-1 | q) = 0$ for odd $n \geq 1$ (see proof of Lemma 3.9 below), we can extend the definition (3.9) to the case $-1 < \alpha = q < 0$. This will give an alternative way to define $\rho_{\nu_{q,q}}$ for $-1 < q < 0$, but both definitions give the same measure.

We need some properties of the Rogers-Szegő polynomials to know when the measure $\rho_{\nu_{\alpha,q}}$ becomes positive.

Lemma 3.9 ([MGH90]). Suppose that $q \in (-1, 1)$.

- (1) If $n \geq 0$ is odd, then $h_n(x | q) \geq 0$ if and only if $x \geq -1$. Moreover, the point $x = -1$ is the unique zero of $h_n(x | q)$ on $\mathbb{R}$.
- (2) If $n \geq 0$ is even, then $h_n(x | q) > 0$ for all $x \in \mathbb{R}$.

Proof. It is known that all the zeros of $h_n(z | q)$ lie on the unit circle $|z| = 1$. See [MGH90] or [S05, Theorem 1.6.11]. Note that the result was obtained for $q \in [0, 1)$, but the proof can be extended to $q \in (-1, 1)$ without any modifications.

By definition, one can see

$$\left[\begin{matrix} n \\ \ell \end{matrix} \right]_q = \frac{(1 - q^{n-\ell+1})(1 - q^{n-\ell+2}) \cdots (1 - q^n)}{(1 - q)(1 - q^2) \cdots (1 - q^\ell)} > 0,$$

and hence $h_n(1 | q) > 0$ for all $n \geq 0$. Thus, $h_n(x | q) \neq 0$ for $x \in \mathbb{R} \setminus \{-1\}$. It then suffices to show that $h_n(-1 | q) > 0$ for all even $n \geq 0$ and $h_n(-1 | q) = 0$ for all odd $n \geq 1$. The recurrence relation for the Rogers-Szegő polynomials is known to be

$$h_{n+1}(z | q) = (z + 1)h_n(z | q) - (1 - q^n)zh_{n-1}(z | q), \quad n \geq 1.\tag{3.12}$$

See [S05, 1.6.76] (note that formula (1.6.76) has an error of a sign). It is easy to see that $h_0(-1 | q) = 1 > 0$, $h_1(-1 | q) = 0$, so by induction and (3.12) one can show $h_n(-1 | q) > 0$ for all even $n \geq 0$ and $h_n(-1 | q) = 0$ for all odd $n \geq 1$. $\square$

We need the following lemma in proof of Theorem 3.11 for the non-existence part of a radial Bargmann measure.

Lemma 3.10. Let μ be a signed measure on $\mathbb{R}$ with compact support and let ν be a nonnegative measure on $\mathbb{R}$. If μ and ν have the same finite moments of all orders, then $\mu = \nu$.

Proof. We denote by m_n the moments of μ (and ν by assumption). Since μ is compactly supported, say on $[-R, R]$,

$$|m_n| = \left| \int_{[-R, R]} x^n \mu(dx) \right| \leq \|\mu\| R^n, \quad n \in \mathbb{N} \cup \{0\},$$

where $\|\mu\|$ denotes the total variation of μ . Therefore, ν is also supported on $[-R, R]$. By Weierstrass' approximation, we have

$$\int_{[-R, R]} f(x) \mu(dx) = \int_{[-R, R]} f(x) \nu(dx) \quad (3.13)$$

for all $f \in C([-R, R])$. This implies that $\mu = \nu$ since, if we use the Hahn decomposition $\mu = \mu_+ - \mu_-$, then (3.13) implies

$$\int_{[-R, R]} f(x) \mu_+(dx) = \int_{[-R, R]} f(x) (\nu + \mu_-)(dx),$$

and hence $\mu_+ = \nu + \mu_-$ as nonnegative finite measures. $\square$

In summary, the complete answer to the unique existence of a radial Bargmann representation of $\nu_{\alpha, q}$ is stated as follows:

Theorem 3.11. *Suppose that $\alpha, q \in (-1, 1)$. The probability measure $\nu_{\alpha, q}$ has a radial Bargmann representation if and only if either (i) $q \geq 0$ and $\alpha \leq q$ or (ii) $\alpha = q \neq 0$.*

In fact, the radial measure is given uniquely by

$$\rho_{\nu_{\alpha, q}} = \begin{cases} -\alpha\delta_0 + (1 + \alpha)\delta_1 & (\alpha \leq q = 0), \\ (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q)^{-1/2} q^{n/2}} & (q > 0, \alpha < q), \\ (q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2; q^2)_n} \delta_{(1-q)^{-1/2} |q|^n} & (\alpha = q \neq 0). \end{cases}$$

Proof. 1. Existence and uniqueness. If $q \in [0, 1)$ and $\alpha \leq q$, then by Proposition 3.6 and Lemma 3.9, the signed measure $\rho_{\nu_{\alpha, q}}$ is in fact a nonnegative measure and becomes the radial part of a Bargmann measure. The case $\alpha = q < 0$ was discussed in Proposition 3.7. Due to Carleman criterion for the moment problem, the inequality given in (3.6) guarantees the uniqueness of $\rho_{\nu_{\alpha, q}}$ for these cases.

2. Non-existence. (1) If $q \in (0, 1)$ and $\alpha > q$, then $\rho_{\nu_{\alpha, q}}$ is not a nonnegative measure and is really a signed measure since $h_n(-\alpha/q | q) < 0$ for odd integers $n \geq 0$ and $q > 0$ from Lemma 3.9. By Lemma 3.10, if a radial Bargmann measure exists, then it must be equal to the signed measure $\nu_{\alpha, q}$. This is a contradiction. Thus, a radial Bargmann measure does not exist.

(2) If $q = 0$ and $\alpha > q = 0$ then by (3.9) $\nu_{\alpha, 0}$ is really a signed measure, and hence by the same argument as above, a radial Bargmann measure does not exist.

(3) Let

$$\beta_k(\alpha, q) := (-\alpha; q)_k [k]_q!, \quad k \geq 0, \alpha, q \in (-1, 1).$$

Given $q < 0$ and $\alpha \neq q$, suppose that there exists a radial part of a Bargmann measure, ρ . Let ρ^2 be the push-forward of ρ by the map $x \mapsto x^2$. Then,

$$\beta_k(\alpha, q) = \int_0^\infty x^k \rho^2(dx) = \int_0^\infty x^{2k} \rho(dx). \quad (3.14)$$

By the way, by Proposition 3.6 it holds that $\beta_k(\alpha, q') = \int_0^\infty x^{2k} \rho_{\nu_{\alpha, q'}}(dx)$ for any $q' > 0$, that is,

$$\beta_k(\alpha, q') = (-\alpha, q'; q')_\infty \sum_{n=0}^{\infty} \frac{(q')^n}{(q'; q')_n} h_n(-\alpha(q')^{-1} | q') \frac{(q')^{kn}}{(1 - q')^k}, \quad q' > 0, \quad (3.15)$$

which is true even for $q' = q$ by analytic continuation.

Now let us consider the signed measure

$$\mu := (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q)^{-1} q^n}, \quad \alpha \neq q < 0,$$

supported on the points $\frac{q^n}{1-q}$ for $n = 0, 1, 2, 3, \dots$. Then by (3.15) for $q' = q$ and by (3.14),

$$\int_{\mathbb{R}} x^k \mu(dx) = \beta_k(\alpha, q) = \int_0^\infty x^k \rho^2(dx), \quad k \in \mathbb{N} \cup \{0\}.$$

By Lemma 3.10, the signed measure μ and the probability measure ρ^2 should be equal. However, the support of μ is not contained in $[0, \infty)$, and hence μ cannot be equal to ρ^2 . This is a contradiction. $\square$

Example 3.12. (1) The radial measure $\rho_{\nu_{0,q}}$ for $q \in [0, 1)$ is of the q -Bargmann [LM95].

(2) The radial measure $\rho_{\nu_{q,q}}$ for $q \in (-1, 1)$ is of the q^2 -Bargmann.

(3) $\lim_{q \uparrow 1} \rho_{\nu_{\alpha,q}}$ is of the classical Bargmann [Barg61][AKK03].

(4) Consider $\alpha = -q^{2\beta}, \beta > 0$. This choice of α is suggested by (3.1) in Remark 3.1. In fact, one can see

$$\lim_{q \uparrow 1} \frac{(1 - q^{2\beta+n-1})[n]_q}{4(1-q)} = \frac{1}{4}(n + 2\beta - 1)n.$$

This limit sequence is the Jacobi sequence of the symmetric Meixner distribution in (3.4), so that $\rho_{\nu_{-q^{2\beta},q}}$ under suitable scaling converges weakly as $q \uparrow 1$ to the radial measure with the density,

$$\frac{2\pi r}{\Gamma(2\beta)} \int_0^\infty h(r, t/4) e^{-t} t^{2\beta-1} dt$$

where

$$h(r, t) = \frac{1}{\pi t} \exp\left(-\frac{r^2}{t}\right), \quad r \in \mathbb{R}, \quad t > 0.$$

This is an integral representation of the radial density for the Bessel kernel measure, which can be also represented by the modified Bessel function [As05][As09].

(5) $\rho_{\nu_{\alpha,0}}$ for $\alpha \in (-1, 0]$ is the radial measure for the symmetric free Meixner distribution. See Remark 3.13 below.

Remark 3.13. Let μ_t be a t -deformed probability measure of a probability measure μ on $\mathbb{R}$ defined through the Cauchy transform G_μ of μ ,

$$\frac{1}{G_{\mu_t}(z)} := \frac{t}{G_\mu(z)} + (1-t)z, \quad t \geq 0,$$

examined by Bożejko-Wysoczański [BW98, BW01]. Krystek-Wojakowski [KW14] discussed the radial Bargmann representation of a t -deformed probability measure μ_t , t -Bargmann representation for short, and obtained necessary and sufficient condition for the admissibility of the representation. The t -Bargmann representation of the Kesten measure κ_t has the form,

$$\rho_{\kappa_t} = \left(1 - \frac{1}{t}\right) \delta_0 + \frac{1}{t} \delta_{\sqrt{t}}, \quad t \geq 1.$$

In [AKW16], the t -Bargmann representation of a symmetric free Meixner law $\varphi_{s,t}$ with two positive parameters s, t is treated and is admitted if and only if $t \geq 1$. In fact, one can see $\rho_{\varphi_{s,t}} = D_s(\rho_{\kappa_t})$ and hence

$$\rho_{\nu_{(1-t)/t,0}} = \rho_{\varphi_{1/\sqrt{t},t}} = D_{1/\sqrt{t}}(\rho_{\kappa_t}), \quad t \geq 1.$$

Therefore, the case (5) in Example 3.12 can be viewed as a t -Bargmann representation, too.

Furthermore, let us state the t -deformed version of Theorem 3.11 for $q \neq 0$ without proof:

Proposition 3.14. *The t -deformed version of $\rho_{\nu_{\alpha,q}}$ for $q \neq 0$ is given by*

$$\left(1 - \frac{1}{t}\right) \delta_0 + \frac{1}{t} \rho_{\nu_{\alpha,q}}, \quad t \geq 1.$$

Remark 3.15. The t -Bargmann representation of ν_q is treated in [KW14] for $q = 1$ and [AKW16] for $0 \leq q < 1$.

Before closing this section, let us give a short remark about relations with the free infinite divisibility. Many of particular examples have so far suggested that the free infinite divisibility of a probability measure implies the existence of a radial Bargmann representation. The converse is not true in general because the Askey-Wimp-Kerov distribution $\mu_{9/10}$ for instance, discussed in [BBL11], is not freely infinitely divisible, but it has a Bargmann representation with a gamma distribution as its radial measure. However, not many counterexamples have been found.

Therefore, we conjecture that the free infinite divisibility of our (α, q) -Gaussian distribution is equivalent to the existence of its radial Bargmann measure:

Conjecture. Suppose that $\alpha, q \in (-1, 1)$. The probability measure $\nu_{\alpha, q}$ is freely infinitely divisible if and only if $\alpha = q$ or $\alpha < q \geq 0$.

This conjecture is guaranteed to be true in the restricted subfamilies $\{\nu_{\alpha, 0} \mid \alpha \in (-1, 1)\}$ ([SY01, Theorem 3.2]), $\{\nu_{0, q} \mid -1 < q < 1\}$ ([ABBL10] and [AH13, Example 3.11] for the free infinite divisibility), and $\{\nu_{q, q} \mid q \in (-1, 1)\}$ (all measures in this family are freely infinitely divisible since they are q^2 -Gaussians).

4 Commutation relations among one-mode (α, q) -operators

Definition 4.1. Suppose that $\alpha, q \in (-1, 1)$ and f is analytic on $\mathbb{C}$.

(1) Let Z be the multiplication operator defined by

$$(Zf)(z) := zf(z).$$

(2) Let D_q be the Jackson derivative given by

$$(D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

(3) The α -deformed Jackson derivative is given as

$$D_{\alpha, q} := \begin{cases} D_q + \alpha q^{2N} D_{1/q}, & q \neq 0, \\ D_0 + \alpha \frac{d}{dz} \Big|_0, & q = 0, \end{cases}$$

where N is the number operator. For $q \neq 0$, we can also write

$$D_{\alpha, q} = D_q + \frac{\alpha}{q^2} D_{1/q} q^{2N}.$$

Remark 4.2. It is easy to check that the α -deformed Jackson derivative is equivalently defined as

$$(D_{\alpha, q} f)(z) = (D_q f)(z) + \alpha (D_{1/q} f)(q^2 z), \quad q \neq 0.$$

For example, if $f(z) = z^n$, $(D_{\alpha, q} f)(z) = (1 + \alpha q^{n-1})[n]_q z^{n-1}$ holds. In fact, the α -deformed Jackson derivative is an analogue of the operator in [BEH15, Theorem 2.5].

Then, one can realize one-mode analogue of (α, q) -operators on an appropriate domain of the one-mode interacting Bargmann-Fock space $\mathcal{B}$ with $\omega_n = (1 + \alpha q^{n-1})[n]_q$ and $\alpha_n = 0$ by

$$a^+ := Z, \quad a^- := D_{\alpha, q}, \quad \text{and} \quad \Phi_n := \frac{z^n}{\sqrt{[\omega_n]!}}.$$

In fact, it is easy to check that

$$\begin{cases} a^+ \Phi_n = \sqrt{\omega_{n+1}} \Phi_{n+1}, \\ a^- \Phi_n = \sqrt{\omega_n} \Phi_{n-1}, \end{cases}$$

hold and the q -commutation relation, one-mode analogue of (A.4),

$$\begin{aligned}[a^-, a^+]_q \Phi_n &:= (a^- a^+ - q a^+ a^-) \Phi_n \\ &= (I + \alpha q^{2N}) \Phi_n,\end{aligned}$$

is satisfied. Let us put $M_{\alpha,q} = I + \alpha q^{2N}$ and then one can get the expression,

$$M_{\alpha,q} = (1 + \alpha)I - \alpha(1 - q^2)ZD_{q^2},$$

due to $(ZD_{q^2})\Phi_n = [n]_{q^2}\Phi_n$.

Therefore one can obtain the following

Theorem 4.3. *Suppose $\alpha \in (-1, 1)$ and $q \in (-1, 1)$. Then the following are satisfied.*

- (1) $[a^-, a^+]_q = M_{\alpha,q}$, $[a^-, M_{\alpha,q}]_{q^2} = (1 - q^2)a^-$, $[M_{\alpha,q}, a^+]_{q^2} = (1 - q^2)a^+$.
- (2) $M_{\alpha,q} = (1 + \alpha)I - \alpha(1 - q^2)ZD_{q^2}$.
- (3) In particular, if $\alpha = q$, then one can obtain a more refined relation, $[a^-, a^+]_{q^2} = (1 + q)I$.

Example 4.4. (1) $\alpha = 0$ implies $[a^-, a^+]_q = I$. Hence $M_{0,q} = I$ commutes with both a^+ and a^- ,

$$[a^-, M_{0,q}]_1 = [M_{0,q}, a^+]_1 = 0.$$

Therefore, the case $\alpha \neq 0$ provides non-trivial commutation relations.

- (2) If $\alpha = -q^{2\beta}$ for $\beta > 0$, then the limiting case of the scaled operator is obtained as

$$\lim_{q \uparrow 1} \frac{M_{-q^{2\beta}, q}}{1 - q^2} = \lim_{q \uparrow 1} \frac{I - q^{2\beta} q^{2N}}{1 - q^2} = N + \beta.$$

Moreover, let us consider the scaled operators,

$$A^\pm := \lim_{q \uparrow 1} \frac{a^\pm}{\sqrt{1 - q^2}}.$$

Then one can get

$$[A^-, A^+]_1 = N + \beta$$

and hence

$$[A^-, N]_1 = A^-, [N, A^+]_1 = A^+.$$

It should be noted that these are the commutation relations for the classical Meixner-Pollaczek polynomials with respect to the symmetric Meixner distribution in (3.4). See [As08].

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A Appendix

Let Σ_n be the set of bijections σ of the $2n$ points $\{\pm 1, \pm 2, \dots, \pm n\}$ with $\sigma(-k) = -\sigma(k)$. Equipped with the composition operation as a product, Σ_n becomes what is called a Coxeter group of type B. It is generated by $\pi_0 := (1, -1)$ and $\pi_i := (i, i+1)$, $1 \leq i \leq n-1$, which satisfy the generalized braid relations

$$\begin{cases} \pi_i^2 = e, & 0 \leq i \leq n-1, \\ (\pi_0 \pi_1)^4 = (\pi_i \pi_{i+1})^3 = e, & 1 \leq i \leq n-1, \\ (\pi_i \pi_j)^2 = e, & |i - j| \geq 2, 0 \leq i, j \leq n-1. \end{cases} \quad (\text{A.1})$$

An element $\sigma \in \Sigma_n$ expresses an irreducible form,

$$\sigma = \pi_{i_1} \cdots \pi_{i_k}, \quad 0 \leq i_1, \dots, i_k \leq n-1,$$

and in this case

$$\begin{aligned} \ell_1(\sigma) &:= \text{the number of } \pi_0 \text{ in } \sigma, \\ \ell_2(\sigma) &:= \text{the number of } \pi_i, \quad 1 \leq i \leq n-1, \text{ in } \sigma \end{aligned}$$

are well defined. Let H be a separable Hilbert space. For a given self-adjoint involution $f \mapsto \bar{f}$ for $f \in H$, an action of Σ_n on $H^{\otimes n}$ is defined by

$$\begin{cases} \pi_0(f_1 \otimes \cdots \otimes f_n) = \bar{f}_1 \otimes f_2 \otimes \cdots \otimes f_n, & n \geq 1, \\ \pi_i(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes f_i \otimes f_{i+2} \otimes \cdots \otimes f_n, & n \geq 2, \quad 1 \leq i \leq n-1. \end{cases} \quad (\text{A.2})$$

The (α, q) -inner product on the full Fock space $\mathcal{F}(H)$ is defined by

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{\alpha, q} := \delta_{m, n} \sum_{\sigma \in \Sigma_n} \alpha^{\ell_1(\sigma)} q^{\ell_2(\sigma)} \prod_{j=1}^n \langle f_j, g_{\sigma(j)} \rangle_H, \quad \alpha, q \in (-1, 1) \quad (\text{A.3})$$

with conventions $0^0 = 1$ and $g_{-k} = \overline{g_k}$, $k = 1, 2, \dots, n$. For example, if one may define the involution as $\bar{f} := -f$, then $g_{-k} = -g_k$. Equipped with this inner product the full Fock space $\mathcal{F}(H)$ is denoted by $\mathcal{F}_{\alpha, q}(H)$ to emphasize on the dependence of the inner product on α, q .

The (α, q) -creation operator $B_{\alpha, q}^+(f)$ is the usual left creation operator on the full Fock space, and the (α, q) -annihilation operator $B_{\alpha, q}^-(f)$ is its adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_{\alpha, q}$. They satisfy the commutation relation

$$B_{\alpha, q}^-(f)B_{\alpha, q}^+(g) - qB_{\alpha, q}^+(g)B_{\alpha, q}^-(f) = \langle f, g \rangle_H I + \alpha \langle \bar{f}, g \rangle_H q^{2N}, \quad f, g \in H. \quad (\text{A.4})$$

The readers can consult [BEH15] for details.

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